Combining transformation of graphs with solutions to absolute value inequalities

Ryan D. Fox
Belmont University

ARTICLE

Abstract: A transformation approach can support students' solving linear inequalities involving absolute value. In particular, the horizontal dilations/compressions and translations of graphical representations of distances from zero along a number line are important tools to emphasize a visual representation of the solutions to absolute value inequalities.

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I must admit to teaching a terrible mnemonic device to solve the inequalities in Figure 1. My high school and college algebra students heard me say the phrases "greatOR than" or "less thAND" to associate the conjunctions needed in the solution. An example of the solution paths can be found in Figure 1.

In the first example (Figure 1, left), because the inequality was less than 17, I would overdramatize to tell students the solution was the intersection of less "thand" the positive value and greater than the negative value: in other words, look for one answer to satisfy two inequalities. For the second example (Figure 1, right), the solution was the union of values "greator" than a larger value or less than a smaller value. My suggestion was to look for two distinct branches in the solution set: greater than the larger value or less than the smaller value. Come test time, however, mnemonics got muddled and student work showed a missing connection between silly names and appropriate solutions.

$$|3x-5|<17$$
$$3x-5<17 \ and \ 3x-5>-17$$
$$3x<22 \ and \ 3x>-12$$
$$x<\frac{22}{3} \ and \ x>-4$$

$$|2x+7|\geq 11$$
$$2x+7\geq 11 \ or \ 2x+7\leq -11$$
$$2x\geq 4 \ or \ 2x\leq -18$$
$$x\geq 2 \ or \ x\leq -9$$

Figure 1. Two absolute value inequalities and their solutions.

Missing from my mnemonic approaches is a focus on conceptual understandings of absolute value and solutions to linear inequalities. A powerful way to think about the absolute value of a number is as a measure of its distance from zero. I use this as a starting point.

In fulfillment of the Purpose of Mathematics statement "Mathematics is the human activity of reasoning with number and shape, in concert with the logical and symbolic artifacts that people develop and apply in their mathematical activity" (Colorado Department of Education [CDE], 2020, p. 4), I connect absolute value to number line representations. If the absolute value of an expression is less than some positive value, then the expression represents all values closer to zero than the given value. Consequently, an absolute value expression greater than some positive value represents the values further away from zero than the given value. To solve absolute value inequalities, I apply the concept of transformations of graphs of functions to distances from zero on a single number line.

To refresh, going from $y=f(x)$ to $y=f(cx)$, graphs undergo horizontal compressions when $c>1$ and dilations when $c<1$. The graph of $y=f(x)=x^3$ contains (2,8); in $y=f(2x)=8x^3$, $y=8$ corresponds to $x=1$. The value of inputs doubled in the new function, the x-coordinates in the new graph are half the value. In $y=f\left(\frac{1}{3}x\right)=\frac{1}{27}x^3$, $y=8$ corresponds to $x=6$; $c=\frac{1}{3}$ tripled x-values for equivalent y-values.

A FIRST EXAMPLE

In $|3x-5|<17$, the absolute value suggests the set of all points inside of 17 units away from the zero, shown in Figure 2. Think of this first step as a parent graph for the upcoming transformations.

Figure 2. Number line representation of |x|<17, created on Desmos (https://www.desmos.com/calculator/dezoto9tsm).

By factoring the expression inside the absolute value, $3x - 5$ becomes $3\left(x - \frac{5}{3}\right)$. Using knowledge of transformations, the constant multiple of 3 suggests a horizontal compression of by a factor of 3. To transform the interval, compress the distances from zero by a factor of 3. Thus, as shown in Figure 3, the interval becomes $(-\frac{17}{3}, \frac{17}{3})$. All points that satisfy $3|x| < 17$ are those points within $\frac{17}{3}$ units of zero. The first transformation is complete!

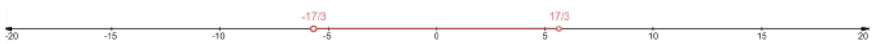

Figure 3. Number line representation of |3x|<17.

The expression inside the parentheses of the factored expression $3\left(x - \frac{5}{3}\right)$ suggests a horizontal shift by $\frac{5}{3}$ units. The interval shifts right by $\frac{5}{3}$ units, to become $(-4, \frac{22}{3})$; the points in the desired solution set are $\frac{5}{3}$ units to the right of the previous intermediate step. An image of the solution set is shown in Figure 4.

Figure 4. Number line representation of $\left|3\left(x - \frac{5}{3}\right)\right| < 17$.

Any point in the interval $\left(-4, \frac{22}{3}\right)$ would be a solution to the inequality $\left|3\left(x - \frac{5}{3}\right)\right| < 17$, matching the solution in the symbolic approach shown in Figure 1.

AN EXAMPLE GOING THE OTHER DIRECTION

In the second example ($|2x + 7| \geq 11$), the absolute value suggests the set of all points that are at least 11 units away from zero. The illustration (Figure 5) represents the points along a number line satisfying this distance requirement.

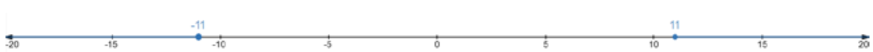

Figure 5. Number line representation of |x|≥11, acting as a parent graph.

By factoring the expression inside the absolute value, $2x + 7$ becomes $2\left(x + \frac{7}{2}\right)$. By the properties of transformations, the transformation from $f(x)$ to $f(2x)$ is a horizontal compression by a factor of 2. Instead of the distance from zero being at least 11 units, now the distance from zero is at least $\frac{11}{2}$ units, as shown in Figure 6. The endpoints have been compressed from -11 and 11 to -5.5 and 5.5, the latter values half the values of the former.

Figure 6. Number line representation of $|2x| \geq 11$.

The other transformation suggested in the factored expression is a horizontal displacement of $\frac{7}{2}$ units to the left. The left endpoint before the displacement was at $-\frac{11}{2}$; with the displacement, the left endpoint is now at -9. The right endpoint before the displacement was at $\frac{11}{2}$; with the displacement, the right endpoint is at 2. The solution set is represented visually in Figure 7.

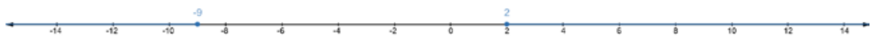

Figure 7. Number line representation of $\left|2\left(x+\frac{7}{2}\right)\right| \geq 11|$.

Again, because all transformations have been completed, the number line in Figure 7 represents the final answer pictorially: the solution set is $(-\infty, -9] \cup [2, \infty)$. This matches the solution in the symbolic approach shown in Figure 1.

CONCLUDING REMARKS

I now have better explanations to absolute-value inequality solutions than memory tricks! I see additional explanations for presenting challenging concepts. I am glad to connect multiple topics in one problem. Linking number line representations to symbolic manipulations can make procedures more meaningful.

This approach addresses a Coherence Connection: "Students develop a notion of naturally occurring families of functions that deserve particular attention … absolute value functions are good contexts for getting a sense of the effects of many of these transformations" (CDE, 2020, p. 161). Students start with an absolute value expression representing a distance from zero and apply transformations to move the solution set along the number line.

Rather than relying on memory tricks, I am excited to apply concepts in different ways to lead to a deeper understanding. In separate presentations I gave to teachers and math majors, the response has been positive and receptive. I invite you to let me know how this approach might and did work in your class!

REFERENCES

Colorado Department of Education. (2020). *Colorado Academic Standards: Mathematics*. https://www.cde.state.co.us/comath/2020cas-ma-p12

Author: Ryan D. Fox is an Associate Professor of Mathematics Education at Belmont University. He teaches content courses for pre-service teachers and general education mathematics courses. His research focuses on the mathematical knowledge useful to teach these courses. Ryan can be reached at ryan.fox@belmont.edu and is on Twitter at @drfoxatbu.

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