Building logarithms as functions

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ARTICLE

Abstract: Students traditionally struggle to conceptualize logarithms as families of functions in their own right. I present an approach to support students' understanding of logarithmic functions. The common properties of logarithms become defining characteristics, which complements the traditional approach of studying logarithms as the inverses of exponentials.

Keywords: logarithms, conceptual understanding, function

I describe a possible approach to building logarithmic functions as a family of functions with their own defining characteristics. My approach follows the historical development of logarithms, as objects in their own right, which were later connected to exponential functions (Maor, 1994).

Typically, curricular materials present logarithmic functions after exponentials, where logarithms are defined as inverses of exponentials (e.g., Felder, 2014). This does support students' understanding of exponential functions and function inverses. Unfortunately, students can be confused about logarithms as a family of functions unto themselves (e.g., Kenney & Kastberg, 2013). This can extend to courses outside of mathematics. For example, in chemistry exponential functions are more useful as the inverses of logarithms, demonstrated by the phrase "antilog" instead of "exponentials" (e.g., Flowers, et al., 2015).

The approach I describe has three parts. First, students will generate lists according to geometric and arithmetic rules. Second, they investigate patterns within and across lists. Third, students justify and generalize patterns to create the defining characteristics of logarithmic functions.

GETTING STARTED

To begin, generate two lists each with 10-15 values. List 'G' (for "geometric") beginning with 1 and created by a common multiplier, and List 'A' (for "arithmetic") beginning with 0 and created by adding the same value. Be creative with the values chosen to generate each list (e.g., whole numbers, rational numbers, irrational numbers). Then, arrange the lists as a table, such as in Figure 1.

List G	List A
1	0
3	7
9	14
27	21
81	28
243	35
729	42
2187	49
6561	56
19683	63
59049	70
177147	77
531441	84

Figure 1.Example where List G uses a multiplier of 3 and List A uses a constant of 7.

Any positive value other than 1 may be used as the multiplier for List G and any value other than 0 for List A. Using a spreadsheet to produce tables automatically is also effective.

PATTERNS WITHIN AND ACROSS LISTS

Now investigate patterns occurring within and across lists. A goal for students is to investigate and reflect on patterns that emerge across the two lists.

Sample prompts for exploring within list patterns for List G include:

- Is the statement, “the product of two values from List G is also a value in List G” always, sometimes, or never true?
- Is the statement, “the quotient of two values from List G is also a value in List G” always, sometimes, or never true?
- Are the statements always, sometimes, or never true for your list alone, or does the truth-value depend on how List G is generated?

These prompts are intended to help students notice that products and quotients of values in List G also appear as elements in List G, even for non-sequential values. In my experience, students usually discuss the common multiplier as a factor of all values in List G to explain this pattern.

Investigate List A with similar prompts, replacing *product* and *quotient* with *sum* and *difference*, respectively. The purpose for these prompts is to become familiar with each list and is intended to be short because patterns emerging across lists are at the foundation of this approach to understanding logarithms. Additionally, discussing prompts about patterns within each list invites students to investigate patterns that emerge from lists of their own creation. Such activity can promote a sense of ownership, to support students as they make sense of structure (Mathematical Practice 7), and construct arguments and critique the reasoning of others (Mathematical Practice 3) (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010).

Next, investigate patterns that connect changes in List G with changes in List A. This can help students notice simultaneous changes in both lists:

- Select two values of List G and locate their product. Then, look at the values in List A that correspond to the three values in List G. What do you notice?
- Discuss patterns you find with others. What patterns work for any list? What patterns work for just your list?

In my experience, students typically describe patterns as, "products/quotients in List G align with sums/differences in List A." Figure 2 demonstrates this pattern by highlighting three values in List G and the corresponding values in List A. Notice that the product of the first two highlighted values in List G is the third highlighted value and that the corresponding values in List A sum to the third highlighted value in List A.

List G	List A
1	0
3	7
9	14
27	21
81	28
243	35
729	42
2187	49
6561	56
19683	63
59049	70
177147	77
531441	84

Figure 2. Example demonstrating that products in List G correspond with sums in List A.

EXPLAINING PATTERNS

After describing patterns between the two lists in words, it is important to promote mathematical justification. When students justify and explain the patterns they notice, it can support their understanding of structure and promotes their generalizing.

So, why do products and quotients in List G correspond to sums and differences in List A? The reason is embedded within the structure of each list. List G can be generalized as the powers of the multiplier, r, where $r^0 = 1$ holds for all non-zero multipliers. Similarly, List A can be generalized as coefficients of the value being added, n, where $0n = 0$.

Figure 3 highlights corresponding rows to demonstrate how products of values in List G correspond with sums of values in List A. This pattern works because List G counts according to exponents while List A counts by coefficients, and the exponents and coefficients are aligned.

List G	List A
$r^0 = 1$	$0n = 0$
r^1	$1n$
r^2	$2n$
r^3	$3n$
r^4	$4n$
r^5	$5n$
r^6	$6n$
r^7	$7n$
r^8	$8n$
r^9	$9n$
r^{10}	$10n$
$\cdots$	$\cdots$

Figure 3. Generalization of all lists, depicting the structure of such lists.

DELVING DEEPER

Expressing the patterns previously established using function notation can help students to understand logarithmic functions as *functions* unto themselves. Figure 4 presents a function table, where the property $f(M \cdot N) = f(M) + f(N)$ is highlighted. This equation is a mathematical re-statement of the pattern that "products get mapped to sums." Similarly, the pattern for quotients can be expressed as $f\left(\frac{M}{N}\right) = f(M) - f(N)$ in function notation.

	x	$f(x)$	
	$r^0 = 1$	$0n = 0$	
	r^1	$1n$	
	r^2	$2n$	
$M \rightarrow$	r^3	$3n$	$\leftarrow f(M)$
	r^4	$4n$	
	r^5	$5n$	
	r^6	$6n$	
$N \rightarrow$	r^7	$7n$	$\leftarrow f(N)$
	r^8	$8n$	
	r^9	$9n$	
$M \cdot N \rightarrow$	r^{10}	$10n$	$\leftarrow f(M \cdot N)$
	$\cdots$	$\cdots$	

	x	$f(x)$	
	$r^0 = 1$	$0n = 0$	
	r^1	$1n$	
$M/N \rightarrow$	r^2	$2n$	$\leftarrow f(M/N)$
	r^3	$3n$	
	r^4	$4n$	
	r^5	$5n$	
	r^6	$6n$	
$N \rightarrow$	r^7	$7n$	$\leftarrow f(N)$
	r^8	$8n$	
$M \rightarrow$	r^9	$9n$	$\leftarrow f(M)$
	r^{10}	$10n$	
	$\cdots$	$\cdots$	

Figure 4. Function table representation of the generalized lists from Figure 3 depicting the product-to-sums (top) and quotient-to-difference (bottom) patterns.

Further exploration can yield the property that exponents are mapped to coefficients $f(M^k) = k \cdot f(M)$. In fact, this property emerges in at least two distinct ways. First, Figure 4 shows that $f(r^1) = 1n$ or equivalently, $f(r) = n$ which can be extended to $f(r^k) = k \cdot n = k \cdot f(r)$. Second, taking any arbitrary input value, *M*, which is a power of *r*, it is possible to show that $f(M^k) = f\left((r^j)^k\right) = k \cdot f(r^j) = k \cdot j \cdot f(r) = (kj) \cdot n$. The second case shows (kj) is both the exponent on *r* in the input and the coefficient on *n* in the output, which means that $(r^j)^k$ and $(kj) \cdot n$ would both appear in the kj^{th} row of the generalized table.

A FAMILY OF FUNCTIONS

Finally, "logarithmic functions" are a family of functions with the properties:

- Products map to sums: $f(M \cdot N) = f(M) + f(N)$,
- Quotients map to differences: $f\left(\frac{M}{N}\right) = f(M) - f(N)$, and
- Exponents map to coefficients: $f(M^k) = k \cdot f(M)$

It is possible to rephrase the properties uncovered across the two lists using more common notation for logarithms (e.g., $\log(M \cdot N) = \log(M) + \log(N)$). This approach can lend itself to additional discussions, such as:

- Consider the domain and range of logarithmic functions, including conversations about why logarithms can produce negative outputs but not take negative values as inputs.
- Explore why the constant multiplier cannot be 1 or 0 and the implications for properties of logarithms.

CONCLUSION

Developing logarithmic functions based on this approach establishes logarithms as functions with certain properties that emerge when juxtaposing multiplicative change with additive change. That is, the properties become the defining characteristics of logarithmic functions.

I recommend implementing this approach in conjunction with traditional approaches that make connections to exponential functions. Alternative approaches to logarithms may in fact strengthen connection-making between properties of exponential and logarithmic functions that could also further support students' understanding of what it means for functions to be inverses.

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