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Mathematicians' Evolving Personal Arguments: Ideas That Move Proof Constructions Forward

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MATHEMATICIANS' EVOLVING PERSONAL ARGUMENTS: IDEAS THAT MOVE PROOF CONSTRUCTIONS FORWARD

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ABSTRACT

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This research is an investigation into the ideas professional mathematicians find useful in developing mathematical proofs. Specifically, this research uses the construct personal argument to describe the ideas and thoughts the individual deems relevant to making progress in proving the statement. The research looked to describe the ideas that mathematicians integrated into their personal arguments, the context surrounding the development of these ideas in terms of Dewey's theories of inquiry and instrumentalism, and how the mathematicians used these ideas as their arguments evolved toward a completed proof.

Three research mathematicians with multiple years of experience teaching real analysis completed tasks in real analysis while thinking aloud in interview and independent settings recorded with video and Livescribe technology. Follow-up interviews were also conducted. Data were analyzed for ideas that participants found useful. Toulmin argumentation diagrams were implemented to describe the evolving arguments, and Dewey's inquiry framework helped to describe the context surrounding the development of the ideas. Descriptive stories were written for each participant's work on each task documenting the argument evolution. Open, iterative coding of each idea, problem encountered, and tool was conducted. Patterns, categories, and themes across participants and tasks were identified.

The mathematicians developed ideas that moved their personal arguments forward that were grouped into three categories according to their functionality: ideas that focus and configure, ideas that connect and justify, and monitoring ideas. Within these three categories were ideas in fifteen sub-types. The ideas emerged through the mathematicians' purposeful recognition of problems to be solved as well as reflective and evaluative actions to solve them. This research implicates that using the full Toulmin model for investigating the process of creating mathematical proof since the modal qualifiers evolve to become absolute as the warrants shift to become based on deductive reasoning. In the instruction of undergraduate students, this work supports teaching content in conjunction with proof techniques and heuristic strategies for problem solving and recommends engaging students in discourse situations that would motivate moving their informal arguments into deductive proofs.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Background of the Problem

In many undergraduate mathematics courses, having students prove statements and understand proofs of statements is a means of conveying information and analyzing student understanding. Both the National Council of Teachers of Mathematics Standards (NCTM; 2000) and the Common Core Standards for Mathematical Practice (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) documents highlight the importance of the development of reasoning and argumentation in students even prior to their enrollment at the university level. The NCTM standards require students to develop and evaluate mathematical arguments and proofs and select and use various types of reasoning methods and proofs. The Common Core Standards for Mathematical Practice require that mathematically proficient students "construct viable arguments and critique the reasoning of others" (p. 6). The ability to construct and understand mathematical proof is instrumental to success in studying mathematics. However, it is well documented that secondary students, undergraduate mathematics majors, and pre-service and in-service mathematics teachers struggle with understanding and constructing viable mathematical arguments (e.g., Balacheff, 1988; Harel & Sowder, 1998; Healy & Hoyles, 2000; Knuth, 2002; Weber & Alcock, 2004).

Undergraduate students learn mathematical proof construction within the context of a mathematics content course or in an Introduction to Proofs course. In an introduction to proofs course, students' struggles with mathematical proof can be complicated by their lack of content knowledge (Tall & Vinner, 1981; Weber, 2001). A challenge for instructors in a content course is that students come into the course with different levels of experience with proofs (Brandell, Hemmi, & Thunberg, 2008), and instructors need to balance instruction in new mathematical content as well as instruction in what mathematical proof is, techniques for constructing mathematical proof, and developing students' abilities to formulate mathematical arguments. More and more, universities are offering an introduction to proofs course as a means of supporting students in learning to construct and analyze proofs. Traditionally in these courses, the instructor lectures about proving techniques and assigns proof construction tasks for homework; the students take notes and try to recreate proofs based on the techniques taught. These techniques may not be effective (Selden & Selden, 2008). Harel and Sowder (2007) indicated that proof courses must also incorporate experiences that would constitute an intellectual need for attending to definitions and other ways of thinking associated with proof construction. Instructors may facilitate more of an inquiry-based classroom, but they must be mindful of the types of scaffolding for their choices for instructional interventions (Blanton, Stylianou, & David, 2009; Selden & Selden, 2008; Smith, Nichols, Yoo, & Oehler, 2009). Harel and Sowder called for a more comprehensive perspective on the learning of teaching of proofs that incorporates mathematical, historical-epistemological, cognitive, sociological, and instructional factors. They stated more research is needed to characterize instructional practices conducive to students formulating an understanding of proof that “is consistent with that shared and practiced by mathematicians of today” (p. 47), indicating the importance of understanding the practices of mathematicians--those who have great experience and abilities in constructing mathematical proof.

Studying mathematicians to describe the practices of the mathematics community (Inglis, Mejia-Ramos, & Simpson, 2007; Weber, 2008; Weber & Mejia-Ramos, 2011), to understand the processes involved with certain mathematical activities, to describe the experience of understanding mathematics (Sfard, 1994), to inform and construct frameworks for use with analyzing student work (Engelke, 2007; Raman, 2003), and to guide the design of effective instruction are common practices in the mathematics education community. Previous researchers have observed that mathematicians and graduate students in mathematics construct proof using both purely formal reasoning and also constructions that are accompanied by informal reasoning (Alcock & Inglis, 2008; Raman, 2003; Weber & Alcock, 2004). This informal reasoning may include the exploration of examples, the rephrasing of the definition of a concept, or other instantiations of concepts. Mathematicians have been shown to use examples for varying but specific purposes including developing understanding, testing conjectures, generating counterexamples and ideas for how to prove the statement (Alcock, 2004, 2008; Alcock & Inglis, 2008). When mathematicians use informal reasoning, they are able to connect their informal reasoning to the formal definitions and concepts, linking their private arguments to a public articulation of an argument (Raman, 2003; Weber & Alcock, 2004). Tall et al. (2012) described that a proof for professional mathematicians as “involves thinking about new situations, focusing on significant aspects, using previous knowledge to put new ideas together in new ways, consider relationships, make conjectures, formulate definitions as necessary and to build a valid argument" (p. 15). While the term "proof" can describe the written product that serves as the mathematician's way to "*display the mathematical machinery* for solving problems and to *justify* that a proposed solution to a problem is indeed a solution" (Rav, 1999, p. 13, italics in original), the process of constructing a proof involves more than the writing of logically valid deductions (Aberdein, 2009). The construction involves informal arguments to find methods to attack the problem as well as incomplete proof sketches (Aberdein, 2009). The construction process may proceed in non-linear stages including the exploration of a problem, the estimation of the truth of the conjecture, and the justification of the statement estimated to be true (Mejia-Ramos & Inglis, 2009). For mathematicians, there is reflection, reorganization of ideas and reasoning that "fill in the gaps" so a proof will emerge (Twomey Fosnot & Jacob, 2009).

I chose to focus on these emergences of proof with the view of the proving process as an evolving argument. A *personal argument* is an evolving, graded (Lakoff, 1987) subset of one's statement image (Selden & Selden, 1995) of the proof situation. The argument is graded in that various elements such as pictures, theorems, statements, algorithms, logical premises and deductions, and so forth may not exclusively be seen to be pertinent to the argument or exterior to the argument; there is a gradation that depends on the degree to which the individual anticipates the ideas do or can "do work" in moving the argument forward. The idea may not be made explicit in the final write-up of the mathematician's proof, but it serves to help the prover get a handle on the mathematics or on how to communicate the mathematical ideas in a logical, mathematically acceptable manner.

A prover determines if an idea moves an argument forward; he or she discerns if an idea is functional as a means of resolving a certain situation. These ideas may include those that convince the prover of the statement's validity, provide insight as to why the statement is true or cannot be true, give insight into ways to communicate a formal proof, and others. While ideas may grant "aha" moments to the prover, they may also include developing a sense of what the statement means, a sense of the implications of the statements, an understanding of the structures, a sense that a line of inquiry will not be fruitful, or a feeling that the actions taken are appropriate and fitting with the other elements in the situation.

Aspects of the statement image such as pictures created and observations and inferences drawn may prove to be more and less central to one's personal argument as the argument evolves. What was extraneous to the personal argument may become more central, and what was once central to one's personal argument may move to the periphery. For example, certain pictures and examples may have been critical to the prover's self-conviction as to why the statement must be true, but it may be the case that the prover abandons these informal representations and the insights gleaned from them when moving to write a formal proof. He or she may instead produce an argument using symbolic manipulations of a general, algebraic instantiation of the definition.

Some ideas that have an impact on the proving process have been identified with varying degrees of specificity by mathematicians and mathematics education researchers. For example, Byers (2007) speaks of the mathematical idea in generality as the answer to the question, “what’s really going on here?” Raman, Sandefur, Birky, Campbell, and Somers (2009) observed critical moments in the proving process in which there were opportunities for a proof to move forward. They identified three moments, the attainment of a key idea (an idea that gives the prover a sense of structural relationship that indicates why the statement is true), the discovery of some technical handles (ways of manipulating or making use of the structural relationship that can support the communication of the key idea), and the culmination of the argument into a correct proof (Raman et al., 2009; Sandefur, Mason, Stylianides, & Watson, 2012). Raman et al. noted that in a single proving episode a prover may attain multiple key ideas and that it may not be possible to connect key ideas to a technical handle or to render a technical handle into a formal proof. They maintained that mathematics faculty may prefer constructing proof that connects their informal key ideas to a formal proof via a technical handle. Although a proof construction process may not involve the attainment of a key idea or technical handle, the work of Raman and colleagues provide empirical evidence that in the proving process there is an attainment of ideas that can push the argument forward.

Statement of the Problem

Mathematicians and mathematics education researchers appear to agree that the mathematician’s proving process includes an attainment of ideas that can motivate a mathematical argument, and the construction of a proof involves a non-linear process of understanding the statement, convincing oneself, convincing a friend, and convincing an enemy (Tall et al., 2012). Students find difficulty differentiating arguments that convince themselves from mathematical proof (Harel & Sowder, 1998; Healy & Hoyles, 2000; Weber, 2010), and at times students do not seek to construct formal proofs that are connected to their informal understandings (Raman et al., 2009). Students have also been shown to struggle with generating and using ideas that can move their arguments toward a rigorous proof (Alcock, 2008; Alcock & Weber, 2010; Brandell et al., 2008).

Little research has been performed that would describe the context around the formulation of the ideas that the prover finds useful and how these ideas influence the development of the mathematical argument. A more thorough account of the state of the literature is given in Chapter II. The phenomenon is not well-described for mathematicians but resonates in authors' accounts of doing mathematics. Looking at the moments where these ideas develop through the perspective of Dewey's (1938) theory of inquiry may grant important information about the context of the generation of these ideas and the purposes that they serve as the argument evolves. Understanding the context surrounding the generation of these ideas may inform the development and creation of experiences for students of mathematics (Harel & Sowder, 2007).

Framing Research in the Context of Dewey's Theory of Inquiry

This research sought to describe the context surrounding the generation of ideas that can move an argument forward. I viewed ideas as tools for performing some sort of work on the proof situation. Byers (2007) posited that ideas emerge from periods of ambiguity. Dewey (1938) likewise described ideas as possible solutions to situations that the individual deems tense and unresolved:

> The possible solution presents itself, therefore, as an *idea,* just as the terms of the problem (which are facts are instituted by observation). Ideas are anticipated consequences (forecasts) of what will happen when certain operations are executed under and with respect to observed conditions. (p. 109, emphasis in original).

The intentional process to resolve doubtful situations through the systematic invention, development, and deployment of tools is inquiry (Hickman, 2011).

The process of active productive inquiry involves the repeating, cyclical pattern of reflection, action, and evaluation. In reflection, the inquirer inspects the situation, chooses a tool to apply to the situation, and thinks through a course of action. After this initial reflection of what could happen, the inquirer performs an action (applies the tool). Either during or after the fulfilling experience, the inquirer evaluates the appropriateness of the selected application of the chosen tools (Hickman, 1990). The process and pattern of inquiry will be described further in Chapter III, but I argue now that viewing the situation surrounding mathematicians' developing personal arguments through the inquiry framework may be informative.

I chose to view personal arguments as progressing if the individual deemed that he or she had incorporated ideas that were useful to achieving the purpose that he or she defined. Similarly, Dewey (1938) defined a proposal, theorem, or course of action as a "tool" if it does work in the inquirer's eyes achieving some sort of purpose. Using the framework, I was able to describe the actions performed by the participants noting if the prover perceived a problem and what they perceived the problem to be. If the prover perceived a problem, I sought to describe process of selecting a tool to apply to the problem, the individual's expected outcome of using the tool, and the individual's perspective of how the action affected the situation. These factors together provided an organization for the context of the situation from the participant's point of view.

Purpose and Research Questions

The purpose of this research was to describe the evolution of the personal argument in professional mathematicians' proof constructions. I defined the *personal argument as progressing* or *moving forward* when the prover generated ideas he or she saw as functional in resolving certain problems encountered and chosen to be solved throughout the proof construction process. Additionally, this research sought to describe the situations surrounding the emergence of ideas through the lens of describing the problem the mathematician endeavored to solve, the tools that he or she applied, and the anticipated outcomes of enacting those tools. The research also described the process of testing the ideas and how the emergence of the idea did or did not change the situation for the mathematician. This research sought descriptive answers to the following questions:

Q1 What ideas move the argument forward as a prover's personal argument evolves?

Q1a What problematic situation is the prover currently entered into solving when one articulates and attains an idea that moves the personal argument forward?

Q1b What stage of the inquiry process do they appear to be in when one articulates and attains an idea that moves the personal argument forward? (Are they currently applying a tool, evaluating the outcomes after applying a tool, or reflecting upon a current problem?)

Q1c What actions and tools influenced the attainment of the idea?

Q1d What were their anticipated outcomes of enacting the tools that led to the attainment of the idea?

Q2 How are the ideas that move the argument forward used subsequent to the shifts in the personal argument?

Q2a In what ways does the prover test the idea to ensure it indeed "does work"?

Q2b As the argument evolves, how is the idea used? Specifically, how are the ideas used as the participant views the situation as moving from a problem to a more routine task?

Definitions of Key Terms

The following definitions are provided to ensure uniformity and understanding of these terms. I developed all definitions not accompanied by a citation.

Argument. "An act of communication meant to lend support to a claim" (Aberdein, 2009, p. 1).

Idea that moves the argument forward. An idea that the individual sees as functional in resolving certain problems encountered throughout the proof construction process.

Personal argument. The personal argument encompasses all thoughts that the individual deems relevant to making progress in proving the statement. It is a subset of the entire statement image (Selden & Selden, 1995) of a proof situation that the individual views as central to his or her aims in developing the argument.

Problem. A situation that the individual recognizes as tense ore unresolved, specifically a task or situation in which it is not clear to an individual which mathematical actions should be applied that the individual has an interest and motivation to solve.

Problematic situation. A situation that the individual recognizes as tense or unresolved, specifically a task or situation in which it is not clear to an individual which mathematical actions should be applied.

Professional mathematician. A professional mathematician is defined to be an individual holding a doctorate in mathematics that is currently teaching and doing research in mathematics.

Proof. A proof is a sub-type of argument that uses deductive-type warrants and the modal qualifier is absolute (Inglis et al., 2007). It is the written end product of an argument meant to convince another of a claim using the language accepted by the mathematics community. It is the individual prover who decides if the argument is a proof or not.

Statement image. The statement image is a construct proposed by Selden and Selden (1995) to describe the mental structure one attaches to certain statements. For this study, I am interested in the statement image of the task statement to be proven. The image includes "all of the alternative statements, examples, nonexamples, visualizations, properties, concepts, consequences, etc., that are associated with a statement" (Selden, 1995, p. 133).

Tool. A theory, proposal, action, or knowledge chosen to be applied to a situation (Hickman, 1990).

Procedures

In order to answer the research questions, three participant mathematicians worked on three or four mathematical proof tasks. The participants were mathematicians who either did research in a field related to real analysis or who taught courses in real analysis at the upper undergraduate or lower graduate level. The mathematicians chose tasks that they found to be problematic, meaning, tasks for which they did not recall a solution or for which they did not know a way to approach. Each participant participated in task-based and follow-up interviews. In the first interview, participants chose tasks and began work on the first task. The second interview consisted of a follow-up interview regarding their work on the first task, and participants will began work on the second and third task. The third interview was a follow-up interview regarding the participant's work on the second and third tasks.

Participants began their work individually in an interview setting that was audio and video recorded using Livescribe technology to record their written work. If they did not complete the tasks in the interview, they continued working on the task on their own, recording their work with the Livescribe technology. I conducted preliminary analyses to formulate questions and identify important moments in the video of the participant's work for use in the follow-up interviews. The follow-up interviews posed the questions formulated in the analyses to clarify participant thinking while they were constructing the proofs.

Data analysis occurred in two major phases: the preliminary analyses to prepare for the follow-up interviews and the primary analyses of the entire data set. The preliminary analyses occurred between interviews for each task for each participant. I identified moments when new ideas seemed to be articulated which acted as markers on the timeline of the argument evolution. Toulmin's (2003) argumentation framework (Toulmin, Rieke, & Janik 1979) informed analysis of the arguments before and after these ideas emerge. This provided a structured description of the elements deemed useful (i.e., objects doing work in the personal argument). I formed hypotheses about what the participant perceived as problematic, the phase of the inquiry process that he or she appeared to be in when articulating the idea, the tools used that influenced the generation of the ideas, and the anticipated outcomes of using these influencing tools. These hypotheses also included how the participant appeared to test the idea that he or she viewed as moving the argument forward and how he or she used the idea throughout the rest of the argument. Finally, the hypotheses informed the formulation of questions and the identification of what video/audio/Livescribe to play back for the follow-up interviews.

The primary analysis began with writing a description of each idea that moved the argument forward and the context surrounding the generation of that idea. I described the argument's evolving structure via Toulmin diagrams formulated in the preliminary analysis and informed by the follow-up interviews. Descriptive stories of each participant's work on each task documented the argument evolution; the development of new ideas that moved the argument forward acted as significant moments in the story. I then conducted open iterative coding of each idea to organize, describe, and link the data. After coding each participant's work on each task, I looked for patterns, categories, and themes across participants and tasks.

Limitations and Delimitations

Any research design can have potential weaknesses outside the researcher's control. In addition, the researcher makes choices about what will and will not be included in the study. This study was limited by sample size and sampling methods, the nature of interview studies and their ability to capture participant thinking, and the types of tasks that participants were able to complete. This research was delimitated to the mathematical proof constructions of mathematicians on real analysis tasks.

Generalizing the findings from this study is problematic as the sample of participants was not large and random. Lincoln and Guba (1985) recommend sampling until the data becomes redundant. In designing this study, I hypothesized that three mathematicians working on three tasks, resulting in nine total tasks could achieve that goal and ended up with participant work on ten tasks. At the conclusion of analysis, it was found that no new codes were needed and that saturation in the data was indeed achieved.

Patton (2002) indicated that interview data has limitations that include "possibly distorted responses" (p. 306), which in this study would be potentially due to personal bias or the emotional state of the interviewee, recall error, or the reaction of the interviewee to the interviewer. This research sought to make sense of the mathematicians' thinking. Internal thinking is not directly observable; I was limited to what participants said in conjunction with what they wrote. To ameliorate this issue, the study used a variety of sources (Patton, 2002) including the follow-up interviews that drew not only upon the participant recollection but also the video, audio, and Livescribe data viewed and interpreted by the participants to test my hypotheses.

The scope of the study was limited to mathematicians' constructing proof on "school-type" tasks, or tasks from textbooks or homework that had been previously solved. This does not completely capture the work of the research mathematician as mathematics research involves posing and investigating novel problems. Simply put, there was not enough time to look at the practice of mathematicians while conducting research in the kind of depth needed to answer the research questions as it would take time for me (a non-expert in the field) to understand the specific field. Additionally, the types of ideas that were the focus of this study may not have emerged in a short interview setting. The ultimate design of this study was based on the assumption that mathematicians could still engage in genuine problem solving on school-type tasks provided they personally identified the task to be problematic.

Mathematicians were specifically chosen based on the assumptions that mathematicians are especially adept at constructing mathematical proof that learning from what mathematicians do can inform teaching. Since mathematicians were the focus of this study, the ideas that they generated may not be the same ideas generated by members of other populations while constructing mathematical proofs. I chose to focus on the subject of real analysis. Ideas generated in this field of mathematics may or may not be indicative of the types of ideas generated in another field.

This study seeks to describe the ideas generated and the emergence of those ideas. Characterizing the emergence involves describing the tools and ways of thinking that participants utilized. However, it is beyond the scope of this study to fully describe all affordances, limitations, and ways of applying the tools observed. The descriptions of the tools used are limited to how they were used on the tasks that the participants chose to complete in this study.

Organization of the Dissertation

This introduction chapter provided a description of the research problem giving an overview of the literature that is expanded in the next chapter. I presented the research purpose to describe how mathematicians' personal arguments evolve in their constructions of mathematical proof. I explained how the research was framed within Dewey's (1938) theory of inquiry and presented the research questions. In the following chapter, there is a presentation of literature related to the research purpose and theoretical framing of the study. The third chapter provides a description of the theoretical perspective guiding this research and the methods for conducting the study. The fourth chapter presents the findings. In the final, fifth chapter, I discuss how the findings of this study relate to the literature and conclusions from the findings.

CHAPTER II

REVIEW OF SELECTED LITERATURE

The purpose of this qualitative study was to describe the evolving personal arguments of professional mathematicians in the construction of mathematical proofs. This literature review along with the research purpose served to provide focus to the theoretical perspective and methods of data collection and analysis (Patton, 2002). In this chapter, I provide an overview of previous research and theoretical conceptions related to mathematical proof constructions, mathematical argumentation, and problem solving, focusing specifically on the practices of professional mathematicians. The first section includes the areas of research on the practices of mathematical professionals. The sections that follow focus on proof construction literature that specifically relates to moving the argument forward including identified difficulties students have in constructing proof, the relationships between informal arguments and formal arguments, and practices that have been identified as useful in the construction of ideas that can move the personal argument forward. Conceptions of relationships between argumentations and mathematical proof in the next section lead to an overview of how past research has used mathematical proof construction as a type of argumentation. Finally, as the formulation of new ideas to solve a mathematical proof task can be conceived as a special type of mathematical problem solving, I describe the identified components in the problem solving process and empirical studies that have related mathematical proof construction to problem solving.

Past Inquiries into Mathematical Professionals

Inquiring into the mathematical practices of professional mathematicians is a long-standing endeavor in the mathematics education research community (e.g., Engelke, 2007; Raman, 2003; Sfard, 1994; Weber, 2008). By studying professionals, research may be better able to describe habits of thinking and reasoning that may be termed successful in order to perhaps characterize what is missing when students are unsuccessful.

In order to describe the experience of understanding mathematics, Sfard (1994) chose to interview working mathematicians. She felt professionals' reflections would provide her with insights that could transcend professional understanding. Researchers have analyzed the mathematics produced by mathematics professionals to inform and construct frameworks for use with analyzing student work or describing mathematical processes (Carlson & Bloom, 2005; Engelke, 2007; Raman, 2003).

More specifically, researchers have sought to describe meanings and practices of proof for research mathematicians. Weber and Alcock (2004) compared undergraduate student and graduate student attempts to prove or disprove two groups are isomorphic; they also compared undergraduates' abilities to instantiate groups to the abilities of professional algebraists. Inglis et al. (2007) analyzed mathematical arguments produced by mathematics postgraduate students using Toulmin's use of arguments and found frequent use of non-deductive warrants to deduce non-absolute conclusions. Savic (2012) studied mathematicians solving mathematical proof tasks outside an interview setting and described their strategies when coming to impasses. Some research has been also conducted into how research mathematicians read and validate mathematical proofs, and these studies have found that the standards for validating proof depend on the authority of the author (Inglis & Mejia-Ramos, 2009; Weber, 2008). Weber (2008) also found that mathematicians' validation standards and strategies can vary and depend on how familiar they are with the mathematical domain in question.

Research has also been conducted in how mathematicians solve problems of other kinds (Carlson & Bloom, 1995; DeFranco, 1996; Stylianou, 2002); however, these problems were not proof tasks. Selden and Selden (2013) point out, "It would be very informative to have research on how advanced university mathematics students or mathematicians actually construct proofs in real time, but such a study has not yet been conducted" (p. 314). This study will contribute to addressing this gap by focusing on the ideas that mathematicians' find useful in moving their arguments forward while constructing mathematical proof. Describing the process of developing, testing, connecting, and utilizing ideas to construct mathematical proof for professionals may be useful for practitioners when finding ways to address student difficulties in developing and using ideas successfully. As elaborated in the next section, two categories of ideas that can be useful in the development of a proof are those that support informal or intuitive arguments and those that support more formal deductive arguments. The next section elaborates on research into the use of informal and formal reasoning in proof construction.

Informal and Formal Arguments

Weber and Alcock (2004) noted that a necessary skill for students to develop is the ability to translate informal intuitions into formal arguments. Selden and Selden (2008) further identified knowledge of and the ability to appropriately use various symbolic representations and an understanding of the logical structure of mathematics as necessary for students to produce proof. Possessing the above attributes contributes to an individual's ability to render informal ideas communicable (Raman & Weber, 2006). The relationships between informal and formal arguments, and the process of moving from the former to the latter have been discussed with varying areas of focus including comparing how professionals navigate between formal and informal reasoning to what novices do when constructing proofs (Raman, 2003; Weber & Alcock, 2004), the identification of reasoning practices that can help students connect these two ideas (Alcock, 2008; Boero, Garuti, Lemut, & Mariotti, 1996), and the identification of salient features of informal and formal arguments (Raman et al., 2009). In this section, I outline how the dichotomy of informal and formal reasoning or arguments in proof construction and writing has been discussed in the literature, the reasoning techniques that have been identified as useful in connecting the formal and informal, and lead into the next section where I discuss how formal mathematical proof has been related to the process of argumentation. Informal understandings, arguments, or reasoning as discussed in the next few paragraphs are understandings grounded in empirical data or represented by a picture. These arguments may or may not directly lead to a formal proof, but may be connected to one.

In a series of investigations, Weber and colleagues (Weber, 2005, 2009; Weber & Alcock, 2004) worked to analyze how individuals with successful experience in mathematical proof and more novice students navigated between formal and informal ways of reasoning. They termed *syntactic reasoning* as thinking based on formalism and logic; the reasoning process involves one starting with definitions and axioms and then using logical deductions to make inferences in a proof production. *Semantic reasoning* is a way of thinking about justifying a statement that considers informal and intuitive representations such as graphs, examples, gestures, and diagrams (Weber, 2009). Weber and Alcock (2004) presented both undergraduates and mathematics graduate students with the task of proving that two groups were not isomorphic. Undergraduate students relied upon the definitions of isomorphism attempting to prove the statement syntactically. The graduate students, on the other hand, applied their rich instantiations of group and isomorphism using reasoning outside the formal system and then were able to connect their informal reasoning back to the formal system. While semantic proof productions often convey the prover has a richer understanding, successful proofs can be produced using purely syntactic reasoning (Weber, 2009). Later research found that even when students attempt to use informal reasoning practices to develop ideas to inform the construction of proof, they are sometimes unable to translate their ideas into a formal proof (Alcock & Weber, 2010).

Raman (2003) investigated the views of proof held by mathematicians, students, and teachers. She identified two aspects of proof construction, the *private* and the *public.* The private argument is "an argument that engenders understanding" (p. 320), and a public argument is one "with sufficient rigor for a particular mathematical community"

(p. 320). In earlier work, Raman (2001) found that university mathematicians and students thought about the public aspects and private aspects of proof in different ways. Mathematicians held the view that there is potential for their private arguments to be connected to a public argument although they could choose to develop a public argument that did not use the ideas from their private argument. Students, in contrast, viewed the public and private aspects of mathematics as separate. To further describe this difference between those experienced with mathematics and novices, Raman (2003) characterized three types of ideas involved in the production of a proof: *heuristic ideas, procedural ideas,* and *key ideas*. A heuristic idea is an idea based on informal understandings and is linked to the private aspect of proof. A procedural idea is "based on logic and formal manipulations" (p. 322); it gives a sense of conviction but not necessarily understanding. A proof based on a procedural idea would be an example of Weber and Alcock's (2004) conception of a syntactic proof production. Because it gives a formal proof, the procedural idea lies in the public domain. Raman (2003) identified a third idea, a key idea, as a heuristic idea that can be mapped to a formal proof; it is thought to link together the private and public aspects of a proof. A complete proof that utilizes a key idea would be reminiscent of a successful proof that utilizes semantic reasoning (Weber & Alcock, 2004). Raman (2003) found that faculty members were more likely than students to construct proofs involving key ideas because students did not have the key idea and because they did not view proof as about key ideas.

Connecting privately held intuitions to a formal proof may be desirable especially as students may at times view the two as disconnected or do not see the differences between them. Harel and Sowder (1998), in characterizing what students view as personally convincing arguments, described a transformational proof scheme as arguments based on a fully developed understanding of the concepts. A transformational proof scheme includes "(a) consideration of the generality aspects of the conjecture, (b) application of mental operations that are goal oriented and anticipatory, and (c) transformations of images as part of a deduction process" (p. 261). Harel and Sowder (2007) advocated for student understanding of proof in line with the transformational proof scheme. Proofs constructed in this manner transform the personal argument into a public one. How this transformation occurs or evolves has not fully been described. However, Raman and colleagues (2009) identified two key moments in the proof construction process, the formulations of *key ideas* (or *critical insights*) and *technical handles,* that may give insight to how this transformation could occur.

Key Idea and Technical Handle

In their observations of proofs that students construct, Raman et al. (2009) found that there were moments that were critical: the moment when students attained a *key idea* (later termed conceptual insight; Sandefur et al., 2012), the moment when students gained a *technical handle* for communicating a key idea, and the culmination of the argument into a standard form. A conceptual insight is an idea that gives a sense of why the statement is true. A technical handle is an idea or a proof procedure that can render the ideas behind the proof communicable. The technical handle may or may not be directly tied to the original conceptual insight that gave an 'aha' feeling. It may be tied to some sort of unformed thoughts or intuition.

Students may be limited in their proof writing abilities because they lack a key idea or because they do not possess tools to link an idea to a formal proof which may be a form of technical handle (Raman, 2003). In their research practice, Raman et al. (2009) "tentatively" distinguished a technical handle in terms of its potential to lead to a correct proof rather than an actual student realization of "now I can prove it." They acknowledged that it may not always be possible to connect key ideas to a technical handle but found that mathematicians are conscious of the possibility of such a connection and may be more likely than students to search for the connection to a technical handle (Raman et al., 2009). Technical handles seemed to be disconnected from key ideas in the minds of students. If student reached a key idea, they appeared to start a discussion about how to do a more "formal" proof, often discussing in syntactic terms and apparently ignoring the key idea formerly reached. Making connections between the ideas that engender understanding of the key concepts and the ideas that can help one to construct a formal proof is difficult for students. Researchers have identified actions that may provide instructors with tools to aid in supporting students' learning to construct proofs. The next section describes some of these identified actions and behaviors, specifically those that can support the development and effective implementation of useful ideas for developing arguments.

Actions That Support the Formulation of Useful Ideas

The Modes of Thinking

Alcock (2008) conducted interviews with four mathematician instructors of an introduction to proofs course. The instructors identified characteristics, practices, and habits that they desired for their students to exhibit when engaging in proof construction. Alcock categorized these into four modes of thinking: Instantiating, Creative Thinking, Structural Thinking, and Critical Thinking.

Alcock (2008) described *instantiating* as an attempt to meaningfully understand a mathematical object by thinking about the objects to which it applies. This is similar to "exploring an example space" or drawing on a rich concept image. Instantiating can involve thinking about known or developed examples of a mathematical idea or object. Instantiating can also mean drawing upon other conceptions of mathematical ideas. For example, picturing a function as shooting the elements of one set to another set (Alcock, 2008) is an act of instantiating. Instantiating is a broad term that encompasses thinking about generic examples and specific objects as well. The actual behaviors may be creating examples with desired properties or listing already known examples and then reasoning about and manipulating them. The utility of example-use in building concept image and facilitating proof construction has been given considerable attention in the literature in the last decade; I elaborate on findings in the next section.

Creative thinking entails examining instantiations to identify a property or set of manipulations that can form the key idea of a proof. This is similar to using examples to look for an insight. Creative thinking is a tool with the end-in-view of looking for properties that give insight into the proof. Instantiation supports creative thinking in that it generates the objects. The purpose of the creative thinking may be to gain a critical insight, to illustrate the structure of the mathematical objects, to show that the result is true in a specific case, or to search for a reason why one could not find a counterexample.

Structural thinking uses the form of the mathematics to deduce a proof. This is not directly related to using examples, but perhaps the exploration of examples can guide the prover to a place to begin thinking about appropriate and applicable theorems. Structural thinking employs syntactic reasoning (Weber & Alcock, 2004). The tools used in structural thinking will most likely be known properties and theorems and the logical structure of mathematics. Additionally, Alcock (2004) notes how structural thinking may inform instantiating and creative and critical thinking. One must know what to instantiate, what the consequences are of certain manipulations, and be able to articulate the results of creative thinking in precise language.

Critical thinking has the goal of checking the correctness of assertions made in the proof. This may occur either syntactically or semantically. Alcock (2008) stated that it is possible to construct viable, rigorous proof without engaging in this mode of thought, but it is a habit that instructors would hope to foster in their students. The instructors in the study described a way of checking one's work- looking for preserved or implied properties. Alcock hypothesized that this would require a more sophisticated knowledge base than just checking examples.

Exploring Examples

Watson and Mason (2005) defined an *example* as a particular case of any larger class about which students generalize and reason, and they defined *exemplification* as using something specific to represent a general class with which the learner is to become familiar. Other characterizations of 'examples' include illustrations or specific cases of mathematical objects, not to be confused with a worked example of a procedure as desired by students (Alcock & Inglis, 2008). Exploring examples has been shown to be a potential source for new ideas in problem solving and proof construction (Alcock & Inglis, 2008; Alcock & Weber, 2010; Sandefur et al, 2012).

Past research has found that exploring example spaces may serve to create instantiations of concepts and develop concept image (Alcock, 2004; Dahlberg &

Housman, 1997; Mason & Watson, 2008; Watson & Shipman, 2008; Weber, Porter, & Housman, 2008). Possessing or creating examples may not be enough (Iannone, Inglis, Meija-Ramos, Simpson, & Weber, 2011). The purposes of examples, the ways examples are generated, and the assumptions and deductions behind them may contribute to how much example generation aids learner understanding and proof construction (Edwards & Alcock, 2010).

Various purposes for using examples in doing mathematical work by both those experienced in mathematics and novices have been discussed in the literature. Notably, Lockwood, Ellis, Dogan, Williams, and Knuth (2012) surveyed mathematicians to determine what kinds of examples they used, how they used them, and for what purposes. Exemplification has been shown to be useful in the construction and communication of mathematical proof in the following ways:

- To understand a statement, definition, object, etc. (Alcock, 2004; Alcock & Weber, 2010; Lockwood et al., 2012; Sandefur et al., 2012)
    - Indicate what is included and what is excluded by a condition in a definition or theorem (Watson & Mason, 2005)
    - Build a sense of what's going on (Lockwood et al., 2012; Michener, 1978)
- Explore behavior and illustrate structure (Sandefur et al., 2012; Watson & Mason, 2005)
- Evaluate the truth of a statement or conjecture by checking inferences (Alcock & Inglis, 2008; Alcock & Weber, 2010; Lockwood et al., 2012)

- To generate arguments
    - “Directly” and “indirectly” (Alcock, 2008; Alcock & Inglis, 2008)
        - Directly by trying to show that a result is true in a specific case hoping the same argument or manipulations will work in general
        - Indirectly by searching for a reason why one could not find a counterexample to the statement
    - To give insight into proving (Lockwood et al., 2012; Sandefur et al., 2012)
    - To understand why the assertion should be true (Alcock & Weber, 2010)
    - Use a specific object to indicate the significance of a particular condition in a definition or theorem (Watson & Mason, 2005)
        - Highlighting the condition’s role in the proof
        - Showing how the proof fails in the absence of that condition
    - To generate counterexamples (Alcock, 2008; Alcock & Inglis, 2008; Lockwood et al., 2012)
    - To generalize (Lockwood et al., 2012)
- To aid in explaining an argument to another (Alcock & Inglis, 2008)
- To indicate a dimension of variation implied by a generalization (Watson & Mason, 2005)
- To indicate something that remains invariant while some other features change (Watson & Mason, 2005)

The list above demonstrates ways in which application of the tool of exploring examples can be useful in developing ideas that an individual may deem as useful in moving the argument forward.

Antonini (2006) classified the types of examples generated by mathematicians and mathematics research students when they were specifically asked to create examples. The three classifications were trial and error, transformation, and analysis. The *trial and error* strategy involved searching a collection of recalled examples from a broader category and testing each example to see if it met the desired criteria. Individuals utilizing the *transformation* strategy modified examples that they viewed as satisfying some of the criteria until they satisfied all of them. The *analysis* strategy occurred when the individual identified properties that the desired object would have, and then recalled or constructed an object with the desired properties. Undergraduates have been found to almost exclusively use the trial and error strategy (Edwards & Alcock, 2010; Iannone et al., 2011).

Teaching students about generic proof production, using well-chosen examples to construct proofs (Mason & Pimm, 1984), may aid in students' development of ideas that can move the proof along. By showing a given statement holds for an arbitrary object from a class of objects may help students to see the structure of the argument (Weber et al., 2008). This reasoning then may be abstracted into a more general, formal proof. When considering students and the use of examples, Watson and Shipman (2008) found that learner-generated examples served as an effective way to introduce mathematical lessons while Iannone et al. (2011) claimed that example generation is not yet well enough understood to be a viable pedagogical recommendation.

Iannone et al. (2011) found that the practice of generating examples did not contribute to valid proof production in real analysis tasks and suspect its contribution from the fact that most students used trial and error. This poses the question that if most undergraduate students use trial and error, is there a gradation in these types of examples into ones that contribute to proof construction and ones that do not? Past work may offer some insight into this question. Edwards and Alcock (2010) suggested mathematical assumptions and deductions within a chosen strategy may be important to consider as there is a dimension of variation from incorrect deductions or assumptions. Weber (2009) cited differences in the purposes of examples, i.e., as a primary strategy versus a last resort for difficult concepts. Sandefur and colleagues (2012) indicated that students found utility in exploring examples during proof construction with a specific purpose in mind. However, Alcock and Weber (2010) found that while students used examples for specific purposes consistent with those described by mathematicians, they could sometimes use examples ineffectively due to an inability to generate examples that satisfy the necessary conditions or the inability to connect their reasoning from examples to the language of formal proof.

When one is developing a mathematical proof, exploring examples, instantiating, and creative, structural, and critical thinking reside within the definition of informal logic or argumentation (Dove, 2009) which is discussed in the next section. These informal arguments may later be conducted to more formal proofs (Raman et al., 2009). When a prover is able to use make these connections, the overall argument is said to have *cognitive unity.* Boero et al. (1996) described a teaching experiment where eighth grade students dynamically explored a situation, made conjectures, and generated proofs of the conjectures. They described cognitive unity as the phenomenon where an individual is able to use the ideas that motivated their conjectures directly in the proofs they construct.

> During the production of the conjecture the student progressively works out his/her statement through an intensive argumentative activity functionally intermingling with the justification of the plausibility of his/her choices. During the subsequent statement proving stage the student links up with this process in a coherent way, organizing some of the justifications ('arguments') produced during the construction of the statements according to a logical chain. (Boero et al., 1996, pp. 119-120)

The generation of the conjectures was viewed as an argumentative activity. Pedemonte (2007) discussed the relationship between argumentation and proof. She defined proof as a particular argumentation and described argumentation in the tradition of Toulmin's argumentation and used Toulmin's (2003) model to compare the structure of students' arguments and the structure of their proofs when investigating topics in geometry. She found that students' participation in argumentation activity aided in the construction of proof. She extended the conception of cognitive unity:

> Nevertheless, cognitive unity as defined by Boero et al. (1996) does not cover all the aspects of the relationship between argumentative conjecturing, proving and proof (as product). The paper has shown the importance of analysing the entire resolution process and not only the conjecturing and proving phases. (p. 39)

In her analyses, she conceived of proof as the end product of an argumentation. The relationship between mathematical proof and the acts of argumentation have been characterized to some extent in the literature.

Proof and Argumentation

This study seeks to conceptualize the process of constructing mathematical proof as a process in which the individual's personal argument evolves from first encountering the task statement to producing a written proof to convince others. With this conception of proof as an end result of argumentation, I elaborate how argument and argumentation have been defined, how they have been related to mathematical proof, and findings of empirical studies that have conceptualized proof as argumentation.

Definitions of Argument and Argumentation

The definitions of an argument can be categorized into two groups: (a) the more narrow conception of argument as a sequence of logical deductions, and (b) the broad conceptualization of argument that encompasses both formal and informal logic. Here are a few definitions of argument presented in the literature. The first two are examples for the first category and the last encompass informal reasoning practices to various degrees in the definitions and are examples of the second category.

> Those of us weaned on formal logic may think this debate is a non-starter because there is a perfectly acceptable definition of argument which is synonymous with the definition of *derivation* available in any textbook on formal logic: an argument is a sequence of statements/sentences/propositions/formulas such that each is either a premise or the consequence of (some set of) previous lines, and the last of which is the conclusion. (Dove, 2009, p. 138)

> An argument in the logician's sense is any group of propositions of which one is claimed to follow from the others, which are regarded as providing support or grounds for the truth of that one. (Copi & Cohen, 1994, p. 5, as quoted in Dove, 2009, p. 139)

> The simplest possible argument consists of a single premise, which is asserted as true, and a single conclusion, which is asserted as following from the premises, and hence also to be true. The function of the argument is to persuade you that since the premise is true, you must also accept the conclusion. (Scriven, 1972, p. 55, as quoted in Dove, 2009, p. 139)

> An argument is a type of discourse or text—the distillate of the practice of argumentation—in which the arguer seeks to persuade the Other(s) of the truth of a thesis by producing reasons that support it. In addition to this illative core, an argument possesses a dialectical tier in which the arguer discharges his dialectical obligations. (Johnson, 2000, p. 168, as quoted in Dove, 2009, p. 139)

> *Argumentation* is a verbal, social and rational activity aimed at convincing a reasonable critic of the acceptability of a standpoint [read: conclusion] by putting forward a constellation of propositions justifying or refuting the proposition

> expressed in the standpoint. (Grootendorst & van Eemeren, 2003, p. 1, as quoted in Dove, 2009, p. 139)
>
> [Define] informal logic as the formulation, testing, systematization, and application of concepts and principles for the interpretation, evaluation, and practice of argumentation or reasoning. (Finocchiaro 1996, p. 93, as quoted in Dove, 2009, p. 138)
>
> Now arguments are produced for a variety of purposes. Not every argument is set out in formal defense of an outright assertion. But this particular function of arguments will claim most of our attention […]: we shall be interested in justificatory arguments brought forward in support of assertions, in the structures they may be expected to have, the merits they can claim and the ways in which we set about grading assessing and criticizing them. It could, I think, be argued that this was in fact the primary function of arguments, and that the other uses, the other functions which arguments have for us, are in a sense secondary and parasitic upon this primary justificatory use. (Toulmin, 2003, p. 12)
>
> An act of communication intended to lend support to a claim. (Aberdein, 2009, p. 2)

While the definitions of argument above differ in whether they include activities that are not formal deductive statements, argumentation theory is a study of argument that emphasizes the aspects that are not prone to deductive formalization. Informal reasoning, deductive logic, and critical thinking are thought to be subfields of argumentation theory (Aberdein, 2009). However, some have stated that there is a clear distinction between argumentative reasoning and deductive reasoning (Balacheff, 1988); while others have not distinguished argumentations and proofs. For example, in their seminal work discussing the proof schemes held by students, Harel and Sowder (1998) conceptualized "proof" as encompassing both deductive and empirical arguments. It may be that argumentation is the pathway between informal conceptions and formal proof. Tall (2004) identified three origins of what he terms *warrants for truth:* (a) the *embodied world*--through perception and action in the physical world, (b) the *proceptual world*--through correct calculation or symbolic manipulation, and (c) the *formal world*--from a set of axioms and basic definitions. According to Mejia-Ramos and Tall (2005), true mathematical proof lives in the formal world. The concept of informal logic has been utilized as a means of conceptualizing the pathways by which one reasons outside formal logic.

In Aberdein's (2009) conception of argument as an act of communication meant to lend support to a claim, proof fits within the definition of argument. He characterizes a species of alleged proof, 'proof*'. These are arguments that either have no consensus on whether they are proof, or they have broad consensus that they are proof. These proofs* include picture proofs*, probabilistic proofs*, computer-assisted proofs*, and so forth. He indicated steps in the process of proving that may require informal argumentation, such as, choosing the problem, choosing the methods to tackle the problem, applying the method to the problem, the review process when the proof is submitted for publication, and the dialectic between the author and the reader of the proof once it is published as mathematicians may "seek to generalize it, extend it, transpose it to a different field, simplify it, or manipulate it in some other way" (p. 2).

Lakatos (1976) raised the question, to what extent is mathematics dialectical? He provided a 'rational reconstruction' of the successive proofs of the Decartes-Euler Conjecture showing that there is continuity between argumentation as a process of statement production and the construction of its proof (relationship between constructing and modifying conjecture and performing trials to prove the statement). Lakatos indicated that argumentation and proof are developed when someone wants to convince oneself or others about the truth of a statement. He gave the indication that math, for mathematicians, is not a formal mathematical system. Math is about humans making arguments and being presented with counter arguments.

In keeping with this conception of mathematics, Dove (2009) argued that proof is more than a finite list of deductive statements citing how mathematicians use *evidentiary* or non-deductive methods (computer-assisted proofs, probabilistic sieves, partial proofs, abduction) using informal logic. He defined informal logic as the application of concepts and principles for the interpretation and evaluation of argumentation and reasoning. He stated informal logic is the logic of mathematical reasoning and that mathematicians use it when they assess mathematical reasoning that is not a proof. It would appear that the construction of mathematical proof requires a combination of informal and formal logic. In the next section, I describe empirical studies that have utilized the argumentation conception of proof construction in their analyses as well as a description of how mathematics and argumentation philosophers have utilized the conception to broaden the definition of proof.

Explorations of Proof as Argument

Aberdein (2009) stated that the study of mathematics practice needs an account of argument and it has largely been unexplored. One framework, specifically, has been a means of analyzing mathematical proof in terms of argument. In 1958, Stephen Toulmin introduced a means of studying non-formal arguments by providing a structure that analyzes the argument into the six components of *claim* (the assertion that is being argued), *data* (the foundations for the argument), *warrant* (that which justifies the link between the data and the claim), *backing* (explains the permissibility of the warrant), *qualifier* (the degree of confidence in the claim), and *rebuttal* (the conditions under which the claim would not hold). A more detailed description of these components is given in the theoretical perspective in the third chapter.

The Toulmin layout has been used across disciplines in the study of arguments and how learning progresses in the classroom (Cole et al., 2012; Krummheurer, 1995). Often these studies use only a condensed version of the full model that only categorizes statements as data, warrants, and claims. In recent years, it has been used frequently to analyze mathematical arguments inside and outside the realm of proof construction. I describe the findings of recent empirical studies that have used Toulmin analyses for mathematical arguments and justifications.

Pedemonte (2007) used the condensed version of Toulmin's model (only data, warrant, and claim) to compare and analyze the structure of arguments associated with French and Italian 12^{th} and 13^{th} grade students' conjectures and the structure of their written proofs. She found that often structural continuity existed between the argumentations, meaning students used the same properties and theorems in the argumentation and the proof. However, there also was structural distance between the two. Specifically, at times, abductive argumentation transformed into a deductive proof, and inductive argumentation transformed to a mathematical inductive proof. She recommended analyzing the entire solution process, not just the conjecturing and proving.

Inglis et al. (2007) found mathematics graduate students used warrants based on inductive reasoning, intuitive observations about or experiments with some kind of mental structure, and formal mathematical justifications when evaluating conjectures about a novel number theory topic. The graduate students provided arguments meant to convince themselves whether or not the conjectures were valid. Inglis and colleagues categorized the warrant-types developed by the participants in terms of their respective backing. Inductive warrants described ways of connecting the hypothesis to the claim that were based on empirical evidence. Structural-intuitive warrants was reasoning based on observations or experiments with some sort of structure. Deductive warrants described reasoning based on formal mathematical justifications. Inglis and colleagues warned against the use of the smaller (data, warrant, claim) version of the Toulmin model emphasizing the importance of considering the modal qualifier because the type of warrant used would affect to what degree the graduate students were convinced that their claims implied the conclusions. Inductive warrants were paired with an appropriate modal qualifier meaning that the graduate students were not *certain* that a conjecture was true by the use of examples. Structural-intuitive warrants gave the individuals direction and reduced uncertainty but at times supported incorrect conclusions. Deductive warrants that included mathematical properties and justifications were the only warrants that gave the participants certainty. This points to the possibility that when constructing mathematical proof, professionals may engage in argumentation that is not formal proof.

Fukawa-Connelly (2014) analyzed an instructor's proof presentation in abstract algebra. He noted that the proof of a statement may involve subproofs or proofs of lemmas. He termed the proofs of the lemmas as local arguments which layer for the global argument of the entire proof. He classified the instructor's spoken statements as data, warrant, backing, qualifier, or conclusion. He described the standards of evidence for coding the instructor's statements. For example, "A statement was classified as a warrant when it linked the data and conclusion in a way that explained how the data supported the conclusion by drawing upon previously demonstrated facts or facts stated as part of the hypothesis" (p. 80).

Fukawa-Connelly (2014) found that the instructor frequently wrote the data and conclusions of the argument, but wrote the warrants, backing, and qualifiers less frequently or not at all. Qualifiers were never written possibly because the instructor only presented completely correct statements. He found that Toulmin analyses were useful for the purposes of making sense of the instructor's writing and dialogue, but the analyses were insufficient in explaining all aspects of the instructor's modeling of proof writing. He called for more research into the teaching of proof writing and the development of theoretical lenses that link the actions of lecture-based teaching to aspects of student mathematical proficiency.

Wawro (2011) coded arguments within whole class discussions of an inquiry-oriented linear algebra classroom. She found the original Toulmin scheme to be insufficient to capture the complexity of some of the arguments observed. Consequently, she developed the expanded schemes of

1. *Embedded structure*: data or warrants for a claim had minor, embedded arguments within them;
2. *Proof by cases structure:* claims were justified using cases within the data and/or warrants;
3. *Linked structure*: data or warrants for a claim had more than one aspect that were linked by words such as 'and' or 'also; and
4. *Sequential structure*: data for a claim contained an embedded string of if-then statements.

The studies summarized above provide insights into this proposed study, both methodologically and conceptually. I utilized the full Toulmin model as recommended by Inglis et al. (2007). Fukawa-Connelly's (2014) standards of evidence for coding each statement proved helpful in analysis. Pedemonte (2007) provided insight into how the structure of the arguments may change when an individual is moving from an informal argument to a formal written proof. The conception of proof construction as argumentation has shown to be useful in analyzing the process. In addition to conceiving of mathematical proof as an argumentative activity, past research has also conceived of creating mathematical proof as a special type of mathematical problem solving (Weber, 2005).

Problem Solving

An individual may enter a proof task without knowing the ideas that warrant the statement's validity or invalidity, or they may not know how to put their ideas together to write the argument in a formal, logical format. In this sense, construction of the mathematical proof can be viewed as a mathematical problem; the formulation of these ideas requires problem solving. A review of the literature related to mathematical problem solving specifically focused on proof problems is warranted.

Definition of Problem

Tasks or situations for which an individual does not recall a solution are often defined as *problems* in the literature (Dewey, 1938; Mason, Burton, & Stacey, 1982/2010; Schoenfeld, 1985). This research is concerned with tasks of the type *very non-routine problems* (tasks that "may involve considerable insight, the consideration of several sub-problems or constructions and the use of Schoenfeld's (1985) behavioral problem-solving characteristics" (Selden & Selden, 2013, p. 305) that the individual has a motivation or interest in entering into solving. Past writings have endeavored to describe the individual's approach to attacking mathematical problems both philosophically and empirically.

Necessary Knowledge for Problem Solving

Schoenfeld (1985, 1992) interviewed mathematicians solving problems and identified four important aspects necessary for success in problem solving: (a) Resources (knowledge of mathematical facts and procedures), (b) heuristics (problem solving strategies); (c) control (having to do with self-monitoring and metacognition); and (d) beliefs (about both mathematics and one's role within mathematics). Schoenfeld argued that thorough descriptions of these categories of problem solving activities are necessary and sufficient for the analysis of an individual's success or failure at solving a problem. He noted that mathematicians exhibited greater control skills than students in problem solving; professionals navigated among the phases of activity without getting bogged down in explorations. The categories of heuristics, control, and affective beliefs resound in other literature surrounding problem solving.

Heuristics include problem solving strategies that may delineate necessary activities or phases of activities relevant in solving problems as well as strategies within each of these activities. Past frameworks have endeavored to describe the phases of solving problems. Dewey (1938) proffered a theory of inquiry, the cyclical process by which an individual identifies a problem, chooses to enter the problem, reflects on the situation, chooses a tool to apply to the situation, evaluates the effect of the tool's application including reflection on the situation after the tool's application to determine if the problem situation has changed, and if more tools should be applied. This framework is used and discussed in detail in the theoretical perspective in the next chapter.

Pólya (1945/1957) presented a model for solving a problem that involved the steps of (a) understanding the problem; (b) developing a plan; (c) carrying out the plan; and (d) looking back. There has been some debate as to whether these steps provide a description of or prescription for solving a problem. While Dewey's (1938) description is cyclical, Pólya's is more linear (Carlson & Bloom, 2005). Other phases of activity in problem solving have been noted and utilized both in analysis and in instruction. A few are *read, analyze, explore, plan, verify* (Schoenfeld, 1985); *entry, attack, review* (Mason et al., 1982/2010); *problem scoping, designing alternative solutions,* and *project realization* (Atman et al., 2007). Problem solving strategies that may be applied while an individual is working within each of these activities or phases may include instantiating objects, manipulating objects to get a sense of a pattern, working backwards, exploiting a related problem, etc. (Burton, 1984; Mason et al., 1982/2010). Authors have noted that students may be limited in their abilities to apply heuristics effectively or to navigate amongst the activities or phases of problem solving. This may be symptomatic of students' difficulty self-regulating, monitoring, and exhibiting control (Schoenfeld, 1992).

Self-regulation, monitoring, and control in mathematical thinking have been noted as crucial. This would include an individual asking themselves questions like "Is this approach working?" or "How does this help me?" (Carlson & Bloom, 2005). Schoenfeld (1992) found that students at times may choose one approach to solve the problem and continue with that approach even if they are not making progress. Carlson and Bloom

(2005) found that mathematicians engaged in metacognition during every phase of the problem solving process and these behaviors appeared to move the mathematicians' thinking and products forward in the solution process. This may imply that identifying ideas that move the argument forward is a metacognitive activity.

Research has repeatedly noted that affective aspects such as belief, attitudes, and feelings play a role in an individual's ability to successfully solve problems and in how the solving process plays out (e.g., Carlson & Bloom, 2005; McLeod, 1992; Selden, McKee, & Selden, 2010). Affective dimensions may cause mental actions and arise from them (Selden et al., 2010). For example, once a problem is recognized, choosing to enter into a problem is an affective choice of the individual (Garrison, 2009; Glassman, 2001; Mason et al., 1982/2010). Carlson and Bloom (2005) noted that while mathematicians experienced negative emotional responses when solving problems, they were able to control them, which contributed to the mathematicians' success. In addition to emotions, beliefs about mathematics play a nontrivial role as they "shape mathematical behavior" (Schoenfeld, 1992). The work of Carlson and Bloom has served to describe how the stages of problem solving relate to the deemed necessary components of resources, heuristics, control, and affective beliefs.

The Multidimensionality of Problem Solving

Carlson and Bloom (2005) built upon the large body of problem solving literature and frameworks working to describe the problem solving behaviors of twelve mathematicians. They conducted personal, task-based interviews with mathematicians solving mathematical problems which they audio recorded. The tasks were chosen to be based on mere foundational content knowledge, challenging enough to engage research

Phase • Behavior	Resources	Heuristics	Affect	Monitoring
Orienting • Sense making • Organizing • Constructing	Mathematical concepts, facts and algorithms were accessed when attempting to make sense of the problem. The solver also scanned her/his knowledge base to categorize the problem.	The solver often drew pictures, labeled unknowns and classified the problem. (Solvers were sometimes observed saying, "this is an X kind of problem.")	Motivation to make sense of the problem was influenced by their strong curiosity and high interest. High confidence was consistently exhibited, as was strong mathematical integrity.	Self-talk and reflective behaviors helped to keep their minds engaged. The solvers were observed asking, "What does this mean?"; "How should I represent this?"; "What does that look like?"
Planning Conjecturing Imagining Evaluating	Conceptual knowledge and facts were accessed to construct conjectures and make informed decisions about strategies and approaches.	Specific computational heuristics and geometric relationships were accessed and considered when determining a solution approach.	Beliefs about the methods of mathematics and one's abilities influenced the conjectures and decisions. Signs of intimacy, anxiety, and frustration were also displayed.	Solvers reflected on the effectiveness of their strategies and plans. They frequently asked themselves questions such as, "Will this take me where I want to go?", "How efficient will Approach X be?"
Executing • Computing • Constructing	Conceptual knowledge, facts and algorithms were accessed when executing, computing and constructing. Without conceptual knowledge, monitoring of constructions was misguided.	Fluency with a wide repertoire of heuristics, algorithms, and computational approaches were needed for the efficient execution of a solution.	Intimacy with the problem, integrity in constructions, frustration, joy, defense mechanisms and concern for aesthetic solutions emerged in the context of constructing and computing.	Conceptual understandings and numerical intuitions were employed to reflect on the sensibility of the solution progress and products when constructing solution statements.
Checking • Verifying • Decision making	Resources, including well-connected conceptual knowledge informed the solver as to the reasonableness or correctness of the solution attained.	Computational and algorithmic shortcuts were used to verify the correctness of the answers and to ascertain the reasonableness of the computations.	As with the other phases, many affective behaviors were displayed. It is at this phase that frustration sometimes overwhelmed the solver.	Reflections on the efficiency, correctness and aesthetic quality of the solution provided useful feedback to the solver

Figure 1. Carlson and Bloom's (2005) multidimensional problem solving framework, p. 67.

mathematicians, allow for multiple solution paths, and sufficiently complex to lead to impasses and affective responses. They cite their audio-recordings and observations of the participants during the interviews as critical for analyzing participants' affective behaviors. From their analyses, emerged a description of the interplay between the phases of problem-solving, cycling, and problem-solving attributes. They discerned four phases of problem solving (*orientation, planning, executing*, and *checking*). Within the planning phase was a subcycle of *conjecturing*, *imagining*, and *evaluating*. In this planning subcycle, the mathematicians hypothetically played out and evaluated proposed solution approaches. The mathematicians would rarely solve a problem by working through the phases linearly. The planning cycle would repeat until the mathematician found a solution approach that could be effective, and participants would cycle through the plan-execute-check phases multiple times within a single problem. Carlson and Bloom noted the resources, heuristics, affective behaviors, and monitoring behaviors exhibited during each phase of the problem-solving cycle that are summarized in a two-dimensional table displayed as Figure 1.

Proof as a Particular Type of Problem Solving

Selden and Selden (2013) considered two aspects of a final written proof: the *formal-rhetorical part*, the part that depends on unpacking and using the logical structure of the statement, associated definitions, and earlier results, and the *problem-centered part*, that depends on genuine mathematical problem solving, intuition, and a deeper understanding of the concepts. They maintain there is a close relationship between problem solving and proof, but having good ideas for how to solve the problem-centered part of the proof is not sufficient for having a proof. They cite two kinds of problem solving that can occur in proof construction: solving the mathematical problems and converting an informal solution to a formal mathematical form.

Weber (2005) considered proof from the perspective as a problem solving task, defining a mathematical problem as a "task in which it is not clear to the individual which mathematical actions should be applied, either because the situation does not immediately bring to mind the appropriate mathematical action(s) required to complete the task or because there are several plausible mathematical actions that the individual believes could be useful" (pp. 351-352). With this lens, Weber was able to describe three qualitatively different types of proof productions: *procedural proof productions, syntactic proof productions,* and *semantic proof productions*. He went on to describe learning opportunities afforded by each type of proof production.

Focusing in on the impasses, incubation, and insight that may occur when one is solving proof problems in mathematics, Savic (2012) described what mathematicians do when they reach an impasse in a proving task. These impasses or points of getting stuck are opportunities for incubation and the generation of new ideas (Byers, 2007). By having mathematicians work on mathematical proofs on their own utilizing Livescribe, Savic was able to observe his participants taking breaks from their work which could be considered incubation periods to recover from impasses. He noted that mathematicians used methods that occurred earlier in the session, used prior knowledge from their own research, used a data base of proving techniques, did other problems from the problem set and came back, and generated examples or counterexamples to recover from impasses. Additionally, they did other mathematics unrelated to the present task, walked around, did tasks unrelated to mathematics, went to lunch, and slept to recover from their impasses. Savic's findings suggest that observing mathematicians' moments of creativity may require data collection outside the traditional clinical interview setting.

Building from the impasse study, Savic (2013) sought to determine if Carlson and Bloom's (2005) Multidimensional Problem-Solving Framework could be used to describe the proving process. He analyzed the Livescribe proof constructions of a professional mathematician and a graduate student. Savic found that for most portions of the transcripts the framework could be used to code and describe the proving process. However, he found some differences including the mathematician cycling back to orienting after a period of incubation to reorient himself to reconsider all of the given information and the graduate student not completing the full cycle of planning, executing, and checking. He suggested the four phases of Carlson and Bloom's framework were important for the proving process. He hypothesized additional problem solving phases could be added to the framework including incubation and re-orientation noting further research as needed.

In the above section, I described the current knowledge of mathematical problem solving because the proof tasks that the participants were asked to complete were genuine problems for the participant as described by Weber (2005). It is within working on these problems that new insights and creativity occur which are the primary interest of this proposed study. The presented frameworks for problem solving agree that problem solving occurs in stages (Carlson & Bloom, 2005; Pólya, 1954) in a more or less cyclical nature (Carlson & Bloom, 2005; Dewey, 1938). It is possible that the solution process for proof construction problems may play out in a slightly different fashion than the phases already identified (Savic, 2013). However, the individual's access to resources and heuristics, their abilities to exhibit self-monitoring, and their attitudes, beliefs, and emotions will play a role in their ability to complete the proof tasks (Savic, 2013; Schoenfeld, 1985, 1992; Selden et al., 2010; Selden & Selden, 2013). While past empirical studies have conceived of proof construction as a particular type of problem solving task, much more is yet to be known. Selden and Selden (2013) identified some areas related to proof as problem solving that could use more research:

> These are: how informal arguments are converted into acceptable mathematical form; how representation choice influences an individual's problem-solving and proving behavior and success; how students' and mathematicians' prove theorems in real time (especially when working alone); how various kinds of affect, including beliefs, attitudes, emotions, and feelings, are interwoven with cognition during problem solving; which characteristics make a problem non-routine (for an individual or a class), that is, what are the various dimensions contributing to non-routineness; and how one might foster mathematical "exploration" and "brainstorming" as an aid to problem solving. (pp. 329-330)

This research sought to add insight to one of these areas, namely how mathematicians prove theorems in real time. Since this research endeavored to describe how personal arguments evolve paying special attention to the tools utilized when new ideas are formed, I hypothesized that this research could also provide insights into how informal arguments are converted into acceptable mathematical form. Such insights may be useful for practitioners because, as described in an earlier section, students have been shown to struggle utilizing their ideas in constructing proof. Students are less likely than those experienced with constructing proof to look for ways to use their informal understandings to develop ideas for how to prove a statement and are at times unsuccessful in identifying the ideas that could potentially be useful in developing an argument and often struggle to connect their informal understandings to their formal proof constructions (Alcock & Weber, 2010; Raman, 2003; Raman et al., 2009).

CHAPTER III

METHODOLOGY

This research sought to describe the evolution of mathematicians' personal arguments as they construct mathematical proof. The following research questions guided the research:

Q1 What ideas move the argument forward as a prover's personal argument evolves?

Q1a What problematic situation is the prover currently entered into solving when one articulates and attains an idea that moves the personal argument forward?

Q1b What stage of the inquiry process do they appear to be in when one articulates and attains an idea that moves the personal argument forward? (Are they currently applying a tool, evaluating the outcomes after applying a tool, or reflecting upon a current problem?)

Q1c What actions and tools influenced the attainment of the idea?

Q1d What were their anticipated outcomes of enacting the tools that led to the attainment of the idea?

Q2 How are the ideas that move the argument forward used subsequent to the shifts in the personal argument?

Q2a In what ways does the prover test the idea to ensure it indeed "does work"?

Q2b As the argument evolves, how is the idea used? Specifically, how are the ideas used as the participant views the situation as moving from a problem to a more routine task?

This chapter describes the theoretical perspective and framing of the study and the methods utilized to collect and analyze data.

Research Strategy

Crotty (1998) indicated four elements composing one's research framework: epistemology, theoretical perspective, methodology, and methods. The researcher's epistemological stance indicates the researcher's assumptions about the nature of knowledge and how we come to know what we know. This research was guided by the epistemology of *constructionism* because the research focuses not only on the activities involved with making meaning (building personal mathematical arguments) but also activities involved with communicating meaning to others (writing proofs). As Crotty indicated, the focus of constructionism "includes the collective generation [and transmission] of meaning" (p. 58). Constructionism emphasizes how culture influences our views of the world. Professional mathematicians work within the culture of the mathematics community. While their ways of making meaning of problems is an activity of the individual mind, it is shaped and influenced by the mathematics community. Moreover, the activity of proof writing involves making public one's own arguments, and a valid proof must be acceptable to the mathematics community at large. I believe we construct knowledge through our interactions with our environments and experiences and these constructions were negotiated in our interactions with society and the community at large.

Since this research was intended to explore how individuals develop mathematical proof under the constructionism epistemology, interpretivism (Crotty, 1998) was the theoretical perspective for this study. Interpretivism was appropriate because this research sought descriptions of the meaning of mathematicians' behavior situated historically and culturally within the mathematics community. My means of enacting the interpretivism perspective was based on John Dewey's (1938) theories of inquiry and instrumentalism (Hickman, 1990) integrated with Toulmin's argumentation theory and a perspective on creative mathematical thinking. The theoretical framework is described in more detail in the next section.

Because the goal of this research was to understand multiple individuals' common experiences of constructing mathematical proofs, the methodology that best framed this research was phenomenology (Creswell, 2007) which "describes the meaning for several individuals of their lived experiences of a concept or a phenomenon" (p. 57). The phenomenon of interest here is the development and incorporation of new ideas to move a personal argument forward. In this study, I sought to understand mathematicians' personal arguments in proof construction. These personal arguments are embodiments of how the mathematician understands the situation encompassing the ideas he or she sees as pertinent to proving or developing a proof of the statement. The data were collected through interviews, observations, and document analysis of written work. The methods will be further elaborated in this chapter.

Theoretical Perspective

The theoretical perspective that framed this research was interpretivism, specifically based on John Dewey's (1938) theories of inquiry and instrumentalism (Hickman, 1990) integrated with Toulmin's argumentation theory and a perspective on creative mathematical thinking.

Theory of Inquiry

John Dewey's (1938) theory of inquiry gives us a means of understanding how knowledge is created and how it is perceived useful in problem-solving situations. In periods of inquiry, one is actively engaged in reflecting on problem situations, applying tools to these situations, and evaluating the effectiveness of the tools. Knowledge in the "honorific sense" (Dewey, 1938) is the outcome of active, productive inquiry (Hickman, 1990). The purpose of this research was to understand how ideas emerge in the proof construction process; ideas that move the proof along can be viewed as a certain kind of *tool* to be applied to the problem situation of developing the mathematical proof.

Hickman (1990), citing Dewey (1938), described how individuals' experiences fall into two categories: everyday experiences and inquirential experiences. In non-cognitive, everyday experiences, our actions and responses to stimuli are immediate because we do not sense a problem. There is engagement in the experience but no reflection (Hickman, 1990). These everyday experiences do not require a conscious recognition of the relationship between actions and their consequences (Glassman, 2001).

Everyday experiences may be "technological" in the sense that the individual applies tools to the situation but not inquirential. For example, a master electrician will apply tools to repairing a broken light switch but choosing which tools to apply and evaluating the effectiveness of the applied tool is not necessary since the actions have become so habitual (Hickman, 1990). As a more educational example of non-inquirential tool-use, Mason and colleagues (1982/2010) indicate that it is possible for students to engage in a certain mathematical exercises and only have the "appearance of thinking" because they merely apply certain rules or techniques given by the teacher.

However, during these non-cognitive experiences (everyday life), problematic situations may occur. There may be a gap, tension, or unexpected outcome. There is an intuitive realization of a pervasive quality that at first we can describe as a problematic situation, but the individual may not be able to describe or acknowledge the existence of a problem. Circumstances provide problematic situations, but the individual makes problems. If these intense and unresolved (Hickman, 1990) situations have been deemed a problem and the individual expresses interest in some objects of the situation, the process of inquiry, the second type of experience, begins. There needs to be a construction and an affective element of interest in actually solving the problem and entry for a problematic situation to become a problem to be solved (Garrison, 2009; Glassman, 2001; Mason et al., 1982/2010).

Inquiry is the intentional process to resolve doubtful situations, the systematic invention, development, and deployment of tools (Hickman, 2011). These doubtful situations are the purposes to which tools are applied. Throughout the entire process, the individual has an "end-in-view" or a desired outcome (Garrision, 2009; Glassman, 2001; Hickman, 2009). These ends-in-view provide tentative consequences for which the inquirer must seek the means (tools and ways to apply tools) to attain them. As inquiry proceeds, the inquirer may modify their ends-in-view. In this framework, I refer to these "ends-in-view" as the purposes to which tools are applied.

The process of active productive inquiry involves reflection, action, and evaluation. Reflection is the dominant trait. The inquirer must inspect the situation, choose a tool to apply to the situation, and think through a course of action. Garrison (2009) described that "a collection of data is the first product in the process of inquiry"

(p. 92). The data are the means or clues to and of something to be attained. The inspection of the traits of the situation involves reflection, a "going outside" the situation to something else to gain leverage for understanding the situation (Hickman, 1990). After the data are collected, a hypothesis or proposition may be formed about what will happen when certain operations or tools are applied to the situation. These propositions are themselves tools (Hickman, 2011). After this initial reflection of what could happen, the inquirer performs an action, applies the tool. Dewey sometimes refers to these actions as "fulfilling experiences" (Prawat & Floden, 1994). Either during or after the fulfilling experience, the inquirer evaluates the appropriateness of the chosen application of the chosen tools (Hickman, 1990). Tools, applications of tools, and evaluation are further described in later sections.

Dewey (1938) describes knowledge in the "honorific sense" as an outcome of inquiry. Knowledge in the "honorific sense" is described as the satisfactory resolution of a problematic situation that yields a warranted assertion (Hickman, 2011). This knowledge may be the construction and understanding of tools or the construction of new problems. At the conclusion of an inquiry, an inquirer may enter and enjoy "non-reflective" experience of the first kind. However, these periods of non-reflection tend to mature when new problematic situations are encountered, and the entire process begins again (Hickman, 1990). The cyclic nature of the inquiry process is represented in Figure 2. Inquiry begins by reflecting on a situation. The cloud refers to considering the perceived problem. Based on the observations, one considers possible tools to apply to the problem and imagines applying the tools to the situation. The cycle is repeated until a tool is chosen. In Action, the individual actually applies the tool to the situation, and evaluates the effectiveness of the tool. The individual returns to reflection as he or she must reflect on the situation after the tool is applied. If the problem is still unsolved or a new problem is perceived, the process begins again.

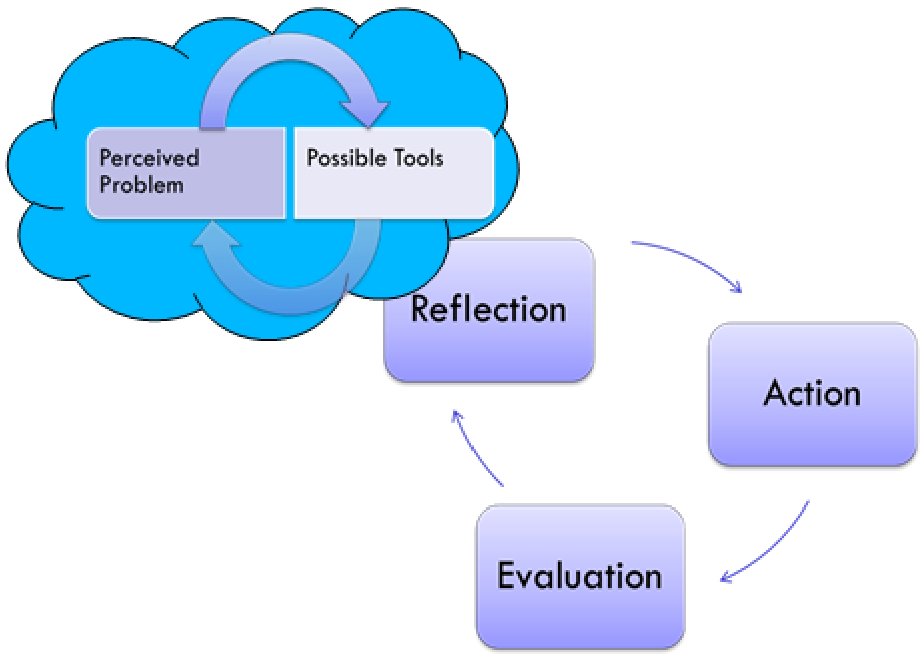

Figure 1. A representation of the cyclic phases of Dewey's theory of inquiry.

The assumptions and conceptions of the inquirer will play a role in determining if situations are identified as problems and how the inquiry plays out. For example, if a student has an empirical proof scheme (where one is convinced by an empirical argument; Harel & Sowder, 1998), he or she may engage in inquiry in determining if a statement is true by actively looking for examples to apply, and inquiry process may be complete from the student's perspective. The student may then present these examples as proof of the statement's validity. The proof may not be correct or sufficient in the eyes of the student's teacher or the mathematics community, but the student has not seen her solution as a problem or her application of presenting a worked example as an inappropriate tool for the situation. Successful inquiry may have occurred in generating these tools (the used examples), but inquiry did not occur in the generation of proof for the statement.

As researchers, we will need to account for an affective dimension in the participants' thought processes. The participant must acknowledge the problem and choose to enter into it (Burton, 1984; Dewey, 1938; Glassman, 2001; Mason et al., 1982/2010). Additionally, personal affect throughout the inquiry process may be noteworthy (Carlson & Bloom, 2005). We may find that even if a participant identifies an aspect of the proof as a problem, or as incomplete, he or she may still choose not to engage in the process of solving the problem. It is probable that engaging in proving will have moments of both non-cognitive and inquirential experiences. Professionals, for example, may engage in habitual practices. It still could be fruitful to describe the application of tools by the professional.

What are tools? A tool is a theory, proposal, action, or knowledge chosen to be applied to a problematic situation. An inquirer reflects on the problem and chooses an appropriate tool to apply to it. During this reflection, the inquirer may contemplate the feasibility of using a tool before applying it to a situation. Even if a tool is chosen and applied, it remains an experiment in the inquiry process (Hickman, 1990) as it may be modified and is evaluated. At times the tool itself may be deemed problematic, i.e., the inquirer may have troubles applying the chosen tool or the tool may not perform in the expected manner. Therefore, the tool must be reflected upon and new tools must be chosen and applied to resolve what is problematic about the tool. Dewey describes how tools (or means to an end) can be successful.

> Present actual means are the result of past conditions and past activities. They operate successfully, or 'rightly' in (1) the degree in which existing environing conditions are very similar to those which contributed in the past to formation of the habits, and (2) in the degree in which habits retain enough flexibility to readapt themselves easily to new conditions. (Dewey, 1938, p. 39)

Tools may be used in ways that are not inquirential in situations that are habitual and routine. For example, professional mathematicians solving a routine problem, such as proving two algebraic groups are not isomorphic, may apply ideas, theorems, and strategies methodically without needing to reflect on what actions to take in order to solve the problem.

Certain tools or actions may be applied to certain situations without the individual reflecting upon the situation or even needing to identify the situation as a problem. For example, experienced mathematicians may immediately try to instantiate a new definition as a means of trying to understand it (Alcock, 2004). The choice of tool used to bring about understanding is immediate. What follows may require reflection and inquiry. The actual act of instantiating a particular object may or may not be difficult for the mathematician, and after instantiating the mathematician may or may not understand the definition better. However, the initial action of trying to instantiate an object to get a better sense of the definition is second nature.

Habits may result from previous, repeated practice of the individual, or they may come from learning from the experiences of others. In the context of mathematics education, students may be taught that particular procedures must be applied to certain contexts. Therefore, when seeing such a problem, they immediately may apply the given procedure. Students themselves may not have engaged in active productive inquiry to gain such knowledge.

Tools may be inappropriate for the situation but forced on the situation anyway (Hickman, 1990). This can occur in two situations: (a) The individual may not be engaged in inquiry but just applying tools to the situation without first reflecting upon their appropriateness and (b) the individual may be engaging in inquiry but has limited experience with the situation.

In order to be considered a tool, an object must be used to do some sort of work. An object may be a tool in certain situations but not a tool in others; technology is context dependent (Hickman, 1990, 2011). Consider the study conducted by Iannone et al. (2011). Two groups of students were introduced to a novel concept in real analysis. One group read through a series of twelve examples of the concept, and the other group was asked to complete twelve example generation tasks involving variations of the concept. All students were then asked to complete four proof tasks related to the same concept; the proof constructions of both groups were given a correctness score by the researchers. According to a statistical analysis, there was no significant difference between the example generation group and the reading group on scores on the four proof tasks. The authors speculated that one explanation for why generating examples did not contribute to valid proof production in real analysis tasks was that most students generated their examples by trial and error (Antonini, 2006) and therefore were not connecting the example generation to the following proof constructions. In their research, Iannone and company introduced students to the strategy of example generation, but the example generation was not performed with the purpose of "doing work" in the context of the problem of proving the given statement. In this sense, for the individual inquirers, example generation was not used as a tool to compete the proof tasks.

In Dewey's (1938) view, intellectual tools derive from the individual's social history (Glassman, 2001). The tools are products of past inquiries of the individual or from the discourse community in which the inquiry resides. For the mathematician, the tools come from the mathematics community at large. For the student in the mathematics classroom, the tools come from past classroom experiences. Sandefur et al. (2012) cited the quality of having the "experience of utility of examples in proving" (p. 15) as an aspect that contributed to students' use of examples in proving; in their past classroom experiences, students learned that generating examples can be helpful in constructing proof. One may not yet be an expert in using a tool, but by performing inquiry on the tool itself, one becomes more adept. He or she can gain knowledge about how to apply the tool and to what situations the tool should be applied.

The environment plays a role in that the limits of the environment are considered in the period of reflection when the inquirer thinks through which tools to apply. For example, if opening a bottle, I may want to use a bottle opener, but I realize that I do not have one, and I would have to drive to the store to go purchase one; so instead, I choose to search for some other tool to do the job. I am aware of the existence of a bottle opener from my past experiences so the tool is available for me to reflect on, but I realize it is unfeasible to apply a bottle opener in this situation. The interview environment may play a role in the tools the individual chooses to use. For example, in the exploratory study, Dr. Kellems first mentioned a theorem would be potentially useful to apply to the linear algebra task. However, Dr. Kellems judged it as inappropriate due to its obscurity and it likely that a proof task posed by two graduate students could be solved by simple means.

Tools utilized in proof construction. The ideas that the mathematician perceives as moving the argument forward are "tools" in that the mathematician sees them as useful in "doing work" in achieving some outcome.

> The possible solution presents itself, therefore, as an *idea,* just as the terms of the problem (which are facts are instituted by observation). Ideas are anticipated consequences (forecasts) of what will happen when certain operations are executed under and with respect to observed conditions. (Dewey, 1938, p. 39)

These ideas as tools were the primary focus of the investigation in this study, but the mathematicians drew upon other tools in developing and utilizing the ideas. For instance, studies have identified certain abilities and knowledge as necessary for students to produce proof: knowledge of and the ability to appropriately use various symbolic representations (Selden & Selden, 2008), the ability to convert between intuitive understandings and formal logical reasoning (Weber & Alcock, 2004), and an understanding of the logical structure of mathematics (Selden & Selden, 2008). The above may be considered tools in that they may facilitate the generation and implementation of ideas perceived to be useful for the construction of the proof.

Purposes of tools' application. As noted earlier, in inquiry, tools are applied with an "end-in-view" or for an intended purpose. According to Dewey, actions performed are a means of ontological change (Prawat & Floden, 1994). There is a situation that the inquirer deems problematic. The tool is applied to reorganize the situation to relieve the tension that the problem caused. Purposes may be nested within each other as there may be problems within problems. For instance, one may choose to apply a computation to an example (Tool A) to get a sense of why the statement works

(Purpose A). If there is an error in the computation, one may choose to try to fix the computation error (Purpose B) and would have to apply other tools to fix the error. The same tool may be applied to different purposes, and various tools may be applied to a single purpose. Planning a course of action, gaining understanding, attaining new insights, and formalizing an argument are possible purposes in proof construction.

How can tools be applied? The application of a tool indicates how a tool is used. One tool can be applied in multiple ways. Choosing how to apply a tool occurs during periods of reflection in a similar way to how a tool is chosen. The application is an experiment that is evaluated and may be modified. At times a prover refers to "going down a path", in terms of the methods or heuristics chosen to prove a statement. We can think of the "paths" chosen or proposed as an application of tools, in that in each application they may employ various tools in certain ways depending on the path and also the purpose. The paths are evaluated as being useful or not.

As earlier indicated, a tool may be applied unskillfully or inappropriately. In this case, the tool or the application of the tool may itself become a problematic situation subject to further inquiry and reflection. Else, the inquiry may not be brought to a productive close; inquiry may be postponed for a period due to frustration (Hickman, 1990). There is the possibility of the failure of artifacts to do their work (to be meaningful). A possible cause of this failure is the neglect on the part of the inquirer to continue to connect the means and the ends; meaning the inquirer is just applying facts and tools with no real purpose or end in view.

As an illustration of how one tool can be applied in different ways, consider the tool of instantiation. Example-use is a tool, but there are different applications of examples. Alcock and Inglis (2008) described four ways that examples may be used in evaluating and proving that have been identified in the literature: (a) generic examples (reasoning with an example that can be generalized to a particular class of mathematical objects), (b) crucial experiments (checking a conjecture against an example that has no special properties), (c) naïve empiricism (justifying a conjecture by checking it against a small number of examples), and (d) using counterexamples to refute conjectures. One may instantiate an example and may manipulate it, use the example to test a conjecture, use examples and non-examples to discern and expose properties of the mathematical objects, or use an example to pilot a proposed manipulation.

Evaluation of tools. Systematic inquiry features periods of evaluation either after the application of the tool or while the tool is being applied (Prawat & Floden, 1994). Tools are tested against the circumstances, and the circumstances are tested against the tools (Hickman, 1990). For instance, a certain tool may not be appropriate for a given circumstance, or the tool may provide additional insight into the traits of the circumstance. Carlson and Bloom (2005) characterize similar mental actions of "reflecting on the effectiveness of the problem-solving process and products" (p. 48) and call it "monitoring".

Instances of evaluation can occur before applying a tool, while the tool is being applied, and after the tool has been applied. Before applying a tool, an inquirer considers if applying a tool is feasible or will be useful; he or she thinks through possible plans of attack. While the tool is being applied or after it is applied, "the worth of the meanings, or cognitive ideas, is critically inspected in light of their fulfillment" (Prawat & Floden, 1994, p. 44).

The criteria used to evaluate tools and applications are products of the discourse community (Prawat & Floden, 1994); the decisions made in an evaluation may vary depending on the community in which the prover participates. The criteria can be revised and are subject to change. Certain tools and ideas may be accepted or rejected based on the values the prover possesses (Dewey, 1938).

As Hickman (2011) indicated, technology is context dependent; this means the conditions the prover is in will play a role in the evaluation of the inquiry at hand. For example, it is likely that the same mathematician's proving process will vary depending on his or her audience, time constraints, etc. What the prover deems as useful or resolved will vary if she is proving just to convince herself, to convince a colleague, or to convince a classroom full of students. These situations in fact may entail wholly different problems for the inquirer.

It must also be noted that evaluation is in fact a value judgment (Dewey, 1938). Two tools may be equally effective, but an individual may choose one over the other due to personal preference. It has been noted that some individuals use examples in proof construction more than others (Alcock & Inglis, 2008; Alcock & Simpson, 2005). This may be due to personal values, styles, and preferences.

What are the consequences after the tool is applied for the prover? After applying a tool, the prover evaluates the effectiveness and usefulness of the tool and re-inspects the situation. Regardless of the outcome of the application of the tool, the prover will have gained new information. It makes sense that three possible outcomes of an evaluation are the problem has found to be unresolved, the problem has been resolved, or the problem has changed.

If the prover has deemed that the situation is not yet resolved, he or she must reflect again on the problem in light of the new information and may choose a different tool, attempt to apply a tool in a different way, or modify the predicted outcome (proposition) of applying such tool. The prover may not change the chosen tool or application, but may deem that the tool itself is problematic and engage in inquiry into the chosen tool or its application.

In evaluating, the prover may find that the current problem is resolved. Dewey (1938; Hickman, 1990) indicates that knowledge or judgments are consequences of successful inquiry which may be the knowledge of if the statement is or is not true, knowledge of how the given statement may relate to other objects and statements in the previously known mathematical structure, a more formulated concept image, a greater proficiency in applying various tools, and so forth. Dewey (Hickman, 1990) indicates there is an enriched period or a feeling of satisfaction from a completed inquiry. These enriched periods mature and then may become unstable, and therefore the inquiry process may begin again.

Upon evaluation, the prover may have deemed that the application of the tool changed the situation. In this case, the prover may not consider the situation as problematic and therefore go back to "everyday" experience where the situation may become problematic. The prover may be aware of the problem but choose not to enter inquiry in the sense of a moral judgment (Dewey, 1938). The prover may enter the problem, beginning a new cycle of inquiry in which the tools, and the knowledge constructed in previous inquiries are available if deemed useful.

I have attempted to give a description of Dewey's (1938) theory of inquiry, working to specifically apply it to the problems of constructing mathematical proof. The next section describes how this research took the perspective that the process of constructing mathematical proof consists of more than writing formal deductive statements but also includes informal argumentation. Toulmin's theory of argumentation is used to describe an arguments structure, and I cite other literature that outlines the relationship between argumentation and mathematical proof. Lastly, there is a description of how these theories have led to the decision to include the term "personal argument" as a descriptor for the prover's concept image of the problem of constructing a mathematical proof.

Argumentation in Proving

This research is interested in the development and implementations of ideas that the prover deems useful during the process of constructing mathematical proof. It is understood that when the proof problem is unfamiliar, the process involves more than the writing of the logical statements that form the proof product. There is a process and it includes "thinking about new situations, focusing on significant aspects, using previous knowledge to put new ideas together in new ways, consider relationships, make conjectures, formulate definitions as necessary and to build a valid argument" (Tall et al., 2012, p. 15). The proof construction process can be seen as involving the practice of *argumentation* (Aberdein, 2009; Inglis et al., 2007; Lakatos, 1976; Pedemonte, 2007). Argumentation has been defined in a variety of ways.

> The whole activity of making claims, challenging, them, backing them up by producing reasons, criticizing those reasons, rebutting those criticisms, and so on. (Toulmin, Rieke, & Janik, 1979, p. 13)

> An act of communication intended to lend support to a claim. (Aberdein, 2009, p. 1)
>
> Argumentation is a verbal, social and rational activity aimed at convincing a reasonable critic of the acceptability of a standpoint by putting forward a constellation of propositions justifying or refuting the proposition expressed in the standpoint. (Grootendorst & van Eeemeren, 2003, p. 1)

Argumentation can be thought of as the activity of supporting claims to convince oneself, convince a friend, and convince a skeptic (Mason et al., 1982/2010). Mathematicians have developed an internal skeptic in that they not only seek to convince others of the statement but to make sense of the mathematics itself (Tall et al., 2012).

Theory of argumentation. Toulmin (1958; Toulmin et al., 1979) developed an approach to analyzing arguments that focuses on the semantic content and structure. For Toulmin, reasoning refers to "the central activity of presenting the reasons in support of a claim" (Toulmin et al., 1979, p. 13). His view of argument includes more that the practice of logical deduction.

Toulmin's scheme classifies statements of an argument into six different categories. The claim (C) is the statement that an assertor wishes to argue to their audience. The grounds (G) (at times termed "data" (Inglis et al., 2007; Pedemonte, 2007) are the foundations on which the argument is based. The warrant (W) is the justification that the grounds really do support the claim. Backing (B) presents further evidence that the warrant appropriately justifies that the data support the claim. The modal qualifiers (Q) are statements that indicate the degree of certainty that the arguer believes that the warrant justifies the claims. The rebuttals (R) are statements that present the circumstances under which the claim would not hold. This structure is illustrated in the diagram in Figure 3.

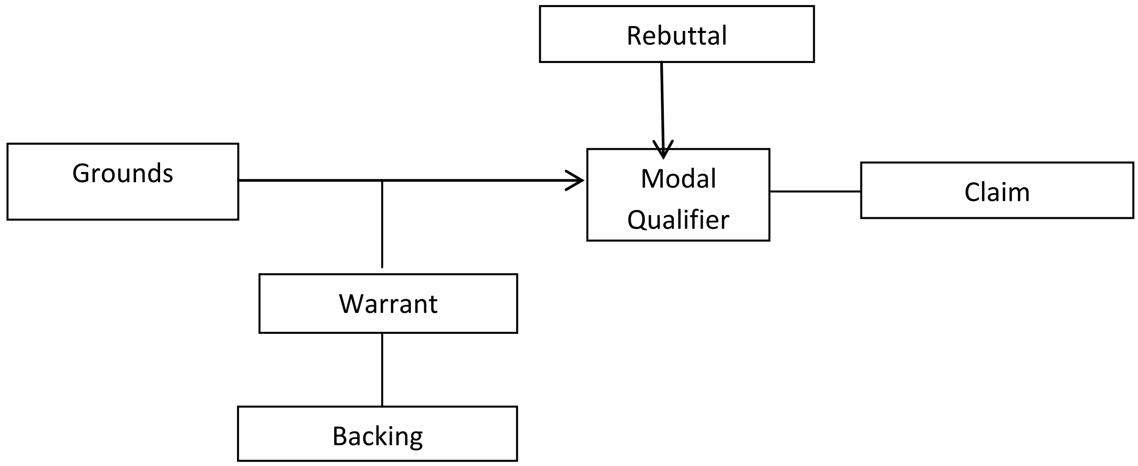

Figure 2. Toulmin argumentation structure.

The diagram in Figure 4 is an example of an argument that Oakland is a shoo-in for the Super Bowl (C) (Toulmin et al., 1979, p. 88). Oakland has the strongest and most-balanced offensive and defensive squads (G), and only a team that is strong in both offense and defense can be tipped for the Super Bowl (W) according to past records (B). One presumes (Q) that Oakland is a shoo-in for the Super Bowl (C) unless Oakland is plagued by injuries or the other teams do some quick and costly talent buying or there is a general upset of the form book (R). This sports example shows us that an argument outside formal logic can be mapped to this structure.

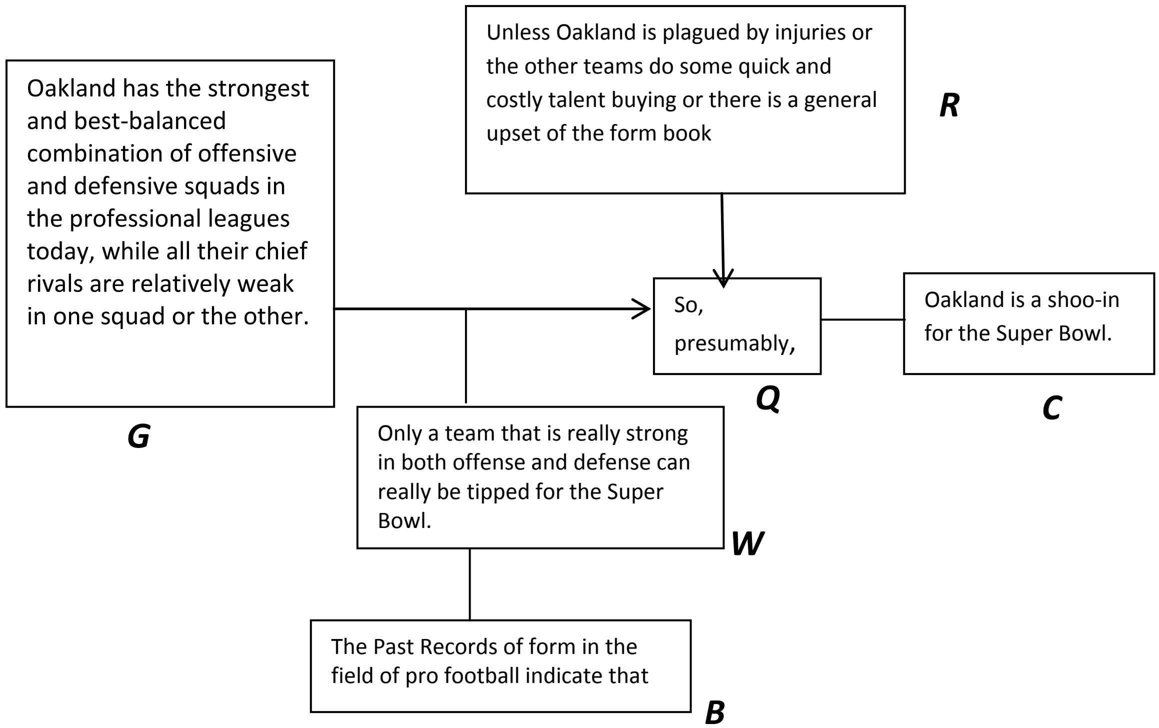

Figure 3. Example of Toulmin layout.

The Toulmin argumentation structure gives us language that can be used to describe how the mathematicians progress in their arguments. Researchers have made use of this structure (Inglis et al., 2007; Pedemonte, 2007) to determine how mathematicians and students use warrants and qualifiers in their arguments, to describe a relationship between proof and argument, and to determine the degree to which they find arguments convincing. The diagram in Figure 5 illustrates an example in which this argumentation structure is implemented to analyze a mathematics graduate's argument against the conjecture, "If p_1 and p_2 are prime, then the product p_1p_2 is abundant" (Inglis et al., 2007, p. 6).

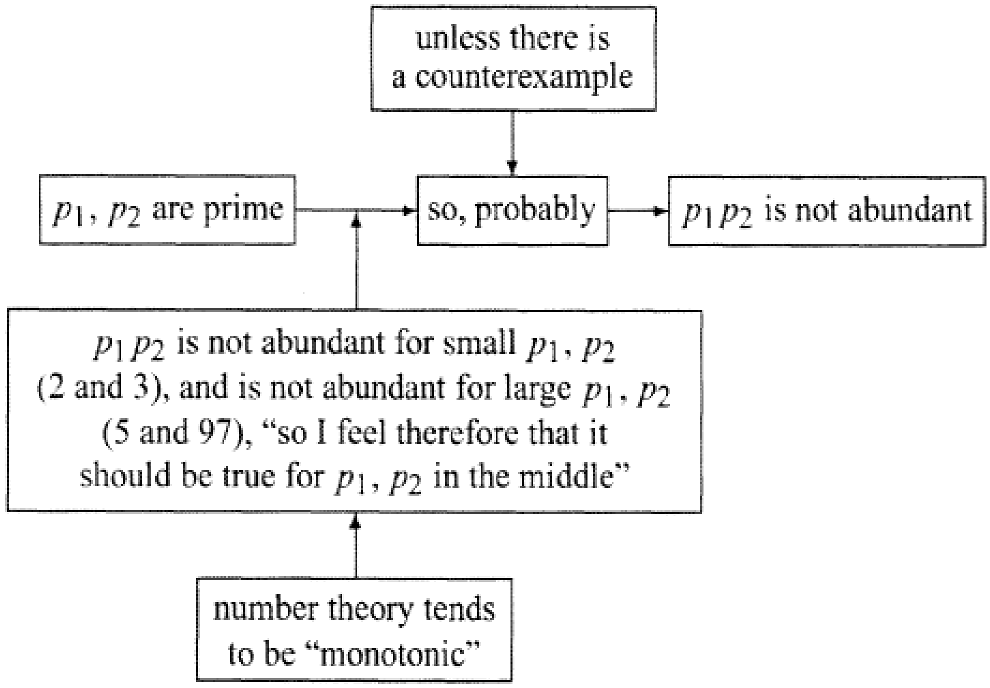

Figure 4. Toulmin analysis of mathematical argument (Inglis et al., 2007, p. 8).

I made use of Toulmin's structure to describe the roles that various aspects played for the mathematicians in their evolving arguments and to give structural descriptions of in what ways the individual's personal argument evolved. Inglis et al. (2007) emphasized the importance of considering the modal qualifier because the type of warrant used would affect to what degree the arguer would be convinced that their claims implied the conclusions. This points to the possibility that when constructing mathematical proof, professional mathematicians may engage in argumentation that is not formal proof. For this reason, this research made use of Toulmin's full model.

Personal argument. Aberdein (2009) used the term *argument* to mean an act of communication intended to lend support to a claim. Aberdein stated that proofs fit within this definition but provided for mathematical practices in the process of proving that go beyond the product that may also be considered argument. Aberdein included choosing research problems, choosing methods to tackle problems, incomplete proof sketches in applying these methods, and the dialectic between readers of the proof and the author. Thurston (1994) noted that in addition to logic and deduction, human thinking includes human language, vision, spatial sense, kinesthetic sense, intuition, association, and metaphor. Formal proof is a subset of argumentation. The informal arguments that lead to formulating a conjecture may be used in the construction of the proof (Boero et al., 1996; Lakatos, 1976; Pedemonte, 2007).

Since argument has been described as encompassing both informal and formal arguments and also arguments to convince oneself or another, this research seeks to describe the proving process as an evolving personal argument. This research adopts Aberdein's (2009) definition of argument as any act of communication meant to lend support to a claim. The argument evolves in that new ideas are incorporated and utilized. I use *proof* to denote the written end-product meant for an "other". A proof is a sub-type of argument that uses deductive-type warrants and the modal qualifier is absolute (Inglis et al., 2007). The *personal argument* encompasses all thoughts that the individual deems relevant to making progress in proving the statement. It is a subset of the entire concept image of a proof situation.

Tall and Vinner (1981) described a concept image as the total cognitive structure that is associated with the concept. It can include mental pictures, instantiations of concepts, and personal definitions. The concept image is built up through experiences, and it is not always coherent, different stimuli can activate different aspects of the concept image. Tall and Vinner term the evoked concept image the portion of the concept image that is activated at a particular time. An individual may evoke conflicting concept images, but only when conflicting aspects are evoked simultaneously may there be any sense of confusion. Tall and Vinner describe the concept definition as the form of words used to specify that concept. The personal concept definition is the individual's reconstruction of a definition, and the formal concept definition is the one accepted by the mathematical community at large. Professionals in a given mathematical field may have multiple personal concept definitions that are consistent with the formal concept definition (Alcock, 2008). As an extension of the construct of concept image, John and Annie Selden (1995) conceived of a *statement image,* a rich mental structure that includes concepts, examples, visualizations, and so forth that one associates with a statement.

The statement image involves the individual's total cognitive structure associated with the proof situation. There may be aspects of the statement image that are evoked at particular times. There are also aspects of the statement image that the individual may perceive to be more or less central to his or her aims in moving the argument forward. The personal argument is the particular subset of the statement image which the individual views as central to his or her aims in developing the argument. The focus of this study is to describe how mathematicians' personal arguments evolve in that we are looking to see how they incorporate and use new ideas that they view as better enabling their arguments forward.

The personal argument is a subset of the entire statement image, but its boundary may be fuzzy. The individual may determine as some aspects of the statement image as central to the personal argument, but there may be aspects that lie on the periphery.

George Lakoff (1987) promoted the idea that “human categorization is essentially a matter of both human experience and imagination-of perception, motor activity, and culture on the one hand, and of metaphor metonymy, and mental imagery on the other” (p. 8). Lakoff observed that some categories do not have gradations of membership while others do. For example, some men are neither clearly tall nor clearly short; they are tall to some degree. Categories that have somewhat fuzzy boundaries are termed graded categories. In these categories, there are central members whose are deemed to fully belong to the category.

Taking these ideas from Lakoff (1987), a personal argument is a graded category. The individual is the one who categorizes certain ideas and aspects of the concept image of the proof situation as relevant or useful to him or her in developing the argument. The category is graded in that there may be different degrees to which the individual incorporates ideas. Some ideas may be peripheral and others more central; as the argument develops an ideas degree of centrality to the argument may change. This study seeks to describe this evolution of the personal argument.

Mathematical Creative Thinking

I believe that the development, testing, and implementation of new ideas that serve the individual’s aims in producing mathematical arguments are acts of mathematical creativity. These moments where creative thought happens are worth studying and describing with empirical evidence. Indeed mathematicians (Byers, 2007) and mathematics educators (Lithner, 2008) have written about what it means for one to think creatively in mathematics and what aspects of mathematics are creative. Defining creativity can be challenging (Haylock, 1997), but there are characteristics of mathematically creative thought that can be defined and identified.

Lithner (2008) posed a framework distinguishing creative from imitative reasoning. In this framework, reasoning is the way of thinking adopted to produce assertions and reach conclusions. Creative reasoning must fulfill the conditions of *novelty, flexibility, plausibility,* and *mathematical foundation.* Creative reasoning is used in non-routine problem solving or *novel* tasks. It can only be observed if the individual is encountering a situation that he or she deems challenging. Creative reasoning must have *flexibility* in that it admits different approaches and adaptations to the situation. Creative reasoning is shown to be *plausible* because arguments are included to support the strategy choice and why conclusions are true. Creative reasoning is founded on mathematical properties.

Byers (2007), a mathematician, wrote that ideas were the organizing principles of mathematical thought and that

> The creative in mathematics is expressed through the birth of new ideas. These ideas may consist of a new way of thinking about a familiar concept or they may involve the development of an entirely novel concept. An idea is usually at the heart of a mathematical argument but an idea may even entail a new way of looking at a whole area of mathematics. Creativity in mathematics is inseparable from ideas. (p. 191)

Byers illustrated his meaning of mathematical ideas using the idea of a “pattern”. One may grasp a pattern and have some intuition that there is something systematic going on – this is still a preliminary stage. An individual may move on to express the pattern explicitly, giving precision to the intuition. Later, one may need to also determine if the pattern can be generalized to an object. This verifying the validity of the pattern requires another idea that convinces the individual why the pattern is valid. Byers stated that the pattern is an idea but also there is an idea in the verification.

An idea is both the feeling that something is going on and the feeling of "now I understand what's going on." Grasping an idea means looking at things in a certain way. Certainly, one will have difficulty grasping the idea of the Fundamental Theorem of Calculus if he or she does not see $F(x) = \int_a^x f(t)dt$ as a function of x. These ideas emerge from moments of ambiguity.

Dewey's (1938) theory of inquiry (Hickman, 1990) advances the thought that knowledge in the honorific sense is a product of active, productive inquiry into problematic situations. Similarly, Byers (2007) emphasized that there is a relationship between ambiguity and idea. The idea overcomes the barrier that is the ambiguity, but the ambiguity persists in the resolution in that the idea is a product of the ambiguity. After an idea is brought to bear, there is still mathematical work to be done to tease out and make explicit the mathematics contained in an idea. The logical deduction needed to convey the formal written proof is another idea.

Alcock (2008) also gave a description of creativity; she described creative thinking as one of four modes of thinking that students could use in constructing mathematical proof. Alcock interviewed mathematician instructors of introduction to proof courses that indicated reasoning abilities that they desired their students to develop and utilizing in proving statements. She divided this reasoning into four modes of thinking, one of which was *creative thinking*. The purpose of the creative thinking mode is "to examine instantiations of mathematical objects in order to identify a property or set of manipulations that can form the crux of a proof" (p. 78). In this mode, one investigates an example with the goal of finding an argument or sequence of manipulations that will generalize to a proof or one tries to create a counterexample and attempts to identify a reason why it cannot be done. One works with an instantiation with the goal of finding some tools that will enable him or her to get a handle on the argument. According to Lithner (2008), Byers (2007), and Alcock, looking for and formulating mathematical ideas that can be used to solve some sort of mathematical problem are activities of creative mathematical thinking. In other words, creative thinking is looking for ideas. In this research, I take the view that generation of ideas that are used to move the proof forward are moments of mathematical creativity brought into fruition in the proving process.

In summary, this research viewed the generation of new ideas used to solve problems as acts of mathematical creativity. In order for these ideas to be created, an individual must be encountering some sort of ambiguity within a problem. When analyzing an individual's navigation through a problem, I utilized Dewey's (1938) theory of inquiry to conceptualize and frame the process. The specific problems that the individuals encountered were the formulation of mathematical proof; therefore, this research utilized argumentation theory.

The above theoretical perspective was the lens through which I viewed the data in analysis (Patton, 2002). My previous experiences and position also played a role in data collection and analysis as the data were filtered through my experiences and my role as the primary instrument of data collection.

Researcher's Role

At the outset of this research and throughout the inquiry process, I endeavored to consider how my position and past experiences factored into the research. I entered this research with knowledge of the literature on mathematical proof and the theoretical perspective described in an earlier section. In collecting and analyzing data, the data were filtered through my experiences and theoretical viewpoints. Awareness of these biases is critical as is looking for and including data that support opposing viewpoints (Merriam, 2009). In the paragraphs below, I describe my own struggles and experiences with learning mathematical proof and my position within this research.

As an undergraduate mathematics major, I had little trouble in performing calculations and exhibiting understanding of the mathematics at play in my first few mathematics courses. However, when I entered my first proof-based course, linear algebra, I struggled. I did not fully comprehend how to structure a formal argument, and one of my errors was to reverse assumptions and conclusions (Harel & Sowder, 1998). After learning how to construct a proof successfully, my proof constructions often consisted of solely symbolic language. I felt the mathematical sophistication of my proof was inversely correlated to the number of actual words used. I viewed the use of pictures and written explanations as sub-par since pictures were neither generalizable nor mathematically precise, and written explanations had no symbols or equations so could not be viewed as math. Even at the beginning of my graduate career, I would not feel comfortable with a proof if I could not translate it to symbolic language. For example, in a master's level topology class, I was trying to prove the existence of a function with given properties. I knew what the function would do and could draw a picture of it, but I could not write out the algebraic formula. The professor of that course made it clear to me that trying to come up with the formula would be a waste of time. That began a turning point in how I thought about mathematical proof. Instead of relying so much on symbols, I have found that I am better served if I think about the structure of the mathematics as it would relate to the structure of the proof. I now feel more comfortable when my arguments include paragraphs of explanation as opposed to purely using symbols and implication arrows. I have worked on my ability to construct pictures and diagrams that are general enough to serve as a rigorous argument. I can see now how mathematical sophistication has nothing to do with symbolization. My struggles with proof and the ontological shift in my perception of proof made me believe that other novice mathematicians may have similar experiences. Delving into the literature shows that in fact my experiences have indeed been shared by others (Harel & Sowder, 1998; Weber & Alcock, 2004).

Setting and Participants

The choice to study professional mathematicians was made in order to make explicit the context surrounding the generation of ideas that can move the argument forward. Mathematical philosophers have written that the formulation of ideas is a real part of what mathematicians do (Byers, 2007; Tall et al., 2012). The evolution of mathematical arguments as moments of creativity is a phenomenon that may resonate with the mathematician. Students may struggle with other issues such as content knowledge and understanding of logic (Selden & Selden, 2008) and may not make use of tools in a way conducive to the formulation of ideas (Alcock & Weber, 2010). Better understanding what professional mathematicians do may help inform designing experiences for students to better be able to develop, recognize, and utilize ideas when constructing proof.

The settings of this study were offices of mathematicians employed in mathematics departments at four-year universities in the Rocky Mountain Region. I solicited participation from individuals employed at various universities because departments vary in size. Participants in this study were professional mathematicians in the field of real analysis. For the purposes of this study, a professional mathematician was defined to be an individual holding a doctorate in mathematics that was currently teaching and doing research in mathematics. Specifically, the mathematicians conducted research in real analysis or a closely related field or had experience teaching an upper undergraduate or graduate course in the subject of real analysis.

To find participants, I contacted representatives at university mathematics departments and accessed department websites to identify mathematicians who either taught real analysis courses or whose field of research was closely related to real analysis. Email (see Appendix A) served as the initial means of contact with prospective participants. This initial contact email informed the participants that the purpose of the study was to observe and describe their processes in solving mathematical proof problems in real analysis. I explained the data collection procedures and time commitment and requested their participation. I initially contacted ten mathematicians across four universities. While five mathematicians agreed to participate, I selected three based on their schedules. Three mathematicians were the entire sample size because it was hypothesized that three participants completing three tasks would be sufficient to find saturation in the codes and data observed. This hypothesis proved correct as the last few tasks analyzed did not necessitate generating new codes.

For the three mathematicians that agreed to participate, I scheduled an initial interview and then subsequent interviews at each meeting. At my request, the participants provided demographic and personal information including the number of years' experience in teaching and conducting mathematics research and the participant's primary field of research. The three participants' pseudonyms are Dr. A, Dr. B, and Dr. C. Table 1 summarizes demographic and professional information about the participants.

Table 1

Participant Pseudonyms, Years of Experience, and Research Areas

Pseudonym	Years teaching or doing research in real analysis post PhD	Primary research areas
Dr. A	20+ years	Queuing theory; evolutionary game theory
Dr. B	5 years	Applied probability theory
Dr. C	20+ years	Functional analysis

Data Collection

Data were collected from three interviews and participants' work on mathematical proof tasks. All three interviews were audio- and video-recorded, and participant work was recorded via Livescribe notebook technology. It was necessary for participants to solve tasks that were genuine problems in order to observe them formulating new ideas. I defined a situation as a "problematic" if it was unclear to the individual how to proceed. I defined a situation as a "problem" if the situation was problematic and the individual constructed an affective element of interest and entered into the situation with the intention of resolving it (Garrison, 2009; Glassman, 2001; Mason et. al, 1982/2010). As described in the theoretical perspective, only the individual can determine if a situation is indeed a problem for him. Indeed, in the exploratory, pilot study (see Appendices B, C, and D), for each of the three mathematicians, at least one of the tasks was not problematic for them. For this reason, this research had the mathematicians choose their own tasks to solve.

The three participants worked on three or four tasks each. The study examined the work on seven total tasks. Each individual chose one task that he or she saw as a challenge or a genuine problem and also chose one task that he or she anticipated another mathematician in his or her field would find difficult or challenging. In the first interview, the participant solved the task he or she found personally challenging. In the second interview, the participants solved the two tasks chosen by another participant as potentially problematic for a peer. Dr. B solved a fourth task on his own with the Livescribe notebook between the second and third interviews.

Dr. A and Dr. C worked on three tasks, and Dr. B worked on four tasks. Each task is included in Table 2. Dr. A's individual task (Individual A) involved applications of the Lagrange Remainder Theorem. Dr. B's individual task (Individual B) required relating limits of sequences to limit points. Dr. C's individual task (Individual C) asked to determine and justify where a given function was continuous. I call the tasks that the participants identified as challenging for a colleague *peer tasks*. Dr. A presented the Uniform Continuity peer task. Dr. B worked on the Uniform Continuity peer task. Dr. C read the task but did not work on it recognizing it as a routine exercise. Dr. B presented me with formulation 1 of the "additive implies continuous" peer task. Dr. A worked on formulation 1, and Dr. C worked on formulation 2. Dr. C provided the "Extended Mean Value Theorem for Integrals" as a peer task and sent me documents with two different versions of the task (formulation 1 and formulation 2). I chose to give the participants the statement of Theorem 1 as it is required to prove Theorem 2, but I did not want not recalling or knowing the theorem to impede the participants' progress. Dr. A worked on formulation 1 of the extended MVT task which came from one of the resources that Dr. C provided, but Dr. A found a limitation to its formulation. I modified the formulation given to Dr. B (formulation 2) based on the second version provided by Dr. C and the issues found by Dr. A.

I chose the Own Inverse task as a backup task. This task was used for the pilot study after performing rounds of task analyses. Dr. B worked on the Own Inverse task after recognizing an inability to complete the extended MVT for integrals task. When Dr. C declared that the Uniform Continuity task would be a routine exercise, I provided the Own Inverse task to solve.

Table 2

Tasks by Participant

Task Name	Chosen by	Worked on by	Task
Additive implies continuous (Linear)	Dr. B	Dr. A	*Formulation 1*: Define f as *linear* if for every x and y, $f(x+y)=f(x)+f(y)$. Let f be a function on the reals. Prove or disprove that if f is linear, then it is continuous.
		Dr. C	*Formulation 2*: Let *f* be a function on the real numbers where for every x and y in the real numbers, $f(x+y)=f(x)+f(y)$. Prove or disprove that *f* is continuous on the real numbers if and only if it is continuous at 0.
Extended Mean Value Theorem for Integrals (MVT)	Dr. C	Dr. A	*Formulation 1:* *Given Theorem 1: MVT for Integrals*: If *f* and *g* are both continuous on [a,b] and $g(t) \geq 0$ for all *t* in [a,b], then there exists a *c* in (a,b) such that $\int_b^a f(t)g(t)dt = f(c)\int_a^b g(t)dt$. *Prove Theorem 2: Extended MVT for Integrals*: Suppose that *g* is continuous on [a,b], $g'(t)$ exists for every *t* in (a,b), and $g(a)=0$. If *f* is a continuous function on [a,b] that does not change sign at any point of (a,b), then there exists a *d* in (a,b) such that $\int_a^b g(t)f(t)dt = g'(d)\int_a^b (t-a)f(t)dt$.
		Dr. B	*Formulation 2:* *Given Theorem 1: MVT for Integrals:* If *f* and *g* are both continuous on [a,b] and $g(t) \geq 0$ for all *t* in [a,b], then there exists a *c* in (a,b) such that $\int_b^a f(t)g(t)dt = f(c)\int_a^b g(t)dt$. *Prove Theorem 2: Extended MVT for Integrals:* Suppose that *g* is continuous on [a,b], $g'(t)$ exists for every *t* in [a,b], and $g(a)=0$. If *f* is a continuous function on [a,b] that does not change sign at any point of (a,b), then there exists a *d* in (a,b) such that $\int_a^b g(t)f(t)dt = g'(d)\int_a^b (t-a)f(t)dt$.
Own Inverse	Researcher	Dr. B Dr. C	Let f be a continuous function defined on I=[a,b], f maps I onto I, f is one-to-one, and f is its own inverse. Show that except for one possibility, f must be monotonically decreasing on I.

Table 2 continued

Task Name	Chosen by	Worked on by	Task
Uniform Continuity	Dr. A	Dr. B	Let f be continuous on the real numbers, and suppose $\lim_{n\to-\infty} \alpha$ and $\lim_{n\to\infty} \beta$. Show f is uniformly continuous.
Individual A (Lagrange Remainder Theorem)	Dr. A	Dr. A	Let *I* be a neighborhood of the point x_0 and suppose that the function $f: I \to R$ has a continuous third derivative with $f'''(x) > 0$ for all x in *I*. a) Prove that if $x_0 + h \neq x_0$ is in *I*, there is a unique number $\theta = \theta(h)$ in the interval (0, 1) such that $f(x_0 + h) = f(x_0) + f'(x_0)h + f''(x_0 + \theta h)\frac{h^2}{2}$. b) Prove that $\lim_{h\to 0} \theta(h) = \frac{1}{3}$. c)
Individual B (Sequences and Limit points)	Dr. B	Dr. B	Let E be contained in the real numbers, then E is closed if it contains all limits of sequences $\{x_n\}$ with $x_n \in E$ for each n.
Individual C(Determine continuity)	Dr. C	Dr. C	Discuss the continuity of the function. $f(x,y) = \begin{Bmatrix} \frac{x^2-y^2}{x-y} & if\ x \neq y \\ x-y & if\ x = y \end{Bmatrix}$

The data collection phase occurred in cycles for each participant. Participants worked on a task or tasks in an interview setting, continued to work on the task on their own, turned in their at home work that was captured via Livescribe technology (which is described below), participated in a stimulated recall interview of their work, and repeated with new tasks in the next interview. The sequence of interviews and tasks is given in Table 3.

Participants worked on the task on their own for a period of three to six days if they did not complete the proof task to their satisfaction in the interview. Between participant interviews, I conducted preliminary analyses of their completed work both in the interview setting and of their "at home" work to prepare for the follow-up interviews.

Table 3

Sequence of Interviews and Tasks

Participant	Interview 1	Interview 2	Interview 3
Participant A	Choose personal task A and peer task A Work on personal task A	Stimulated recall of personal task A. Work on peer tasks B and C.	Stimulated recall of peer tasks B and C.
Participant B	Choose personal task B and peer task B Work on personal task B	Stimulated recall of personal task B. Work on peer tasks A and C.	Stimulated recall of peer task A and C.
Participant C	Choose personal task C and peer task C Work on personal task C	Stimulated recall of personal task C. Work on peer tasks A and B.	Stimulated recall of peer task A and B.

The Livescribe technology consists of a pen and notebook pair. The camera at the base of the pen captures both the real-time writings and audio. When one turns on the pen and taps the record button on the associated notebook, the pen begins recording audio and will record writings in that notebook. These recordings may be uploaded to a computer through Livescribe software. The Livescribe technology has been utilized to capture the work of participants both within and outside the interview setting when collecting data (e.g., Savic, 2012, 2013).

Interview One

The first interview will consist of two aspects, choosing the tasks and working to solve the tasks. I asked the participant to identify two tasks: one that was personally challenging and one that would be challenging for a colleague. These tasks originated from the textbook that the participant used in teaching courses in real analysis, their recollection of problems that they had assigned in the past, and their knowledge of related tasks that others would find challenging. The protocol used is in Appendix E.

The selection of tasks aspect of the interview was semi-structured. I implemented follow-up instructions and questions to restate directions to the participant or to clarify utterances made by the participant. While the participants were selecting tasks, they may have spent time thinking about the tasks which may be interpreted as beginning to enter the task (Mason et al., 1982/2010) but did not begin attacking the problem until they have chosen the tasks. Twenty minutes were allotted for task selection. After the participants chose the two tasks, the interview transitioned to the second section.

In the second section of the interview, the participant worked to solve the task found to be personally challenging. The entire interview was video and audio recorded. The participant used a Livescribe pen and notebook while working on solving the selected task. A video camera was placed to face the participant and his writing. The Livescribe set up captured writing in real time synced with audio recordings.

Prior to beginning the task, I explained that I would like the participant to think aloud. Additionally, I asked the participant to make note of when his perception of the task changed with the anticipation that this may occur when the participant saw the solution and the task was no longer a problem, or the perceived problem changed. Participants worked on the tasks in the interview setting for up until the end of the one-hour interview. If the participant did not come to a solution that he or she deemed satisfying at the end of the allotted time, then the participant continued to work on the tasks alone, outside the interview setting.

I took pictures of the work that each participant created in the notebook and supplied the participant with a Livescribe pen and notebook to continue the work at home. Participants had the Livescribe notebook for up to six days. I provided both written and oral instructions for how the participant should proceed in this “at home” work (see Appendix E). Anticipating that participants could end up thinking about the tasks on their own time, like while waiting in line for lunch (Savic, 2013), the participants received instructions to return to the notebook to write down and audio record their accounts. This concluded the first interview.

Between the first and second interviews, up to nine days elapsed. I transcribed the video data and reviewed the video recordings along with the Livescribe recordings from the interviews as well as the later collected “at home” work. I conducted preliminary analyses and generated hypotheses and questions for the second interview. The procedures for these analyses are described in the section describing data analysis.

Interview Two

The second interview consisted of two parts: a follow-up interview of the previous interview and work on two additional tasks. The third interview involved follow-up questions for each task from the second interview.

A goal of this study was to understand the mathematicians’ thinking processes while solving a proof-task without disrupting the process itself. An interview technique that has been used to get at what participants “think about the problem not after they have finished working on it, but what their thoughts are *as they think their way through the problem for the first time* (Pirie, 1996, p.7, emphasis in original)” is stimulated recall (Calderhead, 1981; Lyle, 2003; Rowe, 2009). In the stimulated recall interview, videotaped passages of behavior are replayed to individuals to stimulate recall of their concurrent cognitive activity. The typical procedure includes a series of open-ended questions posed to the participant as soon as possible after, or during the viewing of a segment. The questions focus on description of the segment, the participant's thinking, or alternative behaviors the participant could have chosen (Lyle, 2003).

The second interview was scheduled within ten days of the first interview (Lyle, 2003; Pirie, 1996). The interview was video and audio recorded, and participants wrote with a Livescribe pen in a Livescribe notebook. The stimulated recall of the previous interview was semi-structured in that I chose specific video clips to play back and wrote questions. The written questions targeted the participants' explaining their thinking at certain moments, the reasons behind the decisions-made, or their perceptions of the situation including problems they perceived and ideas that they thought could be useful. The interview was flexible in that the participant had the opportunity to pause, rewind, and playback events that he or she deemed important. I asked follow-up questions as needed. No questions written specifically asked participants to identify new ideas and the context surrounding those ideas as Calderhead (1981) emphasized the importance of screening the research goal from the participant. After the first stimulated recall, participants worked on two more tasks, and I did not wish to have my focus on the participants' ideas play a role in their thought processes on these future tasks. The stimulated recall portion of the second interview lasted approximately 40 minutes.

As an example of the stimulated recall protocol, consider Dr. A's second interview. I had previously generated hypotheses about what the ideas were and had written questions about Dr. A's motivation when solving the Lagrange Remainder

Theorem task. I began by playing back the Livescribe recording which provided Dr. A's audio and also the writings that accompanied the audio. After playing a section, I would pause the recording to ask questions about what Dr. A was thinking about, what Dr. A had anticipated, and what motivated the certain actions that we had just viewed. Also, Dr. A would point to sections where he wrote what he deemed as important information, and we would play that section back. Dr. A would make comments, and I would follow-up with clarifying questions.

In the final portion of the second interview, the participants worked on two tasks, one at a time. These tasks were chosen by another participant at the time of that participant's first interview. The protocol was as the task portion of the first interview, but I provided additional resources including a list of definitions and relevant prerequisite theorems that the participant may not recall. The definitions were provided as mathematicians possess multiple instantiations of the terms involved (Alcock, 2008), but it is possible that the participants did not perceive their definitions as equivalent to those given by the author of the book. Similar to the first interview, at the conclusion of the second interview, I provided the participants with a Livescribe pen and notebook to continue work on the unfinished tasks for up to six additional days. I conducted preliminary analyses of the data from the two tasks completed in this second interview as well as the "at home" work to prepare for the third interview.

Interview Three

The third, and final, interview was a stimulated recall of the two tasks completed in the second interview. Procedures for this interview were similar to the first portion of the second interview. This final interview lasted approximately 90 minutes.

Data Analysis

Data analyses proceed in three stages. The first stage was preliminary analysis of interviews 1 and 2 for each participant preceding the stimulated recall of the interview. The second stage was refinement of the previous preliminary analysis and descriptive accounts of the total work on each task by each participant. Finally, analyses across tasks and individuals were conducted. Each like-problem was analyzed across the two participants that worked on it. The work by each participant was analyzed across the tasks completed.

The units of analyses to be studied for the first research question were the ideas that participants saw as moving the argument forward as these critical incidents were what the overall research sought to describe (Patton, 2002). To answer the first research question, I worked to identify and describe these ideas and the context surrounding these ideas including the perceived problem or problems when the ideas are articulated, the mode of inquiry into which the participant is entered, and what tools previously applied contributed to the attainment of the idea. For the second research question, a further level of analysis across the entire task was required to describe how the ideas related to one another.

Preliminary Analyses

Preliminary analyses of each task-based interview were conducted prior to the follow-up stimulated recall interview. The primary goal of these analyses was to hypothesize which moments were significant in moving the argument forward, to hypothesize the features of the situation that contributed to the generation of these ideas, and identify moments where it is necessary for the participants to clarify the motivations for their actions. From this analysis, questions were created for the stimulated recall interview.

In this preliminary analysis, I transcribed the videos and watched the video recordings in sync with the Livescribe recordings. I noted moments where (a) the participant appeared to generate a new idea, to identify a certain tool as useful, or to gain some insight into the problem; and (b) moments where it was unclear what motivated a certain action. Primary evidence for identifying the moments of new insights came from the participants' notations of the appearance of these ideas as per the interview protocol. Additionally, I identified insights or ideas that the individual used to some satisfaction in that the application of the idea or tool solved a perceived problem or led to the generation of a tool that solved the problem. After identifying the moments where it appeared that the participant had generated or used an idea that moved the argument forward, I generated initial descriptions of these ideas.

The moments where these ideas occurred acted as markers of transitions in the timeline of the evolution of the argument. In the next stage of the preliminary analysis, I endeavored to give a structural description of the evolution of the argument by performing Toulmin (2003) analyses on the argument prior to these markers and following these markers resulting in a series of Toulmin diagrams of the personal argument. The Toulmin analysis provided a description of the elements viewed useful by the participant and for which purpose. Additionally, it provided insight into the motives of various tool applications, anticipated outcomes, and the perceived problems for the participant. In these initial analyses, motivations and some thinking were unclear; also, the individual's argument was not complete at intermediate points in the proof construction process. Therefore, at many moments, the Toulmin diagrams did not have complete structures. The definitions given by Toulmin for the components of the argument structure are given in the theoretical perspective, but in Table 4 I provide working descriptions and standards of evidence for each component within the context of the mathematical proof construction process and this study. Although Toulmin's argumentation scheme was used to classify the role of certain statements in the argument, during the argument's evolution, various components such as warrants, backing, and rebuttals for warrants were less articulated than complete statements. For example, the drawing of a picture sometimes acted as a backing for the warrant of a certain claim.

Table 4

Statement Categories of the Toulmin Argument Model

Statement Category	Description
Grounds	Particular facts about a situation relied on to clarify and bolster the claim; may be the hypothesis of the statement to be proved, specific features that point toward the specific claim
Claim	Position being argued for
Warrant	Principle or chain of reasoning that connects the grounds to the claim
Backing	Support or justification for the warrant
Modal Qualifier	Specification of limits to the claim, warrant, and backing
Rebuttal	Stated exceptions to the claim

Once a series of Toulmin models were hypothesized, it provided a characterization of the evolving structure of the argument which facilitated describing the context surrounding the emergence of the ideas that the participant viewed as moving the argument forward. Reviewing the context surrounding the emergence of the idea, I hypothesized what the participant perceived as problematic, the phase of the inquiry process that he or she appeared to be in when articulating the idea, the tools used that influenced the generation of the ideas, and the anticipated outcomes of using these influencing tools. These hypotheses included how the participant appeared to test the idea that he or she viewed as moving the argument forward and how he or she used the idea throughout the rest of the argument. Table 5 describes each aspect that was hypothesized as well as the standards of evidence used to identify each concept. The standards of evidence that required researcher inference are noted in italics. In this preliminary analysis, all descriptions were hypotheses which were tested against the information gleaned from the follow-up interviews. From these hypotheses, I selected segments of the video to replay for the participants and generated questions to gain clarification of the participants' thought processes. These questions included requests for participants to watch a chosen clip of the interview and describe their thinking and motivations for certain actions.

Table 5

Standards of Evidence

Concept	Description	Standards of Evidence
Idea that moves the argument forward	Incorporation of the idea changes the set of ideas that the individual sees pertinent to the personal argument.	The participant notes the moment in the interview or the notebook Comments such as: "I now see…" Line of questioning changes
What the problem is	Specific problem the participant is currently entered into solving	Participant comments in the form of "I want to…" "I don't know how to…" "I don't know why…"
Mode of inquiry	Reflection: Participant has articulated a problem and is thinking of tools to apply to the problem Action: Participant is currently applying a tool or performing some action Evaluation: Participant has applied a tool or is currently applying a tool and is specifically reflecting on the effectiveness of said tool	Reflection: Comments such as "what can I do?"; writing or articulating observations or what is known, instantiating the situation with the purpose of finding a tool to apply to the problem Action: Participant is computing, drawing a picture, articulating a theorem, and so forth. Evaluation: Comments such as "does that help me?"
Influencing tool	In its application, the tool used effected some action or product that the participant used in his generation of the idea that moves the argument forward.	The participant refers to the product or action either verbally or with gestures as he/she articulates the idea. *There is a relationship between the idea and the product of the tools*
Anticipated outcome of influencing tool	The purpose of the applied tool.	What the participant says about why the tool may be helpful.
Testing of idea	Ways the idea are tested to determine its usefulness	Comments such as "Does that work?"; "What does that tell me?"; "Is that helpful?"
Use of idea	How the idea influenced the personal argument	The participant refers to the idea or an outcome of the idea An outcome of the idea is evident later in the argument

Primary Analysis

The analysis of these data borrowed from strategies of grounded theory (Corbin & Strauss, 2008). Analysis began by writing a description of each idea that moved the argument forward and the context surrounding the generation of that idea. I described the argument's evolving structure via Toulmin diagrams formulated in the preliminary analysis and informed by the stimulated recall interviews. Written stories captured not only the moment surrounding the idea but how the idea was tested and used as the argument evolved. I then conducted open iterative coding of each idea to organize, describe, and link the data. Finally, I looked for themes and patterns across participants within the same task and across tasks.

The analyzed data included the interviews of the participants working on each task, the follow-up interviews regarding each task, and the individual work of the participants on the tasks outside the interview. All interviews and Livescribe recordings were transcribed. I created a file incorporating all data pertaining to each task. I conducted a further iteration of analysis on the transcripts in conjunction with the video and Livescribe data to modify the previously generated list of ideas that moved the argument forward. New ideas were added and moments previously identified as significant ideas were combined or deleted based on the complete data set.

After creating a file incorporating all data pertaining to each task, I built upon the primary analyses, identifying the ideas that move the argument forward. For each idea that moves the argument forward, I wrote a description as well as the answers to the following questions:

1. What problematic situation is the prover currently entered into solving when one articulates and attains an idea that moves the personal argument forward?

2. What stage of the inquiry process to they appear to be in when one articulates and attains an idea that moves the personal argument forward? (Are they currently applying a tool, evaluating the outcomes after applying a tool, or reflecting upon a current problem?)

3. What actions and tools influenced the attainment of the idea?

4. What were their anticipated outcomes of enacting the tools that led to the attainment of the idea?

In the preliminary analysis, I hypothesized answers to the above questions. I utilized the data in the follow-up interviews to test my hypotheses and modify the answers to the above questions. I wrote these descriptions of each idea and modified the Toulmin analyses of each idea for each task. Then, I wrote stories to describe the evolution of the argument capturing the participant's complete work on the task, sectioned by the ideas that moved the argument forward. Specifically, I wrote to give thick descriptions about each idea, the problem, tools, anticipations, and mode of inquiry, how that changed the argument structure, and what previous ideas influenced that idea and what happened with the idea as the argument evolved. In this process, some ideas were collapsed into one idea because the prior determinations were results, tools, or plans.

Once the stories of a single task were written across both participants who worked on it, I used strategies of inductive analysis (Patton, 2002) to further analyze, compare, and interpret the tasks to develop themes, patterns, and findings across tasks. The data analysis was inductive in that largely "open and axial coding" (Strauss & Corbin, 1998) was be used to develop a codebook. The codes used emerged out of the data and sometimes drew upon the language used by the participants in the style of "in vivo" coding (Merriam, 2009) or if appropriate, implemented language used in the literature. For example, I drew upon Dewey's (1938) theory of inquiry to describe the mode of inquiry.

Each idea was coded, individually. Specifically, I conducted open coding of each idea that moved the argument forward, the problem situation encountered, the tools that influenced the generation or articulation of the idea, and the anticipated outcomes of said tools. I kept records of the generated codes and decisions made in the research journal and associated files and spreadsheets. The final codebook utilized is given in Appendix F.

I made note of emerging themes and patterns that were apparent in individual tasks for individual participants. Analysis occurred across tasks and participants to determine the limits of those patterns and themes. As the idea-types emerged, I categorized idea-types according to their perceived purpose as the argument moved forward. To answer the first research question, spreadsheets aided to analyze each idea-type across problems, tools, and structural shifts. The sense of the pattern of how the ideas were used and tested came from the writing of the stories, in that it seemed that aspects of the stories repeated themselves.

Strategies for Validating Findings

In this research, I endeavored to establish trustworthiness in the research results. My ability to establish trustworthiness depended on how I conceptualize the study and the strategies used in data collection, analysis, and interpretation (Merriam, 2009). This study investigated people's constructions of reality, and any interpretations of these constructions will never capture an objective truth. However, this study adopted and employed strategies to increase the *credibility* of my findings: *triangulation, reflexivity,* and *peer review.* Descriptions of my use of these strategies are given below. Specifically, I considered the credibility of any assertions made about the thought processes of the mathematicians. Human thought cannot be directly observed. This is what motivated the inclusion of the methods of a think-aloud interview, the direction for participants to note when their perception of the situation changes, and the follow-up stimulated recall interviews. The research findings most likely cannot be replicated as human behavior cannot often be isolated and controlled; therefore, as an alternative to traditional reliability, I instead endeavored to ensure "the results are consistent with the data collected" (Merriam, 2009, p. 221). My strategies for consistency and dependability were triangulation, peer examination, articulation of the researcher's role, and establishing an audit trail. Generalizability in the traditional sense cannot occur in this research as the participant sample was not large and random; however this research endeavored to include enough description of the data and data collection procedures to allow the reader to establish if the findings transfer to another situation. This *transferability* can be aided by the use of rich, thick description.

Denzin (1978) described four types of triangulation: using multiple methods, multiple data sources, multiple investigators, and multiple theories. In this study, I employed multiple data sources and multiple methods. I employed multiple methods of data collection; participants were observed working on tasks in a one-on-one interview setting, and they also will complete tasks on their own. Multiple data sources include the interview transcripts and Livescribe pen work from the interviews, the Livescribe pen work from the participants' individual work, and the follow-up, stimulated recall interviews with each participant for each task (Merriam, 2009). The process of triangulation involved comparing observations from the task-based interviews with the stimulated recall interviews.

Since the researcher is the primary instrument of data collection in qualitative methodologies, I must explain my biases, assumptions, experiences, and dispositions regarding the study. This reflexivity statement was given in an earlier section. Its inclusion contributes to the credibility of this study. Peer examination (Merriam, 2009) or expert audit review (Patton, 2002) will be conducted by the committee of this dissertation. The committee applied critical eyes to assess the quality of data collection and analysis according to the theoretical assumptions of this study. As a means of increasing the consistency and dependability of this study, I maintained a researcher's journal (Merriam, 2009) to record my reflections, questions, and decisions throughout the data collection and analyses phases. I used the journal to construct descriptions of how data were collected, how categories were derived, and how decisions were made throughout the study.

Ethical Considerations

Sensitivity to ethical considerations is important at all stages of the research process (Creswell, 2007). In this particular study, no participants were from vulnerable populations; however, I still considered how the qualitative interview process affected the participants (Patton, 2002). Patton (2002) provides a checklist of issues to consider in designing a qualitative study and highlights consideration of *explaining the purpose*, *promises and reciprocity, risk assessment, confidentiality, informed consent, data access* *and ownership, interviewer mental health, advice, data collection boundaries,* and *ethical versus legal* issues. I explained to participants that the purpose of the study is to observe how mathematicians construct proof. I used this language in an effort to prevent the participants from teaching. The specific research purpose and theoretical framing was not shared in order to facilitate observing the process of proof construction as it occurred. For credibility purposes, I asked participants to note when their perspective of the problem situation shifted, and this may have hinted to them what my focus was.

I foresee no greater risk to mathematicians participating in this study than those they encountered from their typical practice of engaging in mathematics and discussing it with others. Participation in the study did require them to sacrifice time, three sixty to ninety minute interviews and time engaging in the tasks outside the interviews which may have had an impact on their professional and personal commitments. Participants were informed of the time requirements prior to providing their consent to participate. I did not compensate participants or provide tangible incentives.

I endeavored to maintain the confidentiality of all participants by assigning pseudonyms for the participants and their respective universities of employment unless the participants opted to be identified (Patton, 2002). Video data and Livescribe recordings were stored on a password-protected computer. Participants' written work was stored in locked filing cabinets. Data will be maintained for three years or until the publication of the results from this dissertation. I followed Institutional Review Board (IRB) guidelines and requirements for research with human subjects and obtained an expedited IRB approval prior to contacting potential participants and collecting data (see Appendix G). I provided a written description of the aspects of participating in this study to the potential participants and obtained written informed consent. The chair of the dissertation committee was charged with storing the data and consent forms. The dissertation committee chair and I were the only individuals who viewed the video and Livescribe data for the purposes of peer review.

The topics of the interviews were not likely to be emotionally draining for me as a researcher. Even so, the interviews were conducted over a short period of time, and I debriefed with committee members to process all that I observed (Patton, 2002). The topics of the interview did not appear to be painful or uncomfortable for participants as they were talking about their experiences in completing a mathematical task within their own fields of study, a typical practice in their careers. However, I provided participants with the option of ending the interview or discontinuing work on a task. The chair of the dissertation committee acted as my confidant and counselor on matters of ethics as not all issues could have been anticipated in advance.

CHAPTER IV

FINDINGS

Introduction

As stated in Chapter I, the study presented sought to describe (a) the ideas mathematicians form while constructing mathematical proofs that move their personal arguments forward, (b) the inquirential context surrounding the emergence of those ideas, and (c) how the ideas are tested and how the ideas do or do not change the situation for the mathematician. Data were interpreted through the framing of Dewey's theories of inquiry and instrumentalism and a conception of an evolving personal argument whose structure was modeled by a series of Toulmin (2003) diagrams.

Three mathematicians participated in interviews where they solved proof problems in real analysis. Table 6 summarizes demographic and professional information about the participants, Dr. A, Dr. B, and Dr. C (pseudonyms). Dr. A and Dr. C worked on three tasks, and Dr. B worked on four tasks. Table 2 from Chapter III listed each task used in this study and Chapter III provided more details on participants and tasks.

Table 6

Participant Pseudonyms, Years of Experience, and Research Areas

Pseudonym	Years Teaching or Doing Research in Real Analysis Post PhD	Primary research areas
Dr. A	20+ years	Queuing theory; evolutionary game theory
Dr. B	5 years	Applied probability theory
Dr. C	20+ years	Functional analysis

Problems, tools, and the types of shifts in the Toulmin structures modeling the personal argument were open coded then refined into final types as described in Chapter III. Tables in Appendix F provide the definitions or descriptions that were used after the refinement process was completed. Table 40 in Appendix F gives the descriptions of the nine types of problems the participants were entered into solving when they articulated ideas. Table 41 presents descriptions of ten types of tools; more specifically, Table 42 expands on the various purposes of examples observed as participants articulated ideas. Table 43 describes the immediate shifts in the Toulmin structures of the personal argument seen upon articulation of a new idea. This chapter presents the thematic findings in response to the primary research questions. Specifically, I present the findings in three sections: the types of ideas that moved the argument forward, themes regarding inquirential context surrounding the formulation of these ideas, and themes found about how ideas were utilized and tested as the arguments progressed toward a complete mathematical proof.

Ideas That Moved the Personal Argument Forward

The ideas that moved the argument forward were ideas that accompanied a structural shift in the personal argument as could be captured by a Toulmin diagram prior to and following the articulation of the idea, provided the participant means to communicate their personal argument in a logical manner, gave a participant a sense that his way of thinking was fitting, or were explicitly referred to by the participant as a useful insight. Pictures, examples, or individual actions were not included as ideas but as tools. However, the insights extracted from performing and reflecting upon these tools or a collection of tools were included as ideas. Ideas that moved the argument forward also sometimes encapsulated facts, known theorems, and the results of their combination.

Proposed warrants (statements or reasons to connect evidence and claim) and extracted or generated equations were included as ideas even though the participant may not have fully believed the warrant or idea would be true or useful. I chose to include these proposals because at the time that the participants articulated the proposals, they seemed to perceive how they could connect the proposal to their argument. Additionally, as will be described later, the acts of testing proposed warrants led to extracting properties and relationships that were combined to formulate a new idea that moved the argument forward.

Each idea was coded in terms of the work that the idea did for the participant. There were 15 idea sub-types that were grouped into three categories: Ideas that Focus and Configure, Ideas that Connect and Justify, and Monitoring Ideas. An action or evaluation of that action described at one particular moment could solve multiple problems or give rise to multiple feelings. Therefore, multiple idea-types could have characterized a single moment. For example, an insight that provided a *deductive warrant* could also give the prover a sense of *I can write a proof*. In the following sections present these three idea categories and the sub-idea-types within them. I will describe each sub-idea-type using examples, present the salient inquirential contexts surrounding the emergence of those ideas, and how those ideas were used as the participant proceeded toward a routine conception of the task (see Figure 6).

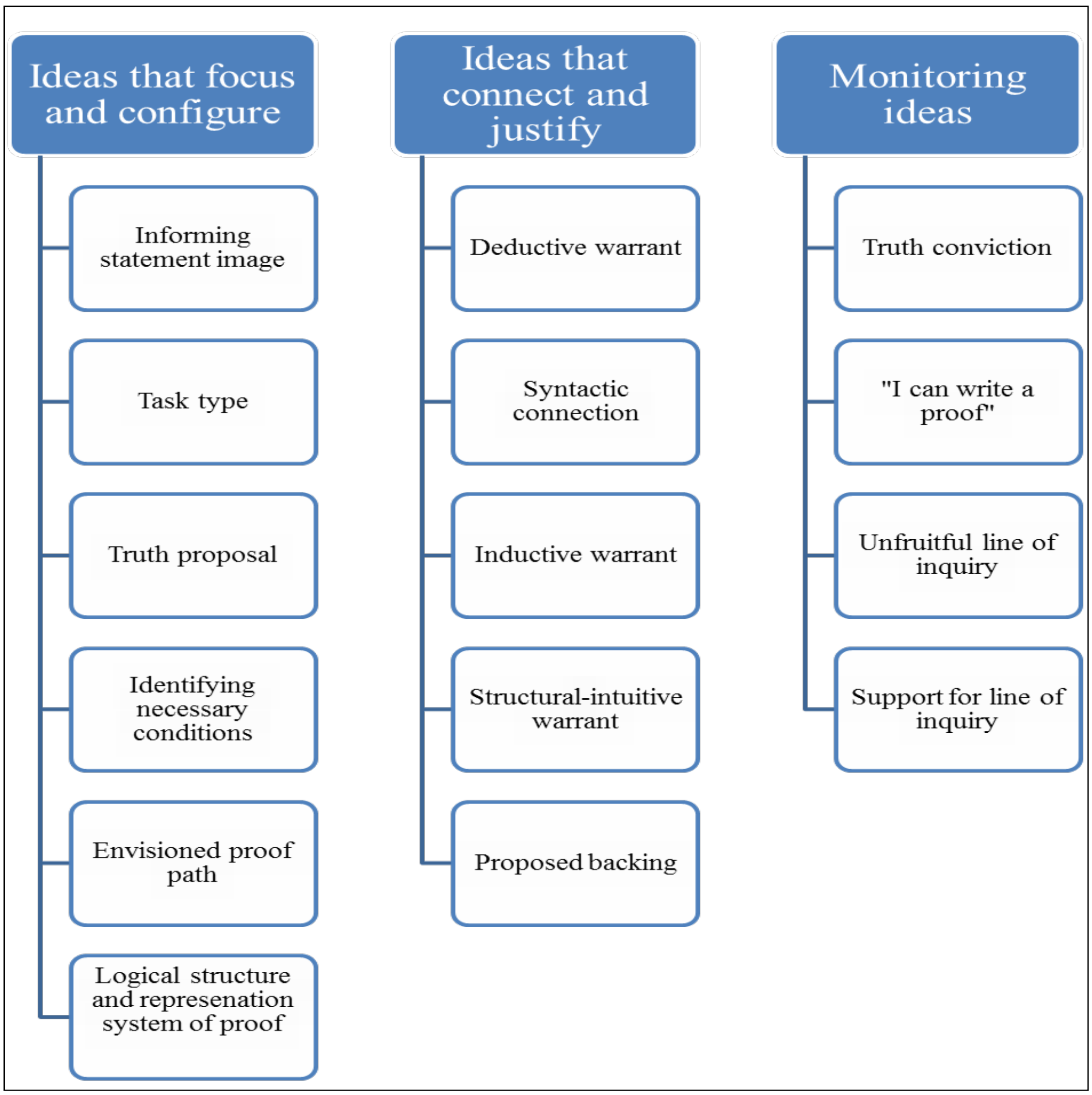

Figure 6. Idea categories and idea sub-types.

Ideas That Focus and Configure

Ideas that focus and configure are the ones that gave the participants a sense of what was relevant, what claims to try to connect to the statement, strategies that would be fitting to achieve connections, and a sense of how to structure and articulate the argument. I have identified six such sub-idea-types under this category.

Informing the statement image. *Informing the statement image ideas* served the purpose of broadening or narrowing the conception of the situation. They were found to be useful by the participants and resulted in added data statements to the entire statement image. Ideas met this sub-type if the participant identified them as relevant statements but did not attribute them to serving some purpose such as acting as a claim or warrant in the central argument or sub-argument to be proven. These ideas acted as added facts in that they were true statements that the participant either believed to be true or justified to be true. I provide two examples of ideas that inform the statement image.

On the own inverse task, both Dr. B and Dr. C combined the assumptions that the function *f* is one-to-one and continuous to discern that *f* must be either increasing or decreasing. The statement that *f* is either increasing or decreasing was added to the given assumptions, and it informed their conception of how the function could behave. It narrowed the class of functions that could fit the conditions on *f*.

I contrast that idea to a statement given by Dr. C on the own inverse task. Upon first reading the statement, Dr. C elaborated the statement that *f* was its own inverse by writing $f\left(f(x)\right) = x$: "It's just another way of saying that *f* is its own inverse. And I knew that that idea was central to the argument that I was going to come up with. But I didn't know how it was going to fit in. I needed to find an algebraic way of making it fit in." The participant viewed this statement as identical to the own inverse condition. The participant stated that this algebraic representation of the condition should be relevant in some way but was not sure how. This added data statement expanded the known assumption of the function being its own inverse.

Table 7 details for which participants and on which tasks ideas that informed the statement image were identified. Some ideas contributed new data statements but were not coded in this category because the ideas contributed other structural changes that were more saliently described in another idea-type. Dr. B had more instances of specifically articulating that ideas informed his perception of what was going on than the other two participants, but all three articulated an idea in this category at least once. New data statements were added to the personal arguments on the Extended Mean Value Theorem for Integrals (MVT) task and the Additive implies Continuous task but their additions were not the salient features of an idea.

As shown in Table 7, the ideas that informed the statement image were generated while the participants were working on a variety of problems with a range of tools. Four out of the seven instances that were coded in this idea-type were articulated when the participants were looking for warrants and while they had been exploring specific examples. Structurally, the articulation of a new idea to inform the statement image added or specified data statements, changed warrant statements, added subclaims, changed qualifiers or rebuttals, and added backing.

Table 7

Summary of Ideas That Inform the Statement Image

Inquirential context and structural shifts surrounding the emergence of ideas that inform the statement image.	
Tasks	Own Inverse (2) UC (2) A-Ind (1) B-Ind (1) C-Ind(1)
Participants	A (1) B(4) C (2)
Problem	No problem (1) Tool Problem (1) Understanding statement & objects (1) Looking for Warrant (1) Looking for (conceptual) warrant (3) Articulating or Generalizing (1) Determining Backing or Validation (1)
Tool	Conceptual knowledge (1) Examples (4) Heuristics & Experience (1) Instantiations & Equivalencies (1) Symbolizing (1)
Shift	Data added (4) Changing warrant (4) Added Subclaim (2) Changed qualifier/rebuttal (1) Added backing (1)

Informing ideas sometimes broadened the perception of all that was included in an object's description, and this broadened view sometimes lead to eliminating the feasibility of proposed warrants or backings for warrants. This idea served as a means of testing a proposed warrant and informing a more narrowed pursuit of a warrant (e.g., why functions cannot oscillate more and more wildly on a compact set).

Participants developed and used informing ideas at all stages of the process of proof construction. They used the ideas in the pictures constructed; as the participants interpreted these pictures, they formulated new warrants and backings. The mathematicians evaluated the statements' usefulness both explicitly and implicitly. Explicitly, participants formulated formal or informal sub-arguments to persuade themselves of the statements' validity or declared a statement a known fact. Implicitly, the statements were tested against their usefulness which was subjective. Participants generated or noted some instantiations of these ideas as important, but the ideas only played a role in the development of the picture. For example, on the Own Inverse Task, Dr. B extracted the fact that if a and b were the endpoints of an interval then either $f(a)=a$ and $f(b)=b$ or $f(a)=b$ and $f(b)=a$ as important evidenced by Dr. B's uttering or writing the idea in some form multiple times and writing it as a known statement in the final proof as well. However, no part of the final argument explicitly depended on this fact. Rather, Dr. B used it repeatedly to structure pictures, and Dr. B worked with the pictures to formulate ideas used to justify claims in the final proof. Some informing ideas, like the endpoint idea above, remained as data statements throughout the entire proof process. Some ideas appeared to drift to the periphery of the personal argument.

Truth proposals. The ideas that are categorized in the *truth proposal* sub-idea type were the participant-generated conjectures about the validity of a given claim. Participants made conjectures during the proof process (e.g. proposing warrants), but only the ones with the specific purposes of determining the truth of the statement given in the task were classified as such (e.g. proposed warrants). For example, in the Additive Implies Continuous task, participants were given the prompt to prove or disprove the claim. Dr. A was given a false version of the task and Dr. C was given a true version. As such, determining the truth of the claim was a potential problematic issue and the decision made would dictate how the participants would proceed. Dr. A went through a series of truth proposals and began the task with the initial inclination that the statement was not true based on his resources of past experiences. However, Dr. A thought about the tool needed to prove the statement was not true, and decided that generating a counterexample would require borrowing ideas from other realms of mathematics besides real analysis and require knowledge beyond that of an undergraduate student in real analysis. The participant's perceptions about the context of the tasks given contributed to deeming pursuing the counterexample inappropriate or infeasible. Dr. A pursued proving the statement was true, that additive implied continuous. Because Dr. A had not yet determined which one was correct, Dr. A used the given and accumulated data to generate proposals of whether the statement was true or false.

As shown in Table 8, the only task that involved truth proposals was the 'prove or disprove' prompt solved by Dr. A and Dr. C. Because Dr. A's version of the task was false and generating a counterexample was perceived to be difficult, Dr. A wavered and made more truth proposals than Dr. C. Specifically, Dr. A set about to try to prove the statement but was unsuccessful in solving the problems of finding warrants to connect additive to continuous. When reaching an impasse, Dr. A would reflect again on the data including the feeling that there is a counterexample, and propose that maybe the statement was in fact false:

> Let me try one more time. Yeah, you need to, what you need to show is that for any epsilon. Okay so need to show, to show continuous, need to show that for any epsilon bigger than zero, there exists a delta bigger than zero so that if x is between zero and delta, then f of x is less than epsilon. Right, that would be

> sufficient, but how do you do that? //Yeah, what I tell people when I'm teaching them how to do proofs is that, you, you know especially things like this, that are wishy-washy, that tell you to prove or find a counterexample, you try one for a while, if you don't get, if you don't get an answer then you try the other one. So, right now I'm at the point where I'm gonna look for, I'm ready to start looking for a counterexample.

It appeared that truth proposals were made when the gathered data pointed toward one side being true or when one did not find success or progress in proving one side.

Table 8

Truth Proposal Occurrences with Counts

Aspect	Instances
Task	Additive implies Continuous (4)
Participant	A (3)
	C(1)
Problem	Determining Truth (3)
	Looking for Warrant (1)
Tool	Connecting & Permuting (1)
	Heuristics & Experience (1)
	Conceptual Knowledge (1)
Shift	Changing claim (4)
	Opening structure (2)

Table 8 summarized the problems encountered and utilized tools when participants made truth proposals. The most frequent problem encountered when making a truth proposal was determining truth, but Dr. A did once change a truth proposal when looking for a warrant. Participants gathered conceptual knowledge, past experiences, assessments of context, and connections amongst known ideas to make decisions.

Making a truth proposal was a choice in how to proceed with the inquiry: whether to try to prove or to disprove. This could dictate the opening structure of the argument. Changing the truth proposal resulted in changing the claim. Dr. A switched which side of the argument for which to argue four times while working on the task while the interviewer was present. The other participant, Dr. C, swapped claims when finding an attempt at generating a counterexample was unsuccessful. The type of task dictated whether these types of truth proposals would be made. The perception of the type of task played a large role in how some of the mathematicians approached constructing their proofs.

Type of task. On some tasks, it appeared that participants made assessments about what tools or ways of approaching generating connections between the conditions and the claim would be fitting. These feelings about how to classify the problem have been deemed *ideas about the type of task.* These feelings were classified as ideas that move the argument forward because they helped the prover identify what sorts of arguments would justify the claim most efficiently.

On the Extended Mean Value Theorem for Integrals (MVT) task, both mathematicians initially had a feeling that proving the Extended Mean Value Theorem for Integrals would involve a specific set of one or more symbolic manipulations combined with an application of the given First Mean Value Theorem for Integrals. Dr. A held an expectation about a certain way to see it: "This kind of looks like the kind of problem that once you see how to do, it then it's easy." In the follow-up interview, Dr. A said that the choices for paths to pursue were guided by a feeling that some manipulation would work: "I was sort of guessing somehow this would work…I think that's normal procedure that you guess that something is going to work, and if it doesn't, you try something else." Dr. B's initial work on the task involved several attempts to make the two equations look similar. Dr. B agreed that these first approaches to the task involved attempts at some sort of symbol manipulation so that the two theorems would look similar to each other. Dr. B also concluded the (never completed) work on the task with the feeling that some symbolic manipulation was appropriate: "I still have a feeling there's a little trick I have to do, just some standard tricks, some rule I have to apply and connect the symbols correctly where I can sort of connect these two theorems."

As shown in Table 9, both Dr. A and Dr. B found assessments about task-type to be useful, on the MVT task. Dr. B also asserted that a construction-type proof would be fitting for his individually chosen task, and when first entering the uniform continuity task, Dr. B spent time discerning the logical structure of the statement in order to determine the claim to justify based on what evidence.

The task-type ideas were generated when participants were first entering the tasks and working to understand the statements and objects. The participants were orienting themselves to the task and reflecting on what was known attending to their conceptual knowledge, connections among known ideas, and instantiations of ideas. The participants employed heuristic strategies such as listing what was known or identifying the hypothesis and conclusion of the statement which informed their choices of how to approach proving the statement. Ideas about the task type influenced how the participants proceeded. Specifically, identifying the task type gave them a direction in what sorts of statements or operations they would need to form to connect the hypothesis to the conclusion.

Table 9

Ideas About Task Type Codes With Counts

Aspect	Instances
Task	MVT (2)
	UC (1)
	B-Ind (1)
Participant	A (1)
	B(3)
Problem	Understanding statements & objects (3)
Tool	Conceptual Knowledge (1)
	Connecting & Permuting (1)
	Examples (3)
	Heuristics & Experience (2)
	Instantiations & Equivalencies (1)
Shift	Changing claim (1)
	Changing warrant (1)
	Data added (1)
	Data repurposed (1)
	Opening structure (2)

Identifying necessary conditions. Sometimes the mathematicians would extract statements or properties that needed to hold in order for the statement to be true; they identified *necessary conditions.* Necessary conditions gave a sense of "The statement can't possibly be true unless this condition is fulfilled." In the additive implies continuous task, Dr. A identified a necessary condition, or a statement determined that must be true in order for the claim to hold, that the function must pass through the origin. Dr. A hypothesized that an argument would involve needing this condition to be true and knew that if it were not true then the statement would have to be false. On the same task, Dr. C also identified the same condition as a necessary condition for the statement to hold.

Table 10 summarizes that both participants Dr. C and Dr. A identified at least one necessary condition. This idea was only articulated five times; each time, the participant was questioning the truth of the statement to be proven or a claim that had been made. Dr. A and Dr. C identified necessary conditions on the Additive implies Continuous task when they had not yet asserted a truth conviction. Also, Dr. A articulated necessary conditions on the Extended MVT task when questioning whether the generated argument containing a string of connected expressions would hold. (Dr. A needed $\frac{g(t)}{t-a}$ to be well-defined on its domain and $g'(a)$ to exist.)

Table 10

Necessary Conditions Codings With Counts

Aspect	Instances
Task	Additive implies Continuous (3)
	Extended MVT for Integrals (2)
Participant	A (3)
	C (2)
Problem	Looking for warrant (3)
	Tool problem (2)
Tool	Conceptual knowledge (3)
	Instantiations & Equivalencies (3)
	Known theorem (1)
	Heuristics & Experience (1)
	Symbolic Manipulation (2)
	Symbolizing (1)
Shift	Added backing (1)
	Added subclaim (1)
	Change qualifier / rebuttal (3)
	Changing warrant (1)

As shown in Table 10, when the participants identified a necessary condition, they were looking for a means to connect the conditions to the claim (looking for a warrant). Dr. A also was inquiring into whether his proof or the statement was incorrect (tool problem). The tools used were either (a) heuristic strategies or symbolic manipulations interpreted with their conceptual knowledge or instantiations of concepts as a means of moving the expression that they had forward, or (b) connections formed amongst their conceptual knowledge and instantiations of concepts in an effort to find a connection between the data and the claim.

The mathematicians envisioned the identified necessary condition as being critical to their truth assessment. If they could prove the necessary condition held true, then they would continue efforts to prove the statement true. If they found it not to hold, then they would have evidence to support disproving the statement. Dr. C said, "At that point, I don't think I knew. I think, well I knew that the answer to the question would depend on what that limit was going to be. If I could figure out what that limit was, I would answer the question."

Structurally, the necessary conditions ideas largely contributed a qualifier or rebuttal to their argument. For example, since Dr. A was fairly certain that the error on the extended MVT task was in the posing of the question and not in the work developed in the interview, Dr. A decided to conclude that the developed argument was acceptable with the rebuttal, "unless g'(a) does not exist." At other times, the necessary condition would contribute another sub-claim and sub-argument. Like in the additive implies continuous task, Dr. A and Dr. C pursued proving the function passed through the origin attempting an algebraic argument before recalling that the additive condition necessitated the function passed through the origin and the recollection of a standard mathematical argument. The proof that the necessary condition contributed another line or sub-argument to their personal argument which resulted in a data statement for proving the statement was true. Both participants hypothesized the statement would factor in as a line in the overall proof. This hypothesized trajectory was a new idea of an envisioned proof path.

Envisioned proof path. An envisioned proof path is an idea that proposes a series of arguments that will lead to a solution. The path may not be complete as there may be missing pieces in the middle, or the formalization of the argument may not be defined. However, there are envisioned trajectories. In his work on the additive implies continuous task, Dr. A articulated that if the function was continuous at zero, then that would be sufficient. Dr. A had already proven that the function passed through the origin but was having trouble connecting the additive property to continuity. In the follow-up, the participant said that from there the proof would be easy and would only be a couple lines. The diagram in Figure 77 shows my interpretation of Dr. A's envisioned proof path. The arrows indicate the warrants or arguments that would connect the statements in the rectangular boxes. Solid lines indicate certainty; dashed borders indicate uncertainty.

As Table 11 shows both participants Dr. A and Dr. B envisioned a path to support a claim. Dr. A had envisioned a proof path on the extended Mean Value theorem task after he had already strung together a series of expressions symbolically. He envisioned that the proof would be a matter of "tying up loose edges" to support that his symbolic manipulations were mathematically sound. Dr. B endeavored to understand the given first MVT in the extended MVT task. He developed an understanding of why the statement was true based on exploring examples of integrals of constant and step functions. Dr. B stated that he could envision how he would prove the statement:

> So now, I'm convinced. I've convinced myself this theorem is true because now the standard machine, and that's building from step functions to continuous functions, should prove this…So that's really good because now I understand the first theorem in the context of step functions… At least at this point, I understand better the statement of the first theorem and how to prove the first theorem.

Dr. B did not try to prove the statement, but felt comfortable moving forward based on the envisioned path.

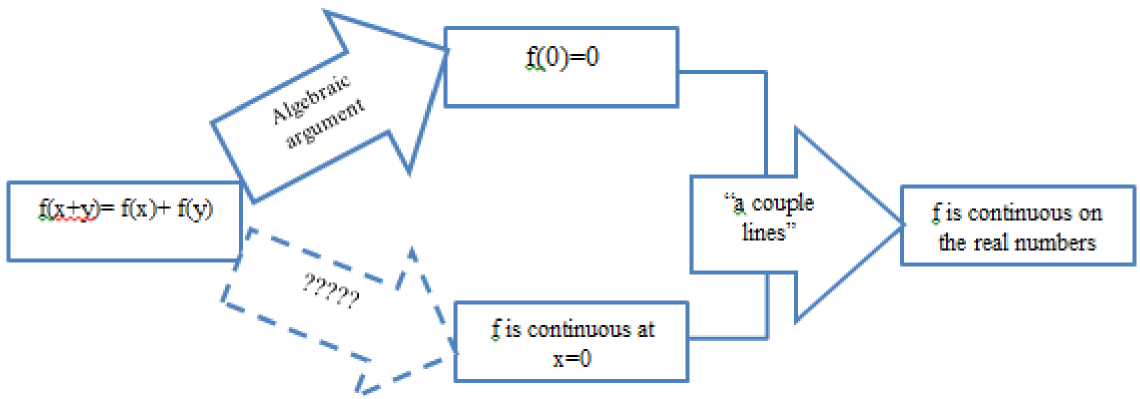

Figure 7. Dr. A's envisioned proof path for the additive implies continuous task.

Table 11

Envisioned Proof Path Codings With Counts

Aspect	Instances
Task	Additive implies Continuous (1) Extended MVT for Integrals (2) UC (1)
Participant	A (2) B(2)
Problem	Looking for warrant (1) Looking for (conceptual) warrant (2) Backing for previous idea (1)
Tool	Conceptual knowledge (1) Connecting & Permuting (2) Examples (1) Instantiations & Equivalencies (1) Other (1)
Shift	Change qualifier/rebuttal (3) Change warrant (1) Data added (1) Data repurposed (1) None (1)

Participants developed envisioned proof paths when looking for justifications using connections between ideas and properties developed from exploring examples, their known instantiations of concepts, and conceptual knowledge. A result of the development of an envisioned proof path was a change to the participants' personal argument structure as a change or addition of a qualifier or rebuttal in that participants could articulate under what conditions they could prove the statement to be true as shown with Dr. A's argument after articulating that showing continuous at zero would be sufficient in proving continuous everywhere (see Table 12).

Table 12

Dr. A's Personal Argument Structure After Articulating His Envisioned Proof Path

Data	Claim	Warrant	Backing	MQ/Rebuttal
Getting a counterexample requires going outside analysis; f(0)=0; f is additive; Continuous means: if y is really small then f(y) must be really small	F is continuous	If continuous at zero, then continuity follows immediately	If y is really small then f(y) must be really small *Envisioned steps*	Truth depends on continuity at 0

The envisioned proof path idea-type "did work" for the participants in so far as the participants' intentions when they developed it. Since the version of the statement given was false, Dr. A was not able to fill in the warrant that showed that *f* was continuous at zero. The envisioned path was abandoned in pursuit of a counterexample. Later Dr. A explained that if provided with the fact that *f* was continuous at zero, then the proof would just need a few more lines. Dr. B did not pursue the envisioned proof path for the First MVT because that was not the intention. The participant was working to get enough of an understanding about why it worked to potentially apply it to proving the Extended MVT. In Dr. B's evaluation, the work achieved the purpose. Dr. B did realize the envisioned proof path while working on the uniform continuity task.

Ideas about formal logic and the representation system of proof. As expected when the task is to write a proof of the situation, the mathematicians made decisions regarding structuring and communicating a formal mathematical argument. Ideas were classified as *ideas about formal logic and the representation system of proof* if they involved decisions in how to either structure or communicate an argument apart from the actual mathematical content of the problem. It should be noted that not all logical decisions were identified as ideas that move the argument forward because many logical decisions were embedded within other ideas or the realizations of these ideas which typically occurred without incident for the participants to note them as important. Other uses of formal logic and the representation system of proof that contributed to the development of other ideas were coded as tools.

The logic and representation system decisions noted as ideas that move the argument forward included decisions on how to structure the argument, what qualifiers to use, writing or rewriting the given task's conditional and hypothesis statements, and discerning what constitutes mathematical proof. As an example consider some of the ideas posed by Dr. C on the Own Inverse task. Early on, Dr. C identified that if a function were increasing and not equal to the identity function, then something would go wrong with the symmetry. The participant spent some time trying to identify what that something was. This led to the idea of pursuing a proof by contradiction attempting to prove the following statement: "Let f be as given, increasing, and not equal to the identity, then f-inverse cannot be f." Later on, Dr. C developed a warrant found to be generalizable and then had the idea to change the argument to prove the statement: "Let f be as given and increasing, then $f(x) = x$." This idea related to how Dr. C planned to structure the argument.

As noted in Table 13, Dr. B expressed more ideas about logical structure than the other two participants, and they appeared on every task that Dr. B worked on. On two tasks, Dr. B decomposed the statement into logical P and Q statements and discerned that it was appropriate to pursue a P implies Q argument. I asked Dr. B if this was typical of his practice, and Dr. B responded that he often found this useful in analysis tasks:

> Doing the P and the Q part? Um, because in analysis, because of the complexity of the statement, there's so many little quantifiers and logical things like that. For me it just helps organize the problem and really isolate the antecedent and the consequence so you can just look at them individually, and individually sort of decode them, you know. So it just helps me, it's difficult for me to look at sort of the original statement. It helps me to visually sort of isolate the statements that, that are involved in the logic we are using. Just on a logical level, yeah. So, that's, I think that's what I isolated those is just to make things more organized on a logical level so I could identify the, I could work with those sort of individually, yeah, in terms of describing them in different mathematical ways if I need to I guess, yeah.

Table 13

Logical Structure Codings and Counts

Aspect	Instances
Task	MVT (1)
	Own Inverse (2)
	UC (3)
	B-Individual (1)
Participant	A (1)
	B (5)
	C(1)
Problem	Backing for previous idea (1)
	Articulate & generalize (1)
	Understanding statements & objects (2)
	Tool problem (1)
	No problem (2)
Tool	Connecting & Permuting (1)
	Heuristics & Experience (4)
	Instantiations & Equivalencies (2)
	Knowledge of sociomathematical norms and logical structure (6)
	Symbolizing (2)
Shift	Changing/Specified Claim (1)
	Changing Warrant (3)
	Data removed (1)
	None (1)
	Opening structure (2)
	Order of presentation (3)

Only one of each of Dr. A and Dr. C's ideas that move the argument forward were classified as ideas about formal logic and the representation system of proof; they were an idea about how to better structure a final argument (Dr. C on the own inverse task) and an evaluation that an argument did not count as mathematical proof and should be explored further (Dr. A on the MVT task).

The formal logical decisions were not the aspects of the task that participants found problematic with the exception of Dr. B's including determining the logical structure of the claim with working to understand the statement. When making the logical decisions, the problems the participants were entered into solving were looking for ways to articulate or generalize a warrant, justify a previous idea, or fix a perceived error in their write-ups.

Dr. C, for example, achieved an algebraic contradiction on the Own Inverse task and moved to writing the details in a formal proof. Dr. C made decisions while carrying this out to change the argument from a proof by contradiction to a proof that if f were as given and non-decreasing, then it must be $f(x) = x$. In making that decision, the participant made the necessary changes to the inequalities in the original algebraic warrant. However, Dr. C applied these decisions or tools expertly knowing the exact outcome each decision would achieve. Dr. C even described that there were instances of writing without thinking which indicates non-inquirential behavior or that no problem was perceived. Dr. C did engage in some checking in rethinking a qualifier used. The participant also checked the argument and realized that it only proved one of two cases and went on to prove that second case.

The structural shifts noted in Table 13 indicate the immediate consequences of this particular idea. Decisions like choosing to argue by cases affected the structure of the personal argument by adapting or changing the warrant, backing, or order that the argument was presented. The implementation of logical structure ideas did not always affect the structure of the personal argument. Ideas about formal logic and the representation system of proof informed how the participants chose to pursue an argument and how they chose to present it.

Summary about ideas that focus and configure. The six above idea-types were grouped together because they all appeared to do work for the participant as far as making decisions about how to begin, how to proceed, and what tools and ideas to use in doing so. Specifically, ideas that informed the statement image broadened, narrowed, or shifted participants' view of what statements, facts, relationships, and so forth were potentially pertinent to completing the task. Combining these ideas with other perceived useful tools enabled participants to make truth proposals, discernments about task-type, and identify necessary conditions. Truth proposals and identified necessary conditions guided the direction of pursuit in that they identified claims to try to connect to the data. Ideas about formal logic and the type of task gave information about potential strategies for development of connections between the data and the claim. Envisioned proof paths and necessary conditions advised what steps would be needed in order to connect to the claim. Ideas about formal logic and envisioned proof paths informed the organization and articulation of the argument. As noted above, the implementation of these ideas did only not affect the organization, data, and claims of the personal argument; focusing and configuring ideas also had influence on warrants and backing as they provided information as to which warrants and backings could be useful and which were inappropriate. The warrants and backings were the means by which the connections sought were achieved and justified.

Ideas That Connect and Justify:
Warrants and Backing

Warrants and backings were the means by which the participating mathematicians connected data with claims. The mathematicians sought specific warrant and backing pairs that would establish connections in ways that could be finally articulated and justified in a mathematical proof. This inquiry sometimes involved looking for and finding symbolic manipulations and syntactic applications of known and given theorems and definitions that could be pieced together to connect the hypothesis and conclusion. Other times, participants needed more information in order to determine what manipulations and theorems should be applied and how they could be applied. So, they first searched for a structural or conceptual reason beyond the symbolic representations that underlie the link between the two mathematical statements. In either instance, participants made conjectures as to which connections would be useful and the connections may or may not have been based on logical deductions.

Inglis et al. (2007) found that when mathematics graduates were making conjectures, they justified them with warrants that were not always based on deductive backing. Inglis and colleagues identified three such warrant-types. *Deductive warrants,* (warrants based on deductive backing), *structural-intuitive warrants* (warrants backed by the prover's intuition and knowledge about the mathematical concepts and their relationships), and *inductive warrants* (warrants backed by the exploration of specific examples). I borrow from Inglis and colleagues to describe the warrant ideas that the mathematicians had while in pursuit of a mathematical proof. I expand upon their classifications to include the warrant-type, *syntactic connections*, in order to include proposed and utilized warrants that serve to connect statements symbolically but do not always attend to the mathematical conditions permitting the manipulation's deployment. Each warrant-type was coded as an idea that moved the argument forward because their proposal, testing, and implementation either led directly the development of a mathematical proof or provided new information deemed useful to the eventual construction of the proof. Examples of these four warrant-types are provided in the sections below.

Typically, proposed and utilized connections between statements fell into these four categories. As such, the backings for the connections were categorized within the warrant type. There were instances where participants proposed new backings to support an already articulated warrant. Also, there were instances where a vague sense of a backing or underlying reason to a connection between the two statements was vaguely articulated prior to the establishment of a warrant to match it. All these ideas were coded as proposed backings.

Table 14 summarizes the four warrant-type ideas and the proposed backing ideas. I will refer to the table in specificity within each subsection and further describe the idea-type, provide examples, summarize the inquirential context surrounding their emergence, and their immediate impact on the personal argument. A later section describes themes regarding the utilization and testing of ideas and the different ways that the warrant-types and backings interacted as the argument progressed toward resolving the deemed problem.

Table 14

Summary of Warrant-Types and Proposed Backing Ideas With Counts

	Deductive	Syntactic Connections	Inductive	Structual-Intuitive	Proposed Backing
Task	Additive implies Continuous (2) MVT (1) OwnInverse(2) UC(2) A-Ind (4) B-Ind (1) C-Ind (3)	MVT (3) A-Ind (3)	MVT (2) OwnInverse (4) UC (1)	Additive implies Continuous (4) Own Inverse (1) UC (3)	Additive implies Continuous (1) OwnInverse (2) UC (1) A-Ind (1)
Participant	A (6) B (4) C (5)	A (6)	B (6) C (1)	A (2) B (3) C (3)	A (1) B (2) C (2)
Problems	Determining truth (3) Looking for Warrant (3) Generalizing/ articulating idea (3) None (6)	Looking for Warrant (symbolic connection) (5) None (1)	Looking for Warrant (conceptual)(6) Looking for Warrant (deductive) (1)	Determining truth (4) Looking for Warrant (deductive) (1) Looking for Warrant (conceptual) (3)	Looking for Warrant (conceptual) (2) Looking for Warrant (deductive) (1) Generalizing/ articulating idea (1) Tool problem (1) None (1)
Modes	Reflection (4) Action (7) Evaluation (2) Unknown (1)	Reflection (4) Action (1) Evaluation (1)	Reflecting (1) Action (3) Evaluation (3)	Reflection (3) Evaluation (5)	Reflection (1) Action (2) Evaluation (2)
Tools	Conceptual Knowledge (6) Connecting & Permuting (9) Heuristics & Experience (4) Instantiations (1) Known Theorem (2) Logical structure (implied in all) Symbolic Manipulation (4) Symbolizing (3)	Conceptual knowledge (1) Connecting & Permuting(4) Known Theorem (2) Symbolic Manipulation (4)	Examples -to see why true (4) -to understand (1) -to test (2) - articulate (1) Conceptual knowledge (3) Heuristics & Experience (1) Instantiations (2) Interruption (1)	Example -to understand (1) -to test (1) Conceptual knowledge (6) Connecting & Permuting (5) Heuristics & Experience (2) Instantiations (2)	Example -to see why (2) Conceptual knowledge (1) Connecting & Permuting (4)
Structural Shifts	Added subclaim (3) Added backing (5) Change claim (2) Change warrant (5) Data added…(3) Data repurposed (6) Opening (1)	Added subclaim (6) Data added (4)	Change backing (2) Change warrant (4) Data repurposed (1)	Added subclaim (1) Change claim (3) Change warrant (5) Data repurposed (1) Opening (3)	Added backing (2) Change MQ/ rebuttal (2) Change warrant (2) Data added (2) Data repurposed (1)
Tested	None (6) Write deductive backing (2) Symbol connections (5) Mathematically hold(2) Unknown (1)	Symbol connections (6) (later) mathematically hold (6) (implied)	Example (6) None (1)	Look for deductive backing (5) Example (2) Context (1)	Example (3) Symbolizing (3) Extend case (1) Symbol connections (1)

Deductive warrants. Participants had the overall goal of developing deductive warrants; that is they worked to develop reasoning why the claim would be true based on generalizable logical statements. Dr. C referred to attaining a deductive warrant as figuring out "how to do it algebraically." An example of the development of a deductive warrant was Dr. C's development of an algebraic contradiction to the own inverse task. Dr. C had been trying to find a reason that could be rendered into a proof why $f(x) = x$ is the only increasing function with the properties of being continuous, bijective on the set $[a,b]$, and its own inverse. After some proposed warrants and backings, Dr. C stated that "It has to depend somehow on the fact that when I reflect over the diagonal line, the piece of the curve that's below the diagonal line ends up above the diagonal." Dr. C then set about trying to utilize this idea in a symbolic way deploying several ideas that he had developed earlier: an idea to argue by contradiction, an idea about how not being on the line $y = x$ meant that a point was either above or below the line, symbolic manifestations of what it meant for a function to be its own inverse, and the developed idea that something happens when the point is reflected across the line to develop the contradiction in Figure 8.

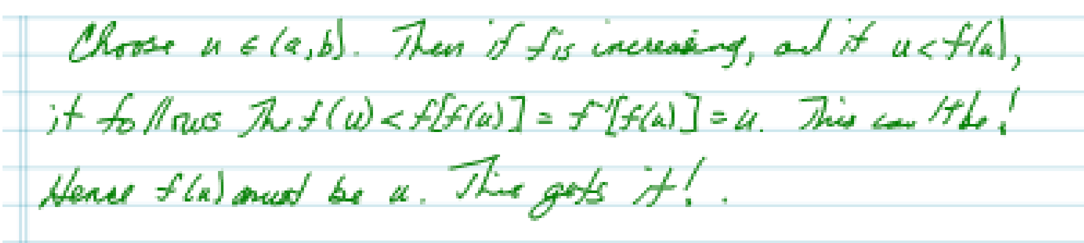

Figure 8. Dr. C's sketch of a deductive argument for the Own Inverse task.

The structure of the personal argument before and after the articulation of the argument is given in Table 15 below. The actual deductive warrant is the idea that there is an x-value, *u*, such that *f(u)* must both be greater than and less than *u*. The backing for the warrant is the series of logical deductions that support it.

Table 15

Dr. C's Personal Argument on the Own Inverse Task Before and After Developing a Deductive Warrant Idea

Data	Claim	Warrant	Backing	MQ/Rebuttal
F is its own inverse so f(f(x))=x and it is symmetric about y=x F is 1-1, continuous, onto, f is either increasing or decreasing Reflecting over y=x moves points below the line above the line Trichotomy principle	F is decreasing except f(x)=x		Something that uses points moving above or below the line y=x The trichotomy principle means points will be below/above the line	
F is its own inverse so f(f(x))=x and it is symmetric about y=x F is 1-1, continuous, onto, f is either increasing or decreasing Reflecting over y=x moves points below the line above the line Trichotomy principle	F is decreasing except f(x)=x	If f were increasing and not f(x)=x, then would get a contradiction that f(u)<u and u<f(u)	Assume u<f(u) (since fIx) is not x), since f is increasing f(u)< f(f(u))=u	"this gets it"

As we can see, Dr. C attended to several data statements. The development of the deductive warrant changed the warrant, the backing, and Dr. C announced a qualifier of certainty that he could develop a proof. It should be noted that the above argument is not a completed mathematical proof. The idea for a deductive warrant came before or during the articulation of an actual proof. The warrant idea is a proposal for a means of connecting the data and claim. It may need to be combined with other ideas to yield a mathematically correct proof.

As was shown in Table 14, each participant developed a deductive warrant idea on at least one task. Some participants developed more than one deductive warrant for an individual task as a means to justify sub-claims within their overall argument. There were two cases where an idea was coded as an (incorrect) deductive argument which meant that the prover had believed that they had deductively justified the claim, but the backings were based on incorrect mathematical statements.

In the example above, Dr. C was addressing the problem of determining symbolically what actually went wrong when the point was reflected over the line (coded generally articulating an idea). The contradiction with the inequalities came while he was applying the above ideas symbolically (coded connecting & permuting, symbolizing, and symbolic manipulation). In general, participants developed deductive warrants by deploying and connecting a variety of known facts, relationships, and heuristics symbolically. Knowledge of logical structure was implied in all cases, just as Dr. C had implied there was still the needed step to write down the case that u was greater than $f(u)$. The problems encountered were actively seeking to develop a connection, seeking to generally articulate an idea, and to determine truth. There were instances when participants developed deductive warrants without perceiving a problem.

Finding warrants that could be articulated generally and in the representation system of proof acted as critical moments in the development of mathematicians' arguments. Upon articulation of the deductive warrants symbolically, participants either tested them or declared they were finished with proving that particular claim. The deductive warrant ideas were tested by writing down and checking over the deductive statements to back them, checking over the symbolic connections against the given data to ensure all connections were allowable, the symbols matched, and that all cases were covered.

The immediate effect on the personal argument was the adding of sub-claims and data statements if several sub-statements were linked together, the changing of the warrant and backing, and the purposing of data statements as warrants or backing. The two instances where there was a change in claim referred to Dr. C's work on the individual task that required assessing the continuity of a given function. The changes in claim were due to the open-ended nature of the task where the person working on the task was to provide the domain on which the function was continuous. Dr. C deductively made assertions about where the function was continuous.

Syntactic connection ideas. As a means of connecting two statements, participants would sometimes search for ways to connect the symbolic representations of the two statements. For example on the extended MVT task, the statement to be proven was an equation between two integral statements, and the first MVT that was given as a known also had a similar looking equation. Based on their assumptions about the *task-*

type, both Dr. A and Dr. B worked to connect the known equations and expressions to result in the an equation of the left-hand-side of the theorem to be proven with the right anticipating it would involve symbolic manipulations that would enable an application of the First MVT.

I term the ideas of useful symbolic manipulations *syntactic connections.* An example of a syntactic connection is the idea Dr. A developed to allow a *(t-a)* factor appear in the product $g(t)f(t)$ by multiplying and dividing *g(t)* by *(t-a)*. These means of connections are a type of warrant as they connect the given evidence to the claim. However, a series of these syntactic connections may need to be strung together to actually connect the evidence to the claim, and they may not always be supportable by deductive reasoning or may not attend to the mathematical objects that the symbols represent. For example, Dr. A found multiplying and dividing by *(t-a)* was useful in manipulating the expression to look in an intended way, but Dr. A did not attend to the fact that dividing by *(t-a)* was potentially problematic until going back to clean up the "loose edges" of the argument.

As was shown in Table 14, Dr. A was the only participant to utilize what I termed as syntactic connections on the Extended MVT task and on Dr. A's own chosen individual task that applied the Lagrange Remainder Theorem which also involved proving a symbolic equation was true. Dr. B did try to find ways to connect expressions symbolically on the Extended MVT but was not successful. The syntactic connections that Dr. A developed either were strung together immediately step-by-step or were left as data statements as other equations were developed. Looking for similarities amongst the gathered statements on the Lagrange Remainder Theorem task, Dr. A employed algebraic substitution to eventually connect all the equations. On The Extended MVT task, once Dr. A had symbolically connected the two expressions to result in the equation to be proven, Dr. A declared that he now knew the steps of the argument but would need to check if the steps were allowed: "I think I know how to do it…I've got the skeleton. I know the steps. I've got to make sure each step is right." Dr. A then worked to write up the proof of the statement. The syntactic connections remained warrants, but Dr. A checked if conditions were met in order to back these syntactic connections. With syntactic connection warrants, the evolving personal argument seemed to consist of a continual adding of sub-claims that would eventually be strung together as proven claims would become data.

Inductive warrants. Inductive warrants are statements meant to connect data to claim based on specific examples. An instance of an inductive warrant was the warrant that convinced Dr. B of the truth of the first MVT. The task as given is shown below.

> **Given: Theorem1- MVT for Integrals**: If *f* and *g* are both continuous on [a,b] and $g(t) \geq 0$ for all *t* in [a,b], then there exists a *c* in (a,b) such that$\int_b^a f(t)g(t)dt = f(c)\int_a^b g(t)dt$.
> **Prove: Theorem 2 – Extended MVT for Integrals**: Suppose that *g* is continuous on [a,b], $g'(t)$ exists for every *t* in [a,b], and $g(a) = 0$. If *f* is a continuous function on [a,b] that does not change sign at any point of (a,b), then there exists a *d* in (a,b) such that $\int_a^b g(t)f(t)dt = g'(d)\int_a^b (t-a)f(t)dt$.

Dr. B had set about trying to understand the two theorems with specific examples of functions. Exploring the first MVT using $g(x) = 1$ and $f(x) = x^2$, Dr. B developed an understanding about what the theorem meant:

> I'm just saying that if I have an area like this then and again the function doesn't really have to be positive, just *g* has to be positive. I'm going to assume *g* equals one right here. It's telling me I can always find an *x* so that the area, there's always going to be a rectangle whose area is the same as my function. That

> makes sense. Okay I can always find a rectangle whose area is the same as my function there.

Dr. B referred to a picture drawn (see Figure 9) as contributing to the rectangle interpretation which was that the 'c' chosen would scale down the rectangle formed by the area under $g(x) = 1$ to a height ($f(c)$) where the area of the new rectangle is equal to the are under $f(x) = x^2$ which was the product of f and g.

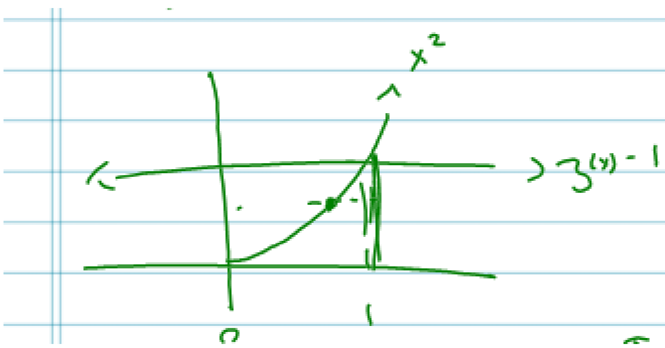

Figure 9. Dr. B's first specific example of the First MVT for Integrals.

> But it was easy for me to see geometrically that, that's what I was thinking back here. In this, um previous picture, in this picture [pointing at screen]. One of these pictures here, I had a line through it. But the thought was that, yeah, all it was saying is that if you have a function like this that the area underneath *g* [meaning *f*] is equivalent to the area of some rectangle [pointing to picture] that you draw across from that. And that's geometrically easy to imagine, you know. And then that was just for a special case for one of the functions.

Dr. B worked to extend the idea about having a rectangle to another case where $g(x) = 1 + x$, but Dr. B was not able to identify a rectangle associated with the non-constant *g(x)*. Therefore, the participant decided to do another example keeping *g* constant. "What if *g* of *x* is two? We're going to slightly complicate, *g* seems to be throwing me off." Dr. B thought about $g(x) = 2$ and $f(x) = x^2$. Dr. B drew the two functions on the interval *[0,1]* as well as their product. Dr. B thought for a while about the areas under the product curve and the area under the constant function (see Figure 10). The participant placed an *'x'* at a point on the function *f*, and concluded that $c = d$. "So I want to say the integral of the function *g* times *f* from zero to one, [shades under *g*f*] I can take two [shades under *g*] this is two, times, alright [marks an x on the *f* curve], and actually in that case, *c* equals *d*."

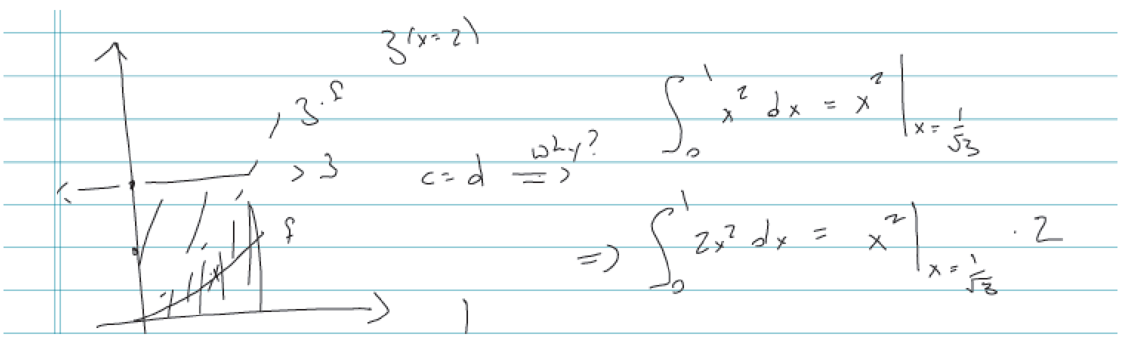

Figure 10. Dr. B's specific example for first MVT (MVT-graph6).

As shown in Figure 10, Dr. B tested why $c = d$, writing the equation for the first MVT for the earlier worked example using $g(t) = 1$ and $f(t) = t^2$ and the equation for the first MVT for the current example, getting the constant that makes both equations true is $\frac{1}{\sqrt{3}}$. Even though Dr. B talked about constants *'c'* and *'d'* which are used in the first and extended MVTs, respectively, these values were the two constants needed for the first MVT for the two pairs of example functions. It is reasonable to assume Dr. B was not thinking about the 'd' in the extend MVT because the participant was not exploring it and in later explorations Dr. B declared it was an attempt to understand the first MVT.

After exploring the examples involving a constant function, Dr. B discerned a grasp on understanding why the first MVT held when one function was constant based on areas of rectangles as well as computations. "That's how that works. You can do this thing for any step function. So if you have *g* of *x* equals constant, then *c* is going to equal *d*." So, the idea evolved from seeing that there is a rectangle whose area is equal to the area under the product curve to the statement that no matter the constant function, the same constant value will make the equation hold. Dr. B alluded to this also holding for any step function and tested this:

> And now, sort of, ideas are starting to sink in here. Because as soon as I can do something for a step function as far as integration theory, you can do other, sort of interesting things. If I had a step function, now this is a step function, so *g* is not continuous in this case, but it is still going to work.

Dr. B generated a step function $g(t)$ and paired it with the function $f(t) = t^2$as before. The participant did not compute the values of the integrals but used properties of integrals to rewrite the statement to be proven as a sum of integrals that could restrict g(t) to constant functions as before (as seen in Figure 11).

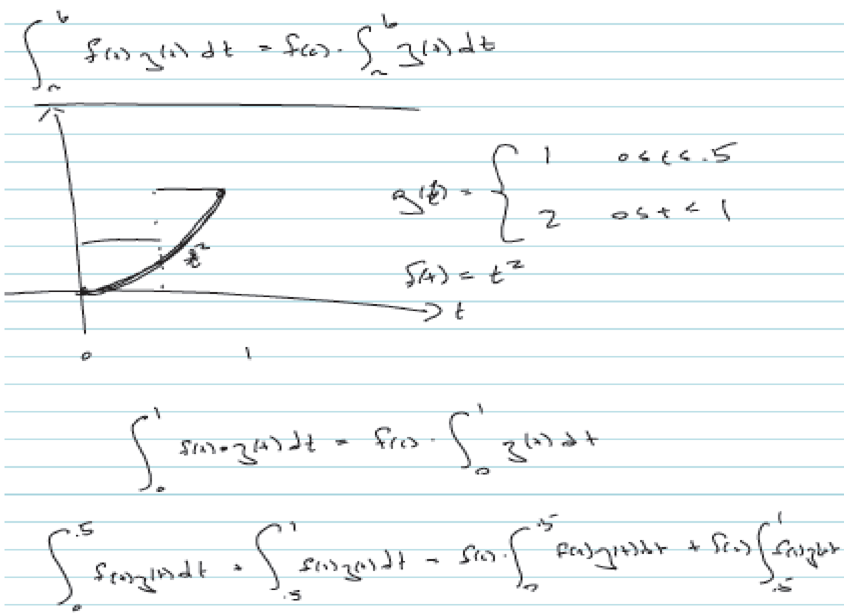

Figure 11. Dr. B's example step function used in exploring the First MVT for Integrals.

To the participant, this equation was enough to achieve personal conviction that the first MVT held for step functions, and Dr. B envisioned a way of proving the theorem would hold for any continuous function using the "standard machine" for integral arguments:

> So now, I'm convinced. I've convinced myself this theorem is true because now the standard machine, and that's building from step functions to continuous functions, should prove this…So that's really good because now I understand the first theorem in the context of step functions.

To address the greater problem of understanding the First MVT, Dr. B set about determining if his proposed idea that the ability to create rectangles of a certain size was the key idea supporting why the statement was true. Because the rectangle idea was developed based on a constant function and then supported by a different constant function, Dr. B proposed that the theorem could be shown to work for step functions with the end in view of using standard integration theory tactics to extend the step functions to all continuous functions. The idea emerged while Dr. B was evaluating the work in testing the idea for constant functions. It proved fruitful when Dr. B developed self-conviction in the idea's viability on step functions by partially exploring an example step function that spanned the interval *[0,1]*. Dr. B did not feel the need to input the functions into the integral computations or to find the '*c*' needed to make the equation true and did not test varying the function *f(x)*. This was because Dr. B's purpose was not to prove the First MVT but to gain enough of an understanding of the First MVT to render it in exploring why the Extended MVT was true. Table 16 summarizes the ending structure of Dr. B's argument supporting the First MVT. It is not a proof as it is based on inductive verifications, but it sufficiently convinced Dr. B and gave the feeling of being able to render the argument into a proof.

In addition to the Extended MVT task, inductive warrant ideas were utilized in the Own Inverse task by both Dr. B and Dr. C, and by Dr. B on the Uniform Continuity task. Dr. B and Dr. C developed inductive warrant ideas, but Dr. A derived none. In fact, Dr. A did not pose any examples while working on any task, but that may have been due to the perceived nature of the tasks that he worked on. Neither Dr. A nor Dr. C utilized specific examples on the additive implies continuous task, and Dr. B only deployed examples on the MVT task because he did not achieve success symbolically while Dr. A did.

Table 16

Structure of Dr. B's Personal Argument Specifically Pertaining to Proving the Given First MVT for Integrals

Data	Claim	Warrant	Backing	MQ/ Rebuttal
Statement of 1st MVT ($g(t) \geq 0$, g and f continuous on [a,b] In 1st MVT, g(t) acts as a weighting function Earlier argument that MVT works or constant g(t)	$\exists c \in [a, b]$such that $\int_a^b f(t)g(t)dt = f(c)\int_a^b g(t)dt$	A standard argument that extends integrals of step functions to integrals of continuous functions	Computations and pictures with $g(x) = 1$ and $g(x) = 2$ while $f(x) = x^2$ on the interval [0,1]	Not a proof

As was shown in Table 14, both Dr. B and Dr. C developed and utilized inductive warrants while trying to achieve some conceptual understanding of why the statement was true or to find a deductive warrant. The examples utilized ranged from examples posed to see why true (used to explore the underlying causes), examples to understand the objects or relationships given in the statement, examples to test to see if another claim holds, and examples to help articulate a vague sense of understanding. The extracted warrants from the examples were based on conceptual knowledge.

The inductive warrants were tested against other examples. For Dr. B, the standard by which the ideas' feasibility was measured was Dr. B's personal understanding or ability to envision how an argument would go. These ideas were used to formulate an envisioned argument of why the First MVT was true. As elaborated in a later section that discusses the evolution of the personal argument, in other cases the inductive warrants were tested to see if the idea extends to other cases and to determine why the extracted property had to work so that an underlying cause could be wielded and articulated in a deductive argument. Structurally, the development of an inductive argument resulted in a new warrant.

Structural-intuitive warrants. A structural-intuitive warrant is to be a statement or idea that the prover proposes could link the data to the claim based on a feeling that is informed by structure or experience. Inglis et al. (2007) observed *structural-intuitive warrant-types* as "observations about, or experiences with, some kind of mental structure, be it visual or otherwise, that persuades them of a conclusion" (p. 12). All three participants utilized a structural-intuitive warrant (Table 4). The warrant-type was used on the additive implies continuous task, the own inverse task, and the uniform continuity task. I describe Dr. A's and Dr. C's structural-intuitive warrant on the additive implies continuous task here.

Upon first reading the statement to be proven, Dr. A expressed a concern that the statement was not true because he remembered that there were things that could be done with unmeasurable functions and the axiom of choice that could produce a counterexample. The proposed warrant that there could be a counterexample was backed by a memory he described as vague and unclear. He was not quite persuaded so his structural-intuitive warrant was coupled with a truth proposal not a conviction.

Dr. C used a structural-intuitive warrant to shed doubt on the truth of the statement when he stated that a function with the additive property would not be continuous on the real numbers because he knew the only continuous linear functions were in the form $y = mx$, and this function was not of that form. Moreover Dr. C knew that proving the additive property was continuous on the rational numbers would involve an inductive argument, but it would not be straightforward for irrational numbers:

> I was thinking about the well-known fact that the only continuous linear functions in the reals to the reals are those of the form y equals mx for some fixed m. And one shows that those are continuous on the rationals fairly easy - linear functions are continuous on the rationals pretty easily by doing some induction.

The participant had used knowledge of structure to connect the additive property to linearity. Dr. C's intuition contributed to the idea that the additive property would not be enough to show continuity.

I did not observe structural-intuitive warrant ideas on any of the individual tasks nor on the Extended MVT task. The majority of the warrants observed on the individual tasks were deductive or syntactic connections. In these tasks, the participants largely proceeded to justify their claims deductively or symbolically with little problem. On the Extended MVT task, Dr. A was able to find syntactic connections so did not have to exit the representation system of proof. Dr. B did explore ideas semantically and used conceptual knowledge and knowledge of mathematical structure, but the conclusions were backed by the exploration of examples deeming the warrants as inductive.

With Dr. A and Dr. C on the additive implies continuous task, these structural-intuitive warrants were starting points when the participants were encountering the problem of determining the truth of the statement. They were reflecting on what they viewed as related. Dr. C mentally compared the additive function to a class of functions (conceptual knowledge), and Dr. A related the conditions on the function to previous experience (heuristics and experiences). Other structural-intuitive warrants drew on connecting knowledge, experience, and strategies to propose a connection between claims specifically in pursuit of a truth determination or a conceptual reason why the statement was true.

On the prove or disprove task, the structural-intuitive warrants informed the mathematicians' initial perceptions that the statements were not true. For the other tasks, since they were not deductive, participants worked to test and generally articulate these warrants. In a later section, I will elaborate some of the ways that inductive and structural-intuitive warrants were tested to see if they could be rendered into deductive warrants. Mostly, in testing their warrants, participants worked to try to generalize or symbolize their intuition or test their intuition on another example to try to extract a general backing.

Proposed backing and proposed (vague) backing. With the non-deductive warrant types, the backing for a statement was implied by the means of developing the warrant and was coupled with the warrant. With Dr. B's exploration of an example to explain why the First MVT worked for constant functions, Dr. B extracted the warrant that one could always find a rectangle backed by the manipulation of the example. The example was the backing. However, participants sometimes proposed a backing separate from a warrant. Participants proposed backings for previously identified non-deductive warrants but also participants proposed a vague sense of what would underlie a possible warrant that may have not yet been articulated. I first describe an instance of a proposed backing for a previously proposed warrant, and then I show how the same participant articulated a feeling about what would be instrumental in any warrant.

Dr. C worked on the Own Inverse task and explored why the identity function ($f(x) = x$) would be the only possible increasing bijective, continuous function that was its own inverse. Based on an instantiation of inverse as being the reflection across the line $y = x$ and some explorations involving drawing and imagining pictures, Dr. C had articulated that any other increasing function would fail to be its own inverse because it would not be the same function when reflected across the line $y = x$. Dr. C was unsure how to articulate why this would happen in a proof and explored both symbolically and with pictures some consequences of being increasing and one's own inverse as shown in Figure 12.

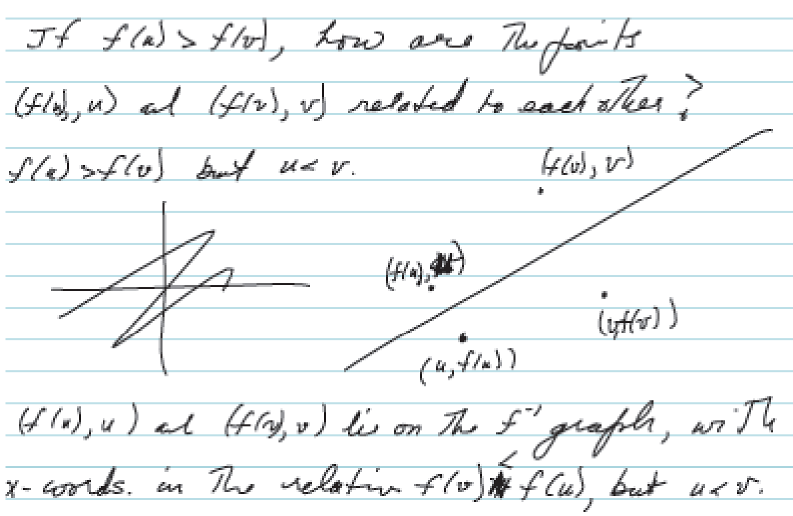

Figure 12. Picture CI-3: Dr. C permuting the logic on the own inverse task.

Dr. C then stated "that doesn't seem to be going anywhere." Dr. C characterized the above work as "permuting the logic of all the conditions I was interested in, trying to find a combination of permutation with those things that I can connect with each other to get an argument." Dr. C declared this work unfruitful because "I'm not seeing anything that I can use in that reformulation of what's going on." Right after declaring that this work investigating as in Figure 12*Figure 1* was not going anywhere, Dr. C stated,

> Oh, but let's see here. The inverse of an increasing function is also an increasing function. And this means that the only way this can work is for $f(x)$ to be x for all x in I. So we can't have f equal to f-inverse if f is an increasing function. That does it. You want me to write it up?

The idea was consistent with his picture CI-3 in Figure 12. Dr. C had declared a general reason why a non-identity function would fail to be its own inverse that could be articulated in a formal write up of the proof. Dr. C felt that this idea was enough to show a proof of the statement using the idea if $f(x)$ is a non-identity, increasing function, then it won't be symmetric across the line $y=x$. The idea, to the participant, served as a reasonable backing for the warrant (contradiction) that the function and its inverse would not be the same function. This instance was an example of a proposed backing idea as it was an idea for backing up an already articulated warrant.

The problem encountered was finding a reason why an increasing function could not be its own inverse that could be generalized. Dr. C was applying a strategy of "permuting the logic" and had been working with picture CI-3 in Figure *1*12 in an effort to find a combination of ideas that could be used for an argument. The participant had almost declared this work unhelpful, but then noticed that the inverse of his increasing function was also increasing. Table 17 illustrates the changes in the personal argument. The backing changed from a picture to a statement that f would not be symmetric because the inverse would be a different function since the inverse of an increasing function is also increasing.

Table 17

Dr. C's Personal Argument Upon the Articulation of a Proposed Backing

	Data	Claim	Warrant	Backing	MQ/Rebuttal
Prior to articulation	F is its own inverse so f(f(x))=x and it is symmetric about y=x F is 1-1, continuous, onto	F is decreasing except f(x)=x	If f was increasing and not the identity, then the inverse would be a different function	Picture CI-2	not an argument
After articulation	F is its own inverse so f(f(x))=x and it is symmetric about y=x F is 1-1, continuous, onto, f is either increasing or decreasing	F is decreasing except f(x)=x	If f was increasing and not the identity, then the inverse would be a different function	Picture CI-3 and associated reasoning that the inverse of an increasing function is also increasing	"that's it", needs to be written up

Dr. C went to work to write up the argument based on this proposed backing but got stuck, stating that what was written down did not match what Dr. C was thinking and the personal thinking was wrong:

> Did I get things backwards up here? I think I did. / It's still correct I think. Somehow. / But I seem to have showed that the inverse of an increasing, or I seem to have thought that the increase of a, the inverse of an increasing function is a decreasing function. And that's not right./ It has to depend somehow on the fact that when I reflect over the diagonal line, the piece of the curve that's below the diagonal line ends up above the diagonal. // And that, I think, is where we have to go.

This was the first time that Dr. C articulated this idea that the points move above and below the diagonal as really important:

> That was the first time I thought, I think that that idea itself was important…I should have seen it when I wanted to think about what happens when *x* and *f(x)* are different. That's the trichotomy principle. Either *x* is equal to *f(x)* or it's less than, or it's greater than. And it was at about that point that I began to see that that was the key to what I needed to do.

The problem was re-assessing why an increasing function would fail to be its own inverse. The participant was looking back at pictures drawn before and was reflecting on what was going on. Dr. C had used the idea that points below the line end up above it implicitly when drawing the picture but did not focus on that being a reason why a contradiction could happen. The participant proposed that a contradiction would be based on a new warrant was not yet fully articulated. Comparing the argument structures in Table 17 to Table 18, we see that the proposed (vague) backing wiped the warrant-slate clean and offered a feeling about what would be important in terms of justifying that warrant. As described in an earlier section, Dr. C moved forward to render this and other ideas into a deductive warrant and eventually a completed proof.

Table 18

Dr. C's Personal Argument After Proposing a New (Vague) Backing

Data	Claim	Warrant	Backing	MQ/Rebuttal
F is its own inverse so f(f(x))=x and it is symmetric about y=x F is 1-1, continuous, onto, f is either increasing or decreasing Reflecting over y=x moves points below the line above the line	F is decreasing except f(x)=x		Something that uses points moving above or below the line y=x The trichotomy principle means points will be below/above the line	

As was shown in Table 15, each participant proposed a backing at least once in every task where non-syntactic thinking led to a deductive argument. Dr. A also proposed a backing in the purely syntactic argument for the individual task when identifying that a particular given condition in the theorem statement gave a hint about what syntactic connection to propose. Proposed backing ideas were articulated while participants were engaged in solving a variety of problems. For four of the instances, participants were either looking for a warrant or a means of generally articulating their already proposed warrant idea. Connecting and permuting data statements and properties was a salient tool contributing to the development of these ideas.

Structurally, the implementation of a proposed backing idea resulted in varied structural shifts in the personal argument. Backing ideas were tested by trying to yield them on specific examples and attempting to symbolize the argument or to extend the case. Dr. A tested the backing for a syntactic connection by seeing if its deployment would yield a useful connection. I explain how these backing ideas potentially could yield rendering a non-deductive warrant into a deductive warrant further in Theme 3 of the results regarding the evolution of the personal argument.

Summary of warrants and backing ideas. In this section, I described how ideas that connect and justify data with claims fell into five categories. Four of the categories were warrant-types paired with their associated backings, deductive warrants, syntactic connections, inductive warrants, and structural-intuitive warrants. The fifth category was for proposed backings or ideas that underlie the connection that were articulated independently from a warrant. For each idea-type, I provided a description of the ideas both generally and within an observed example. In an effort to answer the research questions, I described the inquirential context surrounding the emergence of those ideas.

The warrants and their associated backing were means by which participants found connections between data and claims. Even though non-deductive warrants were not the major goal, their articulation, exploration, and utilization proved fruitful in the attainment of new information and ideas that guided future explorations or helped participants assert truth. In the next section, I will talk about the ideas related to the assessment about the usefulness of the generated ideas or ways of thinking.

Ideas That Monitor the Argument Evolution

Along with ideas that focus and configure and ideas that connect and justify, in the proof construction process, I observed ideas or feelings about the mathematicians' progress on the task. These monitoring ideas guided the mathematicians' decisions in terms of moving the argument forward. There were four idea-types that described the mathematicians' progress on the task: truth conviction, "I can write a proof", unfruitful line of inquiry, and support for line of inquiry. I describe each idea type with examples from participants' work. As these monitoring feelings could be utilized on any type of other idea-type, any type of problem, or on any tool, I focus on describing on how these ideas affected decisions made and the structure of the personal argument. The first monitoring idea-types described are those specific to the proof construction process.

Truth conviction. In the ideas that focus and configure section, I described *truth proposal* ideas that were unique to proving statements whose truth value is unknown. However, achieving a feeling of truth conviction or personal belief as to why a statement must be true is not isolated to any one type of statement. In fact, as shown in Table 19 in all but three instances participants achieved moments where they were convinced why the statement must be true. The three instances where the participants did not overtly state that they had achieved some belief that the statement either was or was not true were Dr. A's work on his individual Lagrange Remainder Theorem task and the Extended Mean Value Theorem task, and Dr. B's work on his individual limits of sequences task. In all of these tasks, participants implied an underlying belief that the proof should work out. In fact, Dr. A asserted that the MVT task-type was one where the symbols should work out.

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Table 19

Moments When Participants Expressed Truth Conviction by Task

Participant	Task	Moments that came with truth self-conviction	Associated ideas
Dr. A			
	Individual (LRT)	Not observed, implicit belief	---
	Additive implies continuous	Assessment that would not be able to prove true	Unfruitful line of inquiry
	Extended MVT	Not observed, implicit belief	---
Dr. B			
	Individual (sequences)	Not observed, implicit belief	--
	Extended MVT	Envisioned how would prove given first MVT	Inductive Warrant Envisioned proof path
	Uniform Continuity	Identifying reasons why showing u.c. on the whole function would not be difficult once established the function was u.c. on a closed interval	Envisioned proof path
	Own Inverse	Identifying the one exception	Structural-intuitive warrant
Dr. C			
	Individual (Determine continuity)	(Incorrect) continuity assertion based on misreading statement	Deductive warrant Can write a proof
		Continuity assertion based on symbolic manipulation and content knowledge	Deductive warrant Can write a proof
	Additive implies continuous	Fulfilling final necessary condition by applying a given condition	Deductive warrant Can write a proof
	Own Inverse	"Heuristically" seeing why the one exception must be f(x)=x	Structural-intuitive warrant

The following examples illustrate moments where the participants achieved a truth conviction. The own inverse problem statement was not typical of the statements of many proof tasks as the own inverse statement had an unknown answer, namely finding what the one exception is. For both Dr. B and Dr. C on this task, finding that the one exception was the identity function occurred along with believing that the statement to be proven was true or having an intuitive idea why the statement is true. These ideas were articulated early on.

> Now if it's its own inverse, that means that's reflected across this diagonal. That's the geometric way of thinking about inverse functions. [draws sketch and line *y*=*x*] So if it started here and here, it would reflect back and forth, like that. It would reflect back and forth and in order to be its own inverse and increasing, so its reflection would be the same thing that you started off with. [runs pen over line *y*=*x*] And you'd have to be right on that line because there's no other way of doing it. So, *f* of *x* equals *x* suffices. (Dr. B)

> Okay. I can see heuristically why this must be so. The exception is *f* of *x* equals *x*. And the other possibility, or the other possibilities all have to have graphs that are symmetric about the line *y* equals *x* because that's the *f* equals *f*-inverse condition. [long pause] But I don't see how to do it / algebraically. (Dr. C)

For both participants, there was an early on recognition what the one exception must be accompanied by some recognition or belief that any other increasing function could not be symmetric over the line *y*=*x* which was a *structural-intuitive* warrant.

Dr. B achieved a feeling of a truth conviction on two other theorems when *envisioning proof paths*. On the uniform continuity task, Dr. B had focused on proving the function would be uniformly continuous on a closed interval in the aims of later extending it to the entire function. In searching for a reason why the function, would be true on the compact set, Dr. B developed reasons why showing uniform continuity on the whole function would not be difficult upon establishing the function was uniformly continuous on a closed interval a fact remembered to be true. As explained earlier, Dr. B worked to understand why the given First MVT for Integrals was true by exploring examples. Dr. B achieved an understanding that he could imagine guiding a proof.

Dr. A was finally convinced that the version of the additive implies continuous task was not true after trying to prove the statement without success or inclination as to how to move forward. On the same task, Dr. C was convinced that the statement was true only upon completing a deductive argument to link the additive property to continuity.

When the participants articulated a truth conviction prior to arriving at a completed deductive argument, they moved to tackle the problems of finding a deductive warrant to prove the statement, generalizing or justifying the ideas that convinced them of the truth, or to continue to try to prove another statement. In the case of the tasks where participants were to make a truth determination, the truth helped the participants to specify the claim and to direct their efforts.

I can write a proof. A moment of transition for participants was when they identified that they had gained enough information and had formulated the connections necessary to communicate their argument as a final proof. I observed all participants achieve this sense at least once on each of the tasks they proved with the exception of Dr. A on two tasks and Dr. B on one task. On the Extended MVT task, Dr. A found issues with how the problem was posed and could only conclude that his argument probably held if another condition was met. On the additive implies continuous task, Dr. A generated the counterexample on his own, away from the Livescribe notebook so I only was able to observe the articulation of the final proof. Dr. B reached an impasse on the MVT task and solved the own inverse task instead.

Table 20 provides descriptions of the moments and idea-types that accompanied the feeling that the participant could write a proof. Note that for three of the instances, the moment that participants were able to write a proof and the moment of truth conviction was the same. On the uniform continuity task, Dr. B achieved self-conviction in the task upon envisioning proof path that depended on proving that continuous functions on closed intervals are uniformly continuous. Dr. B fulfilled that envisioned proof path in that once achieving an argument (although faulty) for the sub-statement, the participant was able to begin and execute a mathematical proof.

Table 20

Moments Where Each Participant Achieved Feelings of "I Can Write a Proof"

Participant	Task	Moments that came with “I can write a proof feeling”	Idea-types
Dr. A			
	Individual (LRT)	Envisioning symbolic connections between current expressions and equation to be proven	Syntactic connection
	Ext. MVT	Not achieved because of conditions on statement	
	Additive implies continuous	Not observed	
Dr. B			
	Individual (Sequence)	The achieved construction of a sequence	Deductive warrant recognition of routine
	Ext. MVT	Not achieved	
	Uniform Continuity	Achieving the (incorrect) argument showing continuous functions are uniformly continuous on closed intervals	(incorrect) Deductive warrant
	Own Inverse	Writing of algebraic contradiction	Deductive warrant Logical structure
Dr. C	Individual (Determine continuity)	(Incorrect) continuity assertion based on misreading statement	Deductive warrant Truth conviction
		Continuity assertion based on symbolic manipulation and content knowledge	Deductive warrant Truth conviction
	Own Inverse	Achieving a proposed backing for a warrant (found issues)	Proposed backing
		Writing of algebraic contradiction	Deductive warrant
	Additive implies continuous	Fulfilling final necessary condition by applying a given condition	Deductive warrant Truth conviction

As expected, when participants stated that they could prove the statement, they moved to try to write down a logical proof. In two instances, Dr. C miscalculated an assessment. On the individual task, Dr. C did not attend to the full conditions of the statement and wrote an argument that was incorrect for the statement as given. When I asked about the argument, the participant found the error and wrote a correct argument. On the own inverse task, Dr. C had asserted through explorations that a non-identity increasing, bijective function would fail to be its own inverse because the inverse of increasing functions were increasing:

> Oh, but let's see here. The inverse of an increasing function is also an increasing function. And this means that the only way this can work is for $f(x)$ to be x for all x in I. So we can't have f equal to f-inverse if f is an increasing function. That does it. You want me to write it up?

Dr. C moved to write up the argument, but as described earlier, found issues with his thinking. Dr. C explored more and eventually wrote an algebraic contradiction that did not serve as a full proof, but provided self-conviction that a logical argument could be written.

Often the identification of when the participant could write a proof was coupled with recognition of routine. As a quick example to this, both Dr. A and Dr. B identified $f(0) = 0$ as a necessary condition for the additive implies continuous statement to be true. The participants set about trying to construct an algebraic argument but both had moments where they remembered that the argument would be simple. Dr. A was convinced that it had to be true because all linear functions pass through the origin and that led to a standard proof by contradiction. Dr. C remembered the standard argument. Most of the proofs that the participants were to write were not as simple and standard as the one described, but the mathematicians knew which tools to use and what language would articulate their ideas. The mathematicians were adept at logically articulating their ideas. That is not to say that they did not make errors. Participants did sometimes slip up quantifiers or miss cases while writing but typically addressed this issue knowing which tools to adapt to correct the problem before completing the task.

Unfruitful line of inquiry. Participants would sometimes abandon a particular line of inquiry because they were not finding it useful in achieving their aims. An unfruitful line of inquiry idea was an idea that persuaded the participant that the tools or actions the participant was pursuing or considering to pursue would not be the best choice for achieving the purpose set. These ideas included feelings that there could be an easier way, that the method would not solve the problem, or that the method was not appropriate given the context or the participants' perceived evaluation of the situation. On the additive implies continuous task, Dr. A generated ideas that fit this category including the idea that the statement ought to be true since generating a counterexample would not be trivial and would require going outside the realm of real analysis. This idea led to pursuing proving the statement instead of generating a counterexample. In the same task, Dr. A determined not to continue pursuing a set of developed equations because it would not be useful in connecting the additive property to continuity for irrational numbers.

In an effort to prove the statement was true, Dr. A had derived the equation $f\left(\frac{m}{n}\right) = \frac{m}{n} * f(1)$. The participant wrote it and then asserted that the equation was probably not true. "Yeah, I have this horrible feeling that this thing isn't true." Dr. A explained in the follow-up interview what he was thinking about when after generating the new statements that led to once again considering the statement to be false.

Dr. A: That's [$f\left(\frac{m}{n}\right) = \frac{m}{n} * f(1)$] true regardless of any of this measurability stuff that I was looking at before.

MT: So after you got this, what were you thinking should come next or was not working?

Dr. A: Well, any real number is going to be really close to m over n and somehow that would be enough. But yeah, that's where I could get stuck is just because a real number is close to m over n, you're still back, you know, even if you're a millionth of an inch away. There's no rule saying that means it's close. That's what you're trying to prove. You can't use what you're trying to prove the thing. And so, it was just, I was just going in circles. And so, yeah, at some point, I, it dawned on me or whatever that it was time to give up trying to prove it and try to look for a counterexample because it, there didn't seem to be any way to get it, you know, break that circular stuff.

While not voiced at the time, Dr. A had evaluated the usefulness of the statement, $\left(\frac{m}{n}\right) = \frac{m}{n} * f(1)$, against what was known about establishing continuity, and had determined that it could only possibly useful for rational numbers. In other words, after evaluating the new relationships established, Dr. A did not find them to be sufficient tools to solve the problem of connecting the linear property to continuity along the real line. Furthermore, Dr. A did not appear to have ideas for new tools. This coupled with the previous idea that a counterexample, while "horrible", could exist, led Dr. A to once again consider that the statement might not be true.

The inquirential context surrounding the determination of an unfruitful line of inquiry was largely an evaluation that the tools deployed or the way of thinking about a task was not fitting. To make these evaluations, participants, like Dr. A above, drew on their conceptual knowledge, connections and permutations of previously identified as relevant ideas, as well as their perceptions of the success they were making against the perceptions of what was fitting with the problem setting.

When a participant encountered an unfruitful line of inquiry, they would change tactics somehow. The additive implies continuous was a prove-or-disprove task so when participants identified an unfruitful line of inquiry, they sometimes would change their claims to argue for the other side of truth. Dr. A had been investigating a proof by induction argument to show that the statement was true but found his argument would only hold for rational numbers. Dr. A identified an unfruitful line of inquiry and also spoke again about a recollection that there might be a counterexample. In this case, structurally, the data generated previously that was not pursued was repurposed as a possible rebuttal to the claim of the statement being true.

Support for the line of inquiry. There were feelings that an approach would be unfruitful, but there were also moments that gave participants a sense that what they were doing was fitting. On the additive implies continuous task, Dr. C had not known for certain that the statement was true but had identified $f(0)=0$ as a necessary condition for it to be true. The participant was able to prove that the necessary condition held which was needed to move forward with the general strategy of determining the value of the limit instantiation of continuity. "If I could figure out what that limit was, I would answer the question. And it was pretty straightforward to figure out what the limit was. It was the additive identity was all I needed to use." Participants could get a sense of support for a line of inquiry along with a feeling that they could write a proof; however, participants could also gain support for a line of inquiry without yet having a feeling that they could write a proof. For example, on the Uniform Continuity task, Dr. B had previously articulated a feeling that proving the statement as given would be similar to proving what Dr. B had characterized as an easier case of the statement (continuous functions on closed intervals were uniformly continuous). As Dr. B worked to determine why the easier case was true, the participant reaffirmed the feeling that the two cases would be similar. "Now first of all, I you know, I'm realizing that there's no difference between showing this on all of *R* than showing this on *[a, b]*." Dr. B articulated this sense but was not yet come ready to write a proof.

Summary for monitoring ideas. Monitoring ideas were used to help participants monitor their progress and push them toward a more efficient solution strategy. Their knowledge of content, what was fitting within the realm of analysis tasks, and their previously posed ideas were standards against which their progress was monitored.

This concludes the section detailing the ideas and types of ideas observed in this study. In the final sections of this chapter, I describe holistically how ideas in the three categories, focus and configure, connect and justify, and monitor, interact in the evolution of the personal argument and how ideas were generated within the perspective of Dewey's Inquiry framework.

Logical Mathematical Inquiry and the Emergence of Ideas

In addition to describing the ideas that move the prover's personal argument forward, this research sought to describe the context surrounding their emergence. The previous section described each idea type and the problems and tools surrounding their development within each type. The next subsection gives a description of the development or lack of development of ideas while engaging in genuine inquiry; later I discuss how engaging in solving different problems with various tools played a role in the types of ideas that were formulated and discuss the exhibited non-inquirential tool use.

To provide context, I first remind the reader of the types of problems entered and tools utilized by the participants in Tables 21 and 22.

Table 21

Problems Encountered by Participants While Proving

Problem Code	Description
Understanding statements or objects	The participant does not understand what the statements mean or the definition o f a object described in the statement of the proof or how the objects in the statement relate and is entered into working understand
Determining truth	Prover is engaged in determining the truth value of the statement
Looking for warrant	Prover is looking for a means to connect the statement to the claim that eventually can be rendered into a proof. I f participants specifically are searching for conceptual reasons why the statement is true or are seeking to connect statements via a symbolic manipulation, then the next two codes were used.
Looking for conceptual reason why true	Prover is trying to find why the statement is true based on conceptual or empirical understandings
Looking for way to connect symbolically	Prover is trying to find means to directly connect symbolic instantiations of statements
Looking for way to communicate/generalize	Prover is engaged in finding a way to communicate or generalize an argument, warrant, backing, or other idea
Looking for backing for previous idea	Prover is engaged in finding general or generalizable support for a posed idea or claim
No problem	Actions are taken or tools are applied without the individual reflecting on the tool to use. The individual indicates that the action taken is “second nature”, “what you’re supposed to do”, “how I usually do it”, etc. The individual may look back at what the action did for him/her but nothing has been deemed problematic prior to that evaluation.
Tool problem	There is an identification and entrance into solving a problem with individual tools or application of these tools, i.e. trouble generating a helpful example, computation issues, etc.

Table 22

Classifications of Tools Utilized by the Participants to Generate Ideas

Tool Type	Description
Conceptual knowledge	Knowledge of relationships among mathematical objects, consequences of actions on objects, or mathematical structure
Known theorem	Specific use of a theorem known to be true
Connecting and permuting	Attending to connecting and rearranging previously generated ideas, definitions, and related concepts
Instantiations and equivalencies	Alternative or Non-formal representations of mathematical concepts or definitions
Symbolizing	Rewriting statements, definitions, or representations in terms of symbols
Symbolic manipulations	Actions on symbolic representations
Example	a particular case of any larger class about which participants generalize and reason
Heuristics and experiences	Rule of thumb, technique that comes with experience
Logical structure	Knowledge of logical structure and the norms of behaving and communicating in the mathematics community
Other	Time, disturbances in the situation, outside resources, etc.

Table 23 presents the ideas generated by participants while encountering each kind of problem. Since participants could have encountered multiple problems at a single time and a single idea could fall under multiple idea-types, summing to get a total idea count is not meaningful. Participants mostly developed ideas while engaged in the problems of looking for warrants, looking for conceptual reasons why true, and determining truth. However, they also developed twenty-two ideas while not perceiving any problem. Table 24 lists the tools that participants deployed that led to the generation of ideas of each sub-type. As can be seen in the table, ideas within each sub-type could have been formed by tools of multiple types. I found it less informative to describe exactly which tools contributed to which idea-type. What was informative was to describe the patterns in how ideas emerged.

Table 23

Ideas Generated by Problem Encountered

		Understanding Statements/ Objects	Determining Truth	Looking For Warrant	Looking For Conceptual Reason Why True	Looking For Way To Connect Symbolically	Looking For Way To Articulate/ Generalize	Looking For Backing For Previous Idea	No Problem	Tool Problem
Focusing and Configuring ideas	informing concept image		C (1)	B (1)	B(3)		B(1) C(1)	B(1)	A(1)	B(1)
	task type	A(1) B(1)		A(1)	B(1)	A(1)				
	truth proposal		A(2) C(1)	A(1)				C(1)		
	logical structure	B(2)					B(1)	A(1)	A(1) B(1) C(1)	B(1)
	identifying necessary conditions			A(1) C(2)					A(2)	A(2)
	envisioned proof path			A(1) B(1)	B(2)	C(2)	C(2)	B(1)	A(1)	A(1)
Connecting and Justifying ideas	inductive warrant			C(1)	B(5)			B (3)		
	Structural-intuitive warrant	C (1)	A(2) C(2)	C(1)	B (4)				A (1)	
	syntactic connections					A(5)			A (1)	
	deductive warrant		C (3)	A (1) C (1)	B (1)		B (1) C (1)	B (1)	A (3) B (2)	C (1)
	proposed backing			C (1)	B (2)		C (1)		A (1)	C (1)
Monitoring ideas	truth conviction	C (1)	B (2) C (1)	A (2) B (2)	B (1) C (1)			C (1)		
	can write a proof		C (2)	B (1)	B (1)		C (2)	C (2)	A (1) B(1)	
	unfruitful line of inquiry		A (1)	A (2) B (2) C (1)	B (2)	B (1)	C (1)	B (2)		
	support for line of inquiry			A (1) C(1)	B (2)			C (1)		

Table 24

Ideas Generated by Tool Type

		Conceptual knowledge	Known theorem	Connecting & Permuting	Instantiations and Equivalencies	Symbolizing	Symbolic manipulation	Examples	Heuristics	Logical structure	Other
Focusing and Configuring ides	informing concept image				B(1)	B (1)		B (4)	A (1)		A (1)
	task type	B (1)		B (1)	B (1)			B (3)	A (1) B (1)		
	truth proposal	C (1)		A (1)			A (1)	C (1)	A (1)		
	logical structure			B (1)	B (2)	B (2)			B (4)	A (1) B(4)	
	identifying necessary conditions	A (1) C (1)	A (1)	A (1) C (1)	A (1) C (1)	C (1)	C (1)			A (1)	
	envisioned proof path	B (1)		B (1)				B (1)			
Connecting and justifying ideas	inductive warrant	B (3)		B (1)	B (2)			B (7) C (2)			B (1)
	intuitive warrant	A (2) C (3)		B (2) C (2)	B (1) C (1)			B (2) C (1)	A (2) B (1)		A (1)
	syntactic connection	A (1)	A (3)	A (5)	A (1)		A (5)				A (1)
	deductive warrant	A (2) B (1) C (2)	A (1)	A (2) B (4) C (5)	A (1)	B (2) C (1)	A (3) C (1)	B (1)	A (2) B (2)	B (1)	A (1)
	proposed (vague) warrant	C (1)		A (1) C (2)				B (2)			A (1)
Monitoring ideas	truth conviction	A (1) B (1)		A (1) B(2)				B (2)			
	can write a proof	B (1)		A (1) B (2)			A (1)		B (2)	B (1)	
	unfruitful line of inquiry			A (1) B (3)			A (1)	B (2)			
	support for line of inquiry	A (1) B (1)		B (1)			B (1)				

In this section, I describe the thematic findings about the context surrounding the emergence of the ideas that moved the argument forward within the perspective of Dewey's theory of inquiry. The ideas emerged from the systematic testing of ideas and tools. I describe this process and some of the notable tools utilized and how the problems that the participants encountered progressed as participants proceeded toward the resolution of an argument. I describe a third theme that participants sometimes deemed a tool to be unsuccessful, but its deployment led to new insights. I finally note some other observed patterns.

Theme 1: Ideas Generated through the Testing of Ideas and Tools

While I was unable to describe a distinct pattern as to which types of problems and tools contributed to which types of ideas, there was a discernable pattern where the participants would propose or articulate an idea or tool, test the tool or idea's usefulness or the usefulness of prior ideas against the consequences of the new idea, and then articulate a new idea or evaluation. The process itself involved the passing through, perhaps multiple times, the inquirential cycle of reflecting, acting, and evaluating. This process necessarily began with an initial idea proposal which was often formulated based on reflection or heuristic strategies for orienting to a proof task. Subsequent ideas were formulated as new ideas were tested. I illustrate this process within the context of Dr. A's work on the additive implies continuous task.

> Define f as *linear* if for every x and y, $f(x+y) = f(x) + f(y)$. Let f be a function on the reals. Prove or disprove that if f is linear, then it is continuous.

Opening ideas. This section elaborates on the generation of the first idea integrated into Dr. A's personal argument then compare his process those of other participants in other tasks. Upon reading the statement, Dr. A recalled that there could be a counterexample based on a memory of previous experience:

> I'm afraid that there's some counterexample using the axiom of choice. Okay, let me, I'll try to prove it. What am I thinking though with the axiom is there might be an unmeasurable linear function. Let's see, what am I thinking? I'm not sure what I'm thinking.

He explained what backed up the idea in the follow-up interview illuminating that the vagueness of a memory could be the reason for the unclear thinking.

> MT: So you initially have this inclination that this wasn't true. What made you-
>
> A: Oh, just some vague memory from a long time ago. Yeah, you know, I guess anything can be analysis. To me this is sort of something that you find in a book on set theory. Or something…I spent some time, I forget when it was, twenty years ago, more than that I think. I was studying set theory for some unknown reason. I think that was when I kind of learned this stuff. Or I knew that these things existed out there. So, yeah, I had this vague recollection that things are not as simple as they sound. So yeah, that's probably cheating too.

This initial idea was a *truth proposal* based on a *structural-intuitive warrant* to solve the problem of determining the truth of the statement. The thought of a potential counterexample occurred as Dr. A reflected upon the statement to be proven against personal experience. The participant had not applied any tools prior to this, but reading the statement led to remembering experiences outside the field of analysis. The other participants on the other tasks also articulated initial ideas while orienting to the task.

Upon reading a task, participants entered into the problems of understanding the statement or determining truth. To achieve these goals, participants would often reflect upon the given statement in conjunction with seen to be relevant conceptual knowledge, experience, and known properties and theorems. The reflections were, at times, coupled with heuristic strategies for orienting to proof tasks such as actively listing the relevant definitions, rewriting and verbally articulating what was known in one's own words, and articulating what to show. In these reflections, participants connected and permuted properties, theorems and definitions, imagined instantiations of concepts and definitions, and performed hypothetical actions. The initial ideas were ideas that informed the statement image, truth proposals, truth convictions based on structural-intuitive warrants, and ideas about the formal logic or task type. The testing and realization of these initial ideas and subsequent ideas was the mechanism by which the argument moved forward.

Ideas generated while testing a developed tool or idea. Dewey's (1938) theory of inquiry allows for evaluation to occur during and after a proposed tool's fulfillment. The following is an example of an idea developed after a full cycle of inquiry. On the Uniform Continuity task, Dr. B had recognized that the given property that the function had limits at positive and negative infinity indicated that or a given epsilon, the real line could be broken into the union of three intervals: $(-\infty, N) \cup [N, M] \cup (M, \infty)$.
Where N and M were the x-values beyond which each output was within half of epsilon of the limit value (see Figure 13). Dr. B anticipated that showing uniform continuity on these tails would be straightforward and decided to assume that continuous functions on closed intervals (like [N,M]) were uniformly continuous.

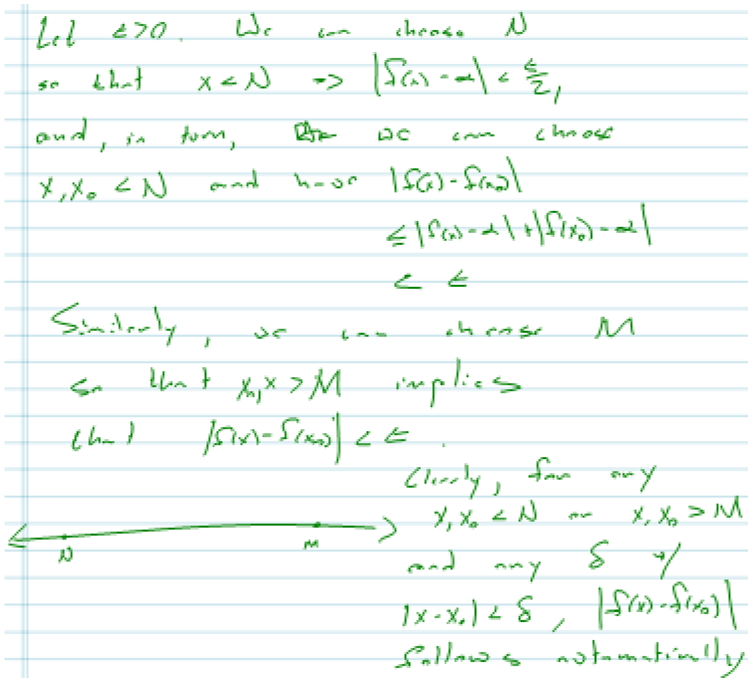


Figure 13. Dr. B's argument that the function is uniformly continuous on its tails.

Dr. B had nearly concluded his proof, showing that the function was uniformly continuous on each of the three intervals. The participant chose to check to see that the argument would support the definition of uniform continuity. As a means of checking, Dr. B walked through where pairs of points x and x_o would come from. Dr. B noted that the argument did not account for when they straddled one of the points N and M. Dr. B had checked silently and then articulated the issue out loud: “And, um, but, / let’s see there’s a little board work case I have to worry about. [pause] Hmm. I think I have a little problem. There’s a little overlap here…if I take x and x-not on each side of N or each side of M, I’m in trouble.”

MT: Let’s see you write this.

[playback going, no audio playing]

MT: About 30 seconds of just quiet.

> Dr. B: and that's when I realized that I had this extra case…I know exactly what I was doing. I was going, okay, I've got this case, this case, because I think I handled the endpoints before that. So I was thinking what else could happen? So I said okay I could have them straddling.

The idea of the issue with how the argument was structured was proposed while Dr. B was evaluating the written argument. When the issue was identified, Dr. B articulated a proposed rebuttal and crossed out what was written previously.

Ideas that moved the argument forward were developed and incorporated into the personal argument via this mechanism where one developed or proposed tools that could potentially be rendered into moving the argument forward; tested the tool or idea against previous ideas, conceptions of the problem, conceptual knowledge, and the problem the prover intended to solve; and then made an evaluation that either established a new idea or devalued a previous idea. As described in earlier sections, the testing process differed across idea-types and problems posed. The problems posed played a role in determining which ideas would be useful. In the next sub-sections, I specifically describe how participants used examples and the tool of connecting and permuting ideas, conceptual knowledge, and instantiations of concepts.

How examples were used. In this study, the term example describes any particular case of a larger case. As shown in Appendix F, participants in this study used examples for four purposes: (a) to understand, (b) to test, (c) to generate a warrant, and (d) to articulate or explain. I provide examples of each of these purposes below.

On the Uniform Continuity task, Dr. B was working to show why a function with finite limits at infinity would be uniformly continuous and drew a picture of an example function that had the conditions set forth by the task statement in an effort to understand the function involved in the problem. "It's totally just important for my visualization in understanding the problem." Drawing this picture led to his idea that this task was a complication of showing that continuous functions on closed intervals are uniformly continuous. This idea was coded as an idea about task type.

Dr. C used a specific example to test whether the piecewise function generated on the Additive implies Continuous task was a counterexample to the statement. Dr. C had generated the piecewise function $f(x) = \{\begin{matrix} 0 & if\ x\ is\ irrational \\ x & if\ x\ is\ rational \end{matrix}$ based on the given condition that the function was continuous at *x=0* and previous conceptual knowledge about additive functions being continuous on the rational numbers but discontinuous on the real numbers:

> Well, I knew that it had to work for the rationals. So I thought I would try something that had one definition in the rationals and something else in the irrationals. And it seemed to me that x in one case and zero in the other case would be the easiest thing to try as a first effort.

To test whether this function was a counterexample, Dr. C looked to see if it satisfied the additive condition, $f(x + y) = f(x) + f(y)$. "It's clear when x and y are both rational because x plus y is also rational and f of x plus y is therefore … *f(x) + f(y)*. But it's not clear for irrationals." He chose to plug in $x = \sqrt{2}$ and $= 2 - \sqrt{2}$. These specific examples of irrational inputs yielded different values for $f(x + y)$ and $f(x) + f(y)$ showing his function was not a counterexample.

As an instance of where participants explored examples to find a warrant, consider Dr. B's work on the Uniform Continuity task. Dr. B had narrowed the task to tackling the problem of proving a function on the compact set [0,1] was uniformly continuous or in Dr. B's words that it could not get steeper and steeper. In a search for a reason why the function could not get steeper and steeper (a warrant based on conceptual understandings), Dr. B drew a picture of a specific function on the set [0,1] and worked to determine why for a specific epsilon equal to 0.5, one could always find a delta. While in the action of thinking about why the example function could not get steeper and steeper, Dr. B articulated the idea that the function would achieve a maximum and a minimum:

> So I wanna be able to, I need to be able to find for any, I need to be able to find an x, a delta such that for any location I might choose, if I go within that of x, it won't vary by more than point 5. Now I'm starting to get to something because one thing I do know is that it has to achieve a min and a max.

At the time, Dr. B anticipated that the remembered idea that the function would have a minimum and maximum would help justify why the function would not get steeper and steeper, but was not sure in what way. So, it became an added data statement with a hypothesis that it could inform the generation of a warrant. Dr. B did test the usefulness of this idea by determining the minimum and maximum fact could be extended beyond compact sets to the given task.

Participants also, at times, utilized examples to articulate or explain their thinking. Dr. B and Dr. C did this on the Own Inverse task. Both participants articulated that they had an intuitive feeling why the statement must be true. After articulating a belief that the statement was true, Dr. B reflected back to try to articulate what exactly gave the truth conviction. "So what have I convinced myself geometrically? … Okay. So first of all, why is *f(x)=x* the only increasing function. So my intuition there, if it wasn't, then if you reflected itself, there would have to be double values of this thing from its reflection." To illustrate and test this idea Dr. B drew picture BI-2 shown in Figure 14. By double values, Dr. B meant there will be two points on the same function $(a, f(a))$ and $(a, f^{-1}(a))$. Dr. B was attending to *f* being its own inverse as symmetry across the line

$y=x$. Picture BI-2 served the purpose of helping articulate what the intuition was, but the participant still had the question, "So why would there be double values on this thing?"

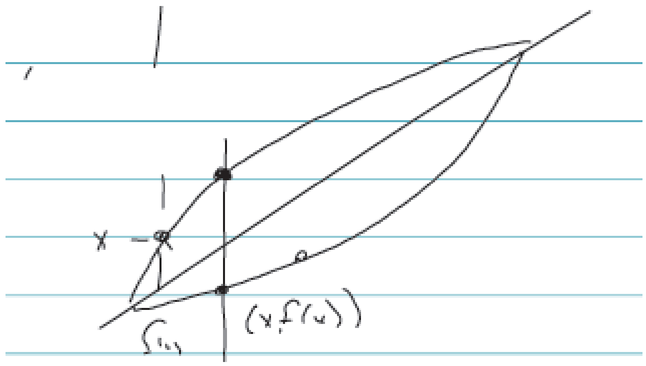


Figure 14. Picture BI-2 drawn by Dr. B on Own Inverse task to assist him in articulating his thinking.

Connecting and permuting ideas, conceptual knowledge, and instantiations of concepts. A tool used by participants that commonly contributed to the generation of new ideas was the action of connecting and permuting given and known ideas and statements. The act of connecting refers to putting statements, facts, and instantiations of concepts, and so forth together to form new ideas. Dr. A demonstrated the connecting while working on the additive implies continuous task when generating the necessary condition idea that the function must pass through the origin.

Dr. A: So at some point it occurred to me that f of zero has to be zero… If it's going to be continuous, then it must be a straight line. Because if it's not a straight line, then it's not going to be linear. So, I was thinking straight line but at first it didn't occur to me that it had to go through zero.

MT: Why was it important that it was zero at zero?

Dr. A: Uh, I think I was just flailing around. Yeah, I, well I was going to try to do something like *g*-. What was I? Yeah. Ah, yeah, yeah, no I was just flailing around. I mean I knew, I guess I knew that whatever the function is, $f(x+y)$ is $f(x)+f(y)$. So if it's going to be continuous, then if y is really small, then they have to be about the same. So you're going to get $f(x+y)$

> is going to be equal to *f(x)* plus something really small. And that's how you're going to show that it's continuous. And uh, so I needed that *f* of zero was zero. If f of zero isn't zero, then if *x* is really small, *f(x)* won't be near zero. Yeah, so I needed *f* of zero to be zero and it didn't occur to me that that was obvious so that you didn't even have to prove it.

Dr. A reflected on the function and the idea of linearity as well as what it meant to be continuous. The participant's conceptual knowledge was rich with pictures of what it meant to be linear and instantiations of continuity at a point. Dr. A, specifically, drew upon a flexible conception of the definition of continuity that involved the notion that a continuous function has the property that adding a small amount, *y*, to an input, *x*, would only move the output a small amount from *f(x)*. Dr. A never wrote it down, but essentially the conception of continuity used was equivalent to the limit definition of continuity: $\lim_{y \to 0} f(x+y) = f(x)$. The instantiation of the definition fit well with the additive property of the function. This definition of continuity was a tool Dr. A applied while reflecting on the problem of what could connect the additive property to continuity. The participant anticipated this tool could be used to gather some more information about *f* that could be applied.

The permuting characterization of this tool comes from Dr. C's descriptions of work on the Own Inverse task. To understand why a function that was one-to-one, continuous and increasing would fail to be its own inverse, Dr. C explored both symbolically and pictorially what would happen if the function were increasing or decreasing. The participant called these actions "permuting the logic":

> What I'm doing here is permuting the logic of all the conditions I'm interested in, trying to find a combination of permutations with those things that I can connect with each other to get an argument. I'm trying to establish that a certain definition can be applied that works. So I'm working with that definition and permuting different pieces of it to see if I can find something that I can use to construct an argument that establishes what I want.

As could be seen in Table 22, participants used connecting and permuting to generate ideas of nearly every type. In the lens of Dewey's (1938) theory of inquiry, I interpret this as participants constantly testing their previously formulated ideas and observations against new observations and previous knowledge of structure and instantiations of ideas with the end-in-view of solving the problems that they pose for themselves.

Theme 2: A Progression of Problems

The salient problem types encountered by the participants were presented above in Table 21. It appeared that the participants transitioned through a series of problems to tackle or tasks to complete in order to finish the construction of the proof. I list them below.

1. Understanding the statement and/or determining truth
2. Determining a warrant of some kind
3. Validating, generalizing, or articulating those warrants
4. Writing the argument formally

Aside from these major problems to solve, the participants also tackled problems parallel to or embedded within these problems such as dealing with a found problem with a tool. Some types of problems described in Table 21 are specifications of the problems in the list above. Varying the task changed the problems the participants entered to yield new ideas. The various tasks determined what kinds of warrants the participant pursued. Also, when a general, deductive warrant was achieved, writing the argument formally was often unproblematic. For the remainder of this section, I will describe the progression of problems within the context of Dr. B's work on the Own Inverse task, generalizing to the other tasks and other participants.

Participants would read the task, and then set to the task of trying to get a handle on what was going on in the problem or to determine the truth of the statement. Dr. B began by writing the statement and then working to draw a picture of the conditions: "Okay, let's think about this one. We have a continuous function on a closed set a,b. And it's one-to-one. I can draw a picture. So I'm going to draw a visual picture of what's going on here. Try to figure out how to prove this." While endeavoring to draw an example function that would be representative of the function in the statement, Dr. B discerned the ideas that the function must be monotonic and where it must start and stop on the interval:

> One hundred percent geometric thinking...It's all part of chipping away at the geometric restrictions. Because that's what I took to this right away because this is my kind of problem. It's saying that f has to have this certain geometric property. So, you know, let's start not literally, I mean in some sense looking at your boundaries, …I'm going to say well, they're telling me it has to have this certain geometric property. Let's draw the most general picture and sort of whittle down. Yeah, and sort of sculpting.

Dr. B had developed ideas to inform an image of what the function in the statement could look like and how it could behave. Continuing his pursuit to understand the statement and objects involved in the statement, Dr. B drew a second picture thinking of what it would mean for the function to be its own inverse. Dr. B articulated an intuitive belief in the statement.

> Show that except for one mathematic- see it's one-to-one, it's onto, and it's its own inverse. Now if it's its own inverse, that means that's reflected across this diagonal. That's the geometric way of thinking about inverse functions. [draws sketch and line y=x] So if it started here and here, it would reflect back and forth, like that. It would reflect back and forth and in order to be its own inverse and increasing, so its reflection would be the same thing that you started off with. [runs pen over line y=x] And you'd have to be right on that line because there's no other way of doing it. So, f of x equals x suffices. That should be our only increasing function. So I know that's what they're looking for. F of x equals x intuitively seems like the only increasing option they're talking about.

In pursuing the problem of understanding the statement geometrically, Dr. B worked to draw pictures of the situation. This drawing involved reflecting upon and connected instantiations of the relevant definitions of one-to-one, continuous, and the inverse of a function. The ideas formulated in this reflection informed the pictures created and what was imagined to happen when one reflected an increasing function over the line *y=x*. The picturing of the situation resulted in an understanding of what the one exception should be which provided self-conviction of the statement's truth.

Participants moved on from working to understand a statement to working to find a warrant that they could use to justify the statement's truth when they developed a personal truth conviction, a truth proposal, or an idea about the task type that pushed them toward pursuing certain lines of inquiry that could lead to a warrant. At times the truth proposal or truth conviction was accompanied by a warrant of some type that participants immediately sought to justify, other times participants moved to find other means of connecting their statement and claim.

Dr. B found what the one exception was and an intuitive belief that no other increasing function would work and then set about looking for a link or warrant between the data and the claim that *f(x)=x* was the only increasing function with the given properties. Dr. B articulated an initial structural-intuitive warrant:

> So now the question is this. So what have I convinced myself geometrically? And how do I prove those things? Okay. So first of all, why is f(x)=x the only increasing function? So my intuition there, if it wasn't, then if you reflected itself, there would have to be double values of this thing from its reflection

Achieving that link, Dr. B set about working to articulate and to test that warrant. The process involved empirically testing a proposed warrant and proposing new warrants based on the evaluations of those tests. The process by which participants worked to validate, to articulate, and to generalize their warrants differed by warrant-type and will be elaborated in another section. If a proposed warrant was tested and found to not be valid or if participants could not wield it into a general algebraic articulation, they cycled back through the problems to either try to understand more about the objects in the statement or they proposed a new warrant which they set about to test.

Once a generalizable warrant was found and written algebraically, the mechanics of writing the argument formally involved the non-inquiriential application of tools as participants did not perceive this task to be problematic. After achieving an algebraic warrant, Dr. B set about writing the formal argument which involved a proof by contradiction with two cases. Dr. B found this writing to be unproblematic, but made errors in his final write up. This occurrence will be described under the theme of auto-pilot actions in a subsequent section.

Theme 3: Tools Deemed of No Use May Be Gateways to New Ideas

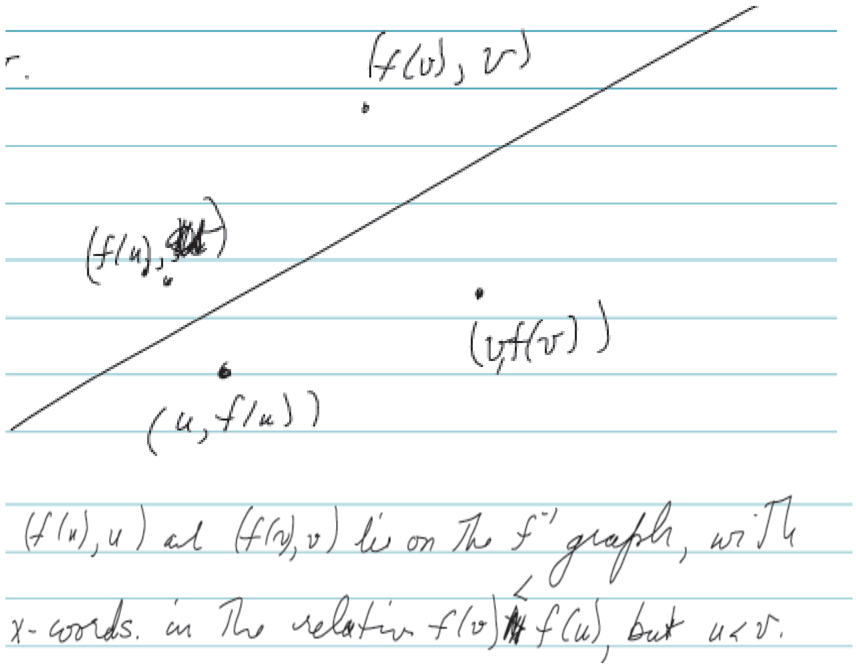
In the pilot study, Dr. Heckert was looking for a reason why the characteristic and minimal polynomials of a pair of 3x3 similar matrices must be equal. To gain some insight, Dr. Heckert constructed an example 2x2 matrix and calculated a matrix to be similar to it. Dr. Heckert found the characteristic polynomials of each matrix, and they were not equal. The participant found a computation error in this construction of the similar matrix. Instead of trying to fix the mistake, the participant determined that working from the examples would not be fruitful. However, looking at the factored form of the polynomials written down, Dr. Heckert started to consider equivalent eigenvalues and found that useful in developing an argument. Dr. Heckert's work with the specific example deemed to be unfruitful led to a development of an idea that moved the argument forward. This phenomenon also occurred in the current study.

In an effort to identify an algebraic or generalizable argument why an increasing, one-to-one, continuous function not equivalent to $f(x)=x$ could not be its own inverse, Dr. C worked on "permuting the logic" of a function being increasing and its own inverse. Dr. C drew an example with two points of a function in an increasing relationship located below the line y=x. The participant also showed their reflection points with the view that the inverse function could not be equal to the original function if the function were to be increasing. Dr. C then moved to understand what would happen if the function was decreasing. "If $f(u)$ is greater than $f(v)$, how are the points $(f(u), u)$ and $(f(v), v)$ related to each other? //Let's see, $f(u)$ is greater than $f(v)$, but u is less than v." However, Dr. C actually drew another increasing function (see Figure 15). I asked about this mismatch.

> Dr. C: I'm not sure what I was up to there because I drew the picture wrong. So I must have been just on automatic pilot thinking about something else but I'm not sure what. Because I've got $f(u)$ clearly smaller than $f(v)$ there. So that picture isn't relevant…I think I was in a trance trying to figure things out.

Dr. C determined that this reasoning was not giving him any new information, but then extracted the idea that the inverse of an increasing function is increasing which he saw as enough to explain why f could not be equal to f-inverse:

> That doesn't seem to be going anywhere. / Oh, but let's see here. The inverse of an increasing function is also an increasing function, and this means that the only way this can work is for f (x) to be x for all x in I. // So we can't have f equal to f inverse if f is an increasing function. / That does it. You want me to write it up?



F*igure 15.* Dr. C's permuting the logic on own inverse task.

Dr. C did not recognize any benefits of his permuting the definition and drew something different from what was spoken. The idea that the inverse of an increasing function is also increasing is consistent with the picture drawn. I asked if the idea came from pictures drawn on the previous page or this one. “I think it came from the whole collection of pictures and just thinking about the relationships of points on one graph to points on the other graph and the symmetry involved.” This picture added to the previous pictures as well as the participant’s thinking about the relationships of the points on *f* to those on *f*-inverse played a role in Dr. C’s finding this idea useful, which was similar to what Dr. Heckert did with his example formulation.

Outlier Themes Involving the Development of Ideas

The earlier half of this section presented some general patterns to how participants’ developed new ideas by reflecting upon, testing, and evaluating proposed tools and previous ideas. In this section, I elaborate upon an outlier case where a participant engaged in active inquiry, but found now new ideas. I also describe how participants were able to proceed routinely without needing to actively reflect upon their actions and consequences of those actions.

Inquiry with no ideas. This section focuses on Dr. B's initial work on the Extended Mean Value Theorem for Integrals task. Dr. B established a need to find a way to link the related equations and determined that some set of symbolic manipulations would connect the expressions. Dr. B tried a series of manipulations but was not successful. This work on the task during the interview was an instance of trying to solve a problem by reflecting on a perceived to be problematic issue, applying a tool to resolve that issue, and evaluating its usefulness. After a series of tools were not deemed useful, Dr. B proposed modifications to the purpose or approach which could be viewed as tools in themselves. Dr. B had first approached the problem of proving the equation by focusing on the problem that the right hand sides of the equation to be proven and the equation given were not the same and worked to manipulate the right hand sides of the equations in order to see how the given theorem could be applied to the equation to be proven. After applying a number of tools, Dr. B reevaluated and proposed a new purpose of trying to directly equate the two right hand sides of the equation but deemed it unachievable. The third tactic was to work to equate the two sides of the equation to be proven by working backwards from the right hand side, directly applying the given theorem in reverse, and then searching for a symbolic manipulation to equate the expression to the left hand side of the equation.

The proposed tools were reflective of Dr. B's way of thinking about the problem. The participant thought that some direct application of the First MVT would enable pulling the function *g(t)* or *f(t)* out of the integral but some manipulation would need to be required to attain the derivative and the factor *(t-a)*. Each tool proposed was aimed at either turning the function into its derivative, attaining the factor *(t-a)*, or enabling the direct application of the First MVT because the overall goal of the participant was to make the left hand side of the equation equal the right hand side of the equation in the Extended MVT. Some tools were reapplied in a different way at different aims, but Dr. B did not spend time working on modifying tools.

Dr. A worked on the same task and also worked to achieve syntactic connections amongst the expressions; he, however, was successful in making progress after less than two minutes of thinking. It is difficult to say what about Dr. B's inquiries did not lead to a useful syntactic connection, but I can note the differences between the two mathematicians' approaches. Both started out with the same anticipation that the First MVT would be applied and that symbolic manipulations would be necessary. There was a difference in approaches in that Dr. A sought to equate the left hand side with the right knowing that at some point the First MVT for Integrals would be utilized. Dr. B, on the other hand, appeared to want to try to directly equate the right hand side of the First MVT with the right hand side of the Extended MVT or to make the two theorems look similar enough in order see how they related. Dr. B attended to the names of the functions and explicitly thought of the named function *f* in the First MVT as analogous to the function named *g* in the Extended MVT and vice versa. Dr. B did find that directly mapping the functions to each other in this way would not yield the desired results, but the perceived analogy persisted through Dr. B's work on the task. Dr. A, on the other hand, did not try to apply the First MVT right away but only after finding an algebraic manipulation and then was flexible in allowing a quotient to act as the function called *f* in the First MVT.

Both Dr. A and Dr. B noticed that the two equations looked different from each other and proposed manipulations that would attend to the *(t-a)* factor and the derivative *g'(t)*. A difference was that Dr. B primarily attended to either getting rid of the appearance of the derivative *g'(t)* or creating a derivative in the First MVT. Dr. A on the other hand initially attended to trying to get the *(t-a)* term to appear on the left hand side of the equation. Dr. B's attention to *g'(t)* was a result of his working to directly correlate the two theorems as opposed to Dr. A's work to move the left hand side of the equation to be shown to the right hand side knowing that the given theorem would somehow be used.

Just as inquiry could proceed without participants making progress on the task, participants could make progress on the task without engaging in inquiry because they perceived no problem. In the following section, I describe how participants worked on the task when they perceived a routine or non-problematic situation.

Auto-pilot actions. Participants recognized a routine by the removal or resolution of the perceived to be problematic and would sometimes make those moments known as they would articulate a feeling of "now I see it" or "I can do it now". This typically was accompanied by the idea that the participant could write a proof. Participants continued to choose tools to apply, but they then knew the tool for the job and did not feel the need to stop and evaluate their tools' effectiveness after applying them because they anticipated the effect that it would have.

More than once, the participants were writing statements and drawing pictures while not thinking about what they were drawing. They admitted to being in a trance or on "autopilot". They did this for one of two reasons: (a) writing statements as a stalling tactic to think about other problems or the next action they need to take, or (b) they had resolved the problem and are just carrying out the actions.

On the own inverse task, Dr. C drew an increasing function when stating an intention to draw a decreasing function. When we watched the play-back together in the follow-up interview, Dr. C was surprised by these actions. In the follow-up, the participant did not appear to remember the thinking or the goals surrounding these actions.

> Dr. C: I'm not sure what I was up to there because I drew the picture wrong. So I must have been just on automatic pilot thinking about something else but I'm not sure what. Because I've got f of u clearly smaller than f of v there. So that picture isn't relevant. I think I was in a trance trying to figure things out. Yeah. Because I've started with something that I'm reasonably sure can't be [points to statement f(u)>u] / Oh, wait a minute. What is going on? What am I doing there? / I want to show that's what happens. / Oh, I'm going backwards. That's what I'm doing. I'm trying to- Okay, yeah. So I am trying to get / somehow a contradiction, I guess. Yeah. / What if f of u is bigger than f of v? What can I say about u and v? But then I drew the picture wrong.
>
> MT: Alright. So, you're asking so what if it is decreasing? [Points to statement 'f(u)>f(v) but u<v'] And then?
>
> Dr. C: Well, if it's decreasing, can I force *u* to be smaller than *v* is what I'm asking myself…I guess. I'm not- I was kind of thrashing around trying to find something that I could use.

Performing actions while thinking ahead could have been a stalling tactic as demonstrated by Dr. C. After establishing that the idea that an increasing function's inverse is also increasing would be enough to justify the statement, Dr. C moved to write up the proof, beginning by writing the utilized assumptions.

MT: Okay. When you were starting to prove, were you planning to prove by contradiction or were you planning to prove that if it's increasing then it has to be f of x equals x?

Dr. C: Well, I think that I didn't really know at this point. What I'm doing, what I'm writing out here is a fairly essential part of a formal proof. I'm also stalling for time while my brain figures out how to work on this.

MT: And what is your brain figuring out?

Dr. C: What's it mean to be off the diagonal.

Dr. C was beginning the formal write-up of the proof but was still working on another problem. Dr. B also demonstrated similar autopilot actions.

The role of interview context and affect. It appeared that the choices that participants made as far as what tools to utilize and what choices were reasonable were influenced by the interview situation and their own affect toward the situation. The interview context played a role in Dr. A's work on the Additive implies Continuous task. Dr. A had the initial feeling that the statement was not true because some counterexample existed. However, Dr. A recalled that generating such a counterexample would draw upon knowledge from mathematical fields beyond real analysis. Dr. A perceived that it would be difficult to remember how to generate such an example; therefore, the participant decided to pursue proving that the statement was in fact true:

> It sure seems like it ought to be true. But the thing is that, you know, to come up with a counterexample you've got to go way off the tracks. It's not, you're talking I think sort of undergrad real analysis, and so the counterexample is not something that you learn about in undergrad real analysis. It's elsewhere.

The perception about the problem being solvable by undergraduate students was most likely because when Dr. A chose tasks for the other participants, he chose tasks that he had previously assigned to his undergraduate students. The perceived difficulty of generating the counterexample did not fit with Dr. A's perception of the context of the interview situation throughout his work on the task.

Affect certainly played a role in participants' decisions as they chose whether to persist or not to persist in solving the problems they encountered. On the Uniform Continuity task, Dr. B made affective decisions when determining whether or not to continue trying to reformulate the proof of the theorem that continuous functions on closed intervals were uniformly continuous, a statement he knew to be true and to be necessary for proving the task statement. Dr. B had earlier determined to not assume its truth and had worked for a period of time to determine why the theorem was true. The majority of Dr. B's work on the task was in pursuit of this warrant; the participant finally wrote an incorrect argument that gave enough self-satisfaction to move on to proving the task statement. When I asked about what Dr. B wrote, it appeared frustration and fatigue (affective elements) played a role in this writing and accepting of the incorrect argument: "Yeah, I don't know what I was. This, this just is, this is just barking up the wrong tree…yeah, and probably this is the point I just got so frustrated I decided to prove the simpler version." By "simpler version," Dr. B was referring to the final write-up where Dr. B made the decision to assume the theorem to be true. In this task, affect (negatively) played a role in that Dr. B wrote and accepted an incorrect sub-argument; however, making the decision not to continue pursuing the sub-proof of this theorem known to be true was prerequisite to the writing of a final proof of the main statement.

This section summarized how the inquirential context played a role in the development of ideas. The problems and tools themselves do not dictate whether ideas are developed, but the interaction amongst the tools, ways that they were applied, the problem and the acts of reflection and evaluation deployed by the prover determine if ideas are generated. The next section provides findings about how ideas found to move the argument forward did in fact move the personal argument forward.

The Evolution of the Personal Argument

I described the ideas that moved the argument forward and I illustrated the context surrounding the formulation of those ideas through the lens of Dewey's (1938) theory of inquiry. In this section, I summarize the observed shifts in the personal argument by category of idea. I depict the findings regarding the conditions surrounding certain types of structural shifts. I conclude the section discussing the interaction amongst the three larger idea-types within the inquirential framework as the personal argument evolves answering how ideas are used and tested.

Shifts in the Personal Argument

Changes in the mathematicians' personal arguments were coded into 10 categories. Table 25 summarizes their descriptions. In the descriptions of each idea-type, I summarized which shifts were attributable to the implementation of the idea. Table 26 compares the shifts across idea-types. With the exception of 'data removed', 'order of presentation', and 'no changes', each structural shift was supported by 10 or more generated ideas. Viewing the gaps in the table by idea category, we see that few focusing and configuring ideas informed data being repurposed, changes in claim or specification, backing being added or changed. Monitoring ideas, though few, were dispersed throughout all shift-types. Syntactic connections largely supported the addition of sub-claims and data as new equations were formulated and incorporated.

Table 25

Observed Structural Shifts

Structural Shift	Description
Opening structure	The structure that the participant begins with when articulating the first idea
Claim changed or specified	The claim of the argument is either changed to a new claim or delimited in some way
Sub-claim added	In addition to attending to justifying the central claim, participants add new claims to proven
Data added, extended or specified	New statements are incorporated into the set of statements that the participant deems as relevant or existing statements are extended to new cases or existing statements are reformulated
Data statements repurposed	Given statements or previously generated ideas are purposed in the argument as claims, warrants, backing, or MQ/rebuttals
Data removed	Previously perceived relevant statements are removed
Warrant added, changed or removed	Warrant is added if none previously existed, replaced by a new warrant, or eliminated as a potential link between statements
Backing added, changed, or removed	Backing statements are incorporated if none previously existed, replaced, or eliminated
Qualifier or rebuttal changed	Qualifier or rebuttal is typically implicit or not present, this code notes when one is specified or removed
Order of presentation	The relevant statements are not changed or deleted but are rearranged or combined with other claim structures
None	No changes

Table 26

Structural Shifts Enabled by Each Idea-Type

		opening structure	Claim Changed or Specified	Sub-claim added	Data added, extended, specified	Data statements repurposed	Data removed	Warrant changed or removed	Backing added or changed	Qualifier or rebuttal changed	none	Order of presentation
Focusing and Configuring ideas	informing concept image	B (1) C (1)	C (1)	B (2)	A (1) B (3)			B (4)	B (1)	B (1)		
	task type	B (1)	B (1)		C (1)	B (1)	B(1)	B (1)				
	truth proposal	C(1)			A (1)					A (1)		
	logical structure	B (2)	B (1)	A (1)			A (1)	B (2) C (1)				B (2) C (1)
	identifying necessary conditions	A (1)		A (3)				A (1)		C (1)	B (1)	
	envisioned proof path			A (1)	B (1)	A (1)		B (1)		B (1) C (2)	B (1)	
	inductive warrant					B (1)		B (5)	B (4)	B (2)		
Connecting & Justifying ideas	intuitive warrant	A (1) C (2)	A (1) B (1) C (1)	B (1)		C (1)		B (3) C (1)				
	syntactic connections	A (1)		A (7)	A (8)							
	deductive warrant	A (1)	C (2)	A (1) B (1)	A (1)	A (2) B (1) C (3)		A (1) B(2) C (2)	A (2) B (3) C (1)			
	proposed (vague) warrant				A (1) B (1)	C (1)		B (1) C (1)	B (2) C (1)	B (1) C (1)		
Monitoring ideas	truth conviction	C (1)	A (1) B (1) C (2)	A (2) B (2)	A (2)	C (3)		B (1) C (1)		B (2)	B (1)	
	can write a proof	A (1)	C (2)	A (1)	A (1)	A (1) C (3)		B (2) C (2)	B (1)	C (1)		
	unfruitful line of inquiry		A (1)		A (1)	B (1)		B (2) C (1)	B (1)	B (1)		
	support for line of inquiry		C (1)	A (3) B (1) C (1)		A (1)					B (1)	

Theme 1: Ideas That Inform Can Be Repurposed

Many data statements were repurposed as warrants or backing for the central argument or sub-arguments. On the Additive Implies Continuous task, Dr. A utilized a previous idea to be data in generating a deductive warrant. When building the counterexample used to disprove the statement, Dr. A utilized the property $\mathrm{f}\left(\frac{\mathrm{m}}{\mathrm{n}}\right) = \frac{m}{n}\mathrm{f}(1)$. This property was a proven claim while the participant was trying to prove the statement was true. The statement also served as evidence when Dr. A concluded that *f* would be true on rational numbers but not for irrational numbers based on a structural-intuitive warrant that irrational numbers could not be close enough to the quotient m/n. So Dr. A's proven claim generated while trying to prove the statement was true became data in arguments that the statement was false based on a structural-intuitive warrant and also a deductive warrant.

The ideas that were purposed were those that ended up being necessary or to underlie the claim. Consequently, many of the statements given as conditions in the statement of the task began as data statements and were repurposed as warrants or backings for warrants.

Theme 2: Claims Could Be Reversed or Specified

On tasks where there was more unknown than how to prove the claim, participants developed ideas to move the argument forward when they were able to specify or determined they would reverse the claim. Dr. B and Dr. C specified the claim when they determined what the one exception was on the Own Inverse task. Dr. C specified a claim on the determine continuity task each time something new was discovered about the function. Dr. A and Dr. C detailed initial truth proposals on the additive implies continuous task. When they found that pursuing the proving the side that they had proposed would be an unfruitful line of inquiry, they reversed the claim and worked to argue for the opposite side. I describe this swapping of claims for Dr. A below.

Because the prompt on the additive implies continuous was to 'prove or disprove', Dr. A needed to determine for which claim to argue. Deciding which was true was a persistent question. Dr. A switched which side of the argument for which to argue four times during work on the task while the interviewer was present. Dr. A called this state of mind, being 'aBayesian' meaning being amenable to a probability distribution informed by the collected data:

> When you're trying to prove something you don't know whether it's true or not, you need to be 'aBayesian'. So what one does is you come up with a prior, and you think, well, I'm ninety percent sure that linear functions are continuous, I'm ninety percent sure. Since I'm ninety percent sure, I'm more than fifty percent sure, I'm going to prove that the functions are continuous. And then I try and I try and I try, and eventually, given all this new data that I've tried for 15 minutes to prove that linear functions are continuous and I can't do it. Suddenly, my posterior distribution has changed. I now think that it's like, uh, eighty percent sure that it's not true. And so what I start to do then is look for a counterexample.

Dr. A began the task with the initial inclination that the statement was not true based on the resources of past experiences. However, the participant thought about the tool needed to prove the statement was not true, and decided that generating a counterexample would require borrowing ideas from other realms of mathematics besides real analysis and require knowledge beyond that of an undergraduate student in real analysis. These perceptions about the context of the tasks given contributed to deeming pursuing the counterexample inappropriate or infeasible. Dr. A pursued proving the statement was true, that additive implied continuous. The participant gathered information about the function, that it passed through the origin and that all outputs for rational inputs would be the input times the function evaluated at one. Reflecting on the gathered information Dr. A decided that these tools would not serve the purpose of showing that the function was continuous on all the real numbers. Because Dr. A saw no means of connecting additive to continuous, he returned to the claim that it was not true suspecting a counterexample.

The consideration of the claim was short lived because Dr. A presented a rebuttal that the counterexample would be hard to attain. Dr. A indicated not remembering how to construct the counterexample just that it existed. So Dr. A once again attended to proving the statement was true. He worked proposing new approaches and articulated that showing that the function was continuous at zero would be sufficient as only a few more steps would show the function was continuous on the entire domain. Dr. A did not find a means of doing this and ended the interview session saying that based on his lack of progress, the statement was most likely untrue, but he would not be able to generate the counterexample in the interview.

It appeared the ‘prove or disprove’ nature of the statement and the requirement to draw upon knowledge outside the realm Dr. A perceived to be relevant to the interview tasks contributed to this switching between opposite claims. The events preceding the claim swaps were not finding appropriate tools to argue the current direction, having a convincing rebuttal, or evaluating progress. Dr. C swapped claims when he found his attempt at generating a counterexample was unsuccessful. Dr. C decided that if the function he generated was not a counterexample, then no function would be a counterexample since all functions that would be continuous on the rationals and discontinuous otherwise would be of the same class of functions of the one he generated.

Theme 3: Testing and Exploring Non-Deductive Warrants

As described earlier, a phase of the proof construction process involved working to test proposed warrants which often meant working to render non-deductive warrants into deductive warrants that could be generally articulated in a proof. The testing process would result in the formulation of new warrant-type ideas as the argument progressed. This section provides descriptions of some ways that syntactic connections were developed into deductive warrants. I explain an instance where a participant cycled through proposing and testing non-deductive warrants until finding an idea that could yielded into a deductive argument, and I illustrate how non-deductive, non-syntactic, informal reasoning was successfully rendered into a deductive argument.

Rendering syntactic connections into deductive warrants. The non-deductive warrant type that was most easily rendered into a deductive warrant was the syntactic connection because their representations both lie within the representation system of proof. Dr. A was the only participant to develop syntactic connection ideas so I will focus on what his work on the MVT task whose statement I present below.

> **Given: Theorem1- MVT for Integrals**: If f and g are both continuous on [a,b] and $g(t) \geq 0$ for all t in [a,b], then there exists a c in (a,b) such that$\int_b^a f(t)g(t)dt = f(c)\int_a^b g(t)dt$.
> **Prove: Theorem 2 – Extended MVT for Integrals**: Suppose that g is continuous on [a,b], $g'(t)$ exists for every t in (a,b), and $g(a) = 0$. If f is a continuous function on [a,b] that does not change sign at any point of (a,b), then there exists a d in (a,b) such that $\int_a^b g(t)f(t)dt = g'(d)\int_a^b (t-a)f(t)dt$.

On the Extended Mean Value Theorem Task, Dr. A began with a conception about the *task-type* that there was some set of connections that would allow the left hand side of the equation to be rendered into the right hand side utilizing the given first MVT and other conditions of the statement. Dr. A proposed a series of three syntactic connections that when strung together made the symbolic expressions match up. The three manipulations were (a) multiplying and dividing by (t-a), (b) applying the given first MVT to the new statement, and (c) applying the regular MVT to the resulting function utilizing the given fact that g(a)=0. His resultant equations are given in Figure.

After writing the work in Figure 16, Dr. A went through checking to see the symbols matched up. This gave Dr. A a feeling about how the argument would go but that this work was not yet a proof, "*Okay, this isn't rigorous, so I think now I know how to do it."* Dr. A then started the task of identifying and securing what he called "loose edges." Upon the formulation of syntactic connections between statements, Dr. A set about checking to see if the manipulations were logically or mathematically valid.

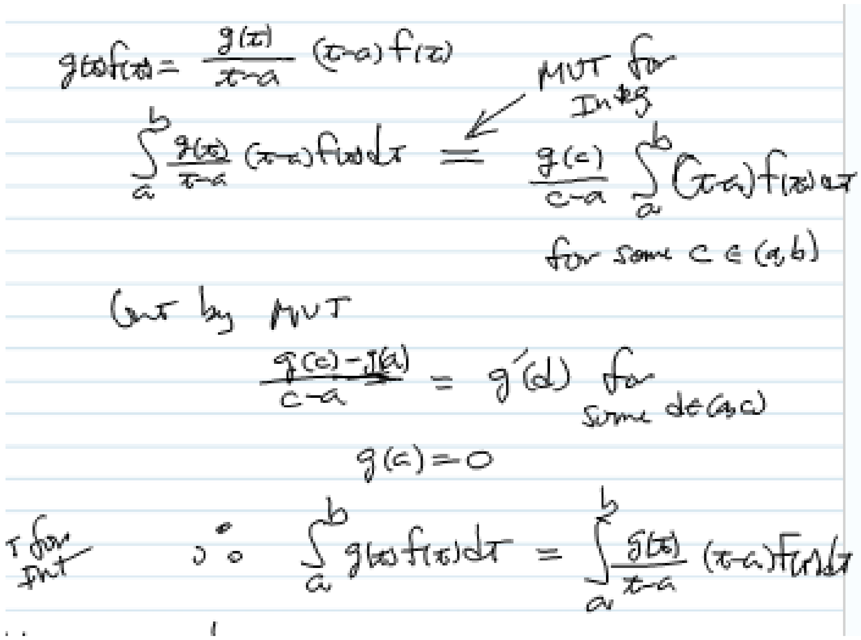


Figure 16. Dr. A's sketch of a proof for the Extended Mean Value Theorem for Integrals.

The participant first wrote a lemma to support the equation $g'(d) = \frac{g(c)-g(a)}{c-a} = \frac{g(c)}{c-a}$ for some *d*. Dr. A attended to the rest of his argument and noted that he had a potential problem with *t-a* in the denominator of the function $\frac{g(t)}{t-a}$ as *t* gets close to *a*. "Okay, so I gotta be careful with *g(t)* over *(t-a)*, because when *t* gets small, I'm dividing by zero." Dr. A was not encountering a problem as he was working to make his work into a proof. Dr. A explained in the follow-up interview that he had not yet realized that the above was an issue until he had entered into checking over the work.

MT: Okay. So were you, were you thinking about this that you have to be careful about this function earlier? Like on the previous page?

Dr. A: No. I don't think so. No, no, I don't think so. And now, I'm wondering if I have to because let's see, ah, because you're integrating, yeah, you're integrating from a to b so I do have to be careful. Yeah. / Yeah, it looks like, yeah, I'm not sure exactly what I was thinking there. But yeah, you have to be careful when you're, if you are integrating from a to b, you've got a problem with this g of t over t minus a. So what I want to do is like call g of t over t minus a, call that h of t or something. And just make sure that h of t is a nice function. So yeah, I mean it's fine except possibly so when yeah, and so taking that some kind of limit, so that's where it's conceivably a problem, and I'm not 100 percent sure it's resolved, but yeah, if there's any problem, that's where it is.

MT: Okay. So this question of it potentially being a problem, it just was right then.

Dr. A: I think it dawned on me right about there.

Dr. A had identified a *necessary condition* for his proof argument to hold. The participant recognized that the rational function would be in the form zero over zero when evaluated at $t=a$ since $g(a)$ was given to be zero, so it would be appropriate to apply l'Hopital's rule to determine the limit as t approaches a and discerned that the limit would be $g'(a)$. Dr. A noted that the statement of the theorem did not say that the function's derivative was defined at its endpoints, so he recognized a problem and either the problem was with his argument or with the posing of the task statement. Dr. A checked over the argument and the wording of l'Hopital's rule, and decided to end the argument with the rebuttal "unless $g'(a)$ does not exist."

Dr. A illustrated a technique to approaching proofs of this type where one proposes a series of manipulations to make the expressions match up and then works to justify the connections logically. How the participant justified his connections relied on ideas that inform planning. His decision to first justify the use of the regular MVT was an idea about formal logic, and he identified a necessary condition that the rational function be well-defined which focused his inquiry. In this example, I demonstrated how some arguments that only illustrate the symbolic connections are incomplete and that leaving an argument as such is problematic. However, the instance also illustrates how syntactic connections can be useful to the prover.

Cycling through non-deductive warrants successfully. All participants proposed some type of non-deductive warrant, but Dr. B proposed the most. Across the four tasks he worked on, he proposed nine inductive or structural-intuitive warrants, while Dr. C proposed four, and Dr. A proposed two. For this reason, I will use Dr. B's work on the own inverse task which involved cycling through a series of proposed warrants and example functions and in the end, developing a deductive backing based on one of the inductive warrants.

> Let *f* be a continuous function defined on I=[a,b], *f* maps I onto I, *f* is one-to-one, and *f* is its own inverse. Show that except for one possibility, *f* must be monotonically decreasing on I.

Prior to first proposing warrant, Dr. B had made some assertions about *f* needing to be either monotonically increasing or decreasing, where it would have to start and stop, and that being its own inverse would mean that it was symmetric about the line y=x. The participant drew a picture of the situation (Picture BI-1 in Figure 17) and articulated a thought that the one exception would be *f(x)=x* because "its reflection would be the same thing that you started off with. And you'd have to be right on that line because there's no other way of doing it."

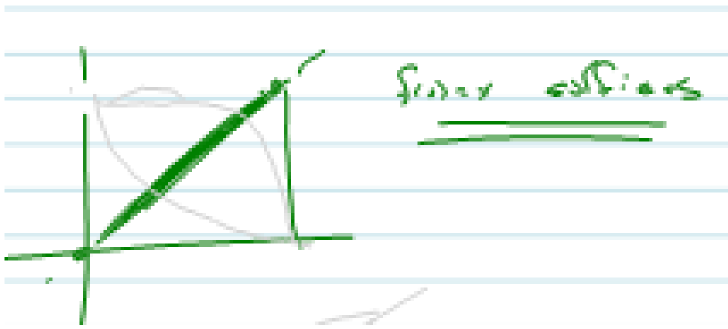


Figure 17. Picture BI-1 by Dr. B in Own Inverse task.

Dr. B had articulated a truth conviction based on a structural-intuitive argument while working through the construction of a general picture. The personal argument structure is given in Table 27.

Dr. B articulated the intuition that gave self-conviction of why *f(x)=x* was the one exception, namely that if *f* was increasing, not the identity, and symmetric about *y=x*, then it would not pass the vertical line test for functions.

> Dr. B: So now the question is this. So what have I convinced myself geometrically? And how do I prove those things? Okay. So first of all, why is *f(x)=x* the only increasing function. So my intuition there, if it wasn't, then if you reflected itself, there would have to be double values of this thing from its reflection.

At this point, Dr. B had not drawn any other pictures besides BI-1 in Figure 17. To illustrate and provide some backing for this idea the participant drew picture BI-2 shown below in Figure 18. By double values, Dr. B meant there would be two points on the same function *(a, f(a))* and $(a, f^{-1}(a))$. Dr. B was interpreting *f* being its own inverse as symmetry across the line *y=x*.

Table 27

Dr. B's Own Inverse Personal Argument Structure Upon Articulating Warrant 1

Data	Claim	Warrant	Backing	MQ/Rebuttal
F is one-to-one, onto, and continuous mapping I to I	F is monotonically increasing from a to b or decreasing from b to a	Monotonic and continuous is the same as one-to-one There's no other way about it	Geometric instantiations of definitions	
F is continuous, monotonically increasing from a to b and its own inverse Geometric instantiation of "own inverse"	F(x)=x	"you'd have to be right on that line (y=x)"	"there's no other way of doing it" Own inverse means "its reflection would be the same thing that you started off with" Picture BI-1	"I've convinced myself" "intuitively seems like the only option"
F is one-to-one, onto, continuous, its own inverse on [a,b] F is either monotonically increasing from a to b or decreasing from b to a.	F is monotonically decreasing except for the one function f(x)=x			

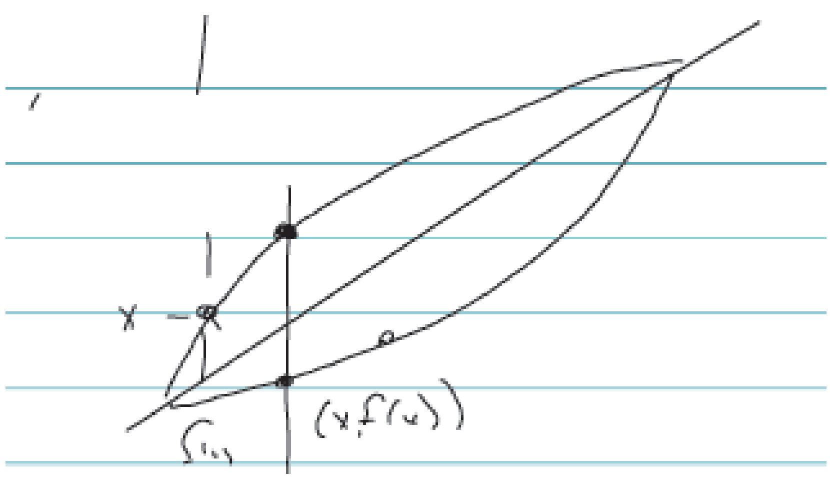


Figure 18. Picture BI-2.

Picture BI-2 (see Figure 18) served the purpose of helping Dr. B articulate what his intuition was. When evaluating this picture against the goal of validating and generalizing the warrant, Dr. B still had the question, "So why would there be double values on this thing?" Dr. B tried working with BI-2 to answer the question but decided he would need to draw another picture to figure out why it would have to happen and drew BI-3 (see Figure 19) with the point $(a, f(a))$ and thought about why $(a, f^{-1}(a))$ would also be a point. "So why does there have to be an f-inverse of a corresponding to, oh, because this is in the same domain."

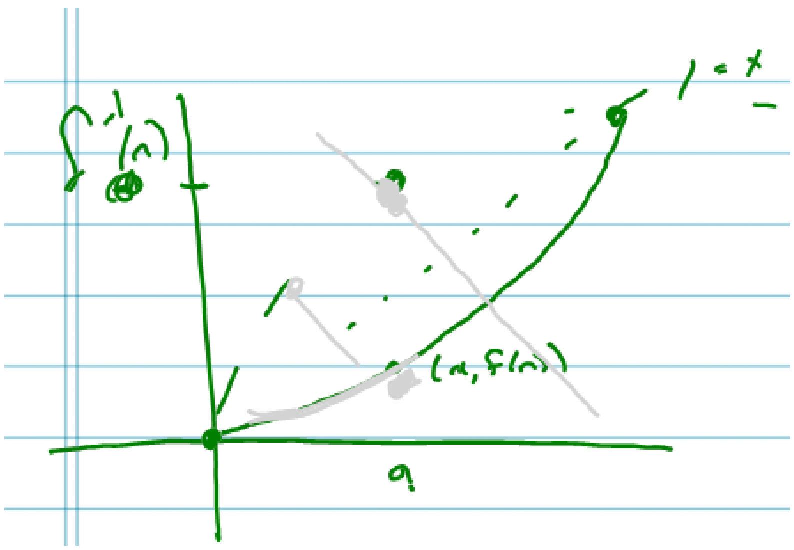


Figure 19. Picture BI-3.

The participant's articulation of the intuition changed the warrant supporting the claim that the only increasing function would be *f(x)=x* (see Table 28). The backing continued to be based on intuition but also now based on some empirical data from the drawn pictures because based on the picture Dr. B thought there was now evidence that this double-value idea would always happen.

Table 28

Dr. B's Own Inverse Personal Argument Structure Upon Articulating Warrant 2

Data	Claim	Warrant	Backing	MQ/Rebuttal
F is one-to-one, onto, and continuous mapping I to I	F is monotonically increasing from a to b or decreasing from b to a	Monotonic and continuous is the same as one-to-one There's no other way about it	Geometric instantiations of definitions	
F is continuous, monotonically increasing from a to b and its own inverse Geometric instantiation of "own inverse"	F(x)=x	Otherwise there would be double y-values from reflection.	Intuition, Picture BI-1, Support from example picture BI-2; picture BI-3	Intuitive self-conviction
F is one-to-one, onto, continuous, its own inverse on [a,b] F is either monotonically increasing from a to b or decreasing from b to a.	F is monotonically decreasing except for the one function f(x)=x			

Picture BI-2 (see Figure 18) served the purpose of articulating an intuitive warrant. Dr. B did not find it useful in determining why the double values phenomenon would always have to occur so he worked with picture BI-3 (see Figure 19). Being convinced by the example, Dr. B determined that the double values idea would serve as a contradiction and worked to articulate this in a symbolic proof by contradiction. However, Dr. B was unable to write down why the function would have double values. Dr. B wrote down these thoughts symbolically and continued working on and adding to picture BI-3. Dr. B worked for a period of time but was interrupted by a knock at the door while trying to articulate why the point on his picture *(a, f(a))* would also have to correspond to a point $(a, f^{-1}(a))$ within the representation system of proof.

Dr. B returned to working on the task explaining that he had been interrupted and worked to reorient himself to what he had worked on before. "So what's going on here? I was just about to get on this. I have this nice picture. And on my picture I know, I can see that if I reflect this type of function, it's not going to be one-to-one." The participant articulated a contradiction that was wholly different from what he had worked on before based on the picture that was meant to justify his previous warrant. When I asked Dr. B about this change, in the follow-up interview, he said that he had probably forgotten what he had done before and was trying to figure out what he was trying to contradict earlier. "I'm just trying. I think there's so many little facts about this I'm trying to think what am I trying to contradict here?...I sort of lost track of what I was doing because of when I got interrupted." The interruption resulted in a new warrant that Dr. B attributed to being based on the picture, so he developed warrant 3, an inductive warrant (see Table 29).

Table 29

Dr. B's Own Inverse Personal Argument Structure Upon Articulating Warrant 3

Data	Claim	Warrant	Backing	MQ/Rebuttal
F is one-to-one, onto, and continuous mapping I to I	F is monotonically increasing from a to b or decreasing from b to a	Monotonic and continuous is the same as one-to-one There's no other way about it	Geometric instantiations of definitions	
F is one-to-one, onto, continuous, and its own inverse	It is possible for f to be decreasing	There are a plethora of examples	Drawn and mental pictures	Absolute
F is continuous, monotonically increasing from a to b and its own inverse Geometric instantiation of "own inverse"	F(x)=x	Otherwise f would not be one-to-one	Picture BI-3	"I can see"
F is one-to-one, onto, continuous, its own inverse on [a,b] F is either monotonically increasing from a to b or decreasing from b to a.	F is monotonically decreasing except for the one function f(x)=x			

Dr. B then tried to figure out why the function, specifically the one in picture BI-3, could not be one-to-one. While exploring Dr. B noted that the point $(a, f(a))$ would connect to its inverse point but did not see how that could happen based on the picture drawn:

> It has to go through something over here. [Draws line connecting $(f^{-1}(a), a)$ to f(x)] // so let me get this right. That function can't be it…So if the inverse function came through there, what would happen? / Right. So I think I see what's going on. If I go through there, it can't do that. So I'm thinking backwards.

This time while exploring, Dr. B noted the inverse point to $(a, f(a))$ and noted that there would be a path from the origin to the inverse point to $(a, f(a))$ (see Figure 20), but he had been thinking of $(a, f(a))$ coming directly along a path to the origin. Dr. B explored for a while wondering about why the two paths would happen and eventually concluded that there would not be a path from *(0,0)* to $(a, f(a))$ otherwise it would result in double values. The idea that there would need to be a path up and then down was a potential backing for his warrant that the function would not be one-to-one, but Dr. B expressed confusion noticing an issue with how he had originally drawn the picture.

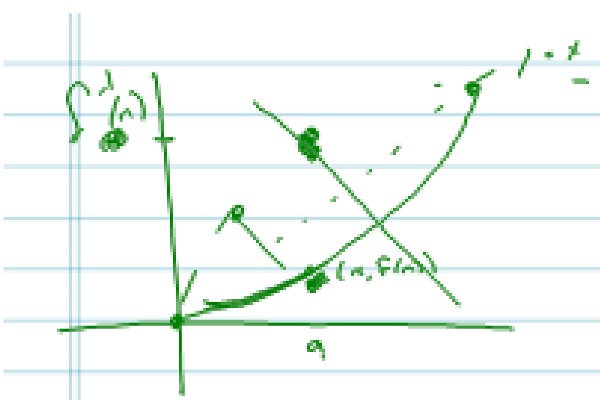


Figure 20. Picture BI-3.4.

Dr. B had not connected points on *f* to their inverse points previously. It seemed that this was a change in the way of thinking about the picture of the function. Dr. B now thought of *f* and *f*-inverse as a single function where previously it appeared there was *f* and *f*-inverse came from reflection. Dr. B was not able to fully articulate the change. His idea here was still a vague idea that something goes wrong when points are connected, but he did assert that he had been thinking about it "backwards" (see Table 30).

Table 30

Dr. B's Own Inverse Personal Argument Structure Upon Articulating An Issue With His Picture

Data	Claim	Warrant	Backing	MQ/Rebuttal
F is one-to-one, onto, and continuous mapping I to I	F is monotonically increasing from a to b or decreasing from b to a	Monotonic and continuous is the same as one-to-one There's no other way about it	Geometric instantiations of definitions	
F is one-to-one, onto, continuous, and its own inverse	It is possible for f to be decreasing	There are a plethora of examples	Drawn and mental pictures	Absolute
F is continuous, monotonically increasing from a to b and its own inverse Geometric instantiation of "own inverse" Picture BI-3 additions	F(x)=x	Otherwise would not be one-to-one	Picture BI-3	"I'm thinking about it backwards"
F is one-to-one, onto, continuous, its own inverse on [a,b] F is either monotonically increasing from a to b or decreasing from b to a.	F is monotonically decreasing except for the one function f(x)=x			

The problem was articulating why he knew the increasing function would not be one-to-one. Dr. B used the picture BI-3 to do so anticipating the picture was "nice" and would reveal why this would happen. However, the picture was originally drawn to justify another warrant. Dr. B even stated that this picture was not going to work. The personal argument shifted in that Dr. B articulated some more justification as to why the function would not be one-to-one, that it would have to connect *(a, f(a))* to its inverse. However, Dr. B was not quite able to articulate this as such just yet. The participant noted as a rebuttal to the argument, that there was something wrong with the picture in that the lines he drew connecting may not actually be there.

Dr. B stated that in his previous pictures he had not fully been using the fact that *f* was its own inverse so he stated, "Instead let's do, supposing that *f* inverse of *x* equals *f* of *x* for all *x*." Dr. B explained, "I finally realized that's the other fact I needed. Because it's its own inverse, right? And that's really what I'm contradicting…Yeah, I think I was like, what am I contradicting here? And I hadn't written this down." Dr. B stated he was not using the fact, but that was the fact that his argument was actually contradicting with the pictures. This idea led to his drawing of a new picture that used this idea where if $(x, f(x))$ was a point on the function, then $(f^{-1}(x), x)$ was also a point on the function. The participant had developed a vague sense about what would be an important reason behind a contradiction. Dr. B introduced idea that could be used to back a warrant; the idea was coded as a proposed (vague) backing. The idea about the two points having to occur was coupled with this proposed (vague) backing about *f* being its own inverse and increasing when Dr. B found a contradiction that $f\left(f^{-1}(x)\right)$ would need to be both greater than *x* and less than *f(x)*. Dr. B had begun drawing picture BI-4 (see Figure 21) knowing that his previous pictures were not capturing the own inverse idea and that lines drawn connecting points may not actually be there. So picture BI-4 only had points at the beginning and end of the interval and at *(x, f(x))* and its inverse point. Then Dr. B thought for a minute and a half about why there would be a problem before drawing a dash above the point *(x, f (x))* corresponding to the point *(x,x)* and declaring "if *f*-inverse…then it can't go through there. Right, okay, I get it." I asked about what the contradiction was that Dr. B seemed to have found.

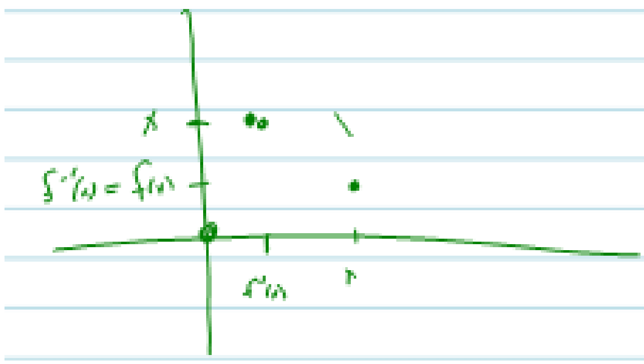


Figure 21. Picture BI-4.2.

> Dr. B: I wanted to show that when I plugged *f(x)* into the original function, that gives two different values that can't be equal actually. Because since *f* equals *f*-inverse, then *f(f(x))* would be *x*. But I mean then that would mean you have a decreasing function because *f(x)* is right there. You can see it geometrically. I think what I realized that I had to have *f(f(x))* be less than *f(x)*. But at this, if I also had that the inverse was equal to the original function, that also tells me *f(f(x))* would also have to be greater than *f(x).* Yeah, so they can be both of those things at once.

Dr. B had articulated a fourth warrant based on an example that was developed based on a (vague) backing (see Table 31). Dr. B worked to articulate this idea and justifying why this would have to happen. In doing so, he focused on *f* having to decrease on some portion if it was its own inverse and was able to render his argument into an algebraic form.

Table 31

Dr. B's Own Inverse Personal Argument Structure Upon Articulating Warrant 4

Data	Claim	Warrant	Backing	MQ/Rebuttal
F is one-to-one, onto, and continuous mapping I to I	F is monotonically increasing from a to b or decreasing from b to a	Monotonic and continuous is the same as one-to-one There's no other way about it	Geometric instantiations of definitions	
F is one-to-one, onto, continuous, and its own inverse	It is possible for f to be decreasing	There are a plethora of examples	Drawn and mental pictures	Absolute
F is continuous, monotonically increasing from a to b and its own inverse Geometric instantiation of "own inverse" Picture BI-3 additions	F(x)=x	Otherwise f(f(x)) would need to be both less than and greater than f(x)	Picture BI-4 F(f(x))=x implies f(f(x)) is greater than x on the picture F(f(x)) would need to be less than x for f to be increasing	Based on picture
F is one-to-one, onto, continuous, its own inverse on [a,b] F is either monotonically increasing from a to b or decreasing from b to a.	F is monotonically decreasing except for the one function f(x)=x			

In summary (see Table 32), Dr. B proposed Warrant 1 based on *structural-intuitive* understanding which utilized picture BI-1 to work through the conditions of the statement to arrive at that understanding. The participant articulated the intuition more specifically as Warrant 2 and used Picture BI-2 to test and articulate it and Picture BI-3 to discern why it would work generally. Dr. B thought he found a reason why and moved to articulate it in a proof but found that he was not able to generally articulate why the function would have double values. Dr. B returned to working with the same picture BI-3 to validate and generalize the warrant. While testing the idea Dr. B was interrupted and upon returning, he articulated the inductive Warrant 3. The participant still needed to justify why it always worked so he continued exploring Picture BI-3.

Table 32

Dr. B's Progression of Warrants on the Own Inverse Task

The change	Warrant	Backing	MQ/Rebuttal
Warrant 1	"you'd have to be right on that line (y=x)"	"there's no other way of doing it" Own inverse means "its reflection would be the same thing that you started off with" Picture BI-1	"I've convinced myself" "intuitively seems like the only option"
Warrant 2	Otherwise there would be double y-values from reflection.	Intuition, Picture BI-1, Support from example picture BI-2; picture BI-3	Intuitive self-conviction
Warrant 3	Otherwise f would not be one-to-one	Picture BI-3	"I can see"
Proposed backing	Otherwise would not be one-to-one	Picture BI-3	"I'm thinking about it backwards"
Warrant 4	Otherwise f(f(x)) would need to be both less than and greater than f(x)	Picture BI-4 F(f(x))=x implies f(f(x)) is greater than f(x) on the picture F(f(x)) would need to be less than f (x) for f to be increasing OR The function will end up decreasing on a portion since (x, f(x)) and (f(x),x) are both points on the graph.	Based on picture

While exploring why the function would not be one-to-one, Dr. B noted a property that the function needed to have to be its own inverse that his picture was not capturing. The participant chose to abandon his picture, BI-3, and reflect once again about what it meant to be its own inverse. He applied the instantiation of the concept by writing the statement $f(f(x))=x$ and used this conception of own inverse along with previous ideas about where an increasing function with the given properties should start and stop to draw Picture BI-4 with the aims of determining a contradiction. The exploration of this picture involved attending to how both points would be on the same function and how the function was meant to be increasing. Through this exploration, Dr. B extracted Warrant 4 which he was able to use in pursuing an algebraic argument. Dr. B's acts of articulating, testing, and exploring examples to generate backing for his warrants are examples of inquiry resulting in new information. Dr. B was able to reason informally and to achieve a formal argument in the end. However, Dr. B's and Dr. C's work on the own inverse task were the only instances where insights gathered from working with non-deductive warrants led directly to the formulation of a deductive warrant.

When are ideas from informal explorations developed into deductive warrants? A goal for Dr. B and Dr. C who explored *inductive* and *structural-intuitive* warrants was to move from these informal understandings into a deductive warrant. As illustrated above, Dr. B actively pursued justifying that his warrants formulated on intuition and examples would always occur. Dr. B explored finding this justification empirically. As described earlier, Dr. C worked to justify his warrants both symbolically and with pictures in acts he called permuting the logic.

Taking the view of the inquiry framework, the participants had gathered, assessed, and connected data statements to formulate a proposed warrant. They evaluated the warrant based on its ability to convince them of the truth and their ability to justify it in a mathematically deductive way. Finding it problematic to do so, they set about either re-exploring to formulate a new warrant or inquiring into and exploring their current warrant until a new insight was found, that new insight could be a reformulation or specification of the warrant, a possible backing for the warrant, or a new unrelated piece of information. Dr. B explored and found an entirely new proposed warrant that the function would fail to be one-to-one. These new insights were tested either by trying to prove the statement or by further exploring. At one point, Dr. C moved to test his ideas by writing the proof symbolically, but while thinking about his next step he noticed something amok with his thinking. While exploring why the function would fail to be one-to-one, Dr. B also noticed something was "backwards" about his thinking and the picture that he had drawn.

At some point during both of their efforts to justify their warrants, they both identified (if vaguely) aspects of the conception of a function being its own inverse that they were not attending to that they should. Dr. B said he was not using the idea that $f(f(x))=x$ and that his pictures were not accounting for how points on the inverse and points on the function must connect. Dr. C said he was not focusing on how points below the line $y=x$ end up above the line and vice versa. They identified that these aspects of the own inverse property would be important in any contradiction that they might get, and this assertion of a proposed backing was critical to their attainment of the deductive warrant.

Informed by this new piece of data, the participants moved to perform more explorations (or actions). This time Dr. B's picture only held the given pieces of information and the aspects that he found useful; Dr. C connected the ideas that his previous explorations had shown to be pertinent including the proposed backing idea. The acts of inquiring yielded new information including a sense that there was something missing from how they had been thinking about the ideas. These missed or unfocused aspects of the own inverse property ended up being central to the mathematical justification of the contradictions that they found.

Relationship Among Idea Categories

This section provides a description of how ideas within the three outlined idea categories, Focusing and Configuring, Connecting and Justifying, and Monitoring were tested and used with each other in the evolution of the personal argument. I first demonstrate the development, testing, and interaction of ideas within the context of a specific task. Later, I summarize how this example compares and contrasts with the work on other tasks by other participants before summarizing the chapter.

Dr. C's evolving argument on the additive implies continuous task. I chose to demonstrate utilizing the additive implies linear task because its directions to prove or disprove provided an open ended format. Dr. C deployed both formal and informal modes of thought, made his monitoring thoughts apparent either within his work on the task or in the follow-up interview, and was not certain he could solve the task until he had developed a deductive warrant so was actively engaged in inquiry as moved forward. The statement as given to Dr. C is presented.

> Let *f* be a function on the real numbers where for every x and y in the real numbers, $f(x+y) = f(x) + f(y)$. Prove or disprove that *f* is continuous on the real numbers if and only if it is continuous at 0.

Dr. C read the statement and articulated a *truth proposal,* "It's certainly continuous on the rationals. I don't believe it for the reals though." The assertion was based on connecting perceived to be relevant content knowledge about how one proves that linear functions (in the form $y = mx$) are continuous:

> I was thinking about the well-known fact that the only continuous linear functions in the reals to the reals are those of the form y equals mx for some fixed m. And one shows that those are continuous on the rationals fairly easy - linear functions are continuous on the rationals pretty easily by doing some induction.

Although not voiced aloud during his work on the task, Dr. C had brought in statements and facts that he perceived to be relevant. These *ideas that informed the statement image* were the basis of his truth proposal. The conjecture was made based on his knowledge of mathematical structure and relationships. It was a *structural-intuitive warrant.* The truth proposal informed Dr. C's decision moving forward. He shifted his inquiry focus from determining the truth of the statement to looking for a way to disprove the statement which he knew to be achieved by developing a counterexample. The structure of his argument is given in Figure 22.

Dr. C chose to develop a function that was continuous at zero and continuous on the rational numbers and discontinuous otherwise based on his earlier structural-intuitive warrant idea. The example would serve as a means of testing his warrant. To test the example, Dr. C knew that the function would need to be continuous at zero and have the additive property. The participant justified that the function was continuous at zero non-problematically as he had designed it to be so. Dr. C tested the additive property with a specific critical example of a pair of irrational inputs whose sum was rational. Dr. C worked through the example and concluded that his function did not satisfy the additive condition.

As a means of testing the structural-intuitive warrant by trying to develop a deductive argument based on the same ideas that informed a warrant, Dr. C deployed the tool of an example in the anticipation that it would serve to disprove the statement. Dr. C found his example tool failed. He evaluated the situation and stated that maybe the statement was true.

Dr. C: It turned out that didn't work. And if the easier ones didn't work, then the harder ones probably wouldn't either. Matter of fact, if the easier one didn't work, then it seemed likely that none of the harder ones would work.

M: Okay. So I was going to ask about that. So after you found that it didn't work, it didn't satisfy it. You paused for a while. Was it because you were trying to think of different examples, or were you convincing yourself that it-

Dr. C: Yeah. I was trying to convince myself that if this didn't work, then nothing would.

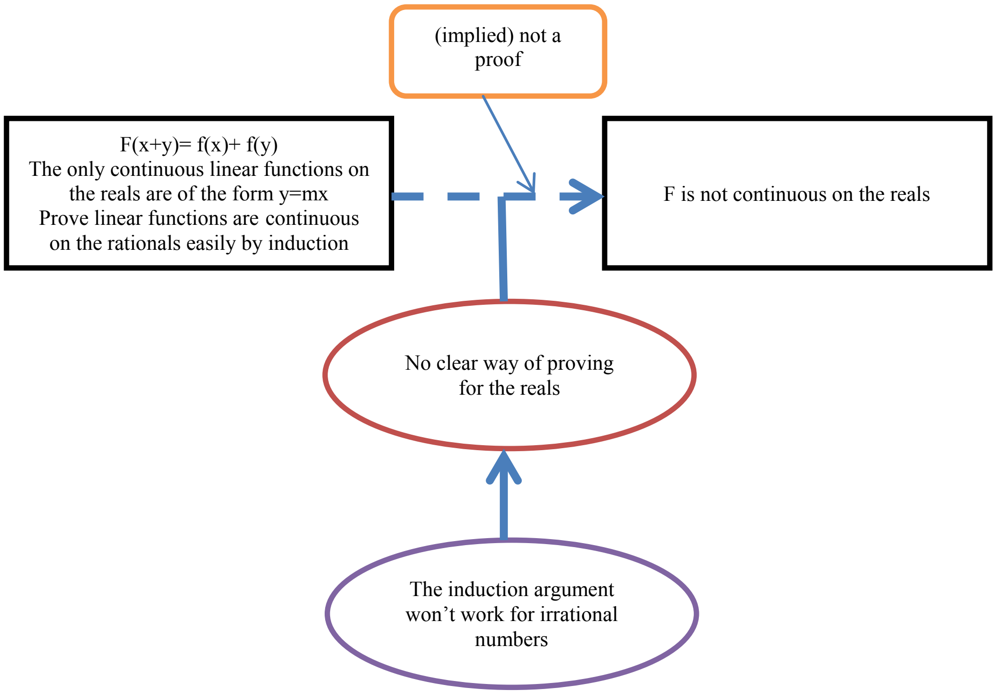


Figure 22. Dr. C's personal argument structure on additive implies continuous task upon articulating a truth proposal.

As can be seen in Figure 23, the new truth proposal informs the claim rectangle on the right. The warrant backed by inductive information provides some connection between the data and claim, but as the warrant is not a deductive type, the connection is not solid and the qualifier is not absolute.

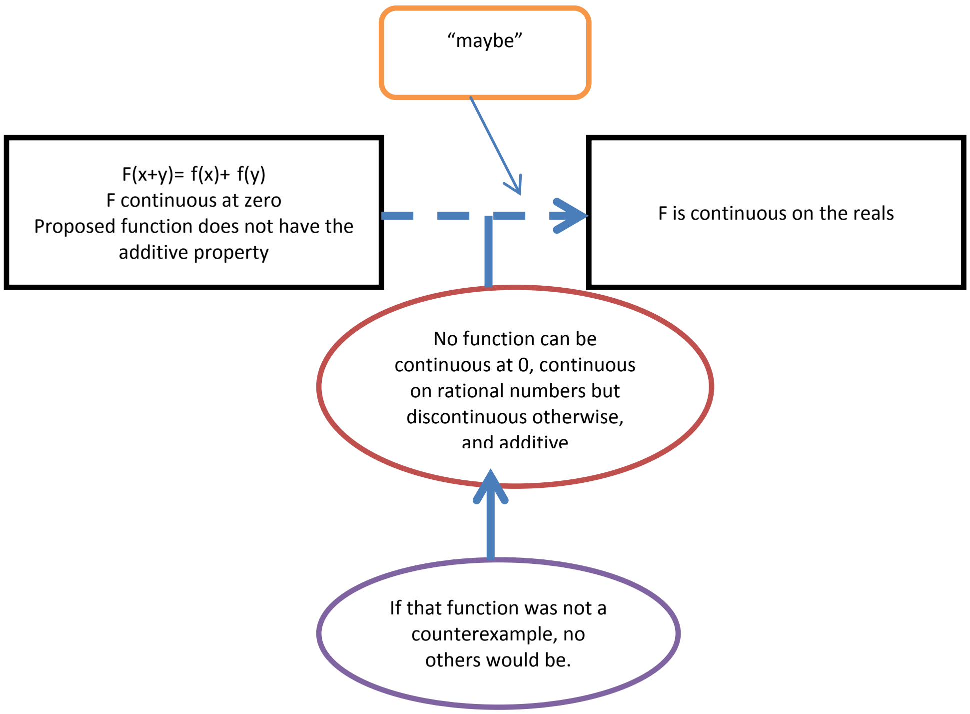


Figure 23. Personal argument structure upon changing the truth proposal based on an inductive warrant.

Dr. C had used the example as a means of testing not only the warrant but the truth proposal it backed. This moment was characterized by a second truth proposal but this time achieved by an inductive warrant, content knowledge about a type of function, and a feeling that pursuing a counterexample would be an *unfruitful line of inquiry*. The tools used and the monitoring idea that there is an unfruitful line of inquiry worked with the inductive warrant to connect the data to the truth proposal.

Dr. C then moved to try to prove the statement was true or to provide an absolute link between the data and the claim. The participant deployed tools of instantiations of the definition of continuous that he deemed to be fitting with the additive property of the function as well as his knowledge of properties of limits to extract the equation given in

Figure 24. Dr. C concluded that the continuity of the function depended on the value of $\lim_{\varepsilon \to 0} f(\varepsilon)$ being equal to zero or not.

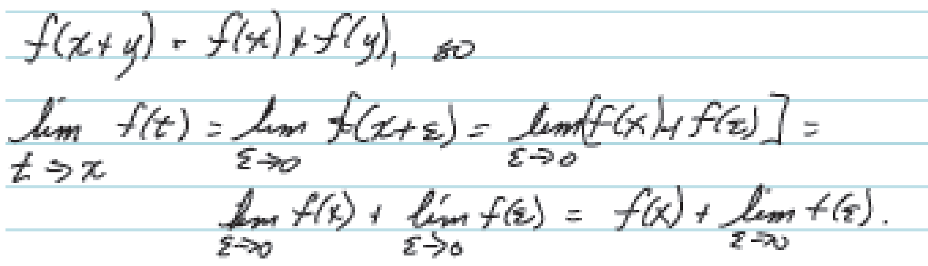


Figure 24. Dr. C's limit equations.

Dr. C had developed a *necessary condition* based on the deployment of tools chosen based on data statements perceived to be useful and a *deductive warran*t. In an effort to fulfill the condition and test his earlier truth proposal, Dr. C determined that the function would need to satisfy *f(0)* the second necessary condition informed his planning. He remembered an algebraic proof that *f(0)=0* which satisfied one aspect and provided *support for the line of inquiry* of attempting to prove the statement true. However, Dr. C still had a condition to fulfill, “So *f* of zero is zero. But that doesn’t mean that the limit as epsilon goes to zero of *f* of epsilon is zero. Does it? Why should it?” Dr. C looked back at his writing of the statement because he “was thinking about how to bring the definition of continuity into the picture. That was returning to what I was given and figuring out how to bring that into the picture to evaluate. Or to show that limit was *f* of zero which is the central question for continuity at the origin.” In an effort to glean more data, Dr. C looked back at the statement and saw that it was given that the function was continuous at zero. “Ah, but we’re given that *f* is continuous at the origin. And we know that *f* of zero equals zero. So the limit as epsilon goes to zero of *f* of epsilon is zero. I think that makes it work." Dr. C symbolically evaluated that his assertions were correct and declared *truth conviction.* This was coupled with a sense that he could now write the proof based on his *deductive warrants.* Because Dr. C's work in proving the task was based on deductive warrants within the representation system of proof, the writing of the proof did not require the formulation of any new ideas (see Figure 25).

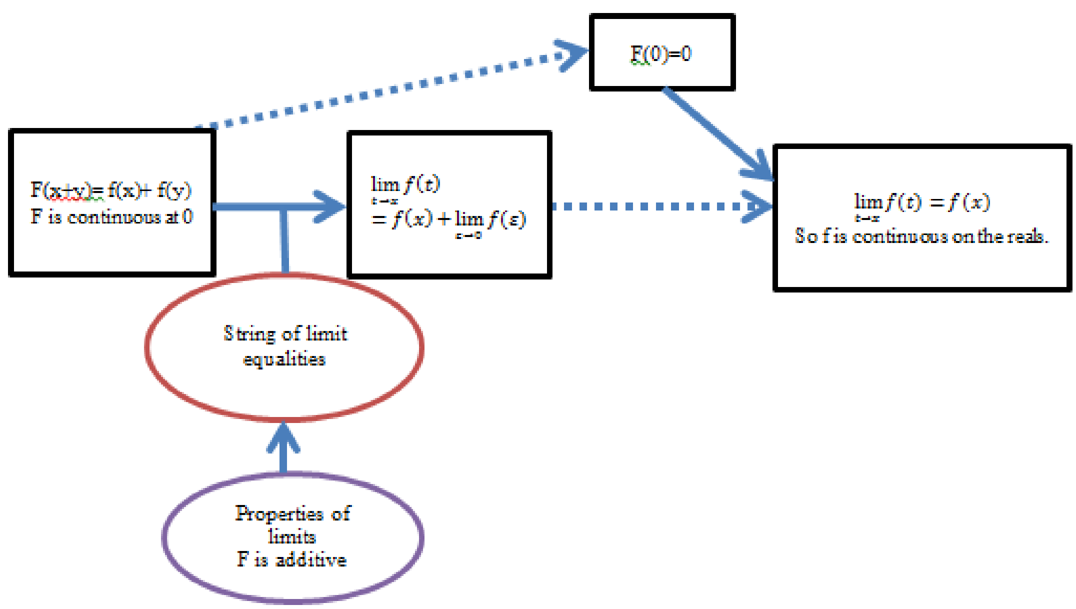


Figure 25. Personal argument structure upon the realization of a necessary condition.

Interaction of ideas for Dr. C. During the task, Dr. C articulated ideas from every idea category as summarized in Table 33. The first idea was to bring in some perceived to be relevant facts to inform the statement image which when combined led to a truth proposal based on structural-intuitive backing. The proposal was tested by a possible counterexample that was generated based on the insights relevant to the structural-intuitive warrant. The possible counterexample function was tested by a numerical example and found to not serve as a counterexample. Conceptual knowledge, an inductive warrant, and a sense of an unfruitful line of inquiry informed a new truth proposal. The truth proposal was tested by the pursuit of a deductive argument. The fulfillment of the argument was achieved by deductive warrants but focused by the identification of necessary conditions whose fulfillment inspired monitoring ideas of support for the line of inquiry which supported the continuation of the proof to the fulfillment of the necessary condition and a truth conviction and the feeling of being able to write a proof.

Table 33

Ideas Generated During Dr. C's Work on the Own Inverse Task

Focusing and Configuring	Connecting and Justifying	Monitoring
• Informing statement image • Truth proposals (2) • Necessary conditions (3)	• Structural-Intuitive warrant • Inductive warrant • Deductive warrant (2)	• Unfruitful line of inquiry • Support for line of inquiry • Truth conviction • I can write a proof

As described above, the idea-types were not always formed independently of other idea-types but sometimes simultaneously in coordination with each other. After Dr. C tested his proposed function and found that it did not serve as a counterexample, he identified a new *truth proposal* based on an *inductive warrant* and an identification of an *unfruitful line of inquiry*. The example was just an example until Dr. C used it to justify his new truth proposal. Dr. C would not have made a new truth proposal if it had not been for his evaluation combining the identification of an unfruitful line of inquiry, the evidence from the example, and conceptual knowledge of types of functions. The identification that searching for a counterexample would be an unfruitful line of inquiry essentially was an idea to pursue proving that the statement was true based on conceptual knowledge and the empirical evidence. No one of these three idea-types was generated prior to the other, their formulation depended on the others, and all three categories of focusing and configuring, connecting and justifying, and monitoring were apparent in this single moment.

Because of this possible amalgam of idea-types into one single moment, linear or even cyclical of patterns defining the steps at which these ideas are formed such as focusing and configuring, then connecting and justifying, then monitoring, repeat are not generally discernable. What is noticeable is a process of proposing or articulating an idea, testing the idea's usefulness or the usefulness of prior ideas against the consequences of the new idea, and then articulating a new idea. The process itself involved the passing through, often multiple times, the inquiriential cycle of reflecting, acting, and evaluating.

Consider again what followed Dr. C's articulation that $f(0)=0$ was a necessary condition. Dr. C tested the truth proposal against this necessary condition. He reflected upon why $f(0)=0$ could possibly be true and began to think about ways of proving it so. Dr. C recalled a standard argument that would prove $f(0)=0$ and applied that argument. In the evaluation of the results of his actions against the statement to be proven, Dr. C declared a monitoring idea of support for the line of inquiry, namely pursuing proving the statement true, but also another necessary condition, that *f* is continuous at zero. Even though there is no discernable pattern amongst the order that the ideas in each category emerge, the idea categories have purposes in structuring the argument and informing the problem-solving pathways.

The focusing and configuring ideas of informing the statement image, truth proposals, and necessary conditions provided the data and claims of the argument structure as well as a proposed pathway. The warrants filled in the links between the data and claim, and the backing, once deductive, made those links sound. We saw in Figure 24 that the identification of the necessary condition that $f(0)=0$ provided another rectangular box claim that was known to be needed to serve as data being somehow connected with other data to support the final claim. This focused Dr. C's inquirential pursuit. He moved to solve the problem of warranting that $f(0)=0$. The monitoring ideas were not always visible in the argumentation structure, but their effects were apparent in the decisions made in that Dr. C abandoned pursuing a counterexample and continued trying to pursue proving true when monitoring ideas were achieved.

Summary of the relationship amongst idea types. Every other participant on each task identified ideas from each of the three idea categories. For each participant, the focusing and configuring ideas informed the rectangle data and claim boxes, warrants and backing, as expected, served as connections between the data and claims, and monitoring ideas, in combination with other means of testing ideas, provided information to the prover as to whether their work was fitting or not.

As seen with Dr. C, the evolution of the personal argument is not linear in identifying focusing and configuring ideas, identifying connections and justifications, and then making monitoring decisions since single moments may characterize multiple idea-types, simultaneously. The process of articulating ideas, testing the new idea or previous ideas against these new ideas, and then proposing new ideas was apparent. The process of testing ideas varied by idea-type as described in earlier sections, but the process involved active, productive inquiry. In general, the ideas were tested against their abilities to do work in solving a perceived problem. As these problems evolved, the ideas were eventually tested against their ability to work with the existing personal argument structure to move the argument into a general, deductive proof.

Summary of Findings

The purpose of this study was to describe the ideas mathematicians developed to move their personal arguments forward, to describe the inquirential context surrounding the emergence of those ideas, and to describe how the mathematicians used and tested those ideas as they proceeded to resolve the proof problem. The mathematicians in this study developed ideas that moved their personal arguments forward. That is, they developed insights, feelings, and statements whose incorporation into the personal argument changed the argument's structure. I grouped ideas into three categories according to their perceived functionality: Ideas that Focus and Configure, Ideas that Connect and Justify, and Monitoring Ideas. The focus and configure ideas included ideas from six sub-types: statement image, task-type, truth proposal, necessary condition, envisioned proof path, and logical structure ideas. The ideas that connect or justify were the proposed warrant and backings for warrants, meaning the proposed reasoning to connect the evidence with the claim. I further classified the proposed warrants based on their respective backings. Monitoring ideas were feelings the participants had about their progress and success in completing the proof tasks and solving their problems.

The ideas emerged through the mathematicians' purposeful recognition of problems to be solved and reflective and evaluative procedures to solve them that incorporated the implementation and development of tools. These tools included rich conceptual knowledge, multiple instantiations of concepts and definitions, purposeful exploration of specific examples, knowledge of heuristic strategies, and the connecting and permuting of ideas and statements. The ideas did not emerge in a linear or cyclical pattern of an idea from each category. However, there was a pattern of working to (a) understand the statement or determine the truth of the statement, (b) find a warrant (or a reason why the statement would be true), (c) validate, generalize, and articulate the warrant, and (d) write the final proof.

As the mathematicians articulated and incorporated new ideas, the structure of their personal arguments changed, notably that data statements could be repurposed as warrants or backing or participants could change their claims. When the mathematicians reasoned informally, they would proceed to test their non-deductive warrants in an effort to progress into formal arguments. In the next chapter, I summarize these findings against the literature and provide possible explanations. In addition, I elaborate on limitations and hypothesize implications of these findings for teaching and research in mathematical proof.

CHAPTER V

DISCUSSION AND CONCLUSIONS

In this chapter, I present a summary of the study and a discussion of the ideas the participating mathematicians developed to move their argument forward, how that development occurred, and how those ideas were utilized and tested as the personal argument evolved. It then presents considerations for the implications of the findings for research and teaching. Finally, it suggests future research paths to investigate further understanding how certain idea-types are tested and to investigate methods for helping students develop and utilize ideas to move their arguments forward.

Summary of the Study

There is general agreement that the mathematician's proving process involves an attainment of ideas that are organized in a way to move an argument toward a mathematical proof (Rav, 1999; Tall et al., 2012). Research is needed to document the context surrounding how mathematicians formulate useful ideas and how these ideas contribute to the development of the argument. The purpose of this research was to describe the evolution of the personal argument in professional mathematicians' proof constructions, to describe the situations surrounding the emergence of ideas that moved the argument forward, and to describe how those ideas are tested and used. Specifically, this research sought descriptive answers to the following questions:

Q1 What ideas move the argument forward as a prover's personal argument evolves?

Q1a What problematic situation is the prover currently entered into solving when one articulates and attains an idea that moves the personal argument forward?

Q1b What stage of the inquiry process do they appear to be in when one articulates and attains an idea that moves the personal argument forward? (Are they currently applying a tool, evaluating the outcomes after applying a tool, or reflecting upon a current problem?)

Q1c What actions and tools influenced the attainment of the idea?

Q1d What were their anticipated outcomes of enacting the tools that led to the attainment of the idea?

Q2 How are the ideas that move the argument forward used subsequent to the shifts in the personal argument?

Q2a In what ways does the prover test the idea to ensure it indeed "does work"?

Q2b As the argument evolves, how is the idea used? Specifically, how are the ideas used as the participant views the situation as moving from a problem to a more routine task?

The research was framed in Dewey's (1938) theory of inquiry, a conception of the proof construction process as involving an evolution of a personal argument, and Toulmin's (2003) model of argument structure. I asked three mathematicians teaching or doing research in the field of real analysis to participate in the study by solving three mathematical proof tasks either while being video-taped or on their own time. Participants recorded all their work in Livescribe notebooks. Participants chose perceived to be problematic tasks, and I contributed one task. In the end, participants worked on three or four total tasks among the seven total tasks included in the study. After the mathematicians completed a task, they participated in follow-up interviews with protocols informed by my preliminary analyses and hypotheses developed from viewing their submitted work prior to the follow-up interview. Informed by the complete data set, I identified each idea that moved the argument moved forward, detailed the Toulmin structure of the argument prior to and following the formulation of that idea, and the inquirential context that contributed to that idea. Open iterative coding of the ideas seen, problems, tools, and shifts in the structural argument preceded analysis for categories and themes across participants working on the same task and then across tasks for the same participant.

Major Findings

The three participants worked on seven total tasks. Results of open-iterative coding discerned that they entered into solving nine types of perceived problems utilizing ten categories of tools (see Tables 40 and 41 in Appendix F). Ideas that moved the argument forward corresponded to structural shifts in the Toulmin diagrams of the personal argument, provided the participant means to communicate their personal argument in a logical manner, gave participants a feeling that their way of thinking was fitting or unfruitful, or were explicitly referred to by the participants as a useful insight into the resolution of an issue that they had identified as problematic.

The framework of logical inquiry was useful in describing the process of generating and testing ideas that moved the argument forward. No tool category or problem was indicative of a single idea category as problems and tools transcended idea category. The interplay of problem, tool, previously articulated ideas, and the individual's perspective determined how ideas were generated.

Ideas That Moved the Personal Argument Forward

As the three participants worked on the proof construction tasks, they developed feelings about how the proof development would go, identified key relationships, and made decisions in order to make progress on the task. These feelings, decisions, and moments of insight were the ideas that moved the argument forward. Some of the ideas identified have been remarked upon by previous literature as “decisions” (Carlson & Bloom, 2005), “resources” (Carlson & Bloom, 2005), and ideas or moments (Raman et al., 2009). Other researchers have commented on the use of various resources, tools or explorations by students and mathematicians to achieve the senses of understanding of a proof situation that characterize some of the ideas that were identified in Chapter IV (e.g., Alcock, 2004; Alcock & Inglis, 2008; Lockwood et al., 2012; Watson & Mason, 2005; Weber & Alcock, 2004). This work, however, contributes to the existing knowledge in this area with the specific purpose of identifying the moments, decisions, and ideas that moved the argument forward from the prover’s perspective, and makes its contribution unique.

The ideas were grouped into 15 idea-types that were further grouped into three idea categories according to their function in moving the personal argument forward. The ideas that *focus and configure* which included ideas that inform the statement image, task type, truth proposals, identified necessary conditions, envisioned proof paths, and formal logic ideas provided insight into making decisions about how to begin, how to proceed, and what tools and ideas to use in doing so. The ideas that *connect and justify* included four types of warrants based on their paired backing as well as proposed backings. These ideas were proposals for how to link the given data statements to the claim statement and how to validate those links. Deductive warrant-types were those sought for the final written arguments, but the articulation, exploration, and utilization of non-deductive warrants proved fruitful in the attainment of new information and ideas that guided future explorations or helped participants assert truth. Participants utilized *monitoring* ideas to evaluate their progress and to make decisions regarding their progress toward a more efficient solution strategy, which were captured by four sub-idea-types.

Ideas that focus and configure. The ideas that focus and configure were those that did work for the participant in terms of where to focus, how to begin, how to proceed, and what tools to utilize. *Ideas that inform the statement image* were statements, relationships, and proven claims that the participant deemed as relevant to the personal argument but had not yet been utilized as a warrant, claim, or backing. These ideas could have been the results of elaborating or connecting given conditions from the task statement or previous knowledge brought into the conception of the task. I have chosen to distinguish these ideas from general tools and conceptual knowledge. These ideas could be wielded as tools (connecting and permuting ideas) and may have resulted from applying conceptual knowledge, but, uniquely, they were statements of fact that the participant deemed as potentially useful or utilized to inform his conception of the personal argument.

Since most work in problem solving has been to identify useful strategies or to describe the cognitive processes involved, the formulation of ideas that inform the statement image have not been characterized by other literature as notable moments. However, decisions about how to use these ideas and others in the focusing and configuring category could be seen as falling into the much broader "resources" problem-solving attribute in Carlson and Bloom's (2005) multi-dimensional problem solving framework. Mason et al. (1982/2010) provide mathematical thinking strategies for students breaking down the problem-solving process into three phases into entry, attack, and review. In the entry phase, students are to ask the questions, "What do I know?, What do I want?, What can I introduce?". Ideas that inform the statement image seem to fall under the "What do I know" question as it encapsulates what is known from the task statement and from past experience. While in the literature these ideas inform how to get started, this research has shown that these ideas can be formulated at every stage of the proof construction process.

On the tasks that instructed to prove or disprove, participants developed *proposals of truth* based upon intuition, experience, or examples. Other researchers have studied the similar constructs of the formulation of conjectures or evaluations of truth (e.g., Inglis et al., 2007; Lockwood et al., 2012; Pedemonte, 2007). More specifically, they studied the reasoning activities involved with formulating these conjectures. This research supports Lockwood and colleagues' findings that proving and disproving are related in that participants may work to prove or disprove simultaneously.

Participants developed ideas about the *type of task* that they were presented on four of the seven tasks. These ideas were accompanied by feelings about what kinds of approaches would be useful or appropriate. The problem solving literature has indicated that identifying problem-type or problems similar to the current problem as potentially useful heuristics (Pólya,1945/1957; Schoenfeld, 1985); mathematicians were noted to perform this behavior as Carlson and Bloom (2005) stated that when orienting to a problem, the mathematicians "scanned knowledge and classified the problem" (p. 67).

The mathematicians in this study identified necessary conditions on the additive implies continuous and extended MVT tasks. That is they identified conditions that they would have to prove in order for the task claim or a claim they asserted to be true. These ideas provided direction for their inquiry efforts and anticipated consequences should they succeed or fail in fulfilling the conditions. The identification of this idea-type as moving the argument forward did not occur on the other tasks. It is possible that this was because the two tasks held the nature where participants worked syntactically (as characterized by Weber and Alcock (2004) to connect given data to an algebraic formulation of a definition. Identifying necessary conditions is related to understanding and unpacking relevant definitions recognized by mathematicians and mathematics educators as a useful step in constructing mathematical proof (e.g. Selden & Selden, 1995; Weber & Alcock).

Both Dr. A and Dr. B achieved moments where they could *envision a proof path*; that is they proposed a series of statements that they could see leading to a solution while looking for a warrant or backing for a warrant. These were identified when participants were stuck on justifying some sub-claim. They had a sense of "if I can show this, then I'm done". Selden and Selden (1995) described how student provers could benefit from unpacking the logic of a statement to develop a *proof framework*, essentially the series of claims needed to be validated in order to achieve a mathematical proof of a given statement. The proof framework nearly encompasses what is meant by an envisioned proof path except that the achievement of an envisioned proof path may require more than the unpacking of the logic of the statement and is accompanied by a sense of how one would justify all other claims provided he/she could justify an initial claim. These envisioned proof paths gave participants a sense of "If I had this, then I'd be done."

Ideas about formal logic and the representation system of proof were decisions made about how to structure the mathematical argument and what logical tools to use including proofs by cases, proving by contradiction, determining qualifiers, and making decisions on what constituted mathematical proof. Ideas of this type are reflective of the theoretical description by Selden and Selden (2013) who classify the proof problem into two parts, the formal-rhetorical part and the problem-centered part. They state that the formal-rhetorical part does not depend on genuine problem solving or a deep understanding or intuition about the concepts involved. The mathematicians demonstrated a fluid knowledge of formal logic and ways of communicating ideas within the norms of the mathematics community. This most likely contributed to these ideas not always being identifiable or seen as meaningful to the participant. While the mathematicians did not encounter problems when making these decisions, a novice, most likely would need to reflect upon these decisions.

The focusing and configuring ideas "did work" for the participants by informing the participants' focus and how to structure both the final argument and also their work while still constructing. While these ideas relate to other constructs in the literature such as resources for problem solving, formulating conjectures, unpacking the logic of the definitions and statements, and the formal-rhetorical aspect of constructing mathematical proof, this research is the first to characterize ideas that inform the statement image, identified necessary conditions, and envisioned proof paths as moments that move the argument forward. Implementation of these ideas influenced the statements of data and claims and the ways in which the participants structured their ideas. Additionally, this study demonstrates that these ideas provided insights into what kinds of warrants and associated backing would appropriately associate the data and claim.

Ideas that connect and justify. Ideas that connect and justify were the statements that participants viewed or proposed could link the statement with the claim as well as the ideas that could back them. As noted in Chapter IV, I adapted Inglis et al.'s (2007) constructs to classify the types of warrants viewed. Inglis and colleagues described how mathematics graduates formulated conjectures based on deductive, structural-intuitive, and inductive backing. Participants in this study also proposed and utilized warrants of these three types, but these findings expand on the work of Inglis et al. by proposing a fourth warrant-type, describing the situation surrounding the formulation of these warrant-types and how participants utilized these warrants as their personal arguments evolved.

The fourth warrant-type was based on observations that Dr. A would propose connections between statements based on symbolic manipulations without attention to the mathematical objects with which they were associated; warrants of this type were termed *syntactic connections.* An example was Dr. A's series of algebraic manipulations to connect the expression $\int_a^b g(t)f(t)dt$ to $g'(d)\int_a^b (t-a)f(t)dt$ on the Extended MVT for integrals task without attending to see if his manipulations were mathematically justified. The naming of the warrant-type was inspired by Weber and Alcock's (2004) classification of a proof production by logically manipulating mathematical statements without referring to intuitive representations as a syntactic proof production. A syntactic connection could be utilized in a syntactic proof production, but the proof production also required its later being connected to a deductive backing.

Dr. A was the only participant to exhibit syntactic connections on the Extended MVT task as well as the Lagrange Remainder Theorem task. Presumably, this was because these two tasks involved specifically justifying symbolic equations. Dr. B, who also worked on the Extended MVT task, did not exhibit this type of warrant, but also did not find any means of warranting the statement to be proven. For Dr. C, the proof of the additive implies continuous task did involve a syntactic (Weber & Alcock, 2004) argument; however, he justified each step beyond the algebraic manipulation. Dr. C's work was therefore classified as utilizing deductive warrants.

In addition to warrant-types, on five occasions, participants proposed backing for a previously generated warrant or a vague feeling about what would back a justification for the claim. The backing ideas were proposed largely while participants were looking for a warrant or a means of generally articulating a previously proposed warrant via connecting and permuting data statements and properties. These ideas characterized as *proposed backings* have not been previously identified in the literature as only few research efforts have utilized the full Toulmin (2003) model that includes backing. However, Raman (2003) referred to a mathematician identifying the "only one thing" that could explain why the claim was true and could be translated into formulas. The utilization of these proposed backings is discussed in a later section.

Participants developed structural-intuitive or inductive warrant-types when specifically searching for conceptual reasons why the statement would be true. The mathematicians formed (or attempted to form) syntactic connections when they suspected that they would need to connect the symbols in the "right way". As will be discussed later, upon articulating a warrant participants would set about the task of either searching for justification for their non-deductive warrants or articulating them generally which led to either formulating new warrants, proposing new backings, or the algebraic translation of their warrant idea. The decisions about the usefulness of the warrants and determinations about the participants' progress were enveloped in the third category of monitoring ideas.

Monitoring ideas. Monitoring ideas guided the mathematicians' decisions and gave them feelings as to whether their current line of inquiry was fitting or not. Two of the monitoring idea-types were specific to the proof construction process, truth convictions and feelings that one could write a proof. The other two, feelings that the actions taken were fitting or would be unfruitful, are relevant to problem solving processes in general.

The *truth convictions* displayed by participants were supported by deductive, inductive, and structural-intuitive warrants. In one case, the conviction that the statement was not true was supported by the participants' inability to find a warrant to support the statement. I purposely separated feelings of truth conviction from truth proposals. A truth proposal was a means of getting started in proving a statement by providing a direction in which to argue; truth proposals were not necessarily accompanied by the participants' belief. A truth conviction, on the other hand, was a feeling that the connections that the participant had developed were enough to justify the assertion to himself. This feeling of personal certainty also distinguishes a truth conviction from a conjecture justified by non-deductive backing as described by Inglis et al. (2007) as well as the types of conjectures developed by the participants in Lockwood and colleagues' (2012) study about how examples were used in conjecturing. Inglis et al. describe how the conjectures made were accompanied by modal qualifiers that indicated incomplete conviction. While the modal qualifiers that accompanied truth convictions in this study often indicated that the participant's argument was not a proof ("intuitively", "I can see", "heuristically"), the participants expressed some sense of belief.

On three of the tasks, participants did not display moments where they developed a truth conviction; instead, they displayed an implicit belief that the statement was true and it was up to them to justify the connection. These three tasks were Dr. A's work on the Lagrange Remainder Theorem task, Dr. A's work on the Extended MVT, and Dr. B's work on the Sequences and Limit Points task. On both the Lagrange Remainder Theorem task and the Extended MVT, Dr. A expressed an idea about the *type of task;* he determined upon reading the task that some series of correctly applied symbolic manipulations should yield the desired result. Dr. B chose his individual task, the Sequences and Limit Points task, from a list of homework tasks for his students. Dr. B's belief in the statement was possibly developed on a previous attempt at the problem.

The moment a truth conviction is attained is related to Raman and colleagues' (2009) characterization of moments when provers identify a key idea or conceptual insight (Sandefur et al., 2012). A conceptual insight is an idea that gives a sense as to why the statement is likely to be true. Therefore, the truth conviction is not the conceptual insight, but the warrant that engendered the truth conviction could be, if the participant was able to express what it was that convinced him as sometimes the sense that a participant achieves could be vague and require more explorations to articulate it (Mason et al., 1982/2010).

When participants' moments of truth conviction were supported by deductive warrants, the ideas that gave a sense of personal truth conviction were the same as those that gave a sense that *one could write a proof*. In tasks where the truth conviction was supported by ideas other than non-deductive warrants, the participants deemed that there was still work to be done before they could write down a proof. The feeling of being able to write down a proof accompanied the participants' development of a deductive warrant or a syntactic connection. In one case, Dr. C stated he could write a proof after proposing a backing that "the inverse of an increasing function was increasing"; however, he was not able to write a proof when he tried and needed to go back to continue exploring. The formulation of these deductive warrants is further discussed in a later section.

Raman and colleagues (2009) term the idea that gives a sense that one can write a proof as a technical handle. Technical handles are the "ways of manipulating or making use of the structural relations that support the conversion of a CI [conceptual insight] into acceptable proofs" (Sandefur et al., 2012, p. 6). Raman and others allow for the conceptual insight that convinced the prover to be a different conceptual insight from the one rendered into a technical handle. I will later discuss how the findings of this study in regards to the evolution of the personal argument and the process of developing ideas that move the argument forward relate to the conceptual insight and technical handle constructs.

Truth convictions and feelings that one was ready to write a proof were monitoring ideas unique to problem solving in proof construction tasks. Feelings that one was engaged in an unfruitful line of inquiry or that the results supported the current line of inquiry were related to the metacognitive behaviors deemed necessary for success in problem solving (e.g., Carlson & Bloom, 2005; Pólya, 1945/1957; Schoenfeld, 1985, 1992). Carlson and Bloom (2005) defined acts of monitoring as "the mental actions involved in reflecting on the effectiveness of the problem-solving process and products" (p. 48). They found that in solving problems mathematicians would "engage in metacognitive behaviors" and "act on their monitoring in ways that moved them forward." Carlson and Bloom defined the actions or decisions made in response to the monitoring behaviors as *control decisions* and *self-regulation.* The identification of unfruitful lines of inquiry or support for a line of inquiry always accompanied a decision to either abandon the line of thought or to keep pursuing it. Certainly monitoring actions in the sense of Carlson and Bloom were present throughout the inquiry process as monitoring is reflective of the ongoing reflecting and evaluating behaviors present in active-productive inquiry. Ideas that move the argument forward that involved monitoring were decisions or determinations made upon evaluating performed actions against the problem one had entered into solving and the current personal argument.

Process of Developing Ideas

I observed that participants would proceed through the solving of the problem or tasks of (a) understanding the statement or determining truth, (b) looking for a warrant, (c) working to validate, generalize, justify, or articulate their warrant, and (d) writing the formal proof (see Figure 26). Within this progression, participants could cycle back to a previous problem or work to solve the problem with an enacted tool. Generally participants did not find writing the formal proof to be problematic once they had identified a generalizable warrant. Specifically, the following aspects have been identified as part of the proof construction process: Understanding the statement or described objects (Alcock, 2008; Alcock & Weber, 2010; Carlson & Bloom, 2005; Savic, 2013); determining the truth of the statement (Sandefur et al., 2012); determining why the statement is true (Raman et al., 2009; Sandefur et al., 2012); translating ideas into analytic language (Alcock & Inglis, 2008; Alcock & Weber, 2010; Weber & Alcock, 2004); and justifying a previous idea (Alcock, 2008; Alcock & Weber, 2010). This research is unique in its specific efforts to identify the problems encountered as participants develop new ideas in the proof construction process at different stages.

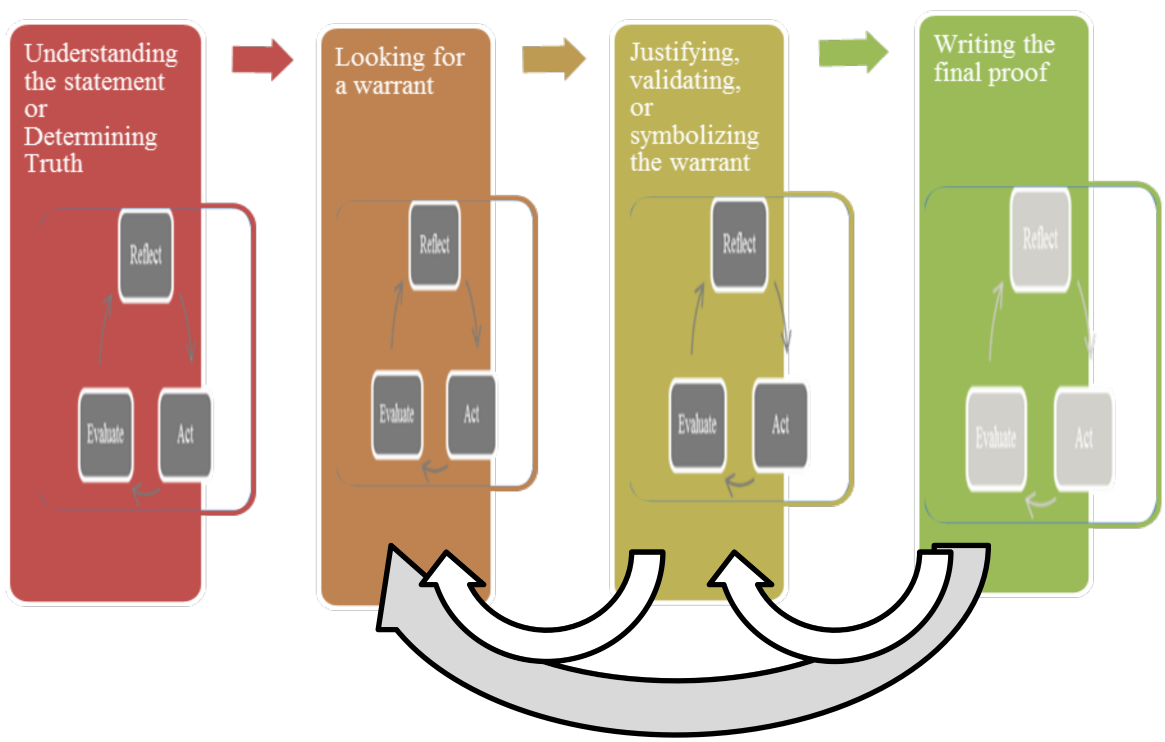


Figure 26. Progression of problems as the personal argument evolves.

Dewey's (1938) framework of logical inquiry was useful in describing the process of generating and testing ideas that moved the argument forward. Participants entered the task focused on the problem of understanding the statement or determining the truth of the statement. They reflected upon what was known, formed connections among the information given in the statement of the task as well as their prior knowledge and experience, and generated new ideas. The act of forming connections would involve drawing pictures, imagining pictures, or connecting definitions. The initial ideas formulated were ideas that informed the statement image, ideas about task type, truth proposals or truth convictions based on structural-intuitive warrants.

Depending on whether the idea formulated solved the problem of understanding the statement or determining truth, the participant would either continue working to understand the situation or they would move to either look for a connection (warrant) between the data and claim or work to generalize or deductively justify their structural-intuitive warrants. To look for a warrant, participants would again perform or imagine actions of drawing and working through examples, performing symbolic manipulations, connecting and permuting the definitions and data statements, and drawing upon alternative instantiations of definitions of concepts and conceptual knowledge. They would perform these actions until they had completed an action, formulated a new idea, or determined that they were engaged in an unfruitful line of inquiry. If no idea was found, participants would either perform more actions of the same type or would reflect upon earlier generated relationships, pictures, and statement in the personal argument and make a change. If a new idea was generated when solving the problem of understanding the statement, it was either a proposed warrant, an additional datum to inform the statement image, an envisioned proof path, or an identified as necessary condition.

The participants would evaluate the statements and relationships developed in their personal argument in light of the new ideas generated, their own conceptual knowledge, and experience. In this evaluation, they would determine if they needed to continue pursuing a warrant and given the new information how they would do so, or they would move on to test the warrant by trying to translate it into a general algebraic form (if it was structural-intuitive or inductive) or to determine why the warrant had to hold (if it was non-deductive). If a participant had developed deductive warrants to connect the conditions of the statement to the claim of the statement, then they moved to write up the proof formally.

The observation that ideas emerged as a result of engaging in genuine inquiry was based on Dewey's (1938) and others' descriptions of the theory of inquiry (Hickman, 1990). As was described in Chapter III, knowledge is the outcome of active, productive inquiry (Hickman, 1990). When engaged in the intentional process to resolve doubtful situations, an individual intentionally and systematically invents, develops and deploys tools (Hickman, 2011). The ideas that moved the argument forward were the tools developed and deployed to resolve the perceived to be problematic situations within their efforts to construct a mathematical proof.

The ideas were generated based on evaluations of the results of performing actions or applying tools to a situation. Carlson and Bloom (2005), drawing upon the work of Pólya (1945/1957) and Schoenfeld (1985, 1992), called these evaluations "strategic control decisions" (p. 64). In their multi-dimensional problem solving framework, Carlson and Bloom formulated a problem solving cycle similar to Dewey's (1938) cycle of reflecting, acting, and evaluating where participants orient to a task, then cycle through planning, executing, and checking actions. The three-part cycle could be repeated to resolve a given issue prior to the participants' cycling forward to repeat the three-part cycle to resolve the next issue. Savic (2013) observed that for mathematicians, the proof construction process was consistent with the multi-dimensional problem solving framework with the exception of participants "cycling back to orienting" and not completing a full planning-executing-checking cycle. The findings of this research support the work of Carlson and Bloom and build upon it by describing the ideas generated while engaged in the process.

Utilized tools. The types of tools deployed that contributed to the generation of ideas (see Table 41) have previously been identified as useful in the proof construction process. What I note here are the salient themes about how the tool-use contributed to the generation of the ideas that moved the argument forward, relating the findings to what has been noted in the literature. These themes included the utility of examples, the connecting and permuting of data statements, and the usefulness of perceived to be useless tools. Additionally, I comment on how the mathematicians made deft choices in the tools they did deploy that could only be explained by their vast experience.

In this study, example-use contributed to the formulation of inductive warrants, structural-intuitive warrants, ideas that informed the statement image, and in one case an envisioned proof path. At times, the inductive and structural-intuitive warrants developed due to the exploration of examples were enough to generate a truth proposal or truth conviction. Dr. B was the only individual to employ examples with the purpose of understanding the statement or objects involved with the statement and the purpose of articulating his thinking. These findings support the work of a number of researchers who have suggested and documented that example use can inform conjectures as to whether a statement is true or not (e.g., Harel & Sowder 1998; Lockwood, Ellis, Knuth, Dogan, & Willliams, 2013) as well as give insights into how to construct a proof (e.g., Lockwood et al., 2013; Sandefur et al., 2012; Watson & Mason, 2005). This work adds to previous findings in further documenting how these ideas informed by examples are tested and connected with other ideas within the personal argument and how they contribute to the evolution of an argument into a formal proof.

Lockwood and colleagues (Lockwood et al., 2012) give a comprehensive framework for describing mathematicians' example activity when exploring and proving conjectures. The framework includes the types of examples explored, uses of examples, and strategies mathematicians employ when using examples. Both Dr. B and Dr. C utilized examples with the purpose of understanding why the statement would be true to get insight into proving and to validate, justify, or test a previous idea. The only example generated by Dr. A was a counterexample successfully utilized to disprove a statement. Examples were utilized in various ways including using a picture in conjunction with generic symbolic explorations, exploring while constructing the picture, and using the picture of a specific case to determine the underlying structure of the situation.

Dr. A's non-proclivity to example-use could be explained by the specific tasks on which he worked as opposed to the tasks that Dr. B and Dr. C completed. Two of his tasks, the extended MVT and the Lagrange Remainder Theorem task, involved the equating of symbolic expressions. Dr. A was able to propose and justify symbolic manipulations to connect the two expressions. The additive implies continuous task involved a type of function that could be difficult to visualize or represent in a picture. Dr. A pictured straight lines as a special case and utilized them to make progress but knew that "weird things" could happen.

As documented by other researchers (Alcock & Weber, 2010; Lockwood et al., 2013), successful utilization of examples in proof construction did not occur without drawing upon other sources of knowledge. The tools of conceptual knowledge, understandings rich with instantiations of concepts and definitions, and the connecting and permuting of known statements not only informed the construction and exploration of examples; they supported the development of ideas from every type. Carlson and Bloom (2005) documented that these heuristics and resources were used and distinctions between how they were used in each of the problem-solving phases.

In some instances of the current study, a participant would deem a tool to be unhelpful in its original purpose. However, its deployment and exploration resulted in added pictures, equations, or insights that contributed to the formulation of new ideas to move the argument forward. I explain this occurrence within the perspective of the theory of inquiry. The participant evaluated a line of inquiry to be unfruitful in solving the perceived problem. This evaluation ended the cycle; the participant, then needed to step back and reflect again on the situation and determine which problem to enter into next. Either in pursuit of solving the same problem or a newly entered problem, the participant would need to once again reflect on the situation (the personal argument and the statement image that encompasses it) and imagine how to connect aspects of the statement image together to apply to the problem. The previously deployed tool resulted in new data which alone did not achieve the original purpose. However, connecting the data to other ideas or reflecting on the data for a new purpose could result in new proposed actions to solve a problem. This reflecting again is reminiscent of Mason and colleagues' (1982/2010) advice to learners that going back to the "Entry" phase (or reflecting) aspect after working on the problem and getting stuck may enable greater understanding of the situation due to achieving more relevant experience.

Certainly, the decisions made by the mathematicians were influenced by their rich experience and knowledge bases. Alcock and Weber (2010) noted student struggles with choosing demonstrative examples to aid in their proving. The mathematicians deftly chose examples informed by the imagined consequences they could evoke. For instance, when choosing an example to test if the function he proposed to be a counterexample fulfilled the additive property ($f(x + y) = f(x) + f(y)$), Dr. C chose values of x and y to be irrational but to have a rational sum not equal to zero because Dr. C had developed the function to be continuous at zero and continuous on a domain restricted to rational numbers but discontinuous on the real numbers. Lockwood et al. (2013) also found that mathematicians' choices of examples were informed by knowledge and experience. Experience contributed to the participants' flexible and rich understandings of relevant definitions. On the additive implied continuous task, both Dr. A and Dr. C utilized an instantiation of the definition of continuity that would capitalize on the additive property of the function. Dr. C knew that showing $\lim_{t \to x} f(t) = f(x)$ would be equivalent to $\lim_{\varepsilon \to 0} f(t + \varepsilon) = f(t)$. This instantiation of the definition proved critical and useful to their generation of ideas and Dr. C's eventual proving of the statement. Individuals who were novice in real analysis could encounter struggles with these tasks that the mathematicians in this study did not.

The role of affect and context. Although not overwhelmingly present in the data, participants' affect and perceptions about the context of the study played a role in the decisions made. Schoenfeld (1985, 1992) described how an individual's mathematical belief system including what mathematics is and one's role in mathematics will contribute to their choices in whether or not they will enter or persevere in a problem. A student problem solver may, due to an affective belief about one's ability, may choose not to enter a problem. The mathematicians in this study displayed affective beliefs that they could solve the problems encountered knowing that if something they tried did not work, then they could always try something else. Largely, the mathematicians' decisions based on affect were appropriate. However, there were times, though, where even for mathematicians, frustration and exhaustion contributed to the participants deciding to discontinue work on a task without completion or, in one instance, writing and accepting an argument that was not mathematically valid without checking it.

As detailed in Chapter III, Dewey's (1938) theory of inquiry posits that decisions made depend on how one perceives the situation and the resources available. The mathematicians in this study held perceptions about how much time they should be spending on problems, that the knowledge required to solve the tasks would be from the realm of real analysis, and what theorems they were allowed to (and should) assume and which ones they would need to prove. These perceptions affected the decisions that they made. For example, even though Dr. A initially thought there could be a counterexample to the additive implies continuous task, he moved to try to prove the statement true based on his perception that generating a counterexample would involve skills and difficulty inappropriate for the tasks encountered in the interview. The participants' sometimes utilized their observations about the interview context to their advantage. On the extended MVT task, both Dr. B and Dr. A determined that if a theorem was given, then it would be utilized in some way in the proof. It may be that experience in mathematical problem solving especially in solving "homework-type" problems has attuned the participants to account for conditions on the situation beyond what is outlined in the task statement.

Inquiry with no ideas. As described in Chapter IV, Dr. B worked on the Extended MVT for Integrals task but achieved no insights into how to prove it beyond a feeling that it would involve some set of symbolic manipulations. In working on that task, the participant displayed the reflective and evaluative nature of active productive inquiry, identifying a target goal based on observations about the situation, proposing and enacting tools, and evaluating their effectiveness. Dr. B engaged in inquiry but yielded no results. It is difficult to say why Dr. B achieved no success, but Dr. A did achieve success utilizing the same overall strategy of finding a symbolic manipulation to connect expressions. As a reminder to the reader, the task as given to Dr. B is restated below.

> Given: Theorem1- MVT for Integrals: If *f* and *g* are both continuous on [a,b] and $g(t) \geq 0$ for all *t* in [a,b], then there exists a *c* in (a,b) such that $\int_b^a f(t)g(t)dt = f(c)\int_a^b g(t)dt$.
>
> Prove: Theorem 2 – Extended MVT for Integrals: Suppose that *g* is continuous on [a,b], $g'(t)$ exists for every *t* in [a,b], and $g(a) = 0$. If *f* is a continuous function on [a,b] that does not change sign at any point of (a,b), then there exists a *d* in (a,b) such that $\int_a^b g(t)f(t)dt = g'(d)\int_a^b (t-a)f(t)dt$.

I looked to the literature for an explanation. Schoenfeld (1992) said the following will help us analyze one's success or failure in problem solving: the individual's knowledge, use of heuristic strategies, behaviors of monitoring and control, and belief systems. Dr. B had a sufficient knowledge base as the task only required one to utilize the given First MVT for Integrals in combination with knowledge from undergraduate real analysis. Dr. B deployed a number of heuristic strategies including listing what was known and applying symbolic manipulations that he knew to typically be useful in integral problems (like integrating by parts and substituting with a limit definition). As a mathematician, we could expect that his beliefs about mathematics would be conducive to success in problem solving.

This leaves his monitoring and control, or the ability to recognize that the participant was headed down an unproductive path. Comparing Dr. A's work on the task to Dr. B's, both participants worked to symbolically connect the statements. Dr. A's focus was on making the left hand side of the equation to be proven to look like the right hand side anticipating that some application of the given first MVT would be used. Dr. B, on the other hand, seemed to focus on relating the two theorems, directly, letting the two functions, f and g, in the first MVT for integrals be the same as the two functions, f and g, in the extended MVT for integrals allowing for possibly swapping their names. This way of thinking pervaded even the participant's work with examples as Dr. B used the same example functions when exploring the first MVT as he did on the extended MVT. Dr. B did not recognize his pursuit as non-productive despite engaging in monitoring or evaluative behavior. In order to move his students into better habits of monitoring and control, Schoenfeld (1985) had his students work in groups, and Dewey

(1938) posits that tools and the standards against which tools are evaluated are products of the social context. Dr. B was working in isolation. It may be that Dr. B needed to discuss this problem with another in order to illuminate that his way of thinking about the two theorems would be unproductive.

Non-inquirential tool use. Participants also engaged in periods where they were completing parts of the task without perceiving a problematic situation. This occurred upon participants' development of a deductive warrant when they moved to write the argument formally. Selden and Selden (2013) described this aspect of the proof construction process as the formal-rhetorical part; they indicate that no real problem solving is needed when completing this part of the proof construction. Indeed once participants had developed a general algebraic argument, writing the argument formally posed no problem. However, this disengagement from the reflective and evaluative nature of inquiry resulted in errors without recognition. Schoenfeld (1992) would term this acting without "control". The first two sub-sections of the major findings provided answers to the first research question outlining the ideas that moved the argument forward and the context surrounding the formulation of the ideas. The next section answers the final research question: How were the ideas used and tested as the personal argument progressed to a routine situation?

The Uses and Testing of Ideas

This study was not the first to utilize Toulmin models to document changes in one's argument (Pedemonte, 2007; Zazkis, Weber, & Mejia-Ramos, 2015). However, it is the one of few to pay special attention to the types of backing involved which has afforded the classification of warrants. This classification of warrants has given some insight into describing how the personal argument can evolve from the informal to the formal; however the development of warrants was not the only structural change informative in this study. Utilizing Toulmin diagrams in conjunction with the conception of the personal argument enabled the identification of the trends of changing or deleting claims, purposing data statements into warrants or backing, and changes to the overall structure of the claim by adding sub-claims or reorganizing the format of the argument. These structural changes informed the discernment of idea-types and idea categories and how those ideas are used to move the argument forward. In an effort to answer the research questions about how ideas were used and tested as the personal argument evolved, I discuss the aforementioned observed structural shifts, but first I provide the general overview of how the ideas from each category were used in the development of the personal argument.

Participants generated ideas that moved the argument forward from each of the three idea categories of Focusing and Configuring, Connecting and Justifying, and Monitoring on every task. The ideas were tested and used together as the personal argument evolved toward a final proof product. The focusing and configuring ideas informed both the data and claim statements as well as the overall structure of the argument in that the integration of some ideas from this category gave the mathematicians insight to what sub-claims they would need to justify and into what logical form to articulate the argument (proof by contradiction, by cases, and so forth). Figure 27 demonstrates how Dr. A’s idea of an envisioned proof path on the additive implies continuous task based on his statement that showing the function was continuous at zero would be continuous informed the structure and provided a new claim statement.

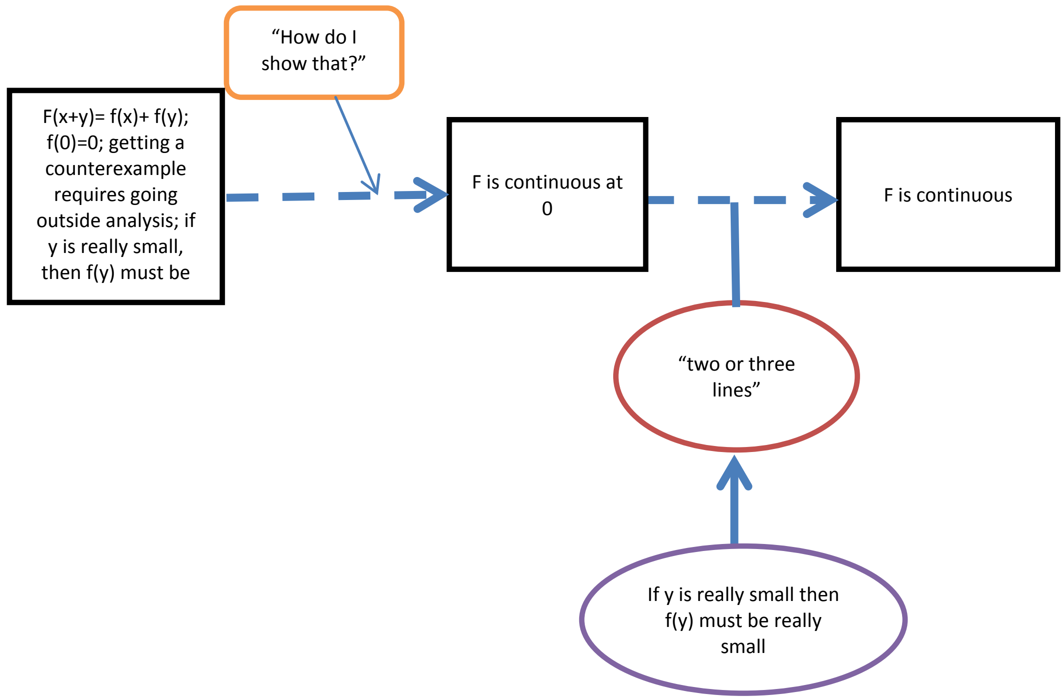


Figure 27. Demonstration of how adding an idea that focuses and configures (envisioned proof path) informs structure.

The Connecting and Justifying ideas included four types of warrants discerned by their coupled backing and proposed backings. These warrant ideas provided the connections hypothesized as needed from the focusing and configuring ideas. Monitoring ideas would determine if the warrant was acceptable (translatable into a formal deductive argument) or if it needed more exploration in order to validate, generalize or justify the warrant. The work to test the idea sometimes resulted in new warrants, new backing, or new focusing and configuring ideas. In light of these new insights, monitoring ideas along with the structural decisions made from the planning ideas would contribute to how the participant would proceed. Ideas were tested in this manner until participants successfully developed a (perceived to be) deductive warrant or chain of deductive warrants to connect the data to the claim. These processes led into the routine writing down of the final formal proof.

Moving non-deductive arguments into deductive ones. The process of utilizing and moving an argument toward a final proof that originally was based on a non-deductive warrant was demonstrated by all three participants on at least one task each. The framing of their efforts in terms of the evolving personal argument described via Toulmin models within Dewey's theory of inquiry provides some insight into answering the mathematics education research community's questions to how and if informal arguments can be converted into an acceptable proof form.

As detailed in Chapter IV, Dr. A moved an argument based on syntactic connections into a deductive proof by first making the symbolic connections and then going back to check if each step could proceed logically. Weber and Alcock (2004) described process of constructing a proof within the representation system of proof (syntactically) as involving choosing a proof framework, listing assumptions, deriving new assertions by applying established theorems and rules of inference, and continuing until reaching the appropriate conclusion. Dr. A's production demonstrates a different pattern specific to syntactic proof productions involving the equating of symbolic expressions. The syntactic connections acted as an in-between tool establishing a path between hypothesis and conclusion whose reasonableness would need to be backed deductively. Dr. A's practice of working to make the symbols look "nice" and then going back to check the manipulations' mathematical integrity seems to be a reasonable practice.

This study teased out this particular instance of a syntactic connection which differed from the process described by Weber and Alcock (2004); however, it could be viewed under the recent study conducted by Zazkis and colleagues (2015). Zazkis et al. found that students who successfully translated informal into formal arguments either (a) attempted to translate the argument into the representation system of proof which they termed “syntactifying,” (b) tried to find a deductive reason for a claim that their informal argument justified termed “rewarranting,” or (c) attempted to add more detail to the proof which they called “elaborating”. Dr. A engaged in elaborating when moving his argument based on syntactic connections into a deductive proof.

Using the language of Zazkis and colleagues (2015), on the own inverse task, it appeared that Dr. B and Dr. C attempted syntactifying but were unsuccessful and proceeded to inquire until they could rewarrant. Dr. B was first convinced of the truth of the Own Inverse task by a structural-intuitive warrant. Testing that warrant empirically, the participant gained new insights and a new warrant. Dr. B tested that second warrant and found that his way of picturing the function was inaccurate. Reflecting again on the situation in light of all he had done previously, Dr. B identified that he was not utilizing the full implications of the given data statement that the function was its own inverse. The missing implication was one Dr. B could symbolize and did so. The participant proposed that this implication would underlie any successful warrant (contradiction). The new proposed backing informed further exploration where Dr. B developed a new warrant that he was able to render into a symbolic form. On the same task, Dr. C also developed warrants based on inductive and structural-intuitive backing, worked to test them, and was unable to render them into a symbolic form. When reassessing, Dr. C identified an implication of the function being its own inverse that he thought would be the underlying reason for any warrant (contradiction). Dr. C, like Dr. B, was able to articulate that implication symbolically. Exploring with symbolic statements, Dr. C was able to produce a contradiction or warrant finally used in the final argument.

The summarized processes of Dr. B and Dr. C seem almost identical through the lens of Zazkis et al.'s (2015) framing; however, their non-deductive warrants prior to their final arguments were not the same. They drew very different pictures, and explored justifying their warrants in different ways. Dr. B realized that he should be using the implication of the function being its own inverse of $f(f(x)) = x$ implying both (x, f(x)) and (f(x), x) would be points on the same line and that would underlie his contradiction. Dr. C, on the other hand, identified the implication involving how reflecting over the line would move points above the line to below the line as being the key. This underlying quality of the property of a function being its own inverse that the participants found translatable into a symbolic form is reminiscent of the *key idea* as it was first described by Raman in 2003. This key idea (not to be confused with the key idea that was later renamed as conceptual insight (Raman et al., 2009; Sandefur et al., 2012) is a "heuristic idea which one can map to a formal proof with appropriate sense of rigor". A faculty member in Raman's study called the idea that she categorized as a key idea, was the "only one thing" about the situation that provided both an explanation for why the claim was true and was translatable into formulas that could demonstrate that the claim was true.

> Prof A: Let's see, an even function. There is only one thing about it, and that is its graph is reflected across the axis. Yeah, and you can be quite convinced that it is true by looking at the picture. If you said enough words about the picture, you'd have a proof. (Raman, 2003, p. 323)

Raman (2003) did not necessarily implement Toulmin's framework so did not classify whether key ideas would serve as warrants or backing in an argument, but I contend that the idea articulated by the faculty member in her study, that the function is reflected across the axis, would act as a backing in a final argument because it does not directly connect data and claim for the task which was proving that the derivative of an even function is odd. The ability to identify that one thing about a given condition that one could informally see as explaining why the statement would be true seems to be a non-trivial task based on the number of researchers who have wondered about the process of converting informal arguments into an acceptable mathematical form (e.g., Boero et al., 1996; Pedemonte, 2007; Raman et al., 2009; Sandefur et al., 2012 Selden & Selden, 2013; Zazkis et al., 2015).

As previously mentioned, in her later work, Raman, in conjunction with colleagues (2009), conceived of the constructs of conceptual insight (which was sometimes also called key idea) and technical handle. A conceptual insight gives a sense of why the statement is true (a truth conviction), and more than one conceptual insight can be attained in proving a given claim. A technical handle is an idea that renders a conceptual insight into a symbolic mathematical form (a feeling that one could write a proof). Sandefur and colleagues (2012) have given some insight into how exploring examples can potentially facilitate students' development of conceptual insights and technical handles, but the literature as of now has not been able to move beyond those descriptions.

The conceptual insight and technical handle constructs seem to be difficult to utilize in research practice. As seen in this study, moments of truth conviction and feelings that one can write down a proof can occur at the same moment, and feelings of truth conviction may not be observed. Between articulating a truth conviction and a feeling that they could write down a proof, participants in this study were sometimes seen to move through multiple warrants, some not always generalizable or valid. If researchers were to try to identify the conceptual insights and technical handles in a given proof construction, would they identify all the proposed warrants as conceptual insights and the means of translating one of those insights into a deductive warrant as the technical handle? Or are warrants that cannot be translatable into a symbolic form merely heuristic ideas (ideas that give personal conviction; Raman, 2003) and conceptual insights are the warrants that an outsider could perceive as having potential to be symbolically represented? Despite the difficulty utilizing the constructs when describing process, the constructs of conceptual insight and technical handle have afforded researchers language to speak about ideas that could move an individual's proof construction forward and provided the inspiration for this study.

Another vein of research regarding the movement of informal argumentation into deductive proof regards *cognitive unity* which describes a situation where arguments developed while evaluating or producing conjectures are translated into a mathematical proof (Garuti, Boero, Lemut, & Mariotti, 1996). Of particular interest are the instances where the initial arguments developed were based on reasoning outside the representation system of proof. For this study, that would be truth convictions or truth proposals based on structural-intuitive or inductive warrants. Researchers have contemplated the conditions for and the obstacles to cognitive unity (Bubp, 2015; Pedemonte; 2007) noting the ways of reasoning (deductively or abductively) when first formulating conjectures influencing whether or not cognitive unity occurred. Only one participant in this study demonstrated cognitive unity for those cases of interest; Dr. A initially had an intuition and memory that there was a possible counterexample to the additive implies continuous task that involved the axiom of choice and eventually produced one. This study suggests that cognitive unity can only occur when the prover happens to be first convinced by arguments that are completely generalizable and translatable into symbolic notation which may dissuade those teaching mathematical proof from trying to evoke this practice. On a brighter note, this study shows that one can begin with a non-deductive warrant, continue to reason outside the representation system of proof, and eventually develop a non-deductive warrant or propose a backing that could support a symbolic deductive argument.

Implications for Research and Teaching

In this section, I discuss the implications of these results for research and for teaching. I argue for the continued use of full Toulmin models to document the evolving argument and for attending to the ideas that the individual finds useful in moving the argument forward. More research is needed in order for these results to inform teaching practices, but I consider some preliminary implications including recommendations for teaching problem-solving within introduction to proof courses and engaging students in testing their arguments.

Value of the Full Toulmin Model

Mathematics education researchers have worked to analyze the mathematical arguments of individuals using a restricted, ternary form of Toulmin's (2003) argumentation scheme, limiting to data, warrant, and claim (e.g. Krummheuer,1995;

Pedemonte, 2007; Zazkis et al., 2015). I argue that in order to understand an individual's argument prior to the articulation of a formal proof and therefore their process in constructing a mathematical proof, the constructs of backing, modal qualifier, and rebuttal need to be included to capture the whole proof construction process. Inglis et al. (2007) also argued this point focusing on the importance of qualifiers; Inglis and colleagues found that the qualifier would signal if the mathematics graduates in their studies were using a non-deductive or deductive warrant.

Authors have dismissed the need for the full model indicating that in mathematical proofs the qualifier is implied as absolute and that a mathematical proof would have no rebuttals. Work in the restricted model tends to try to incorporate the backing into the warrant or assume the warrant is based on deductive reasoning. Considering Dr. B's idea on the own inverse task that he could see from the picture that if a function was one-to-one, continuous, onto, its own inverse, and increasing, then it would have to be $f(x)=x$, otherwise it would not be one-to-one. "I have this nice picture. And on my picture I know, I can see that if I reflect this type of function, it's not going to be one-to-one." Restricting to the ternary model would reduce the full statement to the diagram on the left side of Figure 28. Incorporating the modal qualifier and backing in the diagram to the right of Figure 28 provides the details about Dr. B's personal argument and its progression.

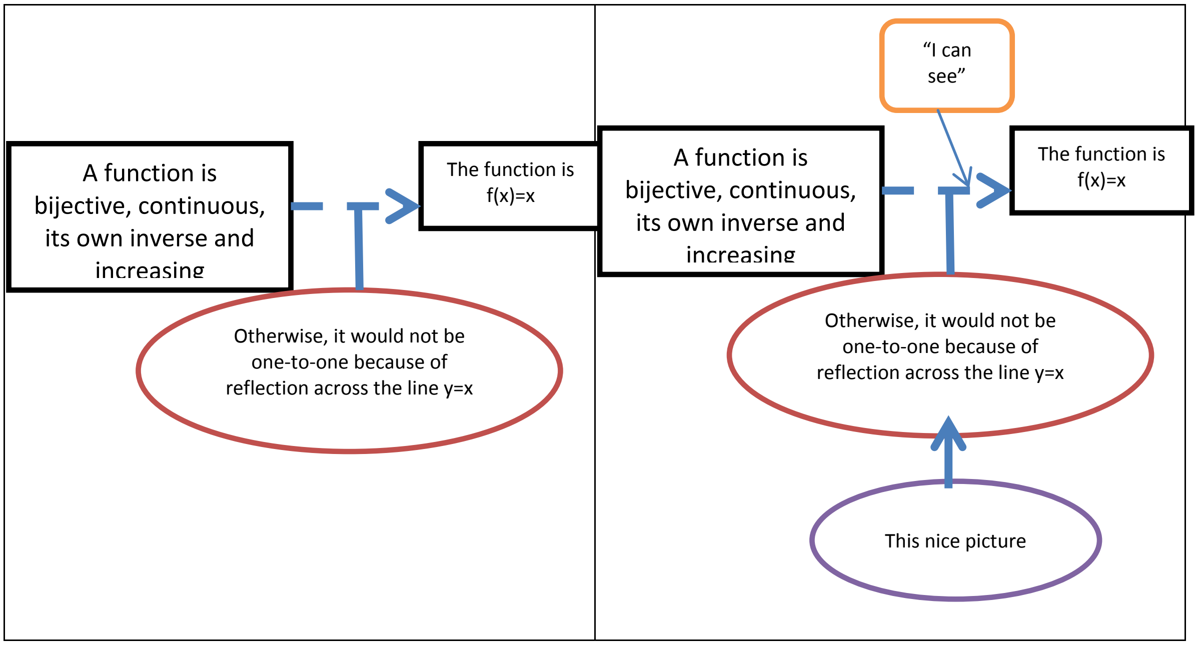


Figure 28. Discernment of Dr. B's argument structure in the ternary versus full Toulmin model.

In addition to providing more information and a better model of the prover's thinking, the full model has explanatory power in that it gave a reason why Dr. B was not able to render this particular argument into an algebraic form. The proposed contradiction that the function would fail to be one-to-one due to reflection is a contradiction that could prove the statement. However, Dr. B's formulation of this contradiction was based on a picture that was not completely general in that it did not capture why the function would always fail to be one-to-one due to its symmetry across the line y=x. Recognizing this, Dr. B abandoned the picture and the one-to-one contradiction and continued on a different path. My understanding of why Dr. B abandoned the one-to-one idea was supported by my specific endeavors to include attention to the backing and qualifiers. I am not suggesting that researchers using the ternary model would ignore or overlook the fact that Dr. B's warrant was based on an example, but if researchers were looking for patterns across large numbers of Toulmin structures, the inclusion of the backing and qualifier could illuminate more patterns explaining the participants' decisions and thinking at various moments while constructing proof.

Related to the recommendation that researchers utilize the full Toulmin model is the recommendation of utilizing language to describe the types of warrants developed. I found adapting Inglis et al.'s (2007) classification of warrant-types to be useful in describing the insights formulated by the participants, how those ideas were tested as the personal arguments evolved, and why participants made the decisions that they did. Earlier in this chapter, I proposed that attending to backing may lend itself to explaining how the understandings based on informal reasoning can be successfully translated into a mathematical proof.

Attending to Ideas that Move the Personal Argument Forward

The invention of the personal argument construct was necessitated by the desire to talk about all the ideas, relationships, concepts, pictures, and so forth that an individual personally judges as important to solving the problem and the perceived relationships and importance of those elements at various points in time. More specifically, it was developed to test and elaborate the hypothesis that there are critical moments where the nature of individual's thinking shifts. Previous proof researchers have used Toulmin models to describe arguments at various moments such as upon the articulation of a conjecture (Inglis et al., 2007; Pedemonte, 2007; Zazkis et al., 2015), at interactive moments within a classroom situation (Prusak, Hershkowitz, & Schwarz, 2012; Wawro, 2011), and upon writing a final proof (Pedemonte, 2007; Zazkis et al., 2015).

Raman and colleagues (2009) have documented that students can come close to a proof from an outsider's (instructor's or researcher's) perspective but not recognize their ideas as useful suggesting that the judgment of whether progress is made is subject to the individual. When investigating an individual's process in constructing mathematical proof alone, we as researchers cannot anticipate all the moments and insights generated that would be significant for the proof construction process. I argue that attending to the moments when ideas are generated that the individual sees as useful breaks the proof construction process into significant events all of which are needed to illustrate the story of the process.

Further Research into the Development and Testing of Ideas

Conceiving of the ideas that move the argument forward as a means of describing the proof construction process is relatively unexplored. Therefore, many avenues of research are open to explore how these ideas develop, how they are tested, and the consequences their development provides for the evolution of the argument. The findings of this study were descriptive and exploratory. Within the realm of these tasks, in the field of real analysis, I found 15 idea-types falling into three major categories according to their function within the conception of the structure of a personal argument. The process of developing and testing these ideas was interpreted within Dewey's theory of inquiry and described in the findings in prior sections.

This research gave descriptive accounts of the tools deemed useful and the purposes and anticipations of the deployment of these tools. Among these tools were conceptual knowledge, rich instantiations of concepts and definitions, the connecting and permuting of known ideas, and the use of examples. Further research into how participants specifically utilized one of these types of tools in developing ideas would provide further insight into the proof construction process. Utilizing a comprehensive framework like Lockwood and colleagues' (2012) categorization of types of examples, uses of examples, and example-related strategies would provide both insight into how ideas evolve but also contribute to the example-use literature. This research would be especially useful if one continued to explore content areas such as real analysis since many studies involving the example-use of mathematicians focus on tasks related to number theory.

Dewey's (1938) theory of inquiry was useful in describing the problem-solving context surrounding the emergence of these ideas. However, other cognitive frameworks may provide more information. For example, instead of describing the problem situation and application of tools surrounding the emergence of ideas, one may be interested in how individuals recognize what knowledge they already possess would be useful and how that contributed to the creation of new ideas. In which case, theories of transfer may be appropriate.

Other variations of studies would include work trying to refine the idea-types or further characterize their development by focusing specifically on one of the idea categories of Focusing and Configuring ideas, Connecting and Justifying ideas, or Monitoring ideas. Varying across mathematical content areas may yield new and clarifying findings to what ideas are useful, how they're developed, and how they are tested, as well as providing insight as to how the proof construction process compares across mathematical content.

Implications for Teaching

Better understanding the practice of professional mathematicians can inform mathematics educators in determining the activities in which students should engage and in making recommendations for goals for student learning. However, precisely how to teach and facilitate the development of these practices and the achievement of these learning goals requires knowing where students start in relation to those goals and what kinds of activities facilitate their progression toward those goals. In the context of the development and utilization of ideas that move the personal argument forward, we know that when constructing proof students struggle in effectively utilizing tools to develop useful ideas (e.g., Alcock & Weber, 2010), may generate new statements or relationships but not recognize their usefulness in moving the argument forward (e.g., Raman et al., 2009), or may utilize ideas in ways not appropriate for the mathematics community (e.g., Harel & Sowder, 1998). More research is needed to recommend best practices for actively facilitating students' abilities to develop, recognize, and utilize the ideas that they can see as useful in moving their arguments toward a mathematical proof. However, I present some preliminary suggestions from this study. .

As found by Carlson and Bloom (2005) in their study of mathematicians engaged in problem solving, I found rich conceptual knowledge greatly influenced the participants' generation of ideas to move the argument forward. There is disagreement amongst those educating mathematics undergraduate students as to whether or not mathematical content should be taught in conjunction with an introduction to proof or if transition to proof courses should only involve content in which all students have workable understandings. Policy documents regarding best practices in mathematics education (CCSM, NCTM) contend that the development of the practice of argumentation should be present in all mathematics classes. Therefore despite where students first learn about proof; instructors of advanced mathematics undergraduate courses should continue to devote some instructional time to developing student capacity to construct proofs. This would not be time wasted as engaging in argumentation can reciprocally facilitate the development of richer conceptual understanding (Harel & Sowder, 2007).

The design of this study was based on the perspective that the formulation and subsequent application of ideas to move an argument forward were acts of creativity that could only emerge from experiencing periods of ambiguity. As such, we cannot expect students to develop the propensity formulate and utilize ideas unless we provide proof construction activities that involve their entering into genuinely problematic situations in the sense of Schoenfeld (1985). Selden and Selden (2008, 2013) assert that mathematical proof has a problem-centered part (the solving of the problem of getting to the given hypothesis to the conclusion) and formal-rhetorical part (unpacking the logical statements and definitions and converting a solution into an acceptable mathematical form), and for students, problem solving can occur in both realms. I suggest that students should experience problem solving in both aspects as the mathematicians in this study not only generated ideas about how to connect the mathematical statements but also about how to communicate their reasoning effectively.

This research suggested that there is value in continuing to teach heuristic strategies for problem solving and proving such as identifying task type, unpacking the logic of a given statement, listing what is known (broadening the statement image), and drawing pictures (to narrow the statement image). All these strategies were employed more than once to result in perceived to be useful ideas. Specifically, they were deployed when the mathematicians were working on the problems of understanding the statement or objects and determining statement truth. Dr. A offered a lesser known heuristic when encountering “prove or disprove” tasks of “being aBeysian”, that is being flexible in moving back and forth between trying to prove or disprove a statement.

When teaching students about proving, it may be of value to attune students to the possibilities of formulating some of the idea-types found in this study so they can hypothesize roles for the ideas that they generate and make evaluations about how to proceed. Having students attend to feelings of personal truth conviction versus feelings of being able to write a proof may require ongoing enculturation of students into the sociomathematical norms of the mathematics community. However, this study suggests attending to and utilizing the reasoning that convinced them of the truth of a statement can be a starting point for finding a warrant that can move the argument forward.

The mathematicians in this study envisioned proof paths and identified necessary conditions while engaging in the task. They also recognized when to purpose elements in their collection of known statements. These ideas necessitated the already explicitly taught practices of reflecting and evaluating as well as attention to understanding the relevant objects and definitions. Opening students to the potential of formulating these ideas that the mathematicians did may provide purpose to the already taught strategies.

Previous literature supports having students engage in working in small groups to solve problems (e.g., Schoenfeld, 1985, 1992) and developing argumentation through classroom discourse and peer interaction (e.g., Prusak et al., 2012). Explicit further research is needed to understand how idea generation and testing patterns would play out in the classroom situation as these findings are limited to an individual context.

Limitations

The findings of the study provided implications for both research and teaching; however, there are limitations to how the findings can be interpreted. I note some of the surprises in data collection and known limitations below including imprecise formulations on two of the tasks that Dr. A worked on, the small sample size, the delimited content area, and the inconsistencies with typical practice for professional mathematicians.

While participants did choose rich tasks for their colleagues to work on, in two cases, the participants provided imprecise formulations of the task statement. One of the formulations of the Extended Mean Value Theorem Task provided by Dr. C did not provide enough conditions for the statement to be true. Dr. B provided a formulation of the additive implies continuous task that he said he knew to be true, but without a necessary condition, the statement was actually false.

I did not identify the issue with the Extended Mean Value Theorem Task prior to presenting the first participant with the task. This participant identified the issue and I rectified the problem for the next participant to work on the task. The first participant also identified that in order for the additive implies continuous statement to be true, the function would also need to be continuous at zero. The participant also identified the formulation of the counterexample would require going outside the realm of real analysis. I decided to rewrite the task to include the necessary condition so that the statement would be true.

As a result of these initially imprecise formulations, participants worked on slightly different versions for two of the three common tasks. Dr. A ended his proof of the Extended Mean Value Theorem for Integrals with a possible rebuttal and not a proof with which he was completely satisfied. Dr. A did eventually formulate a counterexample to his version of the additive implies continuous task but deemed its generation as involving ideas outside the realm of real analysis. Dr. A ended up remembering how to construct the counterexample while on a long driving trip and later wrote it up. This was the only task that had the potential to be correctly solved by a counterexample, and its development was not captured. More research is needed to describe the ideas that would move a personal argument forward when one was developing a counterexample.

An unexpected outcome to the research design of how tasks were chosen and given to participants was that one participant, Dr. A, did not engage in tasks conducive to reasoning outside the formal representation system of proof. Two of the tasks that Dr. A worked on could be solved by making symbolic connections among statements. On the third task, Dr. A worked to both prove and disprove the statement. The work in proving the statement supported by Dr. A's imagining graphical instantiations of concepts and definitions was largely deductive in nature. The interview and Livescribe notebook only captured his memory of the potential of a counterexample and the writing of the formal proof by counterexample. While not a primary focus of this study, the findings related to converting argumentation based on inductive data into formal arguments were based on only two of the participants.

This study does not presume to generalize the findings beyond the context of the interview situation. Since the sample was restricted to three male mathematicians, I provide only descriptive answers to the research questions. I purposefully limited data collection to one mathematical content area, real analysis, to ensure the participants in the study would in fact be able to exhibit expertise not only in proving and problem solving but in proving and problem solving within that realm. Therefore, it may be that themes salient in this study may be less prevalent in other fields of mathematics.

The context of the interview situation provided limitations on the interpretation and generalization of the findings. Working on tasks presented by an interviewer in front of a camera, while speaking aloud, and in the time constraints provided are not representative of a research mathematician's typical practice. The tasks presented genuine problem solving situations for the participants, but they were still "school tasks". As such, participants, informed by their training in school mathematics, brought in their own conceptions about "hints" given by the statement formulation, what were reasonable expectations for a solution, and what theorems they were allowed to assume and which ones they would need to prove. The context was also limiting in that participants worked in isolation. They did not interact with peers and limited utilization of written resources. This is not normative practice for mathematicians or for students.

Further Research

Related to the implications for research noted above, I am interested in designing further studies that utilize the findings of this research effort. Specifically, I wish to continue pursuing how mathematicians develop, test, and reformulate their ideas that Connect and Justify, and I would like to work with others to design teaching experiments where students are guided to move their understandings based on non-deductive warrants into understandings that can be rendered into deductive mathematical proof.

The mathematicians sometimes formulated ideas about what “task type” they were working on and chose either symbolic or informal modes of reasoning in accordance with those judgments In this study, participants pursued or generated syntactic connection warrant-types on two of the seven tasks and inductive warrants and structural-intuitive warrants were found to be useful on multiple, but not all tasks. The formulation of the Own Inverse task, in particular, lent itself to informal argumentations that gave participants a personal sense of why the statement was true but proved challenging to translate into deductive proof. Based on this observation, I hope to design studies that narrow to certain task-types to further study the use and testing of inductive warrants or syntactic connections and how they contribute to the development of a deductive warrant and an eventual formal proof. As part of that research, I would like to further investigate the structural shift where data statements are purposed into warrants or backing as that was seen to be a critical moment for both Dr. A and Dr. C developing an idea that could be translatable into a deductive warrant.

The findings of this research will be useful if eventually, they can inform the teaching of students. It has long been described that a beginner prover struggles with providing arguments based on deductive reasoning as they may find non-deductive arguments to be more convincing and explanatory (e.g., Harel & Sowder, 1998; Healy & Hoyles, 2000; Weber, 2010). More research is needed to lead to a design of a teaching experiment where students are guided to move their non-deductive warrants into deductive warrants that can be translated into a written proof. This would involve instruction where students would encounter a proof problem, formulate an understanding or warrant that they personally find convincing and the teacher and students together test to move students' personal arguments based on non-deductive warrants.

Concluding Remarks

Informed by the writings of mathematicians and mathematics educators that the construction of proof for mathematicians involves the formulation and utilization of ideas this research aimed to provide descriptions of the ideas that mathematicians find useful in moving their arguments forward. More specifically, this research proposed to describe the context surrounding the formulation of those ideas and the subsequent consequences of the incorporation of those ideas into the personal argument.

The utilization of Dewey's (1938) theory of inquiry in conjunction with Toulmin models to describe the progression of the personal argument provided descriptive answers to how ideas are developed and tested and how they were utilized. Mathematicians were found to develop ideas to inform their planning, ideas that served to connect and justify statements, and ideas about their progress. Ideas formulated were tested against their ability to solve the four major problems of understanding the statement or determining truth, providing a warrant, justifying or generalizing a warrant, and writing a final proof. Mathematicians deployed various tools do develop and test ideas including conceptual knowledge, heuristics, exploring specific examples or pictures, and utilizing heuristic strategies.

This work contributes to the proof literature by conceiving of and providing empirical evidence of ideas that move the personal argument forward opening new avenues for research into the process of constructing mathematical proof. Additionally, I describe instances of mathematicians moving their non-deductive arguments into a final deductive proof and provide descriptions of the inquiential activity surrounding those progressions. More research is needed to apply these findings to the introduction to proof classroom, but preliminary suggestions are to engage students in proving activities that present opportunities for problem solving, to continue the teaching of heuristic strategies, and to define some possible purposes of these strategies such as looking for necessary conditions and envisioning a proof path.

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APPENDIX A

PARTICIPANT CONTACT EMAIL

Example Initial Contact Email/ Participation Request

Dear Dr. _______,

I obtained your name from your department as well as the fact that you are currently teaching or have previously taught the course __________________.

As a doctoral candidate in the field of mathematics education, I am interested in observing how expert mathematicians like you engage in the processes of solving mathematical proof problems in the field of real analysis. As such, I would like to conduct a series of 3 interviews with you to learn about how you construct proofs. Two of the interviews will last thirty to sixty minutes, and one interview will take 90 minutes. You will be asked to solve 3 proof problems all together; I will request that you think aloud as you prove and then answer questions about your process after you have completed the proving tasks.

If you are willing and available, we would appreciate your participation. Please let me know if you have more questions.

APPENDIX B

THE EXPLORATORY STUDY

In the spring of 2012, I collaborated with a colleague, Jeffrey King (who will be referred to as King in the rest of this chapter) to conduct task-based interviews with three mathematicians. This data collection and the following analyses served as an exploratory study for the proposed study. The purpose of the phenomenological study was to describe the nature of professional mathematicians' proof construction and proof writing processes. Harel and Sowder (2007) called for a comprehensive perspective on the teaching and learning of mathematical proof; they stated that the goals on the teaching of proof were to "gradually help students develop an understanding of proof that is consistent with that shared and practiced by the mathematicians of today" (p. 47). If these are the goals in teaching proof, then it is necessary to have an accurate and comprehensive understanding of the mathematicians' proof construction processes and the reasoning techniques by which they construct proof. We sought to give insight into these processes by focusing on understanding the mechanisms that lead to new insights. More specifically our research questions in the exploratory study were:

What is the nature of the process of constructing and writing proof for professional mathematicians?

a. What tools and reasoning techniques are used by mathematicians to construct proof?
b. How do professional mathematicians use key idea and technical handle in constructing and writing proof?
c. To what purposes are the identified tools applied?

Theoretical Framing of the Exploratory Study

This exploratory research was guided by the epistemology of social constructivism (Crotty, 1998). We, the researchers, believed we construct knowledge through our interactions with our environments and experiences and these constructions were negotiated in our interactions with society and the community at large. Since this research was intended to explore how individuals think about and talk about mathematical proof under the social constructivist epistemology, interpretivism (Crotty, 1998) was the theoretical perspective for this study.

We applied Dewey's theory of inquiry and tool-use (Dewey, 1938; Hickman, 1990) to analyze mathematicians' proof construction process. Dewey defines inquiry as the intentional process to resolve doubtful situations, through the systematic invention, development, and deployment of tools (Hickman, 2011). A tool is a theory, proposal, or knowledge chosen to be applied to a problematic situation. Throughout the entire inquiry process, there is an "end-in-view" (Garrison, 2009; Glassman, 2001; Hickman, 2009). These ends-in-view provide tentative consequences which the inquirer must seek the means (tools and ways to apply tools) to attain. These ends-in-view may be modified and adapted as the inquiry process proceeds.

The process of active, productive inquiry involves reflection, action, and evaluation. Reflection is indeed the dominant trait. The inquirer must inspect the situation, choose a tool to apply to the situation, and think through a course of action. After this initial reflection of what could happen, the inquirer performs an action, applies the tool. In these actions, the inquirer operates in some way on the situation; she applies a tool to the situation, thus altering it. Reciprocally, during or after the fulfilling experience, the inquirer evaluates the effects and appropriateness of the application of the chosen tools (Hickman, 1990).

We attempted to focus on understanding the mechanisms that lead to new insights. Raman and colleagues (Raman, Sandefur, Birky, Campbell, & Somers, 2009; Raman & Weber, 2006) have developed a model for describing student difficulties for proof production, including the moments of finding a conceptual insight (sometimes termed key idea) and a technical handle. Attaining a conceptual insight gives the prover a sense of conviction and why a particular claim is true. A technical handle is an idea that renders the proof communicable; discovering a technical handle gives the prover a sense of "now I can prove it" (Raman et al. 2009). These constructs characterize moments when the prover creates a new insight, an instance of the invention and deployment of a tool.

Dewey's theory of inquiry and Raman and colleagues' framework of conceptual insight (or key idea) and technical handle provided the primary framing for the exploratory study. As will be described in a later section, the theoretical perspective of the exploratory study needed to be expanded in order to more fully characterize the processes exhibited by the participants. Therefore, a detailed description of the theoretical perspective that will guide the proposed research is given following the description of the exploratory study.

Methods of the Exploratory Study

Participants

The participants, Drs. Nielsen, Heckert, and Kellems (pseudonyms), were chosen according to their diversity among several factors, including years of experience, gender, pure versus applied mathematics research, and primary field of study. Two of the participants were faculty members at a Rocky Mountain region university, which had a department of mathematics that included both mathematicians and mathematics educators. One of the participants was a faculty member at a Pacific Northwest region university, which included mostly research mathematicians but included a small portion of mathematics educators.

Data Collection

Data were collected from a pre-interview questionnaire and task-based interviews (Appendix C). After soliciting participation, we sent an open-ended questionnaire to participants via email. The questionnaire served to elicit routine demographic information, as well as primary field of study, applied versus pure nature of their mathematical research, and teaching experience. The task-based interviews included three proof construction tasks and follow-up questions. These included one task from the field of analysis, one task from the field of abstract algebra, and one task from linear algebra. Having diverse tasks enabled researchers to describe a variety of proof tools and reasoning techniques, as well as the topic-dependence of these tools and reasoning techniques.

To determine the tasks, we generated a list of seven statements requiring proof. The statements came from beginning graduate and upper level undergraduate text books and also tasks used in other studies. We attempted to construct proofs to the tasks individually and piloted the tasks on graduate students studying mathematics education as well as mathematician colleagues. I then created task analyses for each of the tasks noting hypotheses for tools applied to the phases of manipulating, getting a sense, looking for conceptual insights and technical handles, and writing formal proofs. We focused on these aspects because according to Sandefur et al. (2012), during these phases of inquiry in to the development of proof, students were observed to draw upon different tools including the exploration of examples. The three final proofs were decided upon based on varied content area and aptitude for multiple tools used. The tasks are given in the bulleted list below.

- *Linear Algebra Task:* If two 3×3 matrices are similar, then they have the same characteristic and same minimal polynomials.
- *Analysis Task:* Let *f* be a continuous function defined on $I = [a, b]$ *f* maps I onto I, f is one-to-one, and *f* is its own inverse. Show that except for one possibility, *f* must be monotonically decreasing on I.
- *Abstract Algebra Task:* Prove or disprove: S_4 is isomorphic to D_{12} where S_n represents the set of permutations of n elements, and D_{12} the dihedral group with order 24. Note: The members of S_n are bijective mappings from the set {1, 2, . . . , n} onto itself. The group operation in S_n is composition.

We chose tasks from three fields: linear algebra, real analysis, and abstract algebra. The linear algebra task held potential for the participants to use various tools while getting a sense of the situation (equation manipulation, application of definitions, and exploration of examples) and both finding the key idea of the proof as well as developing a technical handle could require the application of various tools as there are multiple ways to solve the problem. We anticipated participants with expertise within the field of linear algebra would use a semantic property-based approach, having a greater interest and more experience. We expected participants whose fields of study were not abstract algebra to pursue more syntactic approaches of equation manipulation due to lack of familiarity or lack of recently constructing similar proofs.

On the analysis task, we anticipated the potential for both symbolic manipulation and the exploration of pictures and examples when participants searched for a conceptual insight. We anticipated the rendering of the ideas into a mathematical proof as potentially problematic.

We chose the abstract algebra task due to its potential to necessitate the use of various tools to enable understanding of the two groups and to discern properties of the groups if the individual is unfamiliar with the two groups in question. We supposed the prover would most likely search for a property that one group had and the other did not that should be preserved under isomorphism and this could require the application of various tools. The order of the tasks was the same for all interviews. Interviews took approximately 90 minutes, where interviewers invoked the think-aloud protocol as described by Patton (2002). The interviewers, King and myself, asked clarifying questions consistent with those described by Weber (2008), as well as reflective follow-up questions pertaining to specific actions taken by the participants or comments they made.

Data Analysis

Interviews were recorded, video-taped, and transcribed, and observation notes were taken by each interviewer. Both interviewers watched all three interviews prior to coding. We noted moments where participants appeared to be applying new tools. In initial analysis, we attempted to describe the tool, the purpose of the tool, and develop codes for each tool. These analyses were not conducive to answering the research questions. Therefore, we developed a coding scheme based on Dewey's theory of inquiry (Dewey, 1938; Hickman, 1990) and Raman and colleagues' characterization of conceptual insight and technical handles (Raman et al., 2009; Raman & Weber, 2006). The coding scheme was largely deductive (Patton, 2002). We included Dewey's theory of inquiry as the research questions regarded individual's tool use in the sense described in that framework. The constructs of conceptual insight and technical handle were included as well due to the purpose of looking for how these ideas are developed and used by participants when constructing proof. Of course, since we had already conducted preliminary analyses, the execution of the coding scheme was informed by the data we had. For example, since instantiation of example objects had been observed, we included potential purposes for the use of examples informed by the literature. If the deductive coding scheme was insufficient in characterizing an event, we generated new codes based on the data. The coding scheme is given in Appendix D.

In applying the coding scheme, we parsed transcripts into "major events," or individual actions or groups of actions involving one purpose or one problem. We coded each major event by type of experience, problem, tool-used, purpose of tools-used, and type of evaluation. We described problems and tools in context for clarity. We then further subdivided major events if we determined that more than one purpose or more than one problem occurred in its duration. Coders added additional codes for problems, tools, and purpose of tools as needed. The researchers coded the interviews individually and then met to discuss and agree upon codings. After we coded the data together, I synthesized the data and provide the summary of results below.

Results from the Exploratory Study

The main themes that seemed to emerge from the data were that there is a progression of problems, and ideas may stem from "failed" tool applications. Mathematicians seemed to progress through the pattern of trying to understand a statement, look for a reason why the statement would be true (key idea), looking for a reason why the statement would be true that could easily be translated to a formal proof (key idea that can be translated to a technical handle), and looking for a means to communicate the proof. The articulation of ideas often required re-checking the arguments that originally convinced them. Also, ideas that the prover deems useful may emerge from actions taken even if the outcomes of the action do not match the mathematicians' expectations.

The Progression of Problems

As consistent with the problem solving literature (Carlson & Bloom, 2005), the mathematicians began their proof constructions by working to understand the statement to be proven and the definitions of the terms given in the proof statement by symbolizing the statement, or instantiating the objects in questions via graphical examples. For example, in the linear algebra task, participants gave instantiations of the term *similar* by giving their own definitions using symbolizations. From the codes, it was apparent that the participants began each problem first attempting to manipulate the premises they were given in order to get a sense of the mathematics they were using. Then, participants generally began applying tools with the purpose of looking for a sense of belief and insight into the reason why the statement is true or false or a key idea.

As a case of this, after reading the analysis task, Dr. Kellems stated, “What I’m puzzled by is why it has to be decreasing.” We interpreted this as articulation of a problem of not knowing why the statement should be true. Dr. Kellems applied the tool of turning “it into a geometry problem” by drawing a picture. He informed his picture by his conceptual knowledge of what it would mean pictorially for a continuous function to be one-to-one and onto: “it can’t go up and down” and for a function to be its own inverse: “it has to be symmetric when I flip it over the line.”

He deemed his picture as fitting the hypothesis and the conclusion but he still needed “to think about why that’s true.” He then drew the identity function and deemed that it was the one exception. Finally he drew another picture including a single point and the line $y = x$. He applied the found property of knowing the one exception and his graphical conception of the graph needing to be symmetric about the line $y = x$. Using these ideas he reasoned,

> I have to have the geometric reflection of that point on my graph, and that forces it to be decreasing, because when I flip a point, well, if I flip a point above that y equals x line across that line, it moves to the right and down. And so there's a geometric argument that it has to be decreasing.

After this geometric reasoning, he asserted, “I think I’m done.” He shifted into articulating an informal argument that consisted of captioning his picture with the thoughts he articulated previously. He made the active choice to not construct an algebraic argument.

> You're asking a very tough question for me, because I would at the moment regard it as a challenge to make this geometric argument rigorous without switching to an analytic argument. And that's because I repeatedly get students in calculus classes and most recently last week in a non-Euclidean geometry class who give me a beautiful geometric argument, short and sweet, start a new paragraph, and say, “mathematically this means that” and give me the algebraic argument. I have no quarrel with saying algebraically or analytically this means that. But to claim

> that the geometric argument is not mathematical really bugs me. And so motivated by that, I would attempt to avoid doing what you're pushing me towards, which is to write down the analytic argument.

Once participants acquired a key idea (or conceptual insight) about the problem, they switched to applying tools with the purpose of looking for technical handle, or a way to communicate the proof (Raman et al., 2009; Raman & Weber, 2006). Consider the following example. Dr. Nielsen had been working on the analysis task.

> Ok so, on the other hand if we start here and if we do something like that, can we make it be its own inverse? It just has to be symmetrical about that point. Ok, now at least I believe the statement. [pause] Ok so it must be monotonically decreasing, so now what could I do to give a proof of that? Well, I could try to just kind of do a straightforward thing, say let c be less than d, and I want to show that, see decreasing, that f of c is greater than f of d.

Dr. Nielsen had grasped a key idea by generating pictures of graphs. His idea was that no matter how the graph looked, if a one-to-one function was not monotonically decreasing, it would not be its own inverse. When he switched to asking himself how to give a proof of that idea, we interpreted it as moving to searching for a technical handle. He proceeded to inquire into symbolic statements to communicate his pictorial idea.

For two of the participants, Dr. Heckert and Dr. Kellems, converting a pictorial or numeric argument to an analytic argument was not always deemed necessary. On the analysis task, Dr. Heckert claimed he would not have continued with a proof beyond his confirmation with a numeric example if we had not asked for such. As described earlier, on the analysis task, Dr. Kellems claimed he was finished after his pictorial argument and did not need to write any more for his geometric argument to be an acceptable proof.

Ideas May Stem from "Failed" Tool Applications

The linear algebra task asked participants to show two similar 3×3 matrices would have the same characteristic and minimal polynomials. Dr. Heckert set about the problem of determining why the statement should be true by thinking through a number of possible tools. He contemplated the feasibility of using the following tools: 1) if he had a theorem for determinants, 2) if he could argue that the polynomials are a property of the transformations in the change of bases, or 3) if he could "do something about row reduction preservation". However, he did not deem any of the tools as being immediately helpful. He transitioned into working a numeric example where he generated two 2 by 2 similar matrices and went about computing their characteristic polynomials to see why they would be the same. He wrote out what the characteristic polynomials were for each of the two matrices; however, the polynomials did not turn out to be the same. Dr. Heckert decided that he must have committed a computation error which was later confirmed by the researchers when analyzing his written work. Dr. Heckert chose not to inquire into where the error occurred because "even without looking for my error, I don't think that that's promising to do it from the definition." (He had viewed manipulating example instantiations of the definition of "similar matrices" as working from the definition.) Dr. Heckert thought back on his past idea on how the polynomial is a property of the transformation itself, but he asked himself, "Why would you even believe that?" He then looked back upon the polynomials which were written in standard form.

> Think of the polynomial as a property of the transformation itself. So how would you make that a proof? Why would you believe that? Certainly it's clear that if this thing has roots, oh I see, so you could go to a field where all the roots are there and then you would do it by eigenvectors - or eigenvalues. Eigenvalues have to be the same, okay, alright.

He recognized the roots would be eigenvalues and that the eigenvalues would be the same for the two matrices. He determined he could construct a proof by reasoning about the eigenvalues. Dr. Heckert then constructed an argument that proved the two characteristic polynomials would be the same, if all the eigenvalues of the matrices were distinct. Dr. Heckert then constructed an argument that required the eigenvalues to be distinct in order for his proof to work, and claimed that he did not care enough to worry about the case where the matrix held repeated eigenvalues.

At the time of formulating the idea of arguing by eigenvalues, Dr. Heckert had chosen to argue that the polynomials were a result of a property that was independent of the bases of the matrices. He had entered into solving the problem of finding a warrant for such a claim. On his paper, he had an example of a characteristic polynomial that was previously generation in the specific example exploration. He referred to that characteristic polynomial when he stated, “Certainly, it’s clear if this thing has roots.” It seemed that the failed example did in fact play a role in his formulation of the idea. Prior to trying the example he did articulate the thought that he may be able to argue that the characteristic polynomial of a matrix was dependent on the linear transformation and independent on the bases, but he abandoned it because he could not immediately identify some sort of backing for his claim. He appeared to have the conceptual knowledge that two similar matrices have the same eigenvalues. He did not, however, mention eigenvalues until he had written out the characteristic polynomials of his example matrices that he had earlier determined were “not going to be useful”. Upon completion of the problem Dr. Heckert explained why he originally determined the example would not be useful.

> It's not because the example didn't work out. It's because there's nothing promising in the pattern here. There's nothing that's telling me why this operation is connected to this operation. I mean for getting the 32, the $\lambda^2 - 5\lambda$. There's no easy way for me to see why that's going to be the same in these two cases.

The example was not useful for his purpose of deducing some structure from the operations performed. The example was useful for finding a warrant for the argument that the characteristic polynomials resulted from a property that was independent of the bases in that having the polynomial written out may have brought to mind other uses for characteristic polynomials, namely the determination of eigenvalues. At the time of the pilot study, we did not follow up with Dr. Heckert to confirm that the written polynomial played that role that we hypothesized. This episode does point to how a tool may be deemed unhelpful for one purpose but may aid in the generation of new ideas if used for a different purpose.

It also bears noting that Dr. Heckert drew upon a significant amount of content knowledge. His instantiations of what it meant for two matrices to be similar included (a) similar matrices occur from changes of bases, (b) similar matrices have equivalent eigenvalues, and (c) similar matrices determine the same linear transformation. Despite the rich content knowledge, the task appeared to be problematic. The content knowledge may have informed him as to why the statement probably should be true; he immediately called to mind that the characteristic polynomials were probably a consequence of some intransient property of a change of basis. However, he needed to perform some investigation in order to determine what that property would be.

Discussion of the Exploratory Study

The exploratory study set about to describe what tools and reasoning techniques were used by mathematicians to construct proof and to specifically describe how the mathematicians used the tools of key idea (or conceptual insight) and technical handle. We observed participants using a combination of semantic and syntactic reasoning when developing proofs. They relied on significant content knowledge, notably rich instantiations of definitions. They drew of pictures and explored examples to familiarize themselves with the mathematical statements and to search for reasons why the statement would be true. Notably the tools applied were meant to address the problems of understanding a statement, looking for a reason why the statement would be true (key idea), looking for a reason why the statement would be true that could easily be translated to a formal proof (key idea that can be translated to a technical handle), and looking for a means to communicate the proof. We, at times, found that participants developed an initial idea that could have been viewed as a conceptual insight but then they searched for additional ideas that could help them communicate why the statement would be true.

For two participants, translating a geometric (or pictorial) argument or an argument using a numeric example to an analytic or formal, deductive argument was not always deemed necessary. On the analysis task, Dr. Kellems made it a point to stop at his geometric argument. Dr. Heckert felt satisfied enough with his argument on the linear algebra task that did not generalize to all cases to not pursue further, and he only wrote a generic, analytic argument for the analysis task because he knew we wished it. Finally, we found that Dr. Heckert explored a pair of similar matrices to determine why they should have the same minimal and characteristic polynomials. Making a computation error and determining the exploration as not useful, he chose to abandon the exploration. However his doing the exploration informed his ways of thinking about the situation which in turn led to a useful idea that he used as the technical handle of his proof.

We at times found that ideas participants found useful were not easily classified as either technical handles or conceptual insights. In the next subsections, I elaborate on issues faced with this particular classification and other observations from the exploratory story that specifically needed more exploration, which I propose to address in this study. Additionally, I discussed some observed limitations in the design of the exploratory study and follow up on how I intend to circumvent these limitations in the proposed study.

Lesson Learned 1: Conceptual Insight (Key Idea) and Technical Handle Constructs

In the exploratory study, the goals were specifically to describe how mathematicians developed and used conceptual insights and technical handles. We found, however, that those two constructs' definitions were not clear enough to be utilized in ways that we could find standards of evidence that would clearly identify an idea as a conceptual insight or a technical handle. We defined a conceptual insight as an idea that gave the prover insight as to why the statement would be true. A technical handle was defined as an idea that enabled the prover to communicate the argument. We were not sure if these two constructs should encompass all insights that were granted from exploring the objects. For example, in Dr. Heckert's construction of the proof of the analysis task, the exploration of the numerical example provided a structure for his final argument. However, it is unclear what would have been classified as the technical handle. In the abstract algebra task, Dr. Kellems spent time exploring the object, the group S_4, it was unclear whether he was looking for a conceptual insight (a reason why S_4 would not be isomorphic to D_{12}) or if he was just familiarizing himself with the object. Either way, he was looking for ideas that would help him proceed into solving the task.

A conceptual insight may appear to be articulated however it is not always enough to convince the prover or it may not be utilized if it is not apparent how it can be rendered into a formal proof. For example, in the abstract algebra proof, Dr. Heckert cited possible reasons why the two groups, S_4 and D_{12}, would not be isomorphic: D_{12} has a non-trivial center and S_4 does not. However, he did not view this idea as being easy to prove. He instead proved the theorem via the argument that D_{12} has an element of order twelve but S_4 does not.

Obtaining a technical handle was not always viewed as necessary, especially if the mathematician felt a geometric argument was sufficient. As described earlier, Dr. Kellems chose to use his informal, pictorial argument as proof citing it as a geometric argument. He did note that translating the argument into a symbolic one would be difficult for him. Dr. Heckert, similarly, stated that on the analysis task, he only wrote out a detailed, symbolic argument for the sake of the interview.

Interviewer: Well after that you sort of said, okay now I guess I'll go through the details of doing this up. Were you thinking that way because you were pretty convinced by..

Dr. Heckert: Oh yeah, I was totally done here in my mind.

Interviewer: So, as a proof, when you're thinking of this as a proof, is that sort of saying like okay because you feel like you have a good sense of the answer or is it because you know if you are trying to present this to somebody else,

Dr. Heckert: Well because there's so many different levels of proof. If I'm trying to get students to be able to go from the concept to the proof which is hard then I really want to do more of this to show them the details but yeah I mean if I was working with somebody and we wanted to check if are we off base here, is this right? We would have been done back there.

Interviewer: So yeah, even if you were doing research, you would have been like good we're done.

Dr. Heckert: Oh yeah, and we wouldn't have written the stuff down. Yeah so this is all about trying to have people be able to write it or have you watch how I would write it up, but I didn't, I didn't need any of this to convince myself from here.

Interviewer: So since you're in an interview with us you're like okay I'll write it up, but if you were by yourself you wouldn't write it up then.

Dr. Heckert: Oh yeah yeah yeah.

Dr. Heckert recognized that "going from the concept to the proof" is a difficult task for students, but he stated that in his personal research, he would not worry about writing the detailed proof for a small result such as the one that we asked him to prove. Conceptually convincing oneself of the truth of a statement would be what mattered. We did not follow up on what situations would promote the need for the writing of the formal proof beyond this conversation; it appeared that the audience of the proof mattered when it came to if he would write out the details of the proof. For the proposed study, it may be important to choose tasks that necessitate the mathematicians' writing of some form of formal proof or to specify that the goal is to have them to work to not only understand the concept and why the statement should be true (the conceptual insight) but also to work to write out the details of the proof so that a student of the course in question would be able to read and understand it.

In algebraic arguments, it was sometimes unclear if an idea was a conceptual insight or a technical handle. Dr. Nielsen completed the linear algebra task by discerning certain manipulations that would enable him to get the equivalence, $\det(A - xI) = \det(P^{-1}AP - xI)$. It is unclear whether we should have classified the manipulations as conceptual insights or technical handles as the manipulations both gave him insight as to why the statement would be true as well as a means for writing out the proof.

The identification of conceptual insights (or key ideas) and technical handles was not always a straight forward process. Additionally, there seemed to be insights gathered by the individuals that were meaningful to the participant but did not fall into the two categories. The proposed study will look into the ideas that the mathematician deems as useful even if it does not fulfill the specific purposes of giving insight as to why a statement is true or direct insight into how to prove the statement without focusing on the categorization into conceptual insights and technical handles.

Lesson 2: Example of an Evolving Personal Argument

Even though describing the evolution of the personal argument was not the purpose of the exploratory pilot study; the data provided insight into how I could describe the progression of an argument into a mathematical proof. I will discuss Dr. Heckert's construction of the proof for the analysis task, and I will explain how certain ideas seemed to emerge that Dr. Heckert viewed as important and helpful, and provides a rational for conducting the proposed dissertation study.

Upon reading the problem, Dr. Heckert drew a picture of a function that matched the problem's criteria. The purpose appeared to be for him to familiarize himself with the situation and to confirm that such a function exists. He stated, "So there is such a function." The domain and range of his initial instantiation was [0, 1]. He then began to draw another set of x and y axes and placed brackets indicating an arbitrary interval because he was not certain the "one exception" could be found on the interval [0, 1], but soon he hypothesized that the interval did not matter. He noted that he had not yet thought of the one exception, "I can't imagine what the exception is now but maybe when I prove it, I'll find it."

He stated his plan would be to prove the statement by contradiction beginning with the assumption: There exists *a*, *b* such that *f(a)* is greater than or equal to *f(b)* and *a* < *b*. He determined proving by contradiction would be useful; therefore it is an idea that can move the proof forward. We did not follow up on his decision process for choosing to prove by contradiction; so we could not determine what about the situation contributed to the choice to argue by contradiction.

On his picture from before (the x and y axis with the arbitrary interval), he drew the line "y = x". After the interview, he stated that he drew the line because the conditions of the statement were that the function be its own inverse and that meant the function would need to be symmetric along the line "y=x": "Well, this is because it's its own inverse. I just wanted it to be symmetric about that and I knew it had to take the interval into itself." After drawing the line, he identified it as the one exception mentioned in the statement of the problem. "Oh, ha! That's the one exception. Okay, so, I don't know why that suddenly came from that picture."

The discovery that the identity function was the one exception was another idea that moved the proof forward. Dr. Heckert was not actively engaged in solving the problem of determining the one exception when he had the idea; he had previously put the problem aside. At the time he was beginning to draw a picture to inform his argument by contradiction. He actively checked each assumption of the problem statement to confirm that the identity map was indeed the one exception.

He returned to his picture, informed that he would be excluding the identity map from the contradictory argument. "Okay, I see if we're not the identity map, we have *a* here and *b* here. And there's something wrong with that because if *f(a) = c, f(b)=d,* the problem with that then is *c* and *d* have to map in reverse. These two connect in some way…It's going to violate one-to-oneness in some way."

The discovery that he would not be considering the identity map informed the conditional. "It seemed clear somehow that what was going on that if I can't stay on that line, then the fact that I'm my own inverse is going to do that to me. And that's going to be the basis of the one-to-one violation."

Here, we see an idea that if the function increased and deviated from the line y =x (did not "stay on that line") on some portion and the function was its own inverse, then the function would not be one-to-one. After the interview, Dr. Heckert stated that this drawing had contributed to a vague picture in his mind of what would happen. "I didn't have this picture in my head yet, but I was picturing that it would do this."

So, it appeared that the personal argument contained a mental picture that the function would have to change direction leading to a violation of one-to-oneness. It is unclear from the transcript if Dr. Heckert had the mental picture prior to his physical drawings or if it came simultaneously with the drawing. If he had the picture before the drawing, the physical actions enabled him to confirm and articulate his hypothesis. If the mental picture emerged as he drew the physical picture, then the picture aided in the generation of the idea. It is probable that the mental picture coincided with his manipulations of the drawing because Dr. Heckert felt the need to "go to the numbers and come back." Later, he explained why he went to the numbers. "Partly, I'm tired and the letters were irritating me, and I wanted to make sure that I was putting them in the right place."

He moved to a numeric example. He created a situation where $f(2) = 1$ which was less than $f(3)=4$. He worked through the situation, finding that $f(1) = 2$ and $f(4) = 3$ since the function was its own inverse. He then applied the Intermediate Value Theorem (IVT) to the situation to get the result he wanted, namely that the function would not be one-to-one. “In this case, we’re already done because by the intermediate value theorem something between hits this point and something between here hits that point.” Prior to working through the numeric example, Dr. Heckert did not mention IVT, but we do not have evidence that he was not thinking of it specifically in the previous mental pictures. Regardless, IVT became a technical tool to ensure that the function would need to pass through a given y-value twice and hence violate one-to-oneness.

At this point, it appeared Dr. Heckert’s argument included his vague picture which was confirmed by his numeric example and backed by the intermediate value theorem. He was personally convinced. He knew what the basic idea of the proof would be.

> So the basic idea of the proof is that if we are not the identity, what happened here is going to happen. We’ll have the intermediate value theorem. Any point in here could be here. Your counterexample to being one-to-one. So something in here has to hit that point, and something in here has to hit that point. Since it’s continuous. So, I’ll write up the annoying details on this one.”

Dr. Heckert transitioned his inquiry to writing an analytic argument that generically translated his mental and numeric pictures into a formal proof. He began by changing his numeric example to one involving general values a and b. In his numerical example, $f(a) < a$. He chose to argue by cases and this was his first case. Even though he described this step as writing up “the annoying details”, the task required him to think, going back to his pictures in order to describe what was happening.

For the analytic argument, he decided to argue by cases. Sometime during the inquiry into the proof whether f(c) was above the identity function or below it became important or useful to him. In the numeric example, he had made f(a) be below the line and f(b) be above the line. The first case maintained that f(a) was below the line, but he only stipulated that $f(b)>f(a)$. So, whether f(b) was above or below the line did not appear to matter. He maintained the use of IVT, just as in the numeric example and the pictorial example that preceded it. In fact he formalized the use of the theorem by using it as a warrant for his claim that the function will violate one-to-oneness. ("So any y between...f(a) and a is the image of some point in [c, a].") Alternatively, the formalized use of the theorem may have been backing for the warrant that the function would need to violate one-to-oneness.

From here, Dr. Heckert continued to work on formulating an argument that incorporated the ideas of (1) arguing by contradiction that one-to-oneness would be violated, (2) the IVT would be necessary to show this violation, (3) using the case where $f(a) < a$ (which was informed by his picture that assumed the function was not the identity). Dr. Heckert identified that the second case where $f(a) > a$ would have an argument “identical to this one.” He stated the third case where $f(a) = a$ was the case of the one exception.

Dr. Heckert’s argument did evolve. He began with a picture or instantiation of the situation to convince himself that such a function existed. He happened across the one exception, but recognizing it helped him to think about cases that were not the identity giving him a picture that a function that was not decreasing and its own inverse would have to violate one-to-oneness. So he knew he would argue by contradiction showing the violation of one-to-oneness. Moving to a numerical example was a means of giving a structure to how the proof should go. While working through the numeric example, he articulated the use of the IVT. His generic argument was a translation from the numeric example into general terms; in his generalizing, he saw the need to differentiate between cases: where $f(a) < a$, $f(a) > a$, and $f(a) = a$. This particular episode highlights the need for exploring the experienced mathematician's proof construction in-depth with a focus on personal argument evolvement.

Lesson Learned 3: A problem is an Individual Determination

Despite having more content knowledge than the students for which the tasks were originally written (upper level undergraduates and lower-level graduates), the mathematicians did seem to engage in genuine inquiry. We had evidence that the individuals did need to reflect upon the problem situations and for the analysis and linear algebra tasks did not immediately know why the statement would be true, and, in the case of the analysis task, Dr. Nielsen and Dr. Heckert both needed to spend time thinking about and trying different approaches to converting their conceptual understandings of why the statement would be true to a symbolic proof.

Dr. Nielsen and Dr. Heckert did not appear to view the abstract algebra task as problematic. Both Dr. Nielsen and Dr. Heckert knew that it would be easier to show the two groups were not isomorphic, most likely through experience. They both appeared to go through a checklist, checking the order of the group, noting properties they knew about the group, and finally choosing to use the fact that D_{12} has an element of order twelve but S_4 does not. The creation and emergence of ideas for these two individuals was not apparent to us because they were able to apply techniques that they had used in the past. Dr. Kellems did not appear to view the linear algebra task as problematic. He began by trying what he termed and advanced formula for determinants.

> I know a property of 3 by 3 determinants that you probably don't, which I'll write down here, which is that for 3 by 3 matrices, the determinant of A is a triple product, it's a trace of something called the Freudenthal product and a Jordan product. So this is fancy algebra that I use in a completely different context, which is really the statement in traditional linear algebra terms that the determinant is related to, and again I'll get this wrong as I've not done basic linear algebra in a long time, but something like the minor, I don't even know how you write the minor. But the determinant is A times something.

He applied the tool of the advanced property but could not recall all the coefficients of the formula. Therefore, he switched to the definition of characteristic polynomial being the determinant of the matrix. *A-xI.*

> And so I know that A cubed minus A squared trace A plus some coefficient which I don't remember the name of, A minus determinant A equals zero. So I'm going to switch gears, and start all over. I'm being much too fancy. Characteristic polynomial... and I'll do it for A is determinant A minus x I *[*$det(A - xI)$*]*. And I know what A is; so that determinant of P inverse B P minus x I *[*$det(P^{-1}BP - xI)$*]*, and that's determinant P inverse B P minus x P inverse P *[*$det(P^{-1}BP - xP^{-1}P$*]*, and now I'm getting to the same place I was trying to get without all the fancy. This comes from doing determinants of 3 by 3 matrices over non associative algebras which is related to my research, so that's why that came up first.

After applying this definition, he seemed to know the manipulations he needed to perform in order to get the equality $\det(A - xI) = \det(B - xI)$ because he did not appear to propose manipulations. Breaking the identity matrix, I, into $P^{-1}P$ seemed merely natural. He stated that the algebra technique was one he used in many contexts and was second nature for him. Although he viewed that we presented the problem as algebraic, the manipulations he performed had conceptual meaning for him.

> That is an algebraic technique that I have used in many contexts, and therefore is second nature for me. In addition, and you heard me use that language from the beginning, you have presented this as a straight algebra problem, but I see immediately that it is about invariances of linear transformations under change of

> basis. And the reason that linear, that that um... well in this case polynomials, that this category of problem is invariant under change of basis is precisely because of that algebraic property. And so for me, the connection between that algebraic statement and the um, the invariance that is associated with the, the geometry if you will of that invariant statement, those are things that are very imbedded in my thinking of linear algebra.

Dr. Kellems's professional practice had dealings in matrix algebra, Dr. Nielsen and Dr. Heckert's research did not reside in that area. Dr. Heckert did not produce an algebraic argument but spent time thinking about properties of linear transformations and eventually gave an eigenvalue argument that worked only if all the eigenvalues were distinct. Dr. Nielsen did give an algebraic argument, but he needed to explore algebraic manipulations and properties before arriving at the manipulation that gave him the equivalence he sought.

A task that is problematic for one mathematician may be routine or familiar to another as observed in the exploratory study. Problems are an individual construction, and the individual is the one who can determine if the task is problematic or not. Professionals in a given field of mathematics may be able to identify if a problem is one he or she can solve easily or if it will require him or her to reflect upon the situation and think through proposed actions. For these reasons, selection of tasks for the proposed study needs restructuring.

Lesson Learned 4: Units of Analysis

The protocol for coding involved looking at each action taken, noting problem perceived at that time if any, the purpose of the action, and the participants' evaluation of the action as being helpful for achieving said purpose. The analysis resulted in a list of actions which was useful in answering the question of what tools are applied. However, using actions as the unit of analysis would not be useful in fulfilling the research purposes of describing what ideas the individuals perceive as useful and in what ways the ideas are used. The proposed study needs to focus on the moments where ideas emerge as opposed to noting what the tools are.

Limitation: Presence of the interviewer. In the pilot study, we performed interviews using the think-aloud protocol (Patton, 2002). We chose to have participants think aloud because we were interested in their thought processes as they occurred. We chose to be in the room with the mathematicians as they worked so that we could observe their actions in real time. By focusing on their actions as they occurred, we strived to minimize having pre-formed hypotheses as to the motives for the mathematicians' actions. The format of the interview did allow for questions from the researchers while the participants were working. Our interrupting questions were limited and usually occurred during a break in the participants thinking or when they were switching between plans of attack. The purposes of these questions were to bring closure to the abandoned plan of attack and to learn motivations for moving on to a new plan. The protocol also included follow-up questions that occurred at the end of each task. We had made notes during the task about what we wanted to ask about as they occurred.

Our questions or the think-aloud protocol may have played a role in how the problem solving evolved. In the linear algebra task, Dr. Nielsen had worked on showing the characteristic polynomials of two similar matrices were equal, but he had reached an impasse. He moved on to the second part of the problem that required him to show the minimal polynomials of the similar matrices were equal. He completed that portion by discerning a property of taking powers of matrices. He returned to the characteristic polynomial portion. One of the interviewers asked him if moving to the minimal polynomial gave him any clarity about what to do for the characteristic polynomial. Dr. Nielsen responded that he did not see something directly related and thought about abandoning the algebraic approach and switching to thinking about it conceptually.

> But there's a big difference; characteristic polynomial you're not plugging in the matrix in for the variable, in for x here, and the minimal polynomial you were. I definitely used that fact here. Perhaps a better reason to move on to minimal polynomial case was just to give myself some time to not think about the other one for a while. And I think that maybe now I would stop doing it algebraically and use facts that I may or may not remember correctly about characteristic polynomials and try a different approach. And they say something or try something like I know something about the roots of the characteristic polynomial and eigenvalues and I know something about similarity there, and that might be a more reasonable approach to try to answer the first one. But that would definitely take some trying to remember my basic linear algebra facts. I don't know if there is anything else. I suppose I could look at the algebraic one more time. It definitely, would think it should work.

However, Dr. Nielsen then decided to look at the algebra "one more time" because "it should work." He thought about the problem algebraically again noting that *A-xI* was a matrix that he wanted to relate to the matrix, $P^{-1}(A - xI)P$ which was an approach different from those he had tried earlier for the characteristic polynomial but related to the manipulations he learned from his explorations of the minimal polynomial as he had earlier discerned that for any matrix *A,* $(P^{-1}AP)^k = P^{-1}AP$. The interviewer's questions that asked Dr. Nielsen if he perceived that the minimal polynomial portion would help with the characteristic polynomial portion may have played a role in Dr. Nielsen's decision to think more about the algebra. His articulation of the situation, specifically, that he would need to think about some possibly forgotten facts from linear algebra and that perhaps there should be similarities between the two pieces may have contributed to his not pursuing the conceptual approach. After completing the task, he did state that he realized that it would be more work to think about properties of eigenvalues.

> I didn't see immediately how doing this sort of a... how using the second problem helped me with the first problem at all. So I really just said ok well, this isn't going to work, I gotta try something else. Although, a second later I realized this did help me. And I was able to use the same trick to get the other one algebraically. And the reason I probably came back to that is I realized that it was going to be perhaps more work to go look up properties of eigenvalues, to remind myself of those facts... It seems like that it was actually harder than I thought it would be.

It is not certain that the nature of the protocol or the question influenced how Dr. Nielsen thought about the problem, but in the future study, I would like to minimize the possibility that the researcher would influence the problem-solving process.

A limitation of the protocol of the exploratory study was that the questions asked during the interview were restricted to what the researchers noted at the time. We concurrently were focusing on the actions and comments made by the participant. We missed opportunities to ask about interesting aspects. Patton (2002) described this limitation of the task-based interview. After viewing the video, transcripts, and written work from Dr. Nielsen's linear algebra task, I noted actions and situations where it was unclear what the purpose of the action was or what he viewed as problematic. For example, Dr. Nielsen proposed a property of minimal polynomials that he said "would be nice". For $p(x)$, the minimal polynomial of a matrix B. Dr. Nielsen was trying to show $p(P^{-1}BP) = 0$. He stated it would be nice if $p(P^{-1}BP) = p(P^{-1})p(B)p(P)$ but then created and worked through a counterexample which eventually led to a discernment of a property involving taking powers of the matrix, $P^{-1}BP$.

> But I'm not sure if we can do that easily at all, because we might have things like P inverse of B P squared plus stuff equals zero. And when you do that, that's doing P inverse B P, so that looks good. Those would be identity matrix, so you would get P inverse B squared P plus stuff equals zero. So that's exactly the thing that we want.

$$(P^{-1}BP)^2 + \cdots = 0$$
$$P^{-1}BPP^{-1}BP$$
$$P^{-1}B^2P + \cdots = 0$$

He extracted the property $p(P^{-1}BP) = P^{-1}p(B)P$. During the interview, we were preoccupied by the fact that Dr. Nielsen discovered a property that allowed him to get the result he wanted. We did not ask him clarify his purpose in creating the above counterexample that led to the property. In analyzing the event, it was unclear if the purpose of exploring the example was to convince himself why his proposed property was untrue or if he had already convinced himself that it was not true and was squaring it out to find some other insight into the problem. Understanding the purpose of his exploration of the counterexample would be important to answering the research questions of the proposed study, specifically to answer the questions about the situations surrounding the emergence of new ideas. It is likely that situations like the above, where individuals perform actions without articulating the purpose of the actions, would occur in another task-based interview. In the current study, I will seek methods that encourage the mathematicians to clarify purposes behind actions taken but minimize potential interference with their problem-solving processes.

Limitation: Graduate Student, Faculty Member Relationship

The participants of the pilot study were professors and the interviewers were graduate students. At times, it appeared that the mathematicians were teaching the interviewers as opposed to articulating their thought processes. The graduate student to faculty member relationship coupled with the researchers' introduction statement to the interview, "We are interested in understanding how you construct proof," may have contributed to the mathematicians taking on a teaching role during the interview.

Rephrasing the introduction to the interview to "I wish to document your thought processes as you construct proof." may help in preventing the "teaching" issue.

Additionally, the mathematicians perceived that there were actions that the student researchers were hoping that the participants would take. The linear algebra task required mathematicians to prove a property of 3x3 matrices. Dr. Heckert suspected that we had crafted the item so that participants would move to exploring the simpler case of 2 by 2 matrices which is just what he did. "Okay so I do have to do what you probably want me to do, which is- So I don't like the three by three because that's too much work, so I'm going to go to the two by two." The other participants also may have suspected there were choices that we expected them to make.

Limitation: Interview Situation Influencing Inquiry Choices

Because of time considerations, the interview situation may have prevented inquiry into certain unresolved issues and influenced what situations were deemed problematic. Dr. Heckert's proof in the linear algebra task showed that the characteristic polynomials of two similar matrices were equivalent only in the case that the matrices had distinct eigenvalues. Dr. Heckert stated that he was not very concerned with resolving the issue for repeated eigenvalues. "If the eigenvalues aren't distinct, I'm not sure I care much, but do you care? But if the eigenvalues are not distinct, I don't care. Because now there's probably some trick I should think about and I probably would just look in a linear algebra book. But, um, but this is the idea."

On the analysis task, Dr. Kellems gave a picture and an oral argument; it served the purpose of communicating his conviction to the interviewers so there was no need for him to write out his proof. He did state that it would be difficult to translate the picture into a symbolic proof. His argument was informal in that he pointed to and motioned with his pictures as he spoke through the argument, only writing down his conclusions. The fact that the interviewers were in the room with Dr. Kellems may have played a role in his choice not to give a formal written argument.

Significance of the Exploratory Study

Despite the above limitations, the results of the exploratory study provide some implications for the teaching and research of proof construction. Participant mathematicians were observed to begin by orienting themselves to the problem, persuading themselves of the reasons why the statement should be true, searching for ideas that may be rendered into a proof, and then finally proceeding to write a formal proof. As such, practitioners may consider introducing proof as the development of an argument. Instead of solely teaching students to write arguments regarding known content and teaching proof techniques, it would be valuable to provide opportunities for students to wrestle with statements whose truth or warrants for truth they do not yet know. As was observed, if a prover is given opportunity to explore the problem, applying various tools, even unsuccessfully, may enable the individual to develop useful ideas.

For the mathematicians, arguments that convinced themselves and proofs that would be acceptable to a general community did not seem to be the same; mathematicians made shifts from convincing themselves to looking for proof ideas. However, they always considered if the ideas that showed them why the statement would be true could be rendered into a formal proof. At times, their ideas could be formalized; other times, mathematicians needed to search for another idea that could be more easily formalized. This confirms the research of Raman and colleagues (Raman, 2003; Raman et al., 2009) who argued that mathematicians are aware of how their personal arguments have the potential to be connected to more formal proof. Encouraging students to search for ways to connect their private arguments to a more public one would be in keeping of developing mathematical habits consistent with the mathematicians of today. It is also necessary to point out to students that some mathematical proofs have more than one possible key idea and some ideas are more easily rendered into proof than others.

As an expansion of the key idea and technical handle framework (Raman et al., 2009; Sandefur et al., 2012), this study has shown that while these two constructs may be observed in the proof constructions of mathematicians, classifying each idea developed or each purpose of an investigation into these two terms can be difficult. It may be more beneficial for researchers to identify ideas that the provers find useful and provide apt descriptions of what the ideas are and how the individual sees them as useful. Instead of classifying the purpose of an action as to search for a conceptual insight or technical handle (Sandefur et al., 2012), researchers may analyze actions surrounding the generations of ideas for their purpose and the problems the applications of such tools they may seek to solve.

This research has served to confirm the ways in which proof construction can be viewed as a special type of problem solving as participants were observed to orient themselves to the task, plan, execute, and check their strategies (Carlson & Bloom, 2005). However, where Carlson and Bloom viewed one task as a problem, we noted multiple problems within one proof construction task as proving requires not just the development of an answer but the development of a personal argument (the development itself may require the resolution of several problems) that also must be translated into a written, logical proof. The study suggests that if researchers seek to describe the development of mathematical proof through the multidimensional problem solving framework of Carlson and Bloom, researchers may wish to consider the existence of multiple problems within the proof construction task. In the following section, I describe the explicit ways in which the exploratory study has informed the methodology of the proposed study.

Implications for Data Collection and Analysis

The lessons learned from and the observed limitations of the exploratory study can inform the design of the proposed study. In this section, I will briefly describe methodological decisions that were directly influenced by the exploratory study including decisions to have the participants choose their own problems, incorporate a follow-up, stimulated recall interviews, and to redefine the units of analysis.

The exploratory study showed that if the researcher chooses the tasks, the tasks may or may not present problematic situations for the mathematicians. For the proposed study, a problem is a situation where the individual cannot recall a situation and resolving the situation requires a "going outside" the situation to think about possible solutions (Hickman, 1990). The definition of problem and problematic is further expanded in the Theoretical Perspective. Only the individual is able to identify a situation as being routine or not. Consequently, the individuals will choose the proof tasks that they will solve.

The presence of the interviewer and the interview situation played a role in how the problem-solving situations unfolded. As opposed to a task-based interview with a think-aloud protocol, the proposed study will involve the researcher video recording mathematicians working on problems as the mathematicians think-aloud. Contrary to the exploratory study, the protocol for the proposed study will not allow for interrupting, clarifying questions or follow-up questions during the proof construction recording. I will act as an observer and recorder of the mathematicians' thought processes. This decision to minimize the researchers' presence is an effort to downplay the possibility that the mathematician will attempt to "teach" the researcher as well as minimize interrupting and changing the mathematicians' thought processes as they unfold. I will still seek to understand the participants' thinking behind their decisions in constructing the proof. This will be partially accomplished by asking the mathematicians to think-aloud as they prove the statements; additionally, I will ask the mathematicians to participate in a follow-up, stimulated recall interview. After the initial, proof-construction interview, I will review the video noting moments where clarification is needed to discern the purposes of actions, what in the situation contributed to choosing an action or not to take an action, what ideas or properties seemed important, and so forth. In the follow up interview, I will play back these moments for the participants, have them watch it, and explain the situation and answer questions I have noted. The stimulated recall protocol will be elaborated in a later section.

The purpose of the proposed research is to describe how personal arguments evolve. For this reason, analyzing by each action taken may not be the most appropriate unit of analysis. Instead, I will begin by noting moments where new ideas are incorporated that the prover sees as useful for moving the argument forward and describe what the ideas are. For each idea, I will describe the problem(s) perceived by the mathematician when the idea is articulated and when it is implemented, the anticipated application of the idea, the other tools that the mathematician used as the idea emerged, and how the idea relates to other ideas. The definition of an idea that moves an argument forward will be further described in the theoretical perspective.

The proposed study focuses on individual's construction of solutions to mathematical proof problems; therefore, theoretical perspective that will be used for this study is a constructivist perspective within an interpretivist framework (Crotty, 1998) utilizing John Dewey's Theory of Logical Inquiry (1938) using argumentation theory (Toulmin, Rieke, & Janik, 1979) and a perspective on creative thinking. The data for this study will consist of data from 3 mathematicians, completing 3 tasks over the course of 3 interviews. Each interview will include a follow-up, stimulated recall of the mathematician's work on the task(s) from the previous interview. Data for this study will be the written work, transcripts, video recordings, and Livescribe recordings of participants' work on the 3 tasks in the interview setting. Participants will complete additional work on tasks outside the interview setting. This work will be recorded via Livescribe pen. Follow-up, stimulated recall interviews will be conducted as well.

The data analyses will proceed in two major phases. The purpose of the first phase of preliminary analyses will be to identify the ideas that move the argument forward, to hypothesize the situations or actions performed by the participant that effected the generation of these ideas, and to create questions for the stimulated recall interview. In the second phase of analysis, I will create descriptions of the individual's proof construction of each task focusing on each idea that moves the argument forward as the unit of analysis. The descriptions will describe the idea, the perceived problem when the idea was articulated, the mode of inquiry entered into when the idea is articulated, and the tools that the participant utilized leading up to the articulation of the idea. Additionally, these descriptions will describe how the ideas are used subsequent to their articulation. After I have given case descriptions of each idea and each task, I will conduct open coding on each task and cross-case analyses across tasks to generate categories and themes to answer the research questions.

APPENDIX C

TOOLS OF DATA COLLECTION FOR EXPLORATORY STUDY

Pre-interview Questionnaire

1. What would you describe as your primary mathematical field of study?
2. What courses have you taught in the past?
3. How long have you been in your current position?
4. Please give a general description of your research, including whether you would consider your research to be applied mathematics or pure mathematics.

Task-based Interview Protocol

- Preliminary Statement
  - Thank you for taking the time to participate in this interview. The purpose of this interview is to explore how you construct and write mathematical proofs.
  - During this interview, we will ask you to construct two proofs. While you are completing these tasks, we will ask you to think aloud. We will also ask clarifying questions as you are working. After you have finished these tasks, we will ask a few open-ended questions.
  - If at any time you feel uncomfortable with the interview, we will stop. Are you ready to begin?
- Observations
  - Evidence of informal reasoning.
- Probing Questions
  - If quiet, ask "What are you thinking?"

  - If participant writes without explaining, interviewer can ask for explanation.
  - Interviewers may ask clarifying questions
- 1st Task – Linear algebra
  - Task: If two 3x3 matrices are similar, then they have the same characteristic and same minimal polynomials.
  - Provided definitions:
    - *Similar*: Two n x n matrices A and B are similar if there exists a matrix P such that $A = P^{-1}BP$
    - *Characteristic*: The characteristic of an n x n matrix A is the polynomial given by *det(A-xI).*
    - *Minimal Polynomial*: The minimal polynomial of an n x n matrix A is the polynomial *p* of the least degree such that *p(A) = 0*.
- 2nd Task – Analysis Task: Let *f* be a continuous function defined on I=[a,b], *f* maps I onto I, *f* is one-to-one, and *f* is its own inverse. Show that except for one possibility, *f* must be monotonically decreasing on I.
- 3rd Task – Abstract Algebra Task: Prove or disprove: S_4 is isomorphic to D_{12}. (S_n represents the set of permutations of n elements, and D_{12} the dihedral group with order 24. Note: The members of S_n are bijective mappings from the set {1, 2, . . . , n} onto itself. The group operation in S_n is composition.)
- Follow-up Questions

- ◦ Can you further explain the use of (picture, diagram, drawing., informal reasoning..)?
- ◦ In what ways do your processes in constructing proof vary across content area?, if at all?

APPENDIX D

EXPLORATORY STUDY CODING SCHEME

Coding Procedures

1. Go through and note major events. These may be individual actions or groups of actions involving one purpose or one problem.
2. Code if the actions are in response to a problem or not. There are 4 recommended codes here. More may be added.
3. Separate into "problem" episodes and periods of non-problems. Give a description of each problem using the initial coding list and adding as needed. Or you may note more specific problems and we may be able to generalize the code from there.
    - Here we will have sections of different inquiries, and non-inquiries
    - We can describe the sequences of problems/non-problems
    - Within each problem we can describe the inquiry involved
    - Within each non-problem, maybe we can describe circumstances that lead to a problem
4. Describe the actions performed utilizing verb and object
    - Identify major tool and how it is applied
        - Note a single tool can be acted upon in a variety of ways
        - Note that other tools may supplement the actions
    - Code the purpose of the action. Use the codes already provided, but you will most likely add to the list or modify/combine codes.
    - Code if evaluation appears to occur or not.
        - If evaluation occurs:

- Code as “planning” or “usefulness” or “both” evaluation
- Code the decision of the evaluation

5. Code the actions (each tool application) as one of the four modes of thinking and note places that are none of the above modes of thinking (we may make a 5th!).

Operating Definitions

Problem/Inquiry

A situation is a problem if it is (1) tense and unresolved for the individual, and (2) the individual acknowledges the issue and begins to reflect on the situation and possible solutions. Sections of transcript will be classified as inquiry, non-inquirential, problem but no inquiry, or other.

- “Inquiry” is happening if the following are occurring:
    - There has been an issue that is deemed problematic
    - The individual has inspected the situation to discern qualities of the situation and has reflected on a tool to apply to the situation with an intended purpose
    - The individual applies the tools chosen with an end-in-view
    - The individual is evaluating (deciding if it’s useful, re-inspecting the situation to see how it changed) during and after the application of the tool
- The situation is “non-inquirential tool-use”; note both experts and novices may have these types of experiences
    - Actions are taken or tools are applied without the individual reflecting on the tool to use

  - The individual indicates that the action taken is "second nature", "what you're supposed to do", "how I usually do it", and so forth.
  - The individual may look back at what the action did for him/her but nothing has been deemed problematic prior to that evaluation
- "Problem but no inquiry" is reserved for situations where:
  - The individual acknowledges that there is something amiss with his/her proof but does not enter into actions to try to solve it
  - The individual may even reflect and describe the problem or the tool he/she would use to resolve it but does not apply the tool and enter the problem

6. We need to have an "other" code. Use it when the individual is not engaged in proof construction

Possible problems. The following is a list of problems that an individual may encounter when proving from the literature and informed from preliminary analyses.

- Don't understand what the statement means
- Unfamiliarity with the mathematical objects
- Unsure if the statement is true
- Trying to find out why the statement is true (looking for a Conceptual Insight)
- Must communicate the argument (looking for a Technical Handle)
- I want to generalize (my arguments, my picture, my example, etc.) (data)
- I want to translate to analytic language (data & lit) [translate back to the representation system of proof
- I want to check to see if what I've claimed (any conjecture) is true

- Problems with individual tools
    - Trouble generating a helpful example
    - Computation issues

Tools

A tool is a theory, proposal, heuristic, or knowledge chosen to be applied to a situation for a specified purpose. To be considered a tool, an object must be used to do some sort of work. Tools may be used in habitual or inquirential actions. A tool cannot be separated with its end-in-view, so note the proposed (perhaps implied) purpose for using each tool. A tool can be applied in different ways; note how the tool is applied. If the subject is engaged in inquiry, make note of evaluating actions made before, during, or after a tool is applied. The following is a list of possible tools we may see.

- Examples
    - Known functions
    - Pictures of functions with specific properties
- Instantiations of mathematical concepts
- Heuristics, algorithms
- Known theorem/property
- Conceptual Knowledge- Knowledge of how things are connected, implications of actions, etc.
- Found theorem/property
- Symbolizing
- Symbol manipulation
- Logical Structure

    - Break into cases
    - WLOG argument

Purposes of Applied Tools

The purposes of applying tools are inseparable from the tools themselves. The purposes are related to the problem to be solved but possible will be more specialized than just "to solve this problem". Note the purpose of using the tool is not necessarily the actual outcome. For instance, one may apply and manipulate a counterexample with the purpose of refuting a conjecture but as a result of the manipulation obtain an insight as to how to communicate the proof. Because of this, unless the participant verbalizes the purpose of using a certain tool, we may need to infer the intended purpose. I can give a preliminary list of possible purposes.

Purposes of using examples. These purposes were specifically identified in the literature and preliminary analyses:

- Understanding- a statement, definition, objects, etc.
    - Indicate what is included and what is excluded by a condition in a definition or theorem
    - Build a sense of what's going on
    - Explore behavior and illustrate structure
    - To indicate a dimension of variation implied by a generalization
    - To indicate something that remains invariant while some other features change
- Evaluate the truth of a statement or conjecture/checking inferences
- Generating arguments

- “Directly” and “indirectly”
  - Directly: trying to show that a result is true in a specific case hoping the same argument or manipulations will work in general
  - Indirectly: searching for a reason why one could not find a counterexample to the statement
- To give insight into proving (looking for a TH)
- To understand why the assertion should be true (Looking for a CI)
- Use a specific object to indicate the significance of a particular condition in a definition or theorem
  - Highlighting the condition’s role in the proof
  - Showing how the statement fails in the absence of that condition

- Generating counterexamples
- As an aid to explain an argument to another (interviewer, student, …)

Purposes of other tools. The following purposes may be observed for other tools.

- Use aspects of mathematical structure (logic, theorems, definitions)
  - To drive the steps of the proof
  - To reduce the complexity
  - To start or structure the argument
  - To inform the manipulations
- Look for a TH: if the individual is seeking a way to make their argument communicable

- Look for a CI: Looking for a conceptual insight is performing actions to gain an insight as to why the statement is true (why examples should work the way they do, why it is not possible to find a counterexample)
- Generalize an argument
- To articulate or communicate an idea
- Translate an idea from one representation to another

Application of Tools

The application of tools indicates the course of action or how the tool is used. One tool can be applied in multiple ways. Choosing how to apply a tool occurs during periods of reflection in a similar way to how a tool is chosen. The application is an experiment that is evaluated and may be modified. When coding applications, most likely you will note the action word associated with the tool or the specific way the tool is used. For instance, an individual may use the tool of logic and formulate a contrapositive of the statement. The tool was logic. The way it was applied or course of action was formulating a contrapositive statement. For the application of examples the literature suggests a pattern in the application.

Examples as tools. If the tool is an example, the hypothesis is that there is a sequence of manipulating, getting a sense of a pattern, and articulating that pattern (MGA). The "manipulating" indicates the various ways that the example is applied to the situation, or the actions that are taken on the example. These are numerous but may include: experimenting (testing against a conjecture), transforming (perhaps while imagining), identifying properties, performing algorithms on, and so forth. Manipulations of these objects may require the application of tools such as conceptual knowledge, structural knowledge, known mathematical properties, and other tools. When one encounters a mathematical object and manipulates it, he or she continues the manipulation until he or she gets a sense of a pattern from the manipulations carried out. Getting a sense is an outcome of a manipulation. It may occur in an instant, but the sense may be vague and require further manipulation in order for the sense to be articulated. An articulation is a representation of the perceived pattern recognized from the manipulations. An articulation may be verbal, diagrammatic, or symbolic. Articulation may occur instantly once one gets a sense, or one may need to perform further manipulations to articulate. If one attains a vague sense of a pattern from the manipulations performed, but is unable to articulate it then "to articulate" becomes a purpose of the inquiry. According to the MGA framework, the sense that is articulated then becomes a mathematical object which can be manipulated in successive cycles since the MGA process is seen as helical. So an articulation may be a **tool** in further inquiry.

Coding application of examples. We will assume applying the tool "using examples" or "specializing" will involve a MGA cycle or cycles if the individual is actually engaged in inquiry. We note how the individual is manipulating the object. We note evaluations occurring during and after manipulating. The vague "getting-a-sense" and articulations may be a subsets of the evaluations.

Evaluation

There are two types of evaluation: planning and usefulness. The evaluation for usefulness is the posing of the questions: How well did the tool or plan of attack work to resolve the initial problematic situation? or How well did this tool help me achieve my intended purpose? Planning poses the question: what does the application of this tool tell me about how to proceed? Planning may entail a re-inspection of the situation to determine how the problem has changed, if it is resolved, or if there is a new problem to address. Instances of evaluation can occur *before applying a tool, while the tool is being applied, and after the tool has been applied*. Before applying a tool, an inquirer considers if applying a tool is feasible or will be useful; she thinks through possible plans of attack. While the tool is being applied or after it is applied, "the worth of the meanings, or cognitive ideas, is critically inspected in light of their fulfillment (Prawat & Floden, 1994, p. 44)." It may be difficult to always observe this. Evaluation may be a thought process occurring simultaneously with the actions performed. The following are evidence of evaluation:

- Verbalization- "that helps me."; "that's not working"; "that's not the problem";etc.
- Periods of quiet thought followed by a course of action or an observation about the problem situation- evidence of planning
- Periods of quiet thought followed by a CHANGE in tool or application- evidence of usefulness evaluation
- BUT evaluation may be a thought process that we don't observe.

The following note possible decisions of *usefulnes*s evaluations:

- Tool application is effective
    - Problem is still unresolved, but I'll keep applying the same tool in the same way.
    - Problem solved: The tool was applied with satisfaction and the problem is no longer an issue. Remember there may be many "problems" within one

proof task." Also "solved" is a judgment made by the prover not the observer or the mathematics community; it is possible that the solution is mathematically incomplete or even incorrect. The prover moves on to the next task which may or may not become a problematic situation. Dewey indicates the resolving of a problem can be accompanied with a feeling of satisfaction or enjoyment and an attainment of new knowledge.

- Tool application is ineffective, Problem is unresolved: When the problem is unresolved, the cycle starts again. The prover inspects the situation including the original problem and the tool itself. She then makes a planning evaluation as to the action to take next. Choices made could be:
    - abandonment of the previous tool and choosing a new one
    - choosing to apply the same tool in the same or a different way
    - determining the tool is problematic and engaging in inquiry to resolve the problem with the tool (this is most likely to happen if evaluation is an interruption during the "fulfilling experience")
    - Exiting inquiry (giving up without satisfaction)
- Problem changed: It may be that the application of the tool changed the nature of the situation for the individual causing something else to be problematic or interesting. In this case, the prover may not deem the situation as problematic and therefore go back to "everyday" experience where the situation may become problematic. The prover may be aware of the problem but choose not to enter inquiry in the sense of a moral judgment (Dewey, 1938). The prover may enter

the problem, beginning a new cycle of inquiry in which the tools, and the knowledge constructed in previous inquiries are available if deemed useful.

Mode of Thinking

The modes are ways of doing tools, or subsets of tools used for specific purposes.

- Instantiating is the attempt to meaningfully understand a mathematical object by thinking about the objects to which it applies. It is using or generating instantiations for the purpose of understanding.
    - Code as instantiating if:
        - Purpose is to understand something (object, statement, definition, etc.)
        - Tool used is an instantiation (example, alternative definition, intuitive conception, etc.) of a mathematical object
- Creative thinking entails examining instantiations to identify a property or set of manipulations that can form the key idea of a proof. The purpose of the creative thinking may be to gain a critical insight, to illustrate the structure of the mathematical objects, to show that the result is true in a specific case, or to search for a reason why one could not find a counterexample.
    - Code as creative thinking if:
        - Purpose is to gain insight into the "crux" of the proof
        - Tools used are instantiations (alternative conceptions of the mathematical objects, examples, etc.)
- Structural thinking uses the form of the mathematics to deduce a proof. Structural thinking employs syntactic reasoning. The tools used in structural

thinking will most likely be known properties and theorems, algebraic manipulations, and the logical structure of mathematics. Additionally, Alcock notes how structural thinking may inform instantiating and creative and critical thinking.

  - Code as structural thinking if:
    - The tools used are related to the "representation system of proof" in the sense of Weber & Alcock (2004)
    - Purposes will most likely vary

- Critical thinking has the goal of checking the correctness of assertions made in the proof. This may occur either syntactically or semantically.
  - Code as critical thinking if the purpose is to check an assertion. Various tools may be used.

Definitions of Tools Used in Exploratory Study

Table 34

Codes Used for Type of Experience in Exploratory Study

Code/Category	Description	Example
Inquiry	There has been an issue that is deemed problematic. The individual has inspected the situation to discern qualities of the situation and has reflected on a tool to apply to the situation with an intended purpose. The individual applies the tools chosen with an end-in-view. The individual is evaluating (deciding if it's useful, re-inspecting the situation to see how it changed) during and after the application of the tool	"Why would that be true? I can't think of a property of the determinants why that would be true, so maybe I'll play with an example to see if I can discern what would make it work that way."
Non-inquirential Tool Use	Actions are taken or tools are applied without the individual reflecting on the tool to use. The individual indicates that the action taken is "second nature", "what you're supposed to do", "how I usually do it", etc. The individual may look back at what the action did for him/her but nothing has been deemed problematic prior to that evaluation.	"Okay we're moving towards things that I can do quicker than other things. ... Um, so it's certainly not true. And um D12 has a non-trivial center and S4 doesn't." [The knowledge of the group structure is a tool used to give a conceptual insight, but it required no inquiry to be found.]
Problematic no Inquiry	The individual acknowledges that there is something amiss with his/her proof but does not enter into actions to try to solve it. The individual may even reflect and describe the problem or the tool he/she would use to resolve it but does not apply the tool and enter the problem.	"This proof isn't finished because there's some annoying little detail that I don't want to deal with."
Other	Use this code when the individual is not engaged in proof construction.	Instances where they are answering our questions, making a demonstration for our benefit, etc.

Table 35

Codes Used for Problems Encountered in Exploratory Study

Code	Description	Example
Understanding the statement	Doesn't understand what the statement means, how the objects in the statement relate, etc.	
Unfamiliar objects	Doesn't understand a definition or an object described in the statement in the proof	
Veracity of Statement	Unsure if the statement is true or not	
Why true	Trying to find out why the statement is true (looking for a conceptual insight)	
Communicating/Articulating/ Generalizing	Attempting to communicate/articulate an argument, insight or thought; this could be looking for a technical handle if they are trying to articulate/communicate the proof	
Translating	Trying to translate back to the representation system of proof (formulate an argument in analytic language)	
Checking	Checking to see if an assertion is true	
Tool Problem	Problem with individual tools or application of those tools, i.e. trouble generating a helpful example, computation issues, etc.	

Table 36

Codes Used for Tools Used in Exploratory Study

Code	Description	Example
Examples (specializing)	Watson and Mason (2005) describe an *example* as a particular case of any larger class about which students generalize and reason, and they describe *exemplification* as using something specific to represent a general class with which the learner is to become familiar.	Specific numbers, representations of functions (graphical, analytic, table), pictures of objects.
Instantiations of concepts	Non-formal representations of mathematical concepts	Function as shooting objects from one location to another.
Heuristics/algorithms	Rule of thumb, technique that comes with experience	Computational shortcuts, modeling
Symbolizing	Rewriting statements, definitions, or representations in terms of symbols	Rewriting "the determinant of A equals the determinant of B" as det(A)=det(B)
Known theorem/property		MVT, monotonically increasing functions are one-to-one
Conceptual knowledge	Knowledge of relationships among mathematical objects, consequences of actions on objects, or mathematical structure	
Found property (trick)	A property/insight/manipulation that was proved or found by the prover	Conceptual insight, technical handle,
A proposal/hypothesis	Dewey tells us an idea/hypothesis that is tested is a tool	Proposed manipulation of symbols
Logical structure	Knowledge of logical structure	WLOG arguments, formulation of a statement, etc.

Table 37

Codes Used for Purposes of Using Examples in Exploratory Study

Code	Description	Example
Understanding	Understanding- a statement, definition, objects, etc. Indicate what is included and what is excluded by a condition in a definition or theorem. Build a sense of what's going on. Explore behavior and illustrate structure. To indicate a dimension of variation implied by a generalization. To indicate something that remains invariant while some other features change.	Start-up examples
Evaluate the Truth	Choosing specific objects to see whether the assertion held for those objects.	Testing a small number of examples. Testing a "critical" example.
Generating Arguments	Using examples to gain insight into building the proof. Directly: trying to show that a result is true in a specific case hoping the same argument or manipulations will work in general. Indirectly: searching for a reason why one could not find a counterexample to the statement. To give insight into proving (looking for a TH). To understand why the assertion should be true (Looking for a CI). Use a specific object to indicate the significance of a particular condition in a definition or theorem. Showing how the statement fails in the absence of that condition.	Prover computes the characteristic polynomial of two generated similar matrices hoping the computation will illuminate why the characteristic polynomials must be equal
Generating counterexamples	Generating counterexamples to prove a statement (conjecture) is false	
Explaining	The prover uses the example not as an aid to explain an argument to another (the interviewer, student, etc.)	"Let me show you why this works"

Table 38

Evaluation Codes Used in Pilot Study

Code	Description	Example
Evaluation	There are two types of evaluation: planning and usefulness. The evaluation for usefulness is the posing of the questions: How well did the tool or plan of attack work to resolve the initial problematic situation? or How well did this tool help me achieve my intended purpose? Planning poses the question: what does the application of this tool tell me about how to proceed? Planning may entail a re-inspection of the situation to determine how the problem has changed, if it is resolved, or if there is a new problem to address. Instances of evaluation can occur before applying a tool, while the tool is being applied, and after the tool has been applied.	Evidence of evaluation: Verbalization- "that helps me."; "that's not working"; "that's not the problem";etc. Periods of quiet thought followed by a course of action or an observation about the problem situation- evidence of planning Periods of quiet thought followed by a CHANGE in tool or application- evidence of usefulness evaluation BUT evaluation may be a thought process that we don't observe Planning: *"Is it easy from here? From the definition?"*
Decision-Tool is effective	Problem is still unresolved, but I'll keep applying the same tool in the same way.	"I think that helps me"
	Problem solved: The tool was applied with satisfaction and the problem is no longer an issue. Remember there may be many "problems" within one proof task." Also "solved" is a judgment made by the prover not the observer or the mathematics community; it is possible that the solution is mathematically incomplete or even incorrect. The prover moves on to the next task which may or may not become a problematic situation. Dewey indicates the resolving of a problem can be accompanied with a feeling of satisfaction or enjoyment and an attainment of new knowledge	

Table 38

Code	Description	Example
Decision-Tool is ineffective	Tool application is ineffective, Problem is unresolved: When the problem is unresolved, the cycle starts again. The prover inspects the situation including the original problem and the tool itself. She then makes a planning evaluation as to the action to take next. Choices made could be:	
	abandonment of the previous tool and choosing a new one	"This is not helping, what else can I try?"
	choosing to apply the same tool in the same or a different way	
	determining the tool is problematic and engaging in inquiry to resolve the problem with the tool (this is most likely to happen if evaluation is an interruption during the "fulfilling experience")	"I can tell that's not going to work, what else can I try?"
	Exiting inquiry (giving up without satisfaction)	"maybe I'll move on to the next problem"
Decision-Problem Changed	Problem changed: It may be that the application of the tool changed the nature of the situation for the individual causing something else to be problematic or interesting. In this case, the prover may not deem the situation as problematic and therefore go back to "everyday" experience where the situation may become problematic. The prover may be aware of the problem but choose not to enter inquiry in the sense of a moral judgment (Dewey, 1938). The prover may enter the problem, beginning a new cycle of inquiry in which the tools, and the knowledge constructed in previous inquiries are available if deemed useful.	

Table 39

Modes of Thinking Codes Used in Exploratory Study

Code	Description	Example
Instantiating	Instantiating is the attempt to meaningfully understand a mathematical object by thinking about the objects to which it applies. It is using or generating instantiations for the purpose of understanding. Code as instantiating if: Purpose is to understand something (object, statement, definition, etc.) Tool used is an instantiation (example, alternative definition, intuitive conception, etc.) of a mathematical object	Draws coordinate axes and a function that is decreasing and concave down that is its own inverse; adds y=x. Then states "so there is such a function"
Creative Thinking	Creative thinking entails examining instantiations to identify a property or set of manipulations that can form the key idea of a proof. The purpose of the creative thinking may be to gain a critical insight, to illustrate the structure of the mathematical objects, to show that the result is true in a specific case, or to search for a reason why one could not find a counterexample. Code as creative thinking if: Purpose is to gain insight into the "crux" of the proof Tools used are instantiations (alternative conceptions of the mathematical objects, examples, etc.)	"So I have, we'll make the interval zero to five. I have f of 2 less than [looks back at previous writing] f of 3. Let's say this is, uh, one and four. So one, and three, four. But then, that says that f of four is, f is its own inverse so f of four is three. And um, in this case, we're already done because by the intermediate value theorem something between hits this point and something between here hits that point. [36:30] So the proof is now a matter of collecting the right places for that to happen. So the basic idea of the proof is that if we are not the identity, what happened here is going to happen."

Table 39 continued

Code	Description	Example
Structural Thinking	Structural thinking uses the form of the mathematics to deduce a proof. Structural thinking employs syntactic reasoning. The tools used in structural thinking will most likely be known properties and theorems, algebraic manipulations, and the logical structure of mathematics. Additionally, Alcock notes how structural thinking may inform instantiating and creative and critical thinking. Code as structural thinking if: The tools used are related to the "representation system of proof" in the sense of Weber & Alcock (2004) Purposes will most likely vary	"Okay so there's I can probably do this without a lot of cases, but um looks like it might be helpful to say case 1 is a is less than…" Is breaking into cases to form the argument.
Critical Thinking	Critical thinking has the goal of checking the correctness of assertions made in the proof. This may occur either syntactically or semantically. Code as critical thinking if the purpose is to check an assertion. Various tools may be used.	"I think that a one-to-one function on an interval attains its maximum and minimum on the endpoints. But I should probably try to prove that."

APPENDIX E

INTERVIEW PROTOCOLS

Interview One

“Thank you for agreeing to participate. The cameras here will record your writing from above as well as you from the front. I will have you write in the Livescribe notebook with the Livescribe pen.

“Our purpose here is to observe you solving mathematics challenging proving tasks. I believe that only you can determine which tasks would be challenging or genuinely problematic and which tasks are not. So you will choose the tasks you will work on. [Researcher hands the professor the book or the professor takes out the book.] Which book are you using? In what course did you use it? How long have you known this book? Please take some time going through the book. Identify one task that you find challenging and one task you believe other mathematicians in your field may find challenging. By challenging, I mean that upon reading it, you do not recall a solution or have an immediate sense of how the proof should go. This may be a theorem that needs proving or a numbered exercise that you would reserve as a challenge problem for your students. [Allow up to 15 minutes for the professor to choose a task. If they deem all the tasks in their book as non-problematic, offer the token textbook.]

“When you identify a task, read it aloud and explain your initial perception of the problem. [Possible follow-up questions: What is your initial inclination of how to solve the problem? How does this problem differ from some of the other tasks surrounding it in the book? Had you encountered this task in designing your course or assigning homework to your students?]

[Have participants choose one task and show it to us. Then have them find a second task that he or she perceives as potentially problematic for a peer. Ask the participants to choose from among the two tasks the one he or she finds challenging.]

“I ask that you think aloud so that I can observe as much as possible. Please articulate what you are thinking including what you find challenging about the problem. It is okay if you cannot articulate what exactly is the issue. If at any point during the problem, you find the task is no longer problematic, please let me know when that happens. Indicate points when you start seeing the situation differently, or any shifts in your perspective of the problem. [Allow up to an additional 20 minutes for the participant to solve. Explain to the participant the amount of time allotted.]

“It looks like we are out of time, but I will request that you continue to work on the task on your own time in this week prior to our next meeting. As during the interview, I ask that you write in this notebook, and think-aloud as you write. You may choose to write out your thoughts in the notebook as well. I would like you to note either in speech or in writing the moments where your perceptions of the situation shift. If you think about the problem while you are away to the notebook, please take time to return to the notebook and record what you thought about.” [Make copies of the work created in the interview. Transfer the Livescribe files to personal computer. Send participant with notebook and pen.]

Interviews Two and Three

Hello, again. This interview will consist of two parts. The first will be reviewing the task you worked on previously; the second will consist of you working on a new task.

Stimulated Recall

Use the laptop to play back chosen sections of the interview. Explain to the participant that he/she can pause the video at any time.

> I have chosen intervals of our last session that I would like us to watch together. I will pause the tape at certain moments and ask you to explain your thinking and the decisions you made. Describe your choices of action and why you chose it. Please elaborate on any element of the situation. I would rather you were honest and say little about a choice you made rather than create an explanation. During playback, feel free to pause or rewind and replay the tape to explain the situation as you see fit.

[Ask for questions. Play back chosen video. Ask questions that will be generated in analyses. Include follow-up questions when appropriate of the form: “Say more about that.” Record time stamps of when the participant stopped, paused, or rewound the video.]

Review of the Participant’s Individual Work

The participant will bring his or her Livescribe pen and notebook to the interview and describe his or her thought processes throughout solving the problem.“Thank you for continuing to work on the task. Where did you work? For how long? Did you think about the task outside of the time you were sitting down with the notebook?

Did you solve the task to your satisfaction? If so, describe the moments when you realized what you were trying would work. If not, explain some things you tried and the results of trying them.”

[If the participants describe moments of gaining new insights or ideas that they viewed as moving their argument forward, ask them about the results of incorporating this idea.]

Work on Task Chosen by Another Participant

Provide the participant with the written proof task. Have available relevant definitions and instantiations of definitions (Alcock, 2008). Provide context to the task including the name and author of the book and the chapter from which the task was derived.

"I ask that you think aloud so that I can observe as much as possible. Please articulate what you are thinking including what you find challenging about the problem. It is okay if you cannot articulate what exactly is the issue. If at any point during the problem, you find the task is no longer problematic, please let me know when that happens. Indicate points when you start seeing the situation differently, or any shifts in your perspective of the problem." [Allow up to an additional 20 minutes for the participant to solve. Explain to the participant the amount of time allotted.]

"It looks like we are out of time, but I will request that you continue to work on the task on your own time in this week prior to our next meeting. As during the interview, I ask that you write in this notebook, and think-aloud as you write. You may choose to write out your thoughts in the notebook as well. I would like you to note either in speech or in writing the moments where your perceptions of the situation shift. If you think about the problem while you are away to the notebook, please take time to return to the notebook and record what you thought about." [Make copies of the work created in the interview. Transfer the Livescribe files to personal computer. Send participant with notebook and pen.]

APPENDIX F

RESULTANT CODEBOOK UPON OPEN CODING

Table 40

Problem Types Encountered by Participants

Problem Code	Description
Understanding statements or objects	The participant does not understand what the statements mean or the definition of a object described in the statement of the proof or how the objects in the statement relate and is entered into working understand
Determining truth	Prover is engaged in determining the truth value of the statement
Looking for warrant	Prover is looking for a means to connect the statement to the claim that eventually can be rendered into a proof. I f participants specifically are searching for conceptual reasons why the statement is true or are seeking to connect statements via a symbolic manipulation, then the next two codes were used.
Looking for conceptual reason why true	Prover endeavors to find why the statement is true based on conceptual or empirical understandings
Looking for way to connect symbolically	Prover endeavors to find means to directly connect symbolic instantiations of statements
Looking for way to communicate/generalize	Prover is engaged in finding a way to communicate or generalize an argument, warrant, backing, or other idea
Looking for backing for previous idea	Prover is engaged in finding general or generalizable support for a posed idea or claim
No problem	Prover applies tools or actions without needing to reflect on choice. The individual indicates that the action taken is “second nature”, “what you’re supposed to do”, “how I usually do it”, etc. The individual may look back at what the action did for him/her but nothing has been deemed problematic prior to that evaluation.
Tool problem	There is an identification and entrance into solving a problem with individual tools or application of these tools, i.e. trouble generating a helpful example, computation issues, etc.

Table 41

Classifications of Tools Utilized by the Participants That Contributed to Idea Generation

Tool classifications	
Conceptual knowledge	Knowledge of relationships among mathematical objects, consequences of actions on objects, or mathematical structure
Known theorem	Specific use of a theorem known to be true
Connecting and permuting	Attending to connecting and rearranging previously generated ideas, definitions, and related concepts
Instantiations and equivalencies	Alternative or Non-formal representations of mathematical concepts or definitions
Symbolizing	Rewriting statements, definitions, or representations in terms of symbols
Symbolic manipulations	Actions on symbolic representations
Example	a particular case of any larger class about which students generalize and reason, and they describe
Heuristics and experiences	Rule of thumb, technique that comes with experience
Logical structure	Knowledge of logical structure and the norms o behaving and communicating in the mathematics community
Other	Time, disturbances in the situation, outside resources, etc.

Table 42

Purposes of Using Tools of Examples

Types of examples utilized	
Examples to understand	A specific example is used to understand a statement, definition, objects, etc.; indicate what is included and what is excluded by a condition in a definition or theorem; build a sense of what's going on; explore behavior and illustrate structure; indicate a dimension of variation implied by a generalization; indicate something that remains invariant while some other features change.
Examples to test	Choosing specific objects to determine i f an assertion held for those objects
Examples to generate a warrant	Using examples to gain insight into building proof: • Directly by trying to show that a result is true in a specific case hoping the same argument or manipulations will work in general, OR • Indirectly by searching for a reason why one could not find a counterexample to the statement, OR • To understand why the assertion should be true, OR • To indicate the significance o f a particular condition in a definition or theorem OR • To generate a counterexample
Examples to articulate or explain	Using an example to more clearly articulate one's own sense of an idea or to explain one's own idea or argument to another or oneself

Table 43

Shifts in Toulmin Structure of Personal Arguments

Structural Shifts in the Personal Argument	
Opening structure	The structure that the participant begins with when articulating the first idea
Claim changed or specified	The claim of the argument is either changed to a new claim or delimited in some way
Sub-claim added	In addition to attending to justifying the central claim, participants add new claims to proven
Data added, extended or specified	New statements are incorporated into the set of statements that the participant deems as relevant or existing statements are extended to new cases or existing statements are reformulated
Data statements repurposed	Given statements or previously generated ideas are purposed in the argument as claims, warrants, backing, or MQ/rebuttals
Data removed	Previously perceived relevant statements are removed
Warrant added, changed or removed	Warrant is added if none previously existed, replaced by a new warrant, or eliminated as a potential link between statements
Backing added, changed, or removed	Backing statements are incorporated if none previously existed, replaced, or eliminated
Qualifier or rebuttal changed	Qualifier or rebuttal is typically implicit or not present, this code notes when one is specified or removed
Order of presentation	The relevant statements are not changed or deleted but are rearranged or combined with other claim structures
None	No changes

APPENDIX G

INSTITUTIONAL REVIEW BOARD APPROVAL

UNIVERSITY *of*
NORTHERN COLORADO

Institutional Review Board

DATE:	May 27, 2014
TO:	Melissa Troudt
FROM:	University of Northern Colorado (UNCO) IRB
PROJECT TITLE:	[606488-1] Mathematicians' Evolving Personal Arguments: Ideas that Move Proof Constructions Forward
SUBMISSION TYPE:	New Project
ACTION:	APPROVED
APPROVAL DATE:	May 27, 2014
EXPIRATION DATE:	May 27, 2015
REVIEW TYPE:	Expedited Review

Thank you for your submission of New Project materials for this project. The University of Northern Colorado (UNCO) IRB has APPROVED your submission. All research must be conducted in accordance with this approved submission.

This submission has received Expedited Review based on applicable federal regulations.

Please remember that informed consent is a process beginning with a description of the project and insurance of participant understanding. Informed consent must continue throughout the project via a dialogue between the researcher and research participant. Federal regulations require that each participant receives a copy of the consent document.

Please note that any revision to previously approved materials must be approved by this committee prior to initiation. Please use the appropriate revision forms for this procedure.

All UNANTICIPATED PROBLEMS involving risks to subjects or others and SERIOUS and UNEXPECTED adverse events must be reported promptly to this office.

All NON-COMPLIANCE issues or COMPLAINTS regarding this project must be reported promptly to this office.

Based on the risks, this project requires continuing review by this committee on an annual basis. Please use the appropriate forms for this procedure. Your documentation for continuing review must be received with sufficient time for review and continued approval before the expiration date of May 27, 2015.

Please note that all research records must be retained for a minimum of three years after the completion of the project.

If you have any questions, please contact Sherry May at 970-351-1910 or Sherry.May@unco.edu. Please include your project title and reference number in all correspondence with this committee.

Dear Ms. Troudt,

Below is feedback from the first reviewer, Dr. Montemayor, who recommended his approval prior to my review:

I am pleased to serve as the first IRB reviewer for your research proposal, "Mathematicians' Evolving Personal Arguments: Ideas that move proof constructions forward." If I read this correctly, this seems to be your dissertation research. Congratulations on reaching this stage of your program!

Congratulations, too, on crafting a very fine narrative for your application. I really appreciate the level of detail with which you addressed every aspect of the proposal (i.e., background, data handling, etc.) And on a personal note, having read through your planned investigation on this topic, I must say -- this research seems fascinating to me!

Seeing no concerns regarding the protection of human participants, I am happy to recommend approval for your work. Your application will now proceed to the IRB co-chair for her final review.

(A minor point: Please know that the IRB co-chair may request that you use a slightly "cleaner" graphic for the UNC logo on your consent forms. I believe you can find templates for this on the IRB website.)

Congratulations again, and all best wishes for your work!

Sincerely,
Mark Montemayor, Ph.D.
School of Music (music education)
IRB member

Following review of your application, I also have no requests for revisions, modifications or additions. Well done on your succinct and thorough application. Please do consider use of a better UNC letterhead graphic through the UNC website. There is no need to resubmit your consent form for further review.

Best wishes with your research. Please don't hesitate to contact me with any IRB-related questions or concerns.

Sincerely,

Dr. Megan Stellino, UNC IRB Co-Chair