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Face-wise Chromatic Number

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Abstract: The chromatic number is a well-studied graph invariant. This is the smallest number of colors necessary to color all the vertices such that no two vertices adjacent to the same edge are the same color. It has a myriad of applications from scheduling problems to cartography. Here we consider what happens when we color vertices with respect to faces instead of edges. That is, two vertices adjacent to the same face must not be the same color. We call this invariant the *face-wise chromatic number* (fwcn). We will see how to compute the fwcn for a variety of graphs and look at connections between the fwcn and the classical chromatic number.

Keywords: *face-wise chromatic numbers*

1 INTRODUCTION

The origins of Graph Theory can be traced back to August 26th, 1735 when Leonhard Euler presented the Königsberg bridge problem to his colleagues (see Figure 1). The Königsberg bridge problem asks whether or not it is possible to cross each of the seven bridges of the town Königsberg exactly once. Euler was certain it was impossible, but there wasn't a valid proof until 1873 when Carl Hierholzer proved it impossible using diagram-tracing puzzles. These puzzles have been around for hundreds of years, some of which involve finding a way to trace a diagram such that one's pencil never leaves the paper or backtracks. It wasn't until the end of the 19th century that Euler's bridge problem was drawn in the form of a graph by W. W. Rouse Ball. Ball represented each area of land as a dot or vertex as we have come to call it and each bridge as a line or edge.

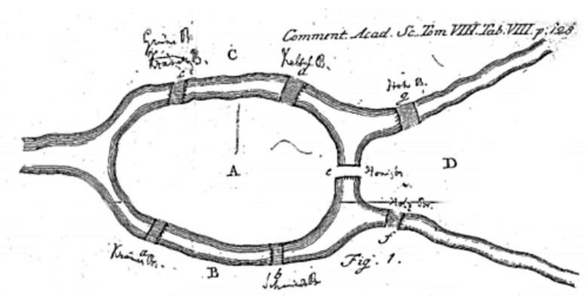


Figure 1. The Königsberg bridges.

In the mid 1800s, Francis Guthrie wondered whether any map could be colored using at most four colors so that no two territories sharing a border were the same color. His brother, Fredrick Guthrie, asked Augustus De Morgan, mathematics professor at University College in London, if he could prove this (which is now known as the Four Color Theorem). De Morgan quickly found himself intrigued and wrote to all of his mathematician colleagues to ask if they could come up with a proof. No proof was found before De Morgan's death in 1871. Alfred Kempe produced a proof in 1879 that was widely accepted but was shown to be incorrect 11 years later by Percy Heawood. While his proof [1] was incorrect, Kempe (1879) did make the important observation:

> If we lay a sheet of tracing paper over a map and mark a point on it over each district and connect the points corresponding to districts which have a common boundary, we have on the tracing paper a diagram of a "linkage," and we have as the exact analogue of the question we have been considering, that of lettering points in the linkage with as few letters as possible, so that no two directly connected points shall be lettered with the same letter. (p. 200)

The Königsberg bridge problem and the four-color problem at first seem to be two completely different problems with very little in common, but they both belong to the area of mathematics called Graph Theory (for a more complete history of the subject see [2].) We can translate both of these problems into graphs and apply what we know about graphs to help us solve them. To do that we must first understand what a graph is.

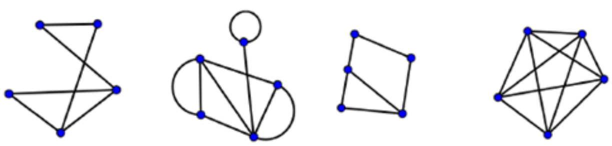
Definition 1. A graph G is a pair of sets (V,E) where V is the set of all the *vertices* in G and E is a set of 2-element subsets of V also known as the *edges* in G.

Definition 2. Two vertices are said to be *adjacent* if they are connected by an edge and edges are said to be *adjacent* if they meet at the same vertex.

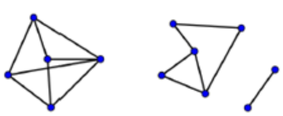
Definition 3. The degree of a vertex is the number of edges adjacent to that vertex.

Definition 4. A path is a sequence of adjacent edges that connect a sequence of vertices.

There are different classes of graphs some of the most important being simple, planar, and connected (see Figure 2).



(a) Simple vs Complex (b) Planar vs Non-planar



(c) Connected vs Disconnected

Figure 2. Some important classes of graphs.

Definition 5. Simple graphs have at most one edge between any two vertices and no vertex is adjacent to itself.

Definition 6. Planar graphs can be drawn in the plane in such a way that no edges overlap or cross each other.

Definition 7. The region enclosed by a planar graph's edges is called a face.

Definition 8. Connected graphs are those which have paths that allow you to start at any vertex and end at any other vertex.

By drawing the Königsberg bridges as a graph (see figure 3), we can prove the problem has no solution because all of the vertices in a graph need to have an even degree for there to be a path that uses all of the edges exactly once (see [3] for a proof).

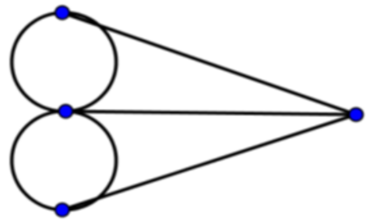


Figure 3. The Königsberg bridge problem drawn as a graph. The vertices represent the land and the edges represent the seven bridges.

What Kempe referred to as a linkage we call a graph today, and while his "proof" was found to be incorrect, Kempe's observation let us look at the four color problem in terms of what is now known as graph coloring. In this paper we will consider a variation of coloring problems for graphs. There are many different ways to color a graph. The most common way to color a graph is to find the smallest number necessary given certain parameters.

Definition 9. The *chromatic number* is the smallest number such that no two vertices adjacent to the same edge are the same color.

Many different classes of graphs have been studied with respect to the chromatic number. Kenneth Appel and Wolfgang Haken finally proved the four-color problem in 1976 with the aid of a computer [4]. Thanks to them we know that while some graphs can have very large chromatic numbers, all planar graphs have a chromatic number no greater than 4.

We refer to these problems as coloring problems for historical reasons, but there are plenty of non-coloring related applications here. For example, one application of vertex coloring is to find a way to store chemicals in a chemistry lab. There are some chemicals that will react poorly if stored in the same cabinet. We can make a graph to help us figure out how many cabinets we'll need and what chemicals can be stored together. The vertices will represent the chemicals and an edge will be drawn between two vertices if those chemicals cannot be stored together. Then we find a chromatic coloring of our graph. Each color represents a cabinet and each chemical with that color should be stored in that cabinet.

Another way to color the vertices is to find the domatic number. A graph's domatic number is the largest number of colors that can be used to color the vertices so that every vertex is adjacent to every color including itself. There are of course many more ways to color the vertices of a graph, and we don't have to just color the vertices.

Edge coloring is coloring the edges of a graph so that no edges meeting at the same vertex are the same color. There are many different types of edge colorings just as there are many vertex colorings. In 1964, Vadim G. Vizing developed a theorem for the edge chromatic number, that is, the smallest number of colors required to color every edge such that no two edges attached to the same vertex are the same color. Vizing's theorem sates that the edge chromatic number is at most the maximum degree plus one.

One application of edge coloring is scheduling. Let's say there is a career fair where 15 companies are holding interviews and dozens of people need to interview with one or more of the companies. Let the vertices represent the companies and the people. We will draw an edge between a company and a person if that person wants to interview with that company. By finding the edge chromatic number we can know the fewest number of time slots needed so that everyone gets a chance to interview for every company they wish to. The different colors will represent the different time slots.

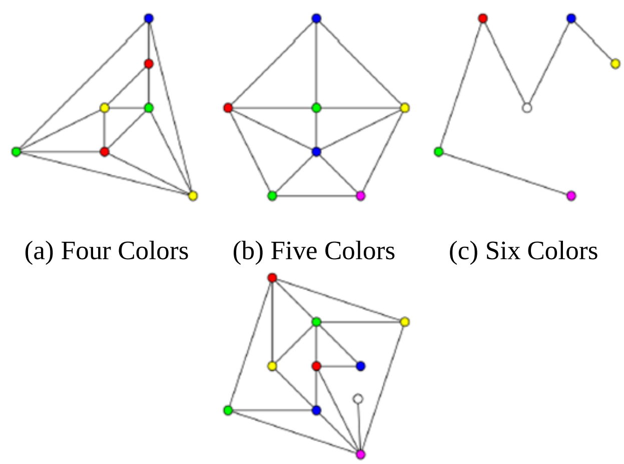
Other types of coloring include: greedy coloring, road coloring, weak coloring, strong coloring, exact coloring, complete coloring, harmonious coloring, and so many more (for more ways to color a graph see [5]). Still there are ways of coloring a graph that no one has yet looked at.

One such way is to color the vertices so that no two vertices adjacent to the same face are the same color. We will call the smallest such number necessary to accomplish this the *face-wise chromatic number* of a graph.

Definition 10. The *face-wise chromatic number* (fwcn) of a graph is the smallest number necessary to color all the vertices of a graph such that no two vertices adjacent to the same face are the same color.

We will only be looking at planar graphs since they are the only type of graphs that have faces. As is usually done with planar graphs, we will consider the surrounding area of the graph a face as well. Our goal is to find a way to easily determine the face-wise chromatic number of any given planar graph. Figure 4 shows a few examples of graphs with various face-wise chromatic numbers.

As you can see in figure 4 *c* and *d*, a graph can have many edges and another graph can have very few, but they both can have the same face-wise chromatic number. Classical vertex coloring has always been related to edges, but when we focus on the faces, the number of edges don't seem to matter which makes our research particularly interesting.



(a) Four Colors (b) Five Colors (c) Six Colors

(d) Six Colors

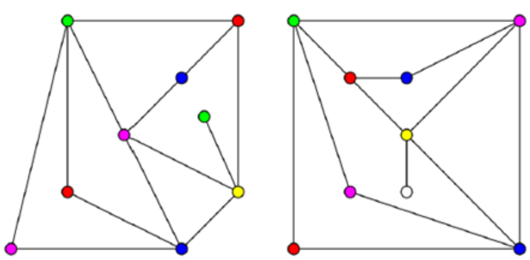
Figure 4. Graphs with face-wise chromatic colorings.

Our goal in this paper is to investigate the fwcn and look for connections to the chromatic number. In section 2 we consider how drawing a graph differently may affect the fwcn. In section 3 we will look to the chromatic number to help us find the fwcn. In section 4 we will mention some interesting questions about the fwcn that needs further research.

2 DIFFERENT DRAWINGS

A graph can be drawn differently and still be the same graph. The number of edges and vertices will remain the same and all the vertices will be connected to the same vertices they were before. The only thing that might be affected is what vertices are adjacent to what faces. What does this mean for the fwcn? As you can see in figure 5, the fwcn depends on the drawing.

What is the largest difference of fwcn for different drawings of the same graph we can make?



(a) fwcn five (b) fwcn six

Figure 5. Different drawings of the same graph.

Proposition 2.1. *The difference between two fwcn of the same graph can be arbitrarily large.*

Proof. Let G be a graph with n vertices. Let v_{2i} be adjacent to v_{2i+1} and both be adjacent to v_0 forming a triangular face. If G is drawn such that it is planar and all of the vertices are fanned out around v_0 (see figure 6), then all vertices are adjacent to the outside face making the fwcn n. If, instead, G is drawn such that it is planar and the face created by v_0, v_1, and v_2 is inside the face created by v_0, v_3, and v_4 which is inside the face created by v_0, v_5, and v_6 ... inside the face created by v_0, v_{n-2}, and v_{n-1}, then v_1, v_2, v_3, and v_4 can be colored the same as v_{4k+1}, v_{4k+2}, v_{4k+3}, and v_{4k} respectively. v_0 will have to be a completely different color giving a fwcn of 5 for this drawing. G has a difference in two of its fwcns of n-5 and since n can be arbitrarily large, the difference can be arbitrarily large.

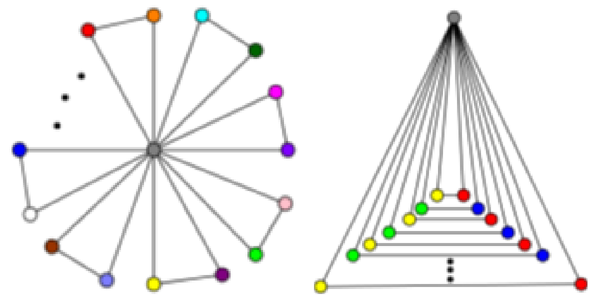


Figure 6.

This result shows that the fwcn depends on the particular drawing of the planar graph, at least for some graphs. Do graphs exist that have the same fwcn for all drawings? Obviously any graph with only one face has the same fwcn no matter how it is drawn, but are there graphs with more than one face that have the same fwcn for all possible drawing?

Proposition 2.2. *Graphs with two or three faces and no vertices of degree one have the same fwcn for all drawings.*

Proof. If a graph has only two faces and no vertices of degree one, then it must be a cycle. A graph that is simply a cycle has all its vertices adjacent to both of its faces resulting in a fwcn equal to the number of vertices in the graph. Let graph G be a graph with three faces and no vertices of degree one (see figure 7). This means G has exactly two vertices with degree three, call them v_1 and v_2, and the rest of its vertices must be degree two. This creates three paths from v_1 to v_2. Paths 1 and 2 create a face A and paths 2 and 3 create a face B. There is also the outside face C bordered by paths 1 and 3.

Take any two vertices in G. If they are on the same path, they are obviously adjacent to the same face and must be colored differently. If they are on paths 1 and 2, they are both adjacent to *A*. If they are on paths 2 and 3, they are both adjacent to *B*. If they are on paths 1 and 3, they are both adjacent to *C*. Thus, no two vertices can be colored the same so the fwcn must be equal to the total number of vertices in *G*.

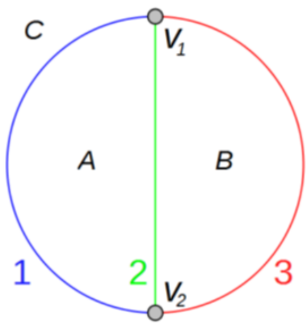
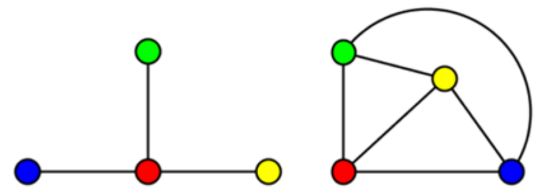


Figure 7. Graph *G*.

3 USING CHROMATIC NUMBER TO FIND FWCN

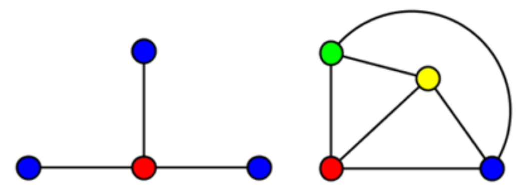
Because we know more about the chromatic number, it might help for us to relate the fwcn to the chromatic number. Since we can only have a chromatic number less than five for planar graphs and we can have fwcn as high as we want (a path for instance), we know that two graphs with the same chromatic number won't necessarily have the same fwcn. Do graphs with the same fwcn have to have the same chromatic number? In figure 8 two graphs are colored with respect to their faces and both have fwcn 4. In figure 9 the same two graphs are chromatically colored yet graph *a* has chromatic number 2 and graph *b* has chromatic number 4. So we can conclude there is no direct correlation between a graph's chromatic number and its fwcn.

However, that doesn't mean that we can't use the chromatic number to help us find the fwcn. Let *G* be a planar graph. Now add edges to connect all the vertices adjacent to a face to all the other vertices adjacent to that same face (see figure 10). Let *G'* be the resulting simple graph.



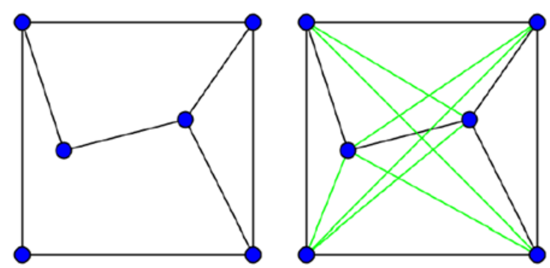
(a) fwcn = 4 (b) fwcn = 4

Figure 8. Face-wise coloring.



(a) chromatic = 2 (b) chromatic = 4

Figure 9. Chromatic colorings.



(a) *G* (b) *G'*

Figure 10.

Proposition 3.1. *The chromatic number of G' is equal to the fwcn of G.*

Proof. We will show that any proper chromatic coloring of *G'* is also a proper face-wise chromatic coloring of *G*, and visa-versa. If two vertices are colored the same in *G'*, they must not be adjacent by an edge which means they are not adjacent to the same face in *G* and thus must also be colored same in *G*. If two vertices are colored the same in *G*, they must not be adjacent to the same face, which mean they are not adjacent by an edge in *G'* and thus must also be colored the same in *G'*. Therefore, the chromatic number of *G'* must be equal to the fwcn of G.

Proposition 3.1 is useful because we can use what we already know about vertex coloring to help us find the fwcn.

Definition 11. A *clique* in a graph is a subset of vertices such that every vertex is adjacent to every other vertex in the subset.

Definition 12. A *perfect graph* is a graph in which the chromatic number of every induced subgraph of *G* is equal to the size of the largest clique (see Figure 11).

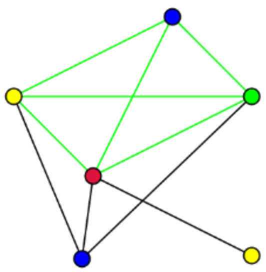


Figure 11. A clique highlighted in green in a perfect graph.

Definition 13. A *chordal graph* is a graph in which every cycle of length for or more has a *chord*, that is an edge that is not part of the cycle. If a graph is chordal, it is perfect [3].

If *G'* is chordal, all we need to do to find the fwcn of G is find the largest clique of *G''*. Finding the largest clique isn't easy (in fact, it is NP-complete) but at least we have a start on finding the fwcn.

Unfortunately, adding the edges doesn't always produce a perfect graph. Figure 12 is a graph that is not perfect but has no more vertices adjacent to the same face that aren't already adjacent to one another. Highlighted in red is a cycle of length four with no chord meaning the graph is not perfect. We will have to find some other way to figure out the fwcn.

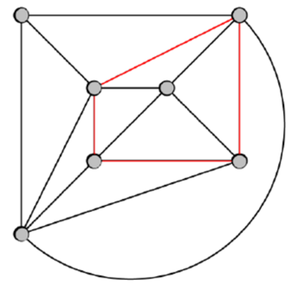


Figure 12.

4 FURTHER QUESTIONS

Even though we answered several questions here, there are still many things we'd like to know. In proposition 2.2 we showed that the fwcn is the same for all graphs with at most three faces and no vertices of degree one. There is an obvious question to consider here - what happens when there are more than 3 faces? Will there always be ways to draw such graphs giving different fwcn, or do some graphs with 4 (or more) faces have fwcn invariant under different drawings? Note also that to prove proposition 2.2, we showed that the fwcn was the same as the number of vertices (in all drawings). So we ask, are there graphs with fwcn less than *n* but which have the same fwcn for all drawings?

In proposition 3.1 we showed that the fwcn of *G* is equal to the chromatic number of *G'*. This was particularly useful when *G'* turned out to be chordal, but that wasn't always the case. What subclasses of graphs don't give a chordal graph when we add the edges? Is there another way we can use finding the chromatic number to finding the fwcn or will we have to try something else altogether? One way to investigate fwcn further would be to write a computer program to find the fwcn for a large collection of graphs, but that approach would only work if there were efficient algorithms for finding the fwcn. In computer science language, we need to know the complexity of finding the fwcn. Proposition 3.1 suggests that finding the fwcn will be difficult, probably NP-complete.

There are two more questions we'd really like to know the answers to. The first being can we find non-trivial bounds on fwcn? Obviously the fwcn can't be larger than the total number of vertices and it can't be any smaller than the degree of the largest face but can we make those bounds tighter? The second question is one that troubles most math research. Does this have any real-world applications? We saw earlier that the chromatic number can be used for many things such as coloring maps and making schedules. What could we use the fwcn for, if anything?

5. REFERENCES

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