5-1-2014

Effects of formative assessment on students' zone of proximal development in introductory calculus

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The Graduate School

THE EFFECTS OF FORMATIVE ASSESSMENT ON STUDENTS' ZONE OF PROXIMAL DEVELOPMENT IN INTRODUCTORY CALCULUS

A Dissertation Submitted in Partial Fulfillment
of the Requirements for the Degree of
Doctor of Philosophy

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School of Mathematical Sciences
Educational Mathematics

May 2014

This Dissertation by: Rebecca-Anne Dibbs

Entitled: *The Effects of Formative Assessment on Students' Zone of Proximal Development in Introductory Calculus*

has been approved as meeting the requirement for the Degree of Doctor of Philosophy in College of Natural and Health Sciences in School of Mathematical Sciences, Program of Educational Mathematics

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ABSTRACT

Dibbs, Rebecca-Anne. *The Effects of Formative Assessment on Students' Zone of Proximal Development in Introductory Calculus*. Published Doctor of Philosophy dissertation, University of Northern Colorado, 2014.

Prior research on formative assessment in classrooms documents a link between formative assessment and increased performance on achievement tests but little is known about how formative assessment helps undergraduate mathematics students improve. The purpose of this mixed methods study was to examine which purposes of formative assessment were relevant to students in two sections of introductory calculus that used a set of in-class labs based upon approximation and students' understanding of limits. The data for the qualitative portion of the project consisted of classroom observations of students' experiences with formative assessment and case studies of nine students. Students' mean achievements on the limits, derivatives, and definite integral labs were compared across participation levels ($n = 54$). Specifically, the researcher examined how asynchronous formative assessments, low stakes assignments completed outside of class for the purpose of feedback and teacher planning, facilitated academic socialization, provided a basis for classroom discussion, allowed for effective student feedback, activated students as learning resources for each other, and increased student ownership of their learning using Vygotsky's Zone of Proximal Development as a theoretical lens. Additionally, since asynchronous formative assessment is a type of participation, the researcher explored how formative assessments could open a dialogue between students and instructors. The findings suggested that the learning trajectory of students was dependent on the regularity of participation in the formative assessments. Although classroom discussion based upon students' questions was less effective as the semester progressed, students who utilized individual written feedback on a draft showed great improvement on their final assignment. There were also indications of attribution and calibration differences between students who participated regularly in formative assessment and those that did not; these differences merit attention in future research.

KEY WORDS: Achievement, Approximation framework, Formative assessment, Ownership, Participation

ACKNOWLEDGEMENTS

I would not have finished my dissertation without the guidance of my committee members and the support of family and friends. I would like to express my deepest gratitude to my adviser, Mike Oehrtman, for having the patience to push me past the limits of what I thought I could do. I would also like to thank Dr. Lahman and Dr. Powers who generously gave their time when I had questions. Special thanks to Dr. Leth for being willing to join my committee at a later point than usual.

Thank you, Debbie, for translating ideas into an intelligible form. Amelia and Andy, thank you for reminding me that there was life outside and away from the computer. Nissa, Danielle, Heath, and Tatiana, thank you for setting the timer and keeping us on track. Dave, thanks for everything. Diana and Avis, thank you for listening. Yuli, Claire, and Jason, your support made finishing much easier than it otherwise would have been.

Tabitha, I thank you for believing in me. Thank you to Dr. Kline, Dr. Grant, Dr. Richter, and Dr. Strom for helping me see what path I should be on. To The Great Bearded Glacier, you were right. Dissertations take much longer than novels.

Thank you Bill, Kathy, Celeste, Mike, Jason, Trevor, Ophia sisters, BRT compatriots, Kelly, Anne, George, and Brian; you all know why. Thank you to the people that got me started: Tyger, Tyger, Josna, Dave, Andrea, Mr. Taylor, and Mrs. Lavery.

Finally, I thank Doris and Dean Douthat, my parents, and my brother, Chris. Your support was invaluable.

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CHAPTER I

INTRODUCTION

Research Problem

> *In terms of systems engineering, present policies in the U.S. and in many other countries seem to treat the classroom as a black box. Certain inputs from the outside-- pupils, teachers, other resources, management rules and requirements, parental anxieties, standards, test with high stakes and so on, are fed into the box. Some outputs are supposed to follow: pupils who are more knowledgeable and competent, better test results, teachers who are reasonably satisfied, and so on . But what is happening inside the box? How can anyone be sure that a particular set of new inputs will produce better outputs if we don't at least study what happens inside?* (Black & Wiliam, 1998)

In their seminal article, *Inside the Black Box: Raising Standards Through Classroom Assessment*, Black and Wiliam argue formative assessments, low stakes assignments given to assess student learning, are a mechanism that can illuminate how students think about the material. Teachers who have been trained how to analyze formative assessments in professional development or as part of their degree program raise students' achievement about .5 standard deviations over control classrooms (Black, Harrison, Lee, Marshall, & Wiliam, 2003; Black & Wiliam, 1998; Clark 2010, 2011; Wiliam, 2007a, 2011; Wiliam & Black, 1996). Although there is a theoretical framework (Black & Wiliam, 2009) that describes the purposes for using formative assessment, this framework was developed on K-12 students; it is unknown how well this framework applies to undergraduates. The purpose of this study was to investigate the feasibility of applying Black and Wiliam's (2009) framework to the undergraduate mathematics classroom.

I examined how the use of formative assessment interacts with students' Zone of Proximal Development (ZPD). Since formative assessments can also increase students' self-efficacy and metacognition (Black & Wiliam, 2009; Clark, 2010, 2011; Pryor & Crossouard, 2005), formative assessments may not dramatically impact students' initial misconceptions; instead, they may help students make connections between topics and pave the way for future independent problem solving on more advanced problems.

The purpose of this dissertation was to investigate which purposes of Black and Wiliam's (2009) framework are relevant to undergraduate mathematics education within the context of introductory calculus. I investigated how formative assessments provide instructors with data about the location of students' ZPD (Vygotsky, 1978), which can help provide scaffolding that moves students forward more efficiently. Formative assessment has several different definitions in the literature; in the next section, I describe one relevant to this study and the context of its development.

Context

The first use of the term *formative assessment* was in 1967 when Scriven made the claim that assessment was a process (Taras, 2009). Formative and summative assessments are processes rather than labels; most assessments have both formative and summative elements and these two processes should be considered to be on a continuum (see Figure 1). By 1971, formative assessment was defined to be assignments completed for feedback rather than a grade; these assessments were considered inferior to summative assessments since it was unclear how formative assessments promote student learning (Taras, 2009).

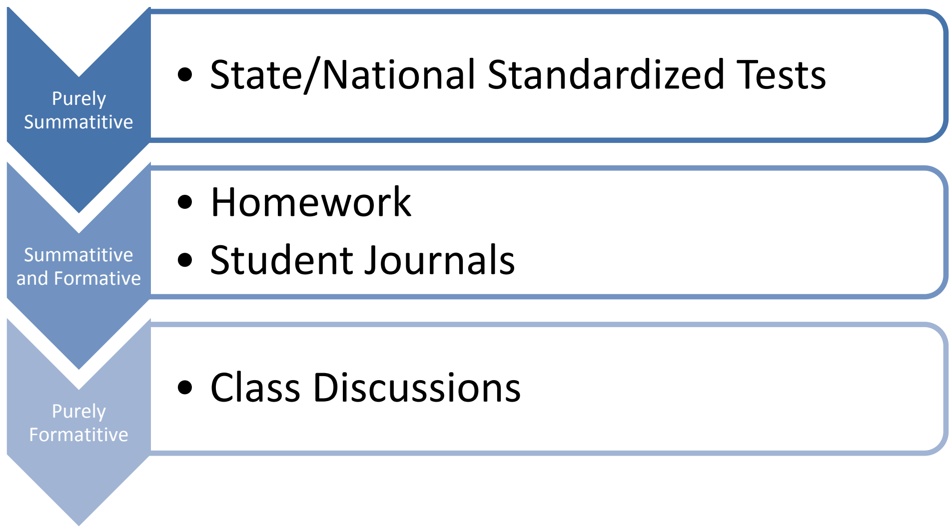

Figure 1. Assessment continuum.

Formative assessment became an active area of research in the late 1990s when Black & Wiliam (1998) published their seminal meta-analysis documenting formative assessment as one of the most effective available instructional techniques for raising student achievement. The modern definition of formative assessment was influenced by both the subsequent work of Black and Wiliam (2009) and French researchers Allal and Lopez (2005); both research groups have worked over the past 10 years to build theoretical frameworks for the measurable and observable benefits of formative assessment. The definition of formative assessment upon which the theoretical frameworks are built is as follows: “Formative assessment is a process used by teachers and students during instruction that provides feedback to adjust ongoing teaching and learning to improve students' achievement of intended instructional outcomes" (Clark, 2011, pp. 165).

To limit the scope of this inquiry to a project reasonable for a dissertation and remove confounding variables, introductory calculus was the only course studied. The introductory calculus classes that participated in this study were taught using the approximation framework, which grew from the research on metaphors for limits in response to earlier research on students' conceptions of limits. The approximation framework is based on the most common strong metaphor for limits students spontaneously use (Oehrtman, 2002, 2003, 2008, 2009). This metaphor helped students to reason about limits in productive ways; the more scientific and structured conception of limits may make instruction easier. The approximation framework is introduced through a series of structured activities scaffolded less each time and designed to help students systematize their informal, unstructured conception of calculus concepts through abstraction (Oehrtman, 2008).

Significance

Understanding the effects of formative assessment in calculus is significant in its own right because while formative assessment has been touted as effective for increasing achievement and understanding (Clark, 2011; Wiliam, 2011) and is highly encouraged in classroom use, none of the quantitative literature that showed formative assessment is effective at increasing students' achievement and conceptual understanding in a broad range of mathematical settings (Black & Wiliam, 2009; Brookhart, Moss, & Long, 2008; Wiliam, 2009) made any claims about the mechanics of how formative assessment works.

This investigation advanced the theory of formative assessment in two ways. First, since the current formative assessment theoretical framework (Black & Wiliam, 2009) was developed on European primary and secondary school students, this investigation was able to confirm and expand upon the framework for the benefit for formative assessment for U.S. undergraduates in introductory calculus. Second, by investigating formative assessment with a mixed methods lens, this dissertation investigated student achievement and provided potential explanations for the achievement boost noted in prior literature.

The fifth construct in Black and Wiliam's (2009) framework was increasing student ownership of the material. This construct is comprised of interest, motivation, self-efficacy, and attribution. Each of these sub-constructs is worthy of study in its own right. There are instruments designed to measure changes in these variables but the instruments are not free. The findings of this dissertation generated hypotheses about students' ownership. These findings suggested which of these aspects of ownership might be worthy of further investigation in a later funded investigation.

This dissertation also contributed to the methodology of formative assessment research. Since formative assessment research is almost exclusively quantitative (Black, McCormick, James, & Peder, 2006; Black & Wiliam, 1998; Clark, 2011; Wiliam, 2011), using QUAL-quan mixed methods design to evaluate the effects of formative assessment on student learning was a novel approach to formative assessment research.

There were also several reasons to investigate calculus pedagogy at the undergraduate level. More students leave Science, Technology, Engineering, and Math (STEM) majors after calculus than any other course (Bressoud, Carlson, Mesa, &

Rasmussen, 2013). One of the benefits to using formative assessment is that there are fewer students earning final grades near the pass/fail cutoff (Cauley & McMillan, 2010), which reduces the DWF rate and provides a firmer delimitation between students who are ready for the second semester calculus and those who are not. Formative assessment is designed to help increase communication with students and instructor, which could help facilitate a sense of connection on the part of the students. Researchers have already made great strides in streamlining formative assessment for large classes (up to 450 students) with a free online software platform (Gasiewski, Eagan, Garcia, Hurtado, & Chang, 2012; Novak, Patterson, Gavrin, Christian, & Forinash, 1999) so implementing formative assessments is a feasible solution, even at large universities.

Inquiry Framework

The act of completing and grading a formative assessment is a social exchange between student and instructor; I chose to frame this study in terms of a social theory of learning. In particular, a Vygotskian (1978) social constructivism perspective allows researchers to focus on how formative assessments aid instructors in deciding if a given task is with the zone of proximal development for students and supports students in the types of activity within their ZPD that can be productive.

While Vygotskian (1978) constructivism globally framed the study, there were local theoretical frameworks for the two major constructs in this study (formative assessment and peripheral participation) and my own researcher stance that I used to frame this inquiry (see Chapter II for a detailed discussion of the theoretical perspective).

Based on earlier literature, which included primarily quantitative research studies, Black and Wiliam (2009) described five main benefits to the instructor, student, and class from the use of formative assessment: clarifying what constitutes successful learning of a particular topic, evaluating where students are at, providing feedback that moves learners forward, activating students as learning resources for each other, and increasing student ownership of the material. This framework became the basis of the coding scheme of my analysis.

Since mathematics is a subject that builds on previously learned concepts, students have to be able to apply their knowledge to new situations in order to be successful. However, applying knowledge to novel problems, particularly in first semester calculus, can be problematic for students. Problems that students are able to solve with assistance, called scaffolding, are said to be in the Zone of Proximal Development for that student. The challenge of providing scaffolding for struggling students as an instructor is determining how much scaffolding is needed; formative assessments might allow us to tighten our focus. The other two characterizations of the ZPD are the Collaborative ZPD, solving a problem in a group no individual member can solve individually, and the conceptual development ZPD. The distinction between these three characterizations is discussed further in the next chapter.

Inquiry Statement

The following research question guided this inquiry:

Q1 What are the functions of formative assessment that scaffold students' peripheral participation and productive engagement in their Zone of Proximal Development for approximation concepts from one context to another in an introductory calculus course?

I investigated which purposes of formative assessment applied to undergraduate mathematics education within the context of Oehrtman's (2008) approximation framework since the activities are designed to keep students within their ZPDs as much as possible.

Delimitations

Given that there is little qualitative research about formative assessment currently published, I set four delimitations to limit the scope of my study. This inquiry did not investigate formative assessment and its relationship with computational skill, daily asynchronous formative assessment, and any verbal synchronous formative assessments completed in class I did not directly observe. I have excluded these lines of inquiry due to lack of support in the literature, lack of relevance to the research question, or lack of feasibility for a dissertation project.

Some research suggested formative assessment can improve computational skills (Brookhart, Moss, & Long, 2010; Gallagher, Bones, & Lambe, 2006; Marcotte & Hintze, 2009); however, I chose not to investigate how formative assessment improves students' computational skills. Most of the literature support for formative assessment is on verbal, in class, formative assessments (Black et al., 2006; Black & Wiliam, 1998, 2009; Wiliam, 2007b) or daily written formative assessments (Dibbs, Glassmeyer, & Yacoub, 2013). However, less frequent formative assessments might be as beneficial to students as long as less than a week passes between the formative assessment and its intervention (Boston, 2002). Since analyzing weekly formative assessments would be more reasonable in scope for a dissertation project and easier to document than a verbal formative assessment, I decided to investigate weekly, written formative assessments over more frequent or harder to document formats, e.g., verbal feedback during group activities.

Finally, given the relative large number of students in my sample, I did not investigate all of the labs students completed throughout the semester. Of the seven labs (quantitative reasoning, exponential growth, limits, derivatives, linear approximation, Newton's Method, and definite integrals), only limits, derivatives, and definite integrals were considered in the analysis. These three main topics in the course and the three labs used the approximation framework most explicitly.

CHAPTER II

LITERATURE REVIEW

Overview

To understand how my dissertation fits into the professional discourse on the Zone of Proximal Development (ZPD) and formative assessment, I investigated the relevant bodies of literature on the topic. After detailing the search and review processes, I synthesize the five areas of literature relevant to my dissertation: formative assessment, the approximation framework for calculus instruction, peripheral participation, the Zone of Proximal Development, and self-monitoring. The chapter concludes with a summary of the substantive research and methodological findings, a discussion of the implications of the literature for the dissertation, and the expected contributions this dissertation made to the literature.

Selection Process

For each area of literature mentioned above, I began by searching Academic Search Premier for recent articles on the topic. Once I obtained these initial articles, I skimmed the initial articles for reference and made note of the researchers who appeared most frequently in the reference list. Next, I used Google Scholar to search for articles, book chapters, and conference papers that cited, were related to, or were authored by researchers who wrote the initial articles. I repeated this process until I was no longer able to generate new relevant references. To finish my literature selection process, I searched the past 10 years of all journals that published two or more of the selected articles as well as the *Journal for Research in Mathematics Education*, *Educational Mathematics*, *Research in Mathematics Education*, *Cognition and Instruction*, and *The Journal of Mathematical Behavior.*

Review of Literature

Once I found the literature, I initially assessed the quality by sorting the material on the basis of peer review. I prioritized peer reviewed material by its publication venue; journals, book chapters, and peer reviewed conference proceedings were considered in that order. For the non-peer reviewed publications, conference papers, action research projects, and dissertations, I contacted the first author to inquire about what additional research was conducted based on the non-peer reviewed source. If there was such a peer reviewed publication, I added it to my literature to review. If no peer reviewed publication was available, I read the peer reviewed references cited in the non-peer reviewed paper and omitted the non-peer reviewed paper.

After finding and winnowing the literature to relevant peer reviewed articles, I systematically reviewed the literature by major category. For each major category, I open-coded each passage using relevance to the research questions as my guide, a process described by Foss and Waters (2007). Then I took the passages I flagged in the initial reading and formed categories; a peer familiar with the literature checked my categorical coding. When several passages discussed the same relevant concept, I included the original citation. In the sections that follow, I present a synthesis of this coding in order of relevance to the research question outlined in Chapter 1: formative assessment, the approximation framework, peripheral participation, the Zone of Proximal Development, and self-monitoring.

Major Works

Formative Assessment

Formative assessment is one of the most effective and cost-effective techniques for raising students' achievement (Al Kadri, Al-Moamary, Magzoub, Roberts, & van der Vleuten, 2011; Black & Wiliam, 1998; Minstrell & Anderson, 2011; Shute, 2008) but much of the research on formative assessment has been conducted on either non-American, elementary, or secondary school students. Nevertheless, this body of literature shaped the project by providing a context to situate this study. After discussing formative assessment's significance in terms of student achievement and cost effectiveness, I briefly synthesize the literature on formative assessment and cognition and the theoretical research on formative assessment. In the remainder of the section, I discuss the research-based best practices for using formative assessment in the classroom including the type, time, and level of feedback appropriate for the undergraduate students in the study.

Using formative assessment as a tool to guide instruction typically raises student achievement .5 standard deviations over a control classroom (Clark, 2010; Taras, 2009). An achievement gain of this magnitude would be large enough to make the United States the top preforming country in the TIMMS study (Taras, 2009). Furthermore, training teachers in the use of formative assessment appears to be more cost-effective at raising student achievement on standardized tests and cumulative final exams than increasing teachers' pedagogical content knowledge or reducing class size (see Table 1); however, this current claim stems mostly from the fact that pedagogical content knowledge is difficult to measure (Wiliam, 2009).

Table 1

Cost-Effect Comparisons for Three Educational Interventions

Intervention	Extra Months of Learning Gained Per Year	Classroom Cost Per Year
30% class size reduction	3	$30,000
Increase teacher content knowledge 2 standard deviations	1.5	Unknown
Formative assessment	6-9	$3,000

Formative assessment has been under-theorized in research to date (Whitelock, 2008) but there are some theoretical articles relating to formative assessment. Effective use of formative assessment is more than providing students feedback to passively absorb; students must actively process the feedback and any interventions based on formative assessment for any long-term learning benefits to occur (Clark, 2010; Whitelock, 2008). Clark (2010) made this explicit when he proposed the six elements that must be present for an assessment process to be considered formative: (a) establish a positive classroom culture, (b) establish clear learning goals for students, (c) alter instruction based on the formative assessments as necessary, (d) use alternative forms of summative assessments to assess student learning if necessary, (e) prompt students to think about their responses, and (f) actively involve students in the learning process. This six-part characterization is based on Black and Wiliam's (2009) theoretical framework of the measurable outcomes of formative assessment. While this is the accepted formative assessment theoretical framework, Black and Wiliam stated that their framework needs to be situated in a broader context to be truly effective:

> The complexity of the situations in which formative feedback is exchanged is such that they can only be understood in terms of the several theoretical perspectives required to explore the issues involved. These might variously illuminate the formative aspects involved, or, more likely, the broader theory of pedagogy within which the formative dimension is located. (p. 5)

Several cognitive constructs appear to improve when formative assessment is used. Students report feeling like they have a more central role in the classroom and understand their role better when completing formative assessments on a regular basis (Willis, 2010). Regular use of formative assessment also leads to measurable gains in students' self-efficacy measured on pre- post- course surveys and might also increase the frequency and quality of metacognitive statements of elementary school students in written reflections (Clark, 2010).

Formative assessment researchers have offered a framework for the best practices for implementing formative assessment successfully in the classroom. The most important best practice is gradualism; formative assessment should be introduced slowly at a limited scale and expanded in scope only after the instructor and class feel comfortable with the current level of formative assessment implementation since formative assessment seems to have a large implementation dip (Black & McCormick 2010; Black & Wiliam, 1998; Clark, 2011; Wiliam 2011). It generally takes instructors a year to fully acclimate to using formative assessment in the classroom (Clark, 2010) but once the basic formative assessment techniques are mastered, it is much easier to add an additional formative assessment to a class (Clark, 2011).

The most difficult aspect of formative assessment to master is when and how much formative feedback to give a student (Black & McCormick 2010; Black & Wiliam, 1998; Clark, 2011; Wiliam 2011). Shute (2008) offered a heuristic for when and how often to offer formative feedback based on learner characteristics (see Table 2). Shute noted that her heuristic was designed for elementary school students and conceded that with increased maturity, older students are more likely to benefit from delayed feedback from asynchronous formative assessment.

Table 2

Best Practices for Formative Feedback

	Timing	Type of Feedback	Purpose of Feedback	Appropriate Detail
Low Achieving Students	Immediate feedback is necessary to correct idiosyncratic thinking	Corrective feedback, that directly points out problematic areas is most effective	To provide scaffolding so students can complete the problem	Highly detailed feedback that directs students to next step in the process
High Achieving Students	Delayed feedback allows students time to reflect and possibly self-correct their own errors	Facilitative feedback in the form of hints, cues and questions	To verify students' thinking and challenge them further	As little detail as possible without frustrating the individual student

Source: Shute (2008).

Not all feedback is formative. Feedback is formative if students are encouraged to engage in reflection on their own solution, are provided with appropriate scaffolding to help them move forward with their task, understand the criteria for success, and are activated as owners of their own learning (Clark, 2011). For instance, telling a student to work harder would not be considered formative feedback because this statement does not scaffold students' improvement. However, giving a student specific strategies he/she could implement in similar future problems would be considered formative feedback (Clark, 2010). Ideally, formative feedback should empower the learner to correct his/her own errors, e.g., directing a student to compare his/her solution with another classmate who was successful (Svinicki, 2010). Criticism should also be avoided as negative feedback can cause an undesirable shift in attribution in students (Black & McCormick, 2010); as students gain experience with formative assessment, there is less danger in negative feedback (Laight, Asghar, & Aslett-Bentley, 2010).

There has been little research on formative feedback with undergraduates but there are several possible benefits specific to this population. First, U.S. undergraduates tend to overestimate their abilities so proscriptive formative feedback, particularly delayed formative feedback, could help students identify areas of weakness before high stakes summative assessments, which could have large adverse effects on their grades (Berlanga, Rosmalen, Boshuizen, & Sloep, 2011). Delayed feedback of at least one class period is preferable for adult learners because it allows time for reflection and appears to facilitate transfer of strategies from one context to another (Shute, 2008); however, formative feedback loses most it its benefit if feedback is delayed by more than a week (Boston, 2002). The final recommendation for adult learners is to use unit tests for formative purposes, particularly when units in the class build on each other (Taras, 2009). I have incorporated these recommendations into the design of the courses I conducted in this dissertation study.

Vygotsky: Major Constructs and Implications for Practice

The Zone of Proximal Development is one of the central constructs of Vygotskian (1978) constructivism--the theoretical perspective I used in this study. In this section, I briefly define the Zone of Proximal Development (ZPD) and discuss two of the major concepts underlying the ZPD: spontaneous and scientific concepts. The section ends with a review of relevant literature on two applications of the Zone of Proximal Development: the collaborative ZPD and how the ZPD influences meta-cognition.

Two characterizations of the Zone of Proximal Development were relevant to this project. The first characterization is that the Zone of Proximal Development represents the difference between what a learner can do independently and what they can do with help (Vygotsky, 1978). The other relevant characterization is that the Zone of Proximal Development is where spontaneous concepts--informal empirically based concepts where the definition is learned last and scientific concepts--formal concepts taught through instruction where the definition is learned first and empirical knowledge comes last interact to produce new knowledge (Vygotsky, 1987). Both spontaneous and scientific concepts, despite the fact that the latter is taught to students, are products of the students' own thinking, not knowledge transmitted to students by adults (Vygotsky, 1987); in fact, these concepts represent fundamentally different types of students' thinking and learning (Karpov & Haywood, 1998).

Spontaneous and scientific concepts have complementary strengths and weaknesses. Spontaneous concepts can be applied by learners as long as they are not consciously asked to; generally, learners cannot verbalize a definition or rule they are using when applying a spontaneous concept when they first use the concept. In contrast, a scientific concept can be verbalized long before it can be applied in a consistent, correct matter (Vygotsky, 1987). Vygotsky (1987) was not a believer in pure rediscovery of scientific concepts in the classroom; he felt students should not have to rediscover thousands of years of human thought in a classroom. Instead, he was an advocate of beginning with a precise verbal definition of a concept and then exploring it with empirical activities (Karpov & Haywood, 1998).

The power of this characterization of spontaneous and scientific concepts is how these two concepts interact with each other. While there is a higher level of conscious awareness of scientific concepts than spontaneous concepts, an increase in student understanding on a scientific concept leads to a rapid increase in conscious awareness of spontaneous concepts students perceive as being related (Vygotsky, 1987). Furthermore, the boundary between spontaneous and scientific concepts is fluid; as students gain more understanding of a spontaneous concept, they can give a precise definition of it or they can begin to use a spontaneous concept scientifically (Vygotsky, 1987).

Spontaneous and scientific concepts are not directly oppositional. In other words, the scientific concept blazes a trail for the refinement of spontaneous concepts and the spontaneous concept provides empirical frames of reference for scientific concepts (Vygotsky, 1987). This interplay between spontaneous and scientific concepts occurs in the Zone of Proximal Development (Lave & Wenger, 1991, Vygotsky, 1987).

Depending on the sophistication of the spontaneous and scientific concepts an individual has formed, the size and the depth of the Zone of Proximal Development varies widely from learner to learner or within the same learner if the domain or context of problem solving is changed (Campione, Brown, Ferrara, & Bryant, 1984). It is also important to remember that there is an affective dimension to the Zone of Proximal Development; students must be willing to participate and accept help in order to advance (Goos, 2004).

How can the Zone of Proximal Development, an abstract construct that is not easily measurable and individual to each student, be used in a classroom setting? Goos (2004) suggested three actions teachers can take that can help students progress through their Zones of Proximal Development: (a) providing scaffolding in the form of hints and additional instruction, (b) allowing students to interact with each other and provide scaffolding for each other, and (c) interweave spontaneous and scientific concepts--even making explicit connections students themselves are missing.

Early research on applying the Zone of Proximal Development indicated that summative assessments, such as unit exams, are relatively poor measures of where students are in the Zone of Proximal Development; group work or individual observations are really the most helpful (Campione et al., 1984). It is also important to remember that scaffolding students through their Zone Proximal Development is a process that should not be a consistent amount of help. It is both proper and expected that the initial stages of the problem solving activity would be heavily supported by the instructor—only as students become more comfortable should they take the lead (Campione et al., 1984). In fact, the amount of scaffolding needed might be a more valuable metric for where students are in their Zone of Proximal Development than their achievement level on a summative assessment (Ferrara, Brown, & Campione, 1986). However, scaffolding does not need to come solely from the instructor; carefully constructed groups where students are of equal status and have the requisite knowledge as a collective may scaffold each other through their individual Zones of Proximal Development (Goos, Galbraith, & Renshaw, 2002). In fact, working in a collective might help students with the poorest starting levels make the most progress; these students might be less reluctant to ask for additional scaffolding from a peer (Campione et al., 1984).

This potential power of students' collaboration forces those intrigued by Vygotskian principles to consider how communities of practice are developed in classrooms (Goos, 2004). When considering how a community of practice develops and functions, one must examine both the practices of the teacher and classrooms as they emerge and are negotiated into common practices over time (Goos, 2004). Ideally, the norms established for group work can help students to function in a collaborative ZPD where the group is more able than the individuals; helping students learn to make the connections between spontaneous and scientific concepts appears to be the most effective mechanism for establishing a collaborative ZPD in a group setting (Goos, 2004). The difficulty of establishing these initial group work norms may account in part for why there has been relatively little research on students scaffolding each other's development (Goos, Galbraith, & Renshaw, 2002). However, there is some indication that this mutual scaffolding may be the true power of group work, particularly if students share how they are thinking as well as their knowledge (Karpov & Haywood, 1998).

While the potential power of collaborative problem solving is considerable, it cannot completely explain how students incorporate the experience of collaborative problem solving into their own individual knowledge, i.e., self-monitoring and metacognition may be the link between group work and the Zone of Proximal Development (Karpov & Haywood, 1998).

While collaborative problem solving can either help or hinder students' learning, it is unclear how much self-monitoring students employ in a group setting (Goos et al., 2002). Poor metacognitive decisions usually lead to an unsuccessful solution but there has been a call for more research about how metacognitive decisions are made in a group setting (Goos et al., 2002). One framework proposed suggests that cognition and metacognition form a dialectic, both in inter- and intra- psychological planes; but to date, this framework has not been empirically tested (Goos et al., 2002).

Vygotsky (1978) theorized that cognition is both socially and meta-cognitively mediated. He admonished that we ignore the metacognitive aspects of learning at our own peril; however, meta-cognitive mediation cannot be the sole focus of a researcher (Karpov & Haywood, 1998). Metacognitive mediation has its roots in interpersonal communication. Students begin to gain self-monitoring skills when they internalize the scaffolding provided by adults (Karpov & Haywood, 1998) but there is still not an adequate theoretical model that explains meta-cognitive processes. The next section summarizes the research on self-monitoring--a construct that has not always, or even usually, been particularly tied to the Zone of Proximal Development.

Self-Monitoring

Self-monitoring appears to be the link between formative assessment and the Zone of Proximal Development; it emerged as a central construct in this dissertation. After defining self-monitoring and explaining why self-monitoring is a valuable skill for mathematics students, I explain how instructors can facilitate self-monitoring and argue how this literature informed the dissertation project.

Self-monitoring is a metacognitive skill—where a learner can accurately gauge their own progress during a problem solving activity as well as the validity of their eventual solution (Gangestad & Snyder, 2000). Self-monitoring is the most researched metacognitive skill, both in mathematics education and educational psychology; the term is sometimes used interchangeably with metacognition (Hoffman & Spartariu, 2008; Schneider & Artlet, 2011). Students with high levels of self-monitoring can accurately describe what they do and do not understand when solving a problem or learning a new concept, which greatly helps instructors provide the scaffolding students need (Perels, Dignath, & Schmitz, 2009; Ramdas & Zimmerman, 2008).

Self-monitoring is helpful for students for several reasons. In addition to the aforementioned link between transfer and self-monitoring (Georghaides, 2000; Grotzer & Mattlefehldt, 2012; Ning & Sun, 2011), there are three other benefits to increasing students' self-monitoring abilities. First, high self-monitoring abilities positively correlate with increased performance in mathematics courses and differences in self-monitoring can explain achievement differences in students with equal aptitudes (Cohors-Fresenborg, Kramer, Pundsack, Sjuts, & Sommer, 2010; Schoenfeld, 1992). Second, students with high self-monitoring abilities are more likely to exhibit high degrees of mathematical motivation even in the face of peer pressure (Hannula, 2006; Hodges, 2008; Kim & Hodges, 2011). Third, students with a high degree of self-monitoring outperform other students on conceptual questions and in novel problem solving situations (Schneider & Artlet, 2011; Sodian & Frith, 2008; Stillman & Mevarech, 2011). It also appears that students with strong self-monitoring skills are more likely to stay in mathematics related majors (Trautwein & Lüdtke, 2007) and are more successful bridging into proofs-based courses (Yen & Lee, 2011).

Two techniques were suggested by the literature for helping students improve their self-monitoring--asking questions they should ask themselves internally and increasing student self-efficacy. Students who were asked verbal self-monitoring questions, a type of synchronous formative assessment, significantly improved on their posttest self-monitoring scores (Schneider & Artlet, 2010; Timm, 2011); students who were given explicit training in self-monitoring could improve their self-monitoring score up to the pre-test score of students with high self-monitoring in the control group (Pennequin, Sorel, Nanty, & Fontaine, 2010). Similar self-monitoring gains have also been observed when students were given feedback, asked to reflect on it, or showed gains in their self-efficacy (Hannula, 2006; Hwang, Chen, Shadiev, & Li, 2011; Lajoie, 2011; Roberts, 2011).

The asynchronous formative assessments I used for this dissertation project asked questions students should ask themselves and might increase student self-efficacy. Further, the self-monitoring a student does to complete a formative assessment requires some conscious awareness of concepts being assessed, which would seem to support scientific reasoning about the concepts. Asynchronous formative assessments might also be considered a form of peripheral classroom participation; in the next section, I discuss this connection in more detail.

Peripheral Participation

Legitimate peripheral participation is one of the pillars of the Zone of Proximal Development as characterized by Lave and Wenger (1991). In this section, I define peripheral participation, explain the types and progression of peripheral participation in mathematics courses, and then discuss what facilitates peripheral participation and how this construct relates to the dissertation study.

Peripheral participation is described as simple low risk activities newcomers to a community take to make contributions (Lave & Wenger, 1991). These activities are still necessary for the community and not simply make-work. As learners gain acceptance into a community of practice, such as a group or a classroom, through peripheral participation, they proceed along a participation trajectory where the learner gradually takes more complex and higher risk activities that are increasingly central to the output of the learning community. Ideally, the learner becomes completely accepted into the community of practice as one of the experts (Lemke, 1997).

Although the role of legitimate peripheral participation has been heavily theorized as an important part of learning, especially within the context of situated cognition (Adler, 1998; Boaler, 1997, 2000; Boaler & Greeno, 2000; Boylan, 2004; Burton, 2002; Goos et al., 1999; Solomon, 2007; Winbourne & Watson, 2008), there have been few non-theoretical publications on peripheral participation in the classrooms. However, Krummheuer (2010) recently published a grounded theory that modeled the specific actions students take in the classroom that could be considered peripheral participation, intermediate participation, or full participation. While this grounded theory was conducted on elementary school students working in small groups on arithmetic problems, the actions seen as peripheral or central (see Table 3) could still arguable apply to older populations of student.

Table 3

Types and Classification of Participation Actions

Participatory Action	Classification
Eavesdropping, over-hearing (across groups)	Peripheral participation
Co-hearing (within groups)	
Relaying information, ventriloquation	
Spokesperson	Apprentice participation
Author, Evaluator	Full participation

Source. Krummheuer (2010**).**

Peripheral participation can be facilitated by giving students specific, defined roles in a community of practice that are low risk, necessary for the group or classroom to function, and give novice learners access and authority to the interactional space (Krummheuer, 2010). Asynchronous formative assessment, which relays information about students' current understandings to the instructor, is a form of peripheral participatory action. Furthermore, as asynchronous formative assessments are worth a minimal percentage of students' final grades and grades on completion, they are low risk activities by design. The information students provide is necessary for the instructor to prep the next class; hence, asynchronous formative assessments are legitimate peripheral participation.

There is little research about transfer and group work so it is problematic in determining how difficult it is for students to apply knowledge learned in a group setting to later work (Barnett & Ceci, 2002). However, students with high self-monitoring skills might learn a great deal from peripherally participating rather than taking a more active role in problem solving (Sodian & Frith, 2008). Overall though, there is no strong, explicit research link between peripheral participation and the other constructs in the research questions in the literature. However, this dissertation study could begin to rectify this lack. In the theoretical perspective section, I discuss the theoretical links between these constructs.

Approximation Framework

The last area of literature to consider is the context within which the entire dissertation study was situated—the approximation framework. This instructional framework for the calculus sequence used students' most common and persistent spontaneous metaphor for limits—approximation—to help students systematize and unify the major constructs in the calculus sequence. After reviewing the literature on students' understandings of limits, I describe the approximation framework, explain how the other constructs in the research question relate to the approximation framework, and describe how literature in this chapter helped to frame the study.

Research about students' understanding of limits has focused on building theory on why the limit concept is problematic for students. Cornu (1991) suggested that the difficulty with limits for students was that limits are defined in terms of a vague process, rather than a concept, which makes abstraction difficult. This aligned with the later work of Cottrill et al. (1996) who found the definition of a limit successfully involved coordinating multiple processes and had an understanding of quantification; there are many opportunities for the students to have difficulty with the definition. Williams (1991, 2001) also found that the limit definition was far more complicated for students than anticipated. However, the data Williams drew most of his conclusions were his interviews so it is difficult to get a holistic picture from them.

More recent research has focused on students' informal understanding of limits rather than the definition of limit itself. Williams (2001) first made the argument that paradigm shift is necessary and analogous to how researchers understand how children complete arithmetic problems. Oehrtman (2002, 2003) continued research on students' informal understandings and found that students used metaphors to help them understand limits. While these metaphors were often mathematically incorrect, they allowed students to reason about and solve problems involving limiting processes (Oehrtman 2002, 2003).

Oehrtman's research (2002, 2003, 2008, 2009) found that students used several metaphors for limits; however, not all metaphors were equally meaningful for students nor did they appear as widely as some of the others. Surprisingly, motion metaphors, which are almost built directly into the approaching language in a limit, were not strong metaphors[1] for students (Oehrtman, 2002, 2003, 2008, 2009). While students used language that seemed to indicate motion, like approaches or closer and closer when they

[1] Strong metaphors are ontologically creative metaphors that change the understanding of both of the concepts that are being compared (Black, 1977).

were asked follow-up questions in their interviews, they denied they thought of anything moving (Oehrtman, 2003, 2009).

On the other hand, the "collapsing dimension" metaphor was an example of a strong metaphor. Students interpreted the physical meanings of h or Δx going to zero in limits and derivatives as the answer collapsing in dimension, e.g., the rectangles in a Riemann sum had zero width (Oehrtman, 2003). Although this metaphor was not mathematically correct, it helped students reason about limits in a wide variety of contexts (Oehrtman, 2003). Another example of a strong metaphor students used was the "infinity as a number" metaphor where the limit at infinity was treated identically to a finite limit point with no thought to how the limiting process worked in that case (Oehrtman, 2009). Generally, when presented with a counterexample to these strong metaphors, students viewed the counterexample as a minor exception rather than a reason to revise their metaphor (Oehrtman, 2009).

The most common strong metaphor for limits students use is the approximation metaphor (Oehrtman, 2002, 2009). In the approximation metaphor, students treat the limit as an unknown value they can approximate with another similar structure, e.g., secant lines or Riemann sums (Oehrtman, 2002, 2009). Students calculate an overestimate and an underestimate to bound their error by the difference between the overestimate and the underestimate (Oehrtman, 2008). The limit exists if the error bound can always be made smaller than any chosen number. The approximation metaphor is uniquely powerful for two reasons: (a) approximation is the most common metaphor students spontaneously use[2] and (b) this metaphor most closely resembles the actual formal definition of a limit.

Although the research about the approximation framework indicated that approximation is an appropriate metaphor to unify limits instruction, the approximation framework is a curriculum that is still under development. The seven labs students completed in the pilot and dissertation project were developed by Dr. Oehrtman. These activities have never been published although an earlier version of Lab 7 Context 2 is used as an example in Oehrtman (2008). The wording of the questions in each lab activity as well as the contexts that frame the lab are revised after each semester as part of the ongoing design experiment to create curriculum based upon the approximation framework. However, the set of labs students complete in a given introductory calculus course changes slightly from semester to semester. The versions of the labs used in this dissertation appear in Appendix A. The figure below describes the process by which the approximation framework helps students engage in reflective abstraction and formalize the approximation framework (see Figure 2). The bottom layer of the diagram represents the individual group activities completed each week. During an approximation activity, each group uses the same mathematics in slightly different contexts. At the end of the group activity, groups present their work to the class. The first level of abstraction we want students to reflect upon is how all of the contexts use the same approximations ideas to solve a variety of problems, which is represented by the horizontal arrows in this layer. Ideally, students then reflect across the activities about a particular concept (the next layer up) and begin to formalize how the approximation framework functions for all of

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[2] Williams (1991) actually found approximation to be the least common metaphor students use.

the activities about a given calculus concept, e.g., derivatives. Eventually, students reflect upon how the approximation framework applies across concepts--the final layer.

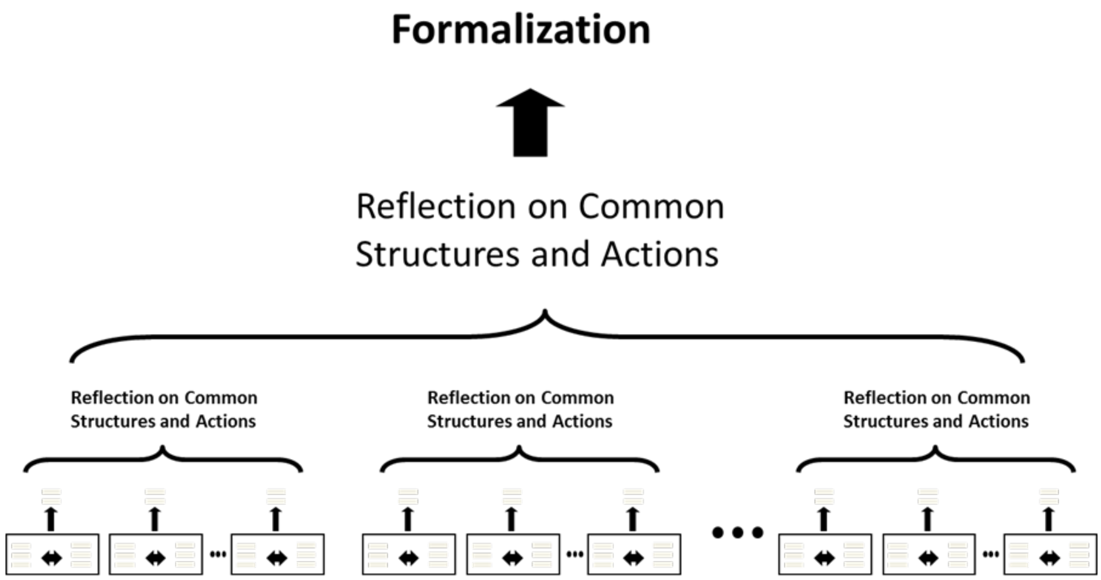

Figure 2. Layers of abstraction.

After two semesters of instruction in the approximation framework, students can still exhibit idiosyncratic thinking about limit concepts (Martin, Oehrtman, Roh, Swinyard, & Hart-Weber, 2011; Oehrtman, Swinyard, Martin, Roh, & Hart-Weber, 2011; Swinyard, 2011). However, students in multiple studies have been able to construct the formal epsilon delta definition for limits in guided reinvention settings after two semesters of the approximation framework, which is an indication of the potential power of the approximation framework.

How did the approximation framework define this study? The approximation framework is a known, research-based, effective way to teach calculus based on how students informally reason about limits. By adding formative assessment into a well-researched instructional framework, it made it easier to see how formative assessment affected students' learning.

As the goal of the approximation framework is to help students systemize their spontaneous concepts on approximation and limits, derivative, and definite integrals, the approximation framework provided a delimiting framework to study the other contexts in the research question; in the study, I examined formative assessment, transfer, self-monitoring, and peripheral participation in terms of the approximation framework. This allowed me to have a clear focus in the investigation. Self-monitoring and peripheral participation are consequences of formative assessment and appeared related to the approximation framework in terms of formative assessment and transfer.

From the literature, it could be inferred that formative assessment is likely to have three effects on students taking an introductory calculus course using the approximation framework as an instructional framework. First, as formative assessment is a form of peripheral participation, students who complete the formative assessments before and after the approximation framework activities give the instructors a snapshot of their current understanding of the approximation concepts, which helps instructors plan the class after the group activities. Second, the act of completing a formative assessment opens a line of communication between student and instructor; it offers a lower risk way for students to answer questions than explicitly asking questions or visiting during office hours. Third, formative assessments give students an opportunity to engage in self-monitoring about their current understanding of the approximation framework, which may scaffold improvements in their self-monitoring. Once systemized, the approximation framework is intended to be a heuristic students can apply to a variety of calculus concepts (Oehrtman, 2003) and applications of a heuristic is a type of far transfer (Barnett & Ceci, 2002). The literature also suggested that formative assessment scaffolds in increases in self-monitoring, which can facilitate far transfer; however, no studies directly linked formative assessment and transfer.

Summary

After briefly discussing implications of this summary, I explain how the dissertation study addressed some of these research needs and built upon the methods of prior literature.

Formative assessment increases student achievement on cumulative or standardized exams. However, even leading formative assessment theorists acknowledge that far more research on how formative assessment benefits students is needed (Wiliam, 2011). Formative assessments of the type in this dissertation study are a form of peripheral participation; situating the inquiry within the context of the approximation framework allowed me to investigate formative assessment in an environment rich in feedback and transfer opportunities. What was unknown with all of these major constructs was how asynchronous formative assessment affected students' self-monitoring, Zone of Proximal Development, and participation levels on the particular population of students I studied—U.S. undergraduates. Knowing how formative assessment affected adult learners would contribute to theory as well as inform practice for instructors who wish to incorporate formative assessment into their classrooms.

This dissertation project contributed to the literature in several ways. First, the results of this inquiry contributed to our understanding of how formative assessment facilitates higher test scores on unit and final exams. Second, I was able to document evidence of changes in self-monitoring and students' Zone of Proximal Development with respect to the approximation framework, which can contribute to the formative assessment theory. Finally, this project contributed to the participation literature since there is limited literature on asynchronous formative assessment as a form of peripheral participation.

The major constructs in the research question have been consistently investigated with the same methods. Recent research on the approximation framework has been investigated through guided reinventions, although the original research on approximation was conducted with similar methods to those in this project. Formative assessment and self-monitoring have both been studied with primarily quantitative methods since the goal was to measure aspects of these constructs. I conducted a qualitative study that built upon the methods used in prior research by collecting students' written work, observing class, and interviewing students.

Overall, while little literature directly linked self-monitoring, formative assessment, and Zone of Proximal Development, I found suggestions that these constructs may be related. Armed with the knowledge of what research has been conducted on the major constructs in my research question, I next turned my attention to what theoretical perspective would most effectively frame the constructs I wanted to examine in the research questions. In the next section, I present this theoretical perspective.

Theoretical Perspective

In this section, I review and synthesize the learning theory I chose, Vygotskian constructivism, together with the theories I used for framing formative assessment, peripheral participation, and my researcher stance.

Vygotskian Constructivism and Legitimate Peripheral Participation

Constructivism is an extremely broad collection of loosely related philosophies that have the same central axiom; all knowledge is constructed by, rather than absorbed by or imparted to, a learner. In mathematics education, two constructivism theories are commonly used: Piagetian and Vygotskian.

Vygotskian constructivism differs from Piagetian constructivism in three major ways. First, Vygotsky (1987) claimed that some concepts, which he called scientific concepts, could actually be taught to learners through formal instruction. In these concepts, the definition is taught first—only later does a student understand how the concept applies in an empirical, everyday sense. For example, consider the concept of density. The formal definition of density is unlikely to be encountered outside of a formal school setting and understanding of what density means comes after the definition is taught. Spontaneous concepts are learned informally through observation and empirical experimentation. For these concepts, the formal definition is learned after an understanding. An example of this is the concept of “brother.” Children can state if someone is or is not a brother before they can state what a brother is. Second, scientific concepts and spontaneous concepts interact with each other by one type of concept serving as a frame of reference for the other (Vygotsky, 1987):

> The strength of the scientific concept lies in in the higher characterization of concepts, in conscious awareness and volition. In contrast, this is the weakness of the child's everyday concept. The strength of the everyday concept lies in the spontaneous, situationally meaningful, concrete applications, that is, in the sphere of experience and the empirical. The development of scientific concepts begins in the domain of conscious awareness and volition. It grows downward into the domain of the concrete, into the domain of personal experience. In contrast, the development of spontaneous concepts begins in the development of the concrete and empirical. It moves towards the higher characterizations of concepts, toward conscious awareness and volition. The link between these two concepts reflects their true nature. This is the link of the zone of proximal and actual development. (p. 220)

Third, learning is development and development has a social origin that is mediated by signs, symbols, and formal instruction (Smagorinsky, 1995; Vygotsky, 1978). Development is defined as physical and psychological maturation, which Vygotsky argued could occur either before learning or due to learning (Vygotsky, 1978). Learning, acquiring new knowledge or skills, can take place with instruction but learning only causes development when the learner incorporates the new knowledge with prior knowledge (Vygotsky, 1978).

So what is learning in Vygotskian constructivism? Learning is the advancement of conceptual formation, which is also the development of new concepts (Vygotsky, 1978):

> The basis for our hypothesis of the *zone of proximal development*. This form of explanation is based in the notion that analogous systems in higher and lower domains develop in contrasting directions. This is the law of interconnection between higher and lower in development. This law was discovered, and has been supported, through our studies of the development of spontaneous and scientific concepts, native and foreign languages, and verbal and written speech. (p. 222)

Learning takes place in the Zone of Proximal Development, which is the difference between what the learner could potentially do and what the learner can do now

(Smagorinsky, 1995). Generally, learning is considered to be what students can do with assistance or scaffolding, which is why Vygotskian learning has been characterized as a form of legitimate participation (Lave & Wenger, 1991; Smagorinsky, 1995). The learner is a peripheral participant because the learner is not being assisted by a more central member of the learning community. As the learner gains knowledge, less scaffolding is needed and the learner becomes a more central participant in the community of practice.

Since learning takes place in the Zone of Proximal Development, it is necessary when conducting research using this theoretical perspective to understand the multiple characterizations of the ZPD. Vygotsky (1978) defined the Zone of Proximal Development as

> the distance between the actual development level as measured by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers. (p. 86)

Note that the Zone of Proximal Development is not a fixed entity; it changes as leaners obtain new concepts (Wertsch, 1983). The ZPD is also thought of as a learner's range of potential learning; it is also understood that the ZPD and learning is influenced by both the social situation and the culture in which the learner is situated (Smagorinsky, 1995).

In their participatory model of learning, Lave and Wenger (1991) characterized the ZPD in three different ways. The first characterization is the one given above—that the ZPD is where students can do problems with scaffolding. The second characterization is that the Zone of Proximal Development is where spontaneous and scientific concepts interact to create new knowledge and advance students' understandings (Lave & Wenger, 1991). The final characterization of the Zone of

Proximal Development defines the ZPD on a more macro level. In this case, the ZPD is where individual actions can be modified for societal and cultural change (Lave & Wenger, 1991). These multiple characterizations of the ZPD give Vygotskian constructivism theoretical power in a wide variety of situations--from tutoring to the classroom and beyond (Smagorinsky, 1995). Vygotsky (1978) argued that no matter how the ZPD is characterized, the most important thing to try and measure is the level of potential learning--what students could do with scaffolding (Wertsch, 1991).

Since scaffolding--assistance given in the form of questions, hints, or instructions --is a key characterization in the ZPD, one characterization of learning in the ZPD has been Lave and Wenger’s (1991) apprenticeship and legitimate peripheral participation model. One of the ultimate goals of the approximation framework is to help students transition more easily into proofs-based mathematics; as such, we can consider these calculus students apprentice upper-level mathematics students. Hence, the characterization of ZPD in this dissertation as type of apprentice-level peripheral participation was both appropriate and relevant. Lave and Wenger initially argued that scaffolding learning is a model of apprenticeship. However, after researching master apprentice relationships, they observed that the apprentices were rarely formally taught their crafts but by the end of their apprenticeship, they were proficient (Lave & Wenger, 1991). In an effort to make the idea of apprenticeship more explicit in terms of learning, Lave and Wenger proposed the term legitimate peripheral participation. Peripheral participation is described as simple low risk activities that newcomers to a community take to make contributions (Lave & Wenger, 1991). These activities are still necessary for the community and not simply make-work. As learners gain acceptance into a community of practice, such as a group or a classroom, through peripheral participation, they proceed along a participation trajectory where the learner gradually attempts more complex and higher risk activities that are increasingly central to the activity at hand. The ideal situation in a classroom setting is that the learner becomes completely accepted into the community of practice as one of the experts (Lemke, 1997). However, this might not completely happen in one semester.

Although the role of legitimate peripheral participation has been heavily theorized as an important part of learning, especially within the context of situated cognition (Adler, 1998; Boaler, 1997, 2000; Boaler & Greeno, 2000; Boylan, 2004; Burton, 2002; Goos et al., 1999; Solomon, 2007; Winbourne & Watson, 2008), there have been few non-theoretical publications on peripheral participation in the classroom. However, Krummheuer (2010) recently published a grounded theory that modeled the specific actions students take in the classroom that could be considered peripheral participation, intermediate participation, or full participation, which was discussed earlier in this chapter.

Overall, legitimate peripheral participation is a vehicle for scaffolding in a learner's Zone of Proximal Development. Peripheral participation can be facilitated by giving students specific, defined roles in a community of practice that are low risk, necessary for the group or classroom to function, and give novice learners access and authority to the interactional space (Krummheuer, 2010). This allows more learners access to their ZPDs without having to individually instruct individual students and suggests how Vygotsky's theories could be used pedagogically (Smagorinsky, 1995). Furthermore, when Vygotskian constructivism is used as a theoretical perspective, the actions of instructors and researchers do not “contaminate” students’ learning; they are simply a part of the scaffolding provided, which has obvious benefits for researchers.

The theory described above relates to the research question in two ways. First, asynchronous formative assessment, which relays information about students’ current understandings to the instructor, is a peripheral participatory action; however, formative assessment also serves as an instrument to evaluate what concepts the class does and does not understand. Based upon the formative assessment, the instructor knows what sort of scaffolding students will need during a group activity before class begins. Furthermore, asynchronous formative assessments are worth a minimal percentage of students’ final grades and grades on completion; hence, they are low risk, high reward activities by design. The information students provide is necessary for the instructor to prep the next class; hence, asynchronous formative assessments are legitimate peripheral participation. Second, the labs are designed to be at the upper end of students’ Zone of Proximal Development; by completing the sequence of the activities in the introductory calculus course, the student formalizes his/her spontaneous concept of what a limit is in terms of the approximation metaphor. Thus, these activities provide a framework that scaffolds students’ thinking about limits from spontaneous concepts to a more structured and organized scientific one. The formative assessments allowed me to evaluate the quality of the scaffolding and help instructors decide what to emphasize in class the next day.

Formative Assessment

The formative assessment theory that framed the design of the formative assessments in my study is one proposed by Black and Wiliam (2009). After explaining the major components of this framework (see Figure 3), I discuss how the formative assessment framework, which was originally created to describe the benefits of synchronous formative assessment in elementary school students, could be used in the context of an undergraduate mathematics course.

	Where the learner is going	**Where the learner is at right now**	**How to get there**
Teacher	1. Clarify learning intentions and criteria for success; understanding and sharing learning intentions and criteria for success; understanding learning intentions and criteria for success	2. Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding	3. Providing feedback that moves learners forward
Peer		4. Activating students as resources for each other	
Learner		5. Activating Students to be the owners of their own learning	

Figure 3. Formative assessment framework (Black & Wiliam, 2009).

The theoretical framework outlined in the above diagram suggests five purposes for using formative assessment in the classroom: (a) defining success for everyone, (b) preparing class based on where the learners are at, (c) providing feedback that scaffolds learning, (d) providing a common base so everyone has a starting place in peer learning activities, and (e) raising student “ownership” of learning. These five purposes are divided into three foci of analysis: teacher, classroom, and individual learner.

The first purpose of formative assessment is to help the class and learners understand what “success” means in the particular activity. Formative assessments give the teacher data on what the students believe the learning outcomes and success mean by initiating the conversation. The instructor can then clarify students’ thinking about goals and objectives; this clarification can either serve as starting point for classroom discussion or as feedback for the individual learner.

The second purpose of formative assessment is to help instructors design and implement classroom tasks tailored to match the current needs of the students. For synchronous formative assessment, generally in the form of oral questioning in class, these adjustments help to guide classroom discussion in a productive direction. Asynchronous formative assessment, which generally takes place before a lesson, is used to make a lesson plan or tentative discussion outline that can be refined in class using synchronous formative assessment. In both cases, the purpose of the formative assessment is to provide the instructor real time information on students' current level of understanding so the lesson can be adjusted to fit the needs of the students.

The third purpose of formative assessment is to provide feedback that moves the learner forward. Black and Wiliam (2009) argued that the feedback from formative assessment (if not the act of formative assessment itself) is the scaffolding that helps learners advance through their ZPD. To provide feedback that scaffolds learning, Black and Wiliam recommended that feedback from formative assessment guide students to better understanding rather than telling students what they did incorrectly.

The fourth purpose for formative assessment is to give students a starting place to talk with each other about the material. The intention is to use the formative assessments as ice-breakers for cooperative learning and peer interaction. The final purpose of formative assessment is to give students ownership of the material. That is, formative assessment is supposed to evaluate and scaffold improvements in how students think and feel about the subject. Constructs included under the umbrella of ownership are included but not limited to metacognition, self-monitoring, attribution, and motivation.

Overall, the purpose of formative assessment seems to be providing a place where students and teachers can map out expectations--a type of peripheral participation. This information can then be used by the teacher to plan lessons and provide students' feedback. Then students can use this information to learn from each other; this initial act of independent learning should have positive meta-cognitive and affective effects. While all five parts of the framework can apply to asynchronous formative assessments, I focused my analysis on parts two and five. In particular, I investigated how formative assessment increased student self-monitoring and how foreknowledge of students' current understandings helped instructors structure class discussions.

Researcher Stance

In addition to the theories that framed the inquiry for the study, my own experience as a researcher, particularly in qualitative research, is important to disclose because I was the instrument of data analysis. In this section, I discuss my researcher stance as it related to the study; I discuss my interest, experience, qualifications, and how my background might have biased data analysis.

I first became interested in formative assessment as an undergraduate working in the mathematics tutoring center. The professor who taught the calculus sequence required that we complete pre-lecture worksheets before every class. These worksheets looked similar to the weekly quizzes we would take at the end of the week and each worksheet was usually returned about a week after we completed it. Those of us who worked in the tutoring center, who easily earned A's in calculus, felt these worksheets were just busywork and a complete waste of time. However, the students we tutored, who were earning high D's to low B's in the classes, insisted that these worksheets greatly helped them prepare for class and understand the material.

Years later in my first qualitative research course in graduate school, I decided to investigate this phenomenon and discovered those pre-lecture worksheets were probably a type of formative assessment. While plenty of literature indicated formative assessment benefitted students in elementary and secondary school, none of the literature could explain why formative assessment was effective for undergraduates.

I now have experience with formative assessment, both as an instructor and a researcher. I have been using and refining asynchronous formative assessments for the courses I have taught in the past four years--introductory calculus, introductory statistics, remedial algebra, and mathematics for pre-service elementary teachers. When I researched formative assessment in undergraduate classrooms, my findings aligned with the research conducted on younger populations of students but I never felt like I completely understood why formative assessment was helpful to my students.

I am qualified to conduct research on how formative assessment benefits students for three reasons. First, I completed all of the necessary coursework and examination requirements to begin dissertation research under the guidance of my chair and committee. Second, I have published articles and presented at peer reviewed conferences on formative assessment (Dibbs & Blasjo, 2011; Dibbs et al., 2013; Dibbs & Oehrtman, 2012; Dibbs & Yacoub, 2010); the project I conducted was a natural extension of my previous work. Third, I have conducted a pilot study, which I describe in the next chapter, which helped me refine the research question and methods I used in this dissertation.

Experience with a particular topic can be a double edged sword when conducting qualitative research since my prior experience could also have made me more likely to see the results I expected to see. I also believe formative assessment benefits students so I am more inclined to analyze the data in a positive light. To ameliorate my possible biases, I took several steps to increase the trustworthiness and credibility of the project, which I have described in Chapter V.

Summary

Now that I have described my research question, examined the literature, and framed my inquiry with an appropriate theoretical perspective, I conducted a pilot study to help solidify the methods for the dissertation. I present the results of this initial inquiry in the next chapter.

CHAPTER III

METHODS

Pilot Study

Purpose

I conducted a pilot study in the fall semester of 2011 to clarify the research question and methods for the dissertation study. I had six goals I wanted to accomplish through the completion of the pilot study: (a) clarify the main research question, (b) develop and refine the formative assessments used in the data collection, (c) decide what data were appropriate and feasible to collect, (d) develop data collection protocols, (e) develop a classroom observation protocol and pilot student interview questions, (f) refine the standards of evidence for the themes that emerged from the pilot study data. Before describing the design of the pilot study in the next section, I briefly describe the rationale and outcomes for each of these goals.

Design

With the goals in mind for the pilot study, I began by gaining access to an introductory calculus class taught by an instructor experienced in teaching approximation framework activities. In this section, I describe the design of the formative assessments and the pilot study, data collection, and data analysis. I met early in the Fall 2011 semester with Dr. Michael Oehrtman, who created the approximation framework activities as part of ongoing design research, to discuss how to develop formative assessments for the first multi-week activity, which began in the third week of classes. The formative assessments used in the pilot study can be found in Appendix B. The formative assessments were emailed to students after class on Tuesdays and were due back to Dr. Oehrtman by email Tuesday night. After conducting a preliminary analysis of the formative assessments, I wrote a brief journal entry on what was successful and unsuccessful about each formative assessment including the data collection method and the wording of the questions. The two introductory calculus instructors besides Dr. Oehrtman and I provided verbal feedback about what was successful in their classrooms but neither section was formally part of the pilot study.

This section of calculus was one of the three grouped classes for the math major first year experience group; all but the eight students who registered late were first semester freshmen who were either secondary education mathematics majors or elementary education majors with a mathematics concentration. The class followed a similar schedule each week (see Table 4). I helped facilitate the group work activity on Tuesdays, along with an undergraduate teaching assistant, and I also observed class on Wednesday.

Table 4

Generalized Weekly Schedule

Monday	Tuesday	Wednesday	Friday
New content is introduced, preparation for group work	Group work activity, formative assessment completed	Formative assessment intervention, group work summary, new content	New content or exam

The data collection for the pilot study began each week on Tuesday. While students worked at their tables, Dr. Oehrtman, the undergraduate teaching assistant, and I circulated through the room and facilitated group work by answering student questions and providing limited guidance when they got stuck. I carried a clipboard with me during the group work and took limited fieldnotes during class; most of my observations were head notes (Wolcott, 2005). After class, I expanded my mental head notes and written fieldnotes into a longer and more cohesive written account; this was done within 48 hours of the group work activity.

Dr. Oehrtman would email the formative assessment to the students later on Tuesday morning and the students needed to email their responses to him by 9 pm that night; these emails were forwarded to me after the deadline. For the content-based questions, I made a note of all of the idiosyncratic thinking students displayed and which questions were left blank. On the open response questions, I noted what connections students did and did not make between the current group activity, the approximation framework, and mathematics learned in other classes. I also noted any questions students asked in their formative assessment. After the analysis, I sent a list of three or four most common student difficulties; this became the basis of the first 15–20 minutes of the next class on Wednesday.

At the end of the unit, I collected a copy of participants' exams. During the third unit, I only collected the conceptual question based on the formative assessment. However, after speaking with participants informally during group work sessions, I realized that knowledge of the approximation framework could appear in the other questions on the exam and copied the entire test for all subsequent tests.

I chose to limit the scope of the pilot study to the two multi-week activities during the second and fifth unit of the semester. These two activities were the longest activities in the semester and were when the two major concepts of introductory calculus, derivatives and integrals were introduced. Also, the approximation framework was intended to establish a unifying link between the differentiation and integration instruction so piloting the formative assessments and analyzing preliminary results during these two critical units was my highest priority.

I analyzed the pilot data in three rounds. In the first analysis, I coded all of the formative assessments for idiosyncratic student responses, explicit evidence of transfer, implicit evidence of transfer, self-monitoring, and peripheral participation since these themes were suggested by the literature review.

In the documents, *idiosyncratic student responses* were defined to be all non-computational errors on the formative assessments. These were first identified and then sorted into categories of similar errors. *Far/near transfer* was only coded on the definite integral lab and tests; this code was used when students correctly used the approximation framework language on the integral lab or when they used the approximation framework as a problem solving heuristic on their exam. Students had to be able to identify that they were using the approximation framework explicitly; otherwise, correct applications of the approximation framework coupled with a statement that there was no relationship between the derivatives lab were considered *implicit transfer*. The final formative assessment code was *self-monitoring (good/poor*). A student was *showing good self-monitoring* if their computational work matched their statement (e.g., an incorrect computation coupled with a statement that the work was probably wrong). Poor self-monitoring was when there was no agreement between the computation and the statement.

After I completed this coding, I analyzed the fieldnotes using the same standards of evidence I developed from the formative assessment coding. I also modified the definition of peripheral participation to include classroom behaviors. Two additional codes were considered for the fieldnotes. A student was a *peripheral participant* in their group if they were listening and on task but not offering solutions during group work. Students were also coded as *peripheral participants* if they took notes during the discussion on Wednesday or asked questions on their post lab. A student was considered a *central participant* if he/she led his/her group or asked questions during lecture.

Pilot Study Results

The research question I used in this section was the original research question I used to guide the analysis of the pilot data; they are slightly different from the research question in the first chapter. After stating the original research question, I briefly discuss the results of the preliminary analysis of the pilot study data.

> Q1 What are the functions of formative assessment as outlined by Black and Wiliam (2009) that scaffold students' peripheral participation, self-monitoring and far transfer of approximation framework concepts (Oehrtman, 2008) from one context to another in an introductory calculus course?

To my surprise, one area where formative assessment seemed to have no effect was on the type and frequency of initial student errors. Typical student misconceptions about identifying over- and underestimates, limits, and confusing error bounds with ranges of values appeared at least once per paper for all students earning a C or lower (approximately the bottom 10 students) in the class. However, the instances of idiosyncratic student thinking began to disappear from future formative assessments and exams for those students who attended class on the intervention day with mistakes on their formative assessments (five students). Students I talked to credited their improvements to the intervention lecture based on the formative assessments rather than the formative assessments themselves. This intervention was based on students' formative assessment responses but was not actually part of the formative assessment as Dana told me in an informal conversation after class: "I don't know how much doing the formative assessment helps me understand, but it sure does let me know what I don't know. Then the next day, I hear something and then, I know I have to really listen to that part."

On the first formative assessment, 8 of 20 (40%) students were able to answer the content question correctly and 28 of 35 (80%) answered the same content question correctly on the exam. The students who missed the test question did not turn in formative assessments. Students answered the test question based on formative assessment correctly more frequently than on the formative assessment, which could indicate evidence of near transfer. Data collection on whether or not students displayed evidence of far transfer of the approximation framework is still ongoing. However, based on the comments from students, it is possible that the combination of group activities, formative assessment, and intervention lecture facilitated transfer for future learning. When students' misconceptions were pointed out in a way that allowed students to avoid public admissions of their misunderstandings and followed by a lecture intervention, students significantly outperformed in later transfer tasks of a control group without any intermediate feedback (Barnet & Ceci, 2002). Hence, evaluating students' current level of knowledge and skills and providing feedback that moved learners forward appeared to be the two main functions of formative assessment at play in this question but these two functions appeared to be intrinsically tied to increasing students' ownership of the material.

Preliminary Implications and Reflection

The pilot study had three major implications for the dissertation study. I anticipated that learning trajectories would be a central theme of this project but that was not supported by the pilot data. This construct was removed from consideration in the larger study. Based on informal conversations with students, formative assessment seems to scaffold students' self-monitoring of what they do and do not understand as well as connections to prior material. This type of peripheral participation had two consequences for the class: it helped facilitate connections between concepts (Georghaides, 2000; Grotzer & Mattlefehldt, 2012; Ning & Sun, 2011) and made interventions more effective. After working with this somewhat unwieldy theoretical perspective throughout the pilot study, with the guidance of my committee I reworked the dissertation study to consider a simpler central construct--the Zone of Proximal Development. The theoretical framework given in the prior chapter reflected those revisions.

As the pilot study progressed, I realized several aspects of the pilot study needed to be changed before I implemented the dissertation study on a larger scale, beginning with the research question. I realized the research question I used for the pilot study was too broad in scope; so I changed the main research question and removed the subquestions originally in the dissertation proposal to reflect narrower, more do-able goals. Also, since students' actual learning trajectories were not qualitatively different from students learning the approximation framework without formative assessment, I changed the research question to eliminate learning trajectories as a construct.

As I grappled with altering the research question for the dissertation study, I realized I also needed to adjust the data collection. In particular, I needed to collect more data if I wanted to conduct a credible study on students' Zone of Proximal Development, peripheral participation, and self-monitoring; I needed to revise the formative assessments. I decided to collect students' homework assignments in the dissertation study and conduct student interviews since having these data in the pilot study would have helped me find a more complete answer to the research question. The formative assessments, which were called prelabs and postlabs in the next semester, were revised; prelabs and postlabs were written for all seven labs assigned. These new formative assessments appear in Appendix C. Analyzing the data for the pilot study helped me think about how I wanted to analyze the data in the dissertation project. I did not use the pilot framework in the dissertation; the new coding framework is described later in this chapter.

Methods

Overview

After the pilot study, I used the following research question for the dissertation:

> Q1 What are the functions of formative assessment that scaffold students' peripheral participation and productive engagement in their ZPD for approximation concepts from one context to another in an introductory calculus course?

The remainder of this chapter describes the methods used to answer this research question. After explaining the methodological framework, I discuss the design, data collection procedures, instrumentation, data analysis, expected results, work plan, and limitations of the dissertation study.

Methodological Framework

Before detailing how I conducted the dissertation study, I need to situate this research into a methodological paradigm that aligns with both the theoretical framework and the research question. I collected both detailed case study data from nine participants and achievement data from the whole class. I used a QUAL-quan simultaneous mixed design in this study (Creswell & Clark, 2007). This design was defined as a primarily qualitative study with supplemental quantitative analysis where the quantitative and qualitative data were collected in the same research cycle.

The data for this project were all collected in the same semester, hence the simultaneous designation. The quantitative data provided a larger context for the qualitative data. Through the statistical analyses of how all of the students performed throughout the semester, the context allowed me to determine how representative the case study students were of their peers. The combination of qualitative and quantitative data allowed for a broader base of evidence from which to answer the research question.

I used three data collection methods to answer the research questions: document analysis, observations, and interviews. A document analysis of students' written work allowed me to see how, if at all, students progressed through their ZPD with regard to the approximation framework changes throughout the semester. I also conducted observations the day of the lab and the day after. However, the best intentions of data collection cannot result in a well-executed project without a solid design for a study. In the next section, I provide an overview of the setting and basic procedures of the dissertation project.

Study Design

The dissertation study was conducted at a mid-sized doctorate granting university in the Rocky Mountain region. The university enrolls approximately 11,000 undergraduates and 2,000 graduate students every year. Sixty percent of the undergraduate population was female and 20% of the undergraduates self-identified as a member of an ethnic minority. The university was originally a normal school and education is still a common major on campus. The five most popular majors were interdisciplinary studies (elementary education), business administration, psychology, dietetics, and English language. Eighty-eight percent of the undergraduates were residents of the state in which the university is located and 55% of these students were the first students in their families to attend college. The university had a first year retention rate of 70% and 46% of students who enrolled at the university graduated with a Bachelor of Arts or Bachelor of Science degree.

Three different introductory calculus courses are offered at the institution: Calculus I, Calculus for the Life Sciences, and Topics in Calculus. The latter two courses are intended to be terminal mathematics courses for biology majors, biology pre-professional majors, and business majors, respectively. Calculus I was the only course included in the dissertation study. This class is intended for all other majors who are required to take calculus. Most of the students enrolled in this class major in elementary mathematics education, secondary mathematics education, mathematics, chemistry, meteorology, or geology; however, occasionally business majors or biology majors intending to pursue graduate work enrolled in Calculus I instead of the suggested topics courses for their majors. A few graduate students from other disciplines enroll in Calculus I each year to complete the admissions requirements for their programs as well. The gender distribution of Calculus I was similar to the university proportions but there were generally fewer minority students enrolled in calculus. Approximately half of the students enrolled in Calculus I had prior experience with the course content--either by taking Advanced Placement Calculus and failing to earn the credit or by failing the course at this or another post-secondary institution.

Calculus I, a four credit course, is the first course in a three 4-credit course calculus sequence but only some science majors, mathematics majors, and mathematics minors continue on to the second course. In the first semester, after reviewing pre-calculus concepts, the introductory calculus course covers limits, derivatives, derivative shortcut rules, selected derivative applications, definite integration, the fundamental theorem of calculus, and an introduction to differential equations. The second semester course covers techniques of integration, applications of integration, and sequences and series; this course is a terminal course except for mathematics and meteorology majors. There is a Gateway exam over the shortcut formulas in both of the first two semesters of the calculus sequence; the first semester Gateway is on differentiation and the second is on integration. The third semester of the calculus sequence is over multivariate calculus.

Three sections of Calculus I were offered in the semester I conducted the dissertation study; this is a typical number of sections offered in a spring semester. Two of the sections are offered at the same time; while I collected documents from all three sections, only the two sections I observed were included in the study. The instructors of record for these two sections were a full time non-tenure track faculty member and a graduate student. Both instructors had equal experience with the calculus curriculum and formative assessment. Neither the members of the dissertation committee nor I were involved in the instruction of the courses.

This course used the fifth edition of *Calculus* by Hughes-Hallet et al. (2009). I have included a sample syllabus and course schedule of introductory calculus in Appendix D. The introductory course covered the first, second, and eighth sections of chapter 1, all of chapters 2 and 3, and selected applications in chapters 4, 5, and 6. Enrollments for introductory calculus depended on the size of the classroom in which the course was scheduled but most classes had approximately 40 students enrolled at the beginning of the semester. The percentage of students who historically earn an unsatisfactory grade or withdraw from the course is 33%, which is slightly below the estimated national average (Ganter, 2006).

During the week, instructors lectured over new material on Mondays and Fridays. On Tuesday, students worked in groups on the approximation framework activity that week; an undergraduate teaching assistant and I helped the instructor facilitate the group activities by circulating through the room, asking probing questions of students' understandings, and providing hints when groups got stuck. Students completed a formative assessment that night; the class on Wednesday spent part of the class on discussing the formative assessment and the rest of the time covering new material. In addition to the weekly formative assessments, students completed 20 WeBWorK assignments throughout the semester, prepared a written report of their own answers to the approximation framework activities, and had five chapter exams and the final.

Instructors met once a week to discuss the schedule and activities for the next week. Students' individual reports of the approximation activities were group graded during the weekly coordination meeting. Instructors wrote their own unit test but the final exam was written and graded by all of the calculus instructors.

Data Collection

The following sections detail the accessible and target study population, data collection procedures, and data handling procedures. Overall, the primary data sources were artifacts but I collected several types of artifacts, observations, and interviews in order to place the document analysis into context.

Data sources. I used five different data sources to help answer the research question: formative assessments, lab write-ups[1], final exams, observation fieldnotes, and student interviews. After describing each type of data in detail, I argue why each type of data was necessary to fully understand the research question.

The asynchronous formative assessments, which were called prelabs and postlabs during the semester, students completed were central to the dissertation project. A copy of the formative assessments I used in conjunction with the approximation framework activities can be found in Appendix A; these formative assessments were changed based on the results of the pilot study. A typical formative assessment appears in Figure 4; these assignments took students approximately 15 minutes to complete. The first part of each formative assessment asked students about the content covered in the Monday class and explored during the group activity on Tuesday--in this case, the first two questions. These questions allowed me to understand how well the class understood the initial

[1] For the purposes of this study, a lab write-up is defined to be an individual student's written responses to all of the items in the lab directions.

example and how well they applied the content of the lecture to their group work problem. The final question asked students to reflect on their current understanding of the material and invited them to ask questions they still had about the material.

Directions: Answer the following questions to the best of your ability. Responses need not be lengthy, but should answer all parts of the question. Please type your answers into this word document and email it back to [Me] at [Your.instructor@unco.edu] by [9 pm tonight].
In Activity 4, we were given information about the NASA Q36 Robotic Lunar Rover. Specifically, it can travel up to 3 hours on a single charge and has a range of 1.6 miles. After *t* hours of traveling, its speed is $v(t)$ miles per hour given by the function

$v(t) = \sin\sqrt{9-t^2}$. One hour into a trip, the Q36 will have traveled 0.19655 miles. Two hours into a trip, the Q36 will have traveled 0.72421 miles.

Consider the following table of velocities:

Time *t* in hours	0	.5	1	1.5	2
Velocity *v*(*t*) in mph	0.14 112	0.18 252	0.30 807	0.51 715	0.78 675

Assuming the speed at the beginning of each half hour, we would determine the Q36 traveled

$\frac{1}{2}(0.14112)+\frac{1}{2}(0.18252)+\frac{1}{2}(0.30807)+\frac{1}{2}(0.51715)=0.57443$ miles.

Assuming the speed at the end of each half hour, we would determine the Q36 traveled

$\frac{1}{2}(0.18252)+\frac{1}{2}(0.30807)+\frac{1}{2}(0.51715)+\frac{1}{2}(0.78675)=0.89725$ miles.

1. Use the information above to answer the following questions.
   a. What is the unknown value we are trying to approximate?
   b. In the context of this problem, what does the value mean?
   c. What are the approximations?
   d. Identify an approximation that is an overestimate. How do you know it is an overestimate?
   e. Write down a formula for the error (in words or math symbols)
   f. What is a bound on the error?
2. Now consider situation your group worked on today.
   a. What is the unknown value we are trying to approximate?
   b. In the context of this problem, what does the value mean?
   c. What are the approximations?
   d. Identify an approximation that is an overestimate. How do you know it is an overestimate?
   e. Write down a formula for the error
   f. What is a bound on the error?
3. Write a short paragraph that answers the following two questions. What mathematical concepts or phrases used so far this week do you recognize from calculus? From other mathematics courses?
4. What do you understand about approximating distance traveled? About your group's context? What questions do you still have about the material?

Figure 4. Typical formative assessment (definite integrals, week 13).

After the formative assessments were analyzed, the next documents I collected from the participants were their written homework assignments—the lab write-ups. These assignments were individual write-ups of the group work students completed Tuesday. These assignments were generally due the Friday after the last day of the group work activity. In their individual write-ups, which are analogous to a lab report in science, students were asked to explain their work in verbal, graphical, numerical, and algebraic representations; these assignments were graded by the instructors. For the purpose of this research, I re-graded all of the lab write-ups on a rubric more suited to answer the research question. This rubric is discussed further in the data analysis section in this chapter.

In addition to the three different types of written assignments I collected from students, I also observed their classes twice a week. I observed the classes on Tuesday—when the participants worked on their group activities and on Wednesday—when the intervention based on the formative assessment analysis and instructor debriefing was conducted. I describe the observation protocols I used for those two days in the data collection activities section.

The final source of data for this dissertation study was semi-structured student interviews, which were primarily used in the third analysis. I interviewed six participants from the three sections twice—nine participants in the first interview and six in the second. The interviews lasted approximately 20 minutes each time. The interviews contributed to understanding the impact of formative assessment by exploring the students' perspective. This provided a check to the analysis and increased the credibility of the findings, thereby strengthening the dissertation study.

Study population. The accessible population of the study consisted of two sections of introductory calculus in the spring semester of 2012 at the research site described earlier. The inclusion criterion for each section of calculus in the study was instructor experience with formative assessment. Since formative assessment is difficult to implement successfully the first time it is implemented (Wiliam, 2007b), I analyzed the formative assessments for the instructors throughout the semester. Since the approximation framework was not used in introductory calculus at any institutions within driving distance, these three sections represented the maximum accessible population for the dissertation; but these two sections, which contained 66 students, were more than sufficient to conduct a qualitative study.

The target population was students in calculus classes using the approximation framework; incorporating formative assessment had the potential for great benefits to students' acquisition, retention, and transfer of the approximation framework within and between contexts, which could change how the approximation framework is implemented in first semester calculus. However, the benefits of formative assessment, such as self-monitoring and facilitation of transfer, could benefit any calculus student; hence, some themes might be applicable to any calculus course.

Sampling procedures. The sample for the quantitative portion of the study was the study population described in the prior section. Three sections of calculus were offered in the semester the data were collected but two were offered at the same time. Of the two sections, I chose the instructor who had previously taught calculus.

For the qualitative portion of the study, I solicited participants from both sections of introductory calculus using the approximation framework to which I had access. I purposefully chose to interview participants at all achievement levels; participation emerged as a critical variable only in later analysis. In the third week of classes, I asked all students to consent to photocopying of their formative assessments, homework assignments, and tests.

During the derivatives unit, I conducted the preliminary data analysis in order to purposefully select participants. First, for each section of introductory calculus with participants in the study, I created a spreadsheet of students who had consented to be interviewed using their pseudonyms. I used this spreadsheet to track the grades students received from their instructor on the documents I collected, which gave me an estimate of their current grade in calculus. In addition to the grades on assignment, I made a note of the initial code of each formative assessment for completeness. I selected nine participants, five from one class I observed and four from the other, to invite to interviews. Two participants were earning an A at the time of the first interview, one was earning a B, three were earning a C, and three had either a D or an F.

After I selected the initial participants, I wrote a letter inviting them to meet with me for their first interview. These initial letters were distributed the week after the exam, with interviews conducted during the next two weeks. At the conclusion of the first interview, I scheduled a second interview with all participants and I sent an email reminder and confirmed the second interview verbally as the second interview date approached.

Since the second participant interviews were scheduled after the final day to withdraw from classes, I was not able to interview three participants for a second time. Two of the participants, one who earned an A and one who earned an A-, had work conflicts and could not find the time to schedule an interview even after three follow-ups. The third participant, one of the three who failed the course, stopped attending class after the first week of the derivatives lab and did not respond to any requests for an interview.

Data collection activities. While I collected the data throughout the entire semester, the focus was on the third, fourth and seventh lab because these activities are the most critical for acquisition of the approximation framework. The content of these labs--limits, derivatives, and definite integrals--are the core concepts in calculus and these have the only three week activities in the semester, which allow students ample time to engage with the material. Hence, the richest data were obtained during these labs. During the first two weeks of the semester, the main research goal was to establish the role in the classroom. During the derivatives lab, I conducted the first interviews and analyzed the first two labs.

Regardless of what unit the class was completing, I conducted several data collection activities every week: facilitating group work, initially analyzing formative assessments, debriefing the instructors, and observing the intervention based on the formative assessments. Each week, I also collected one or more documents in the form of formative assessments, homework assignments, or exams. I also interviewed participants twice during the semester. Each data collection event is described in the following sections.

Facilitate and observe approximation framework activities. On Tuesdays, I facilitated and observed the group work activity in the two sections of introductory calculus. I had the same role in the classroom as the instructor and undergraduate teaching assistant—I assisted groups by asking questions that were intended to redirect unproductive lines of thinking, gave hints, or checked for group understanding. The role as a group facilitator did not include tutoring, sharing the answers, or making explicit connections between concepts for students.

During the time I facilitated the groups, I also collected observation data. I used the group work observation protocol (see Appendix E) to help streamline taking fieldnotes I expanded after class. The codes I described in this observation protocol were based upon observation notes from the group facilitation of the pilot study. At the top of each page, I filled out the date and page number of the observation protocol. When I filled out each line of the protocol, the first thing I did was note the time of the observation. Next, I entered a code for the type of observation activity: (a) on task behavior (OTB), (b) formative assessments (FA), (c) connections (C), (d) peripheral participation (PP), (e) central participation (CP), and (f) leadership. I also used a seventh code to indicate unusual group activities. Each line of the protocol had one code. I described the standards of evidence for those criteria in the data analysis section. At the beginning of each group work activity, I filled out a seating chart and assigned each group a number. I noted the number of the group I observed in the next cell of the observation protocol. In the description cell, I wrote very brief notes that helped me expand the protocol along with the headnotes into code able fieldnotes as soon as possible after the observations were completed. In the final cell of each row, I made preliminary classifications of the observation by circling the appropriate words in the cell. I filled out one line of the observation protocol each time a new code was needed.

Observation of classroom intervention. When I observed the class immediately following the group activity, I arrived at class as close to the beginning of the passing period as possible. As students came to class, I made note of where they sat on a dated seating chart. Since students did not typically use computers to take notes in mathematics classes at the research site, I took the initial observation notes on paper; by conducting an activity similar to the one the students were doing, I remained as unobtrusive as possible.

I started taking fieldnotes 10 minutes before the beginning of class because the pilot observations suggested students occasionally talked about the formative assessments before class began. I observed for the first five minutes and then wrote notes on the initial impressions of the observations during the next five minutes. I have included the observation protocol I used in Appendix E. On it, I noted the time, an initial code for the actions I observed, which group I observed, and brief notes on what I saw. In the space below the first line, I took five minutes to expand the brief descriptions into more detailed notes that I could expand after class. The focus during the observations was on capturing typical vignettes of student behavior and recording a broad play-by-play of each class. I then expanded the raw fieldnotes into a longer narrative form (Emerson, Fretz, & Shaw, 1995) within 48 hours of the observation.

Document collection. I collected three different types of documents for the study: formative assessments, lab write-ups, and unit exams. I collected students' formative assessments as part of the initial analysis of these documents. The second type of document I collected from students was their lab write-ups. These assignments were collected in class by the instructor. After class, I made photocopies of each assignment. I collected and copied students' exams immediately after the exam period ended so I could obtain clean copies for analysis. I followed the procedures outlined in the data handling section to sanitize these data and prepare them for later analysis. Although I collected students' entire exams, only the common questions and common objective questions were ultimately used in the analysis and then only for triangulation (see Chapters IV and V).

Interviews. I interviewed nine participants for the first interview and six of them again on the second interview. I sampled five participants from the first class and four from the second. I used the spreadsheet I described earlier to select these participants. I used a semi-structured interview format during both interviews; the interviews lasted approximately 20 minutes (Patton, 2002). The first interview was scheduled after the second test--the sixth week of classes and the second interview was scheduled after the fifth test--the 13th week of classes. During the interviews, I followed the interview protocols I created; a copy of these protocols can be found in Appendix F. I asked all participants the same basic questions; however, due to the nature of semi-structured interviews, each participant might be asked different probing questions. I audio-recorded the interviews, which were then transcribed.

Participants and Classroom Setting

I focused mainly on the students I interviewed since I had the richest data available from them. In this section, I provide a brief description of each student and descriptions of the room. Students in groups that are numbered are in Section 3 and students in groups that are lettered are in Section 1. I did not list a table or group designation for the students in Section 1 because their groups changed for every lab.

- George: A student in Section 1. He was a sophomore majoring in biology. Normally he would be taking the bio-calculus course but he thought taking regular calculus would help his chance at getting into medical school. He was included as an interview participant since he was the leader of every group in which he participated and was one of the students willing to centrally participate in class. He completed only the first interview. This was his first time taking calculus. George is Caucasian and a first generation college student.

- Charles: A student in Section 1. He was a sophomore elementary education major with a concentration in mathematics. He was included as an interview participant because he was willing to be a central participant and was one of the few students in the second two rows of tables to show consistent engagement in the class. He completed both interviews. He took calculus in high school. Charles is Caucasian.
- Sandra: A student in Section 1. She was a non-traditional student in her third year and was majoring in Chemistry. Sandra is approximately 30 years old and was conditionally admitted into the Chemistry master's program. She was selected as an interview participant because she was consistently near the median scores and was identified by her instructor as outspoken and articulate. She completed both interviews. She had never had calculus before this course and considered herself bad at math.
- Kaitlyn: A student in Section 1. She was a freshman. She began the semester as an elementary education major with a concentration in mathematics and switched to a pure mathematics major by Week 6 of this introductory calculus course. She was selected as an interview participant

because her work was consistently good and she seemed to acquire the approximation framework more quickly than the other students. She only completed the first interview and had never had calculus before this course. Kaitlin is Caucasian.

- Lisa: A student in Section 3, Table 2. She was a freshman majoring in meteorology. She had never had calculus before taking this course. She was included as an interview participant for several reasons. First, she was legally blind and many of the other students used visual metaphors when talking about understanding. Second, her group was very diverse in terms of ability and they collaborated better than the other groups; I wanted to understand their group from all perspectives. Third, she struggled with the material even though she was better than most at the algebra; she ultimately failed the course. She completed both interviews. Lisa is Caucasian and a first generation college student.
- Leonard: A student in Section 3, Table 2. He was a sophomore Chemistry Education major. He was included as an interview participant because his group was so interesting and because he was one of the few students earning a D who was willing to interview with me. He had never taken calculus before. He completed both interviews. Leonard is Caucasian.
- Emily: A student in Section 3, Table 2. She was a freshman Mathematics major (secondary education). She was selected as an interview participant for several reasons. First, she was one of the few students in the class who took the pre-calculus course at the research site. Second was her group

membership. Third, she seemed to have a deeper understanding of the approximation framework than most of the other students. Finally, as a Korean-American, she was one of the few students in the class who was not Caucasian. She completed both interviews.

- Tre: A student in Section 3, Table 8. He was a graduate student taking calculus as a requirement of his conditional admittance to the physical therapy master's program. He was included as an interview participant for four reasons: (a) his group membership, (b) the unusual reason why he was taking calculus, (c) he asked questions during class when he did not understand, and (4) he was one of the two African American students enrolled in calculus this semester. Tre only completed the first interview.
- Brandon: A student in Section 3, Table 8. Brandon was a junior Physics major. Brandon was not selected as an interview participant. However, both of his group members credited him with being the leader of their group even though Eva finished the semester with a B+ and he finished the semester with a C. Brandon is Caucasian.
- Eva: A student in Section 3, Table 8. She was a sophomore majoring in Biology. She was selected as an interview participant because of her group membership and because she deferred to Brandon even though she had the highest grade in the group. She is Mexican-American and the only Latina student enrolled in any section of calculus the semester the data were collected. She completed both interviews.

Figure 5 is a map of the classroom in which almost all observations took place. Section 1 always met in this room and Section 3 met in this room every day except when they did labs. On those days, Section 3 was in a classroom with individual desks that students turned toward each other to create spaces to collaborate.

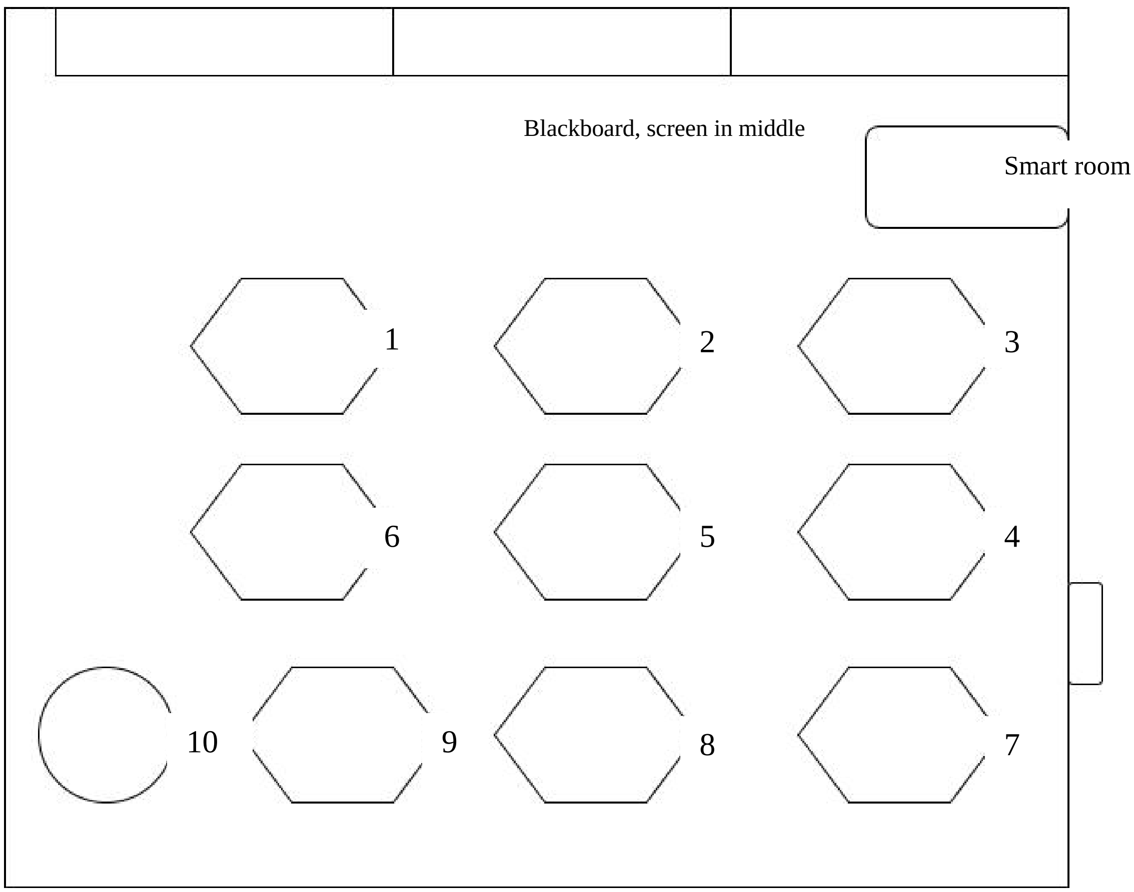

Figure 5. Classroom map.

Data Handling

There were three major stages in which I needed to manage data to ensure the confidentiality of the participants: collecting the raw data, removing participants' identifying characteristics from the raw data for analysis, and storing the raw and sanitized data. In the paragraphs that follow, I describe the steps I took to ensure participants' confidentiality was preserved when I sanitized and stored their data.

Each type of data was stripped of identifying participant characteristics as soon as possible after collection. When I recorded observation notes, I used participants' initials rather than full names; raw observation notes were primarily headnotes with enough written information to help me organize the later write-up (Wolcott, 2005). I expanded the raw fieldnotes on the same day of observation; when the fieldnotes were expanded, I replaced participant initials with his/her pseudonym. Participants' exams and homework assignments were copied as soon as possible after completion and before they were graded. I made both a hard copy and a pdf file of the tests. I covered the participant's name on the test and replaced it with his/her pseudonym during photocopying. Participants' formative assessments were saved with their pseudonym after I graded them. Identifying characteristics were omitted from the summary report sent to their instructor; using email was an acceptable risk to participants since the email was contained on the University's internal server.

I stored both a hard and an electronic copy of the raw and de-identified data. The original hard copy of the raw data was in locked storage to which only I had access; electronic copies of the raw data were encrypted, saved on a flash drive, and stored in a locked desk drawer. Hard copies of the raw data will be destroyed upon completion of the study and the electronic copy will be destroyed after dissemination of the results. I stored hard copies and electronic copies of the sanitized data in the same way but the sanitized data were not stored with the raw data.

Instrumentation

Since the central data sources of the dissertation study were asynchronous formative assessments and unit exams, it is worth briefly discussing the quality of these instruments. I used Thorndike and Thorndike-Christ's (2011) Reliability, Validity, Standardization, and Practicality (RSVP) instrument. In the paragraphs that follow, I explain how the formative assessments met the criteria or, in the case of standardization, why not meeting the criteria failed to be detrimental to the study.

As the intent of formative assessments is to capture a brief snapshot of students' current understandings, inter-rater reliability is the most important form of reliability. To make certain I was coding the analysis consistently, I took two different steps to ensure I was consistently rating the formative assessments in the same manner. I developed and used standards of evidence tables as a rubric to code the data. I outline the initial standards of evidence tables I used based on the pilot data in the following section. The coding procedures section after that explains how I changed the standards of evidence tables as new themes emerged. I employed similar strategies when coding the other documents, exams, and homework write-ups to ensure I was rating the documents in a consistent manner.

The lab write-ups were analyzed quantitatively in the second analysis. The reliability of those assessments must be at least 0.6 to be considered reliable enough to analyze (Thorndike & Thorndike-Christ, 2011). The KR-20 values for the three summative labs—the limits lab, the derivatives lab rewrite, and the definite integral lab—were 0.83, 0.72, and 0.78, respectively, so this criterion was satisfied. Since the initial derivate lab write-up had a very large variance in student scores, it had a KR-20 value of 0.16, so this assessment was not used in the quantitative portions of the analysis.

The second criterion of high quality data collection, standardization, is less crucial in a qualitative study. We urged students to take no more than 15 minutes on the formative assessment. The fact that the assessments were graded for completion and rarely resembled prior work students could use as a template helped minimize the temptation to provide answers other than the students' current thinking. The exams covered the same sections using similar problems with different numbers, a compromise between standardization and test security. However, with two sections of introductory calculus taking the exam on the same day at the different times, it was an unfortunate but necessary compromise. Since I was not attempting to incorporate any quantitative components into the study, this lack of standardization should not negatively impact the results.

Validity, the third criterion of high quality data collection instruments, was the most important facet of instrumentation for the dissertation study--if the formative assessments are not valid, any conclusions drawn about how to alter instructions to suit students' current needs would be fundamentally flawed. Creating formative assessments with a high degree of face validity was one of the central goals of the pilot study. I designed the formative assessments I used in the dissertation study after consulting with Dr. Oehrtman, an expert in the approximation framework, to capture key ideas from each activity. After each formative assessment was completed in the pilot study, I journaled about what was and was not successful about the formative assessments and modified formative assessments for data collection in the dissertation study.

The final criterion of high quality data collection instruments was practicality. I balanced the levels of reliability, standardization, and validity of the data collection instruments against what was practical for this study. Furthermore, the asynchronous formative assessments were a highly practical means to gauge students' current understanding. The initial analysis of the formative assessments for the instructor debriefing took me under 30 minutes for the pilot after I factored out the data collection activities unique to the study. This additional grading time is not overly burdensome for faculty interested in adopting formative assessment in their classrooms; this time could be shortened further if technology such as the Just in Time Teaching platform or course management software was used. Low levels of additional grading burden are one factor that helped formative assessment be more widely adopted in undergraduate mathematics classrooms.

For the quantitative analyses of the whole class data, I conducted a preliminary analysis of the data to confirm that the assumptions for the statistical tests were met. The results of the normality tests for each sample used in an ANOVA appear in Table 5. Although one set of scores was not normal and these samples were not random, ANOVA was robust to these assumption violations and was still an appropriate analysis. Normality was checked using the Shapiro-Wilks Test and all analyses were conducted with SPSS.

Table 5

Normality Tests for Analysis of Variance

	Limits, Items Discussed in Class	Limits, Items Not Discussed in class	Limits, all items	Definite Integrals; Items Discussed in Class	Definite Integrals; Items Not Discussed in Class	Definite Integrals, All Items
Regular	0.53	0.69	0.76	0.03*	0.23	0.12
Sporadic	0.13	0.38	0.28	0.19	0.49	0.30
Nonparticipant	0.17	0.07	0.12	0.72	0.41	0.51

*Significant non-normality

The data for the derivatives lab sufficiently satisfied the assumptions of an ANCOVA. For the ANCOVA analysis of the derivatives lab, there was no need to test for multicollinearity since only one covariate was used. The homogeneity of variance assumption was satisfied ($p = 0.21$), as was the homogeneity of regression slopes ($p = 0.302$). All of the covariate and dependent variable samples were sufficiently normal (see Table 6). The skewness of the scores suggested that the distributions were sufficiently symmetrical; the kurtosis values were also acceptable (skew = 0.78, kurtosis = -0.866). A more detailed description of the assumption check may be found in the previous chapter. I used the score on the first submission as the covariate and the score on the second submission as the dependent variable. There was a significant linear relationship between the covariate and the dependent variable ($r^2 = .41$) and the homogeneity of regressions condition was satisfied ($p = .306$). Finally, a Shapiro-Wilks test indicated that the residuals were also sufficiently normal ($p = 0.08$), and error terms were uncorrelated ($p = 0.293$).

Table 6

Normality Tests for Analysis of Co-Variance

	Initial Submission (Covariate)	Revised Submission (Dependent Variable)
Regular	0.33	0.52
Sporadic	0.45	0.56
Nonparticipant	0.12	0.75

Data Analysis

Quantitative analysis. The 54 students in the two sections of calculus were grouped into three participation levels; the justification for this decision may be found at the beginning of Chapter IV. There were 23 students classified as Regular participants in formative assessment (completed at least 7/12 [58%] formative assessments throughout the semester); 15 students classified as Sporadic participants in formative assessment (completed at least one but no more than six formative assessments all semester), and 16 students were classified as Nonparticipants in formative assessments (completed zero formative assessments). It is worth noting that although there were differences in how often these students completed the formative prelabs and postlabs, the proportion of students at each participation level completing the lab write-ups was not significantly different until the definite integral lab ($p = 0.60,\ 0{,}25,\ 0.002$, respectively). The difference on the final lab was the Nonparticipants in formative assessment, most of whom were mathematically eliminated from passing the course at the time, turned in the definite integral write-up at a lower rate than the other two participation levels.

Since the lab write-ups were graded by each instructor separately, I had no way to verify inter-relater reliability. The solution to this was to re-grade all of the student work before analysis. Since each question on the lab asked students to produce several representations of the approximation framework, I created the 20-item code sheet seen in Figure 6. All of the items were scored as 1 (correct) or 0 (incorrect/blank). Items where the only mistake was a transcription error and where the student calculated the correct answer but wrote the number down incorrectly in the answer were considered correct. The total score was the sum of the 20 items and the scores on the items discussed/not discussed in class were calculated as sub-scores.

	Contextual	Graphical	Algebraic	Numerical
Unknown Value				
Approximation				
Error				
Error Bound				
Method to Achieve Desired Accuracy				

Figure 6. Code grid for quantitative and case study analysis.

For the limits and definite integral labs, three different ANOVAs were used: differences in mean total score across participation levels, differences in mean total score across participation levels on the questions discussed in class, and differences in mean total score across participation levels on the items not discussed in class.

On the derivative lab, students were given feedback on their initial submission and asked to correct their errors. For this lab, I used an ANCOVA to determine if there was a difference in mean total score across participation levels on the rewritten derivatives lab after taking the score on the initial write-up into account. I excluded all students from this analysis who did not complete both an initial and a rewritten derivatives lab.

Coding procedures. When I coded the data, I used the following procedures to conduct the analysis. I analyzed the data three times using three slightly different theoretical lenses: (a) collaborative ZPD, (b) scaffolding ZPD, and (c) the spontaneous/scientific ZPD.

Since I analyzed all of the formative assessments as they were collected, the most natural initial analysis of the data was chronological. After developing the revised standards of evidence, I conducted an initial analysis of the data for all three characterizations of the ZPD. These standards of evidence were developed throughout the data collection process and the following fall semester.

During the subsequent fall semester, I attempted to contact participants via email for additional member checks but I also made use of peer checks and the expertise of my dissertation committee. By the end of the second round of coding, I had an understanding of the research question at a macro classroom level; I began to structure the results chapter at this time. During the analysis, it became clear that frequency counts were not sufficient to convey any meaningful patterns in the data. Thus, I conducted an item analysis to investigate if there were any underlying patterns in the frequency tables. An explanation of this analysis may be found in Chapter IV.

The standards of evidence in the following section were developed through an iterative open coding process (Corbin & Strauss, 2008) through which I operationalized the constructs in the theoretical literature and attempted to find examples within the documents of the section of calculus not used in the final analysis and the observation data from the pilot. These data were used as a new pilot study so as not to bias the analysis of the dissertation data. After conducting this analysis, I used the expertise of my committee to refine the definitions. There were six iterations before the final standards of evidence were developed.

Standards of evidence. I used two different analyses to answer the research question. Each analysis corresponded with one of the two characterizations of the Zone of Proximal Development most commonly found in the data: ZPD as scaffolding and the ZPD as an interaction between students' spontaneous and scientific conceptions of a topic, i.e., the approximation framework. The third characterization of the ZPD, solving problems as a group that the individual group members could not, was rarely seen throughout the semester. I discuss why this was the case at the beginning of Chapter IV.

To determine where students received scaffolding that enabled students to complete a task successfully that they could not do independently, I began with an error analysis of students' final lab write-ups ($n = 66$). Only the three labs directly a part of the approximation framework[2] were used in this analysis. After classifying the portions of the lab in the appropriate cell of the approximation framework, I classified each of the student responses in the 20 cells as either correct or incorrect[3].

After analyzing what students could and could not successfully complete on their final reports, I next analyzed the fieldnotes for the days the instructors provided the additional scaffolding. Here I noted which students were present. Next, I analyzed the

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[2] Lab 3 (Limits), Lab 4 (Derivatives), and Lab 7 (Definite Integrals)

[3] All questions with the exception of numerical approximation and numerical desired accuracy are one step responses. On those two cells, transcription errors were also counted as correct.

formative assessments on which the scaffolding was based. I used two codes on these formative assessments: problems and problematic issues (see Table 7).

After coding the formative assessments, I checked the fieldnotes from my observation of students' lab activity during class to triangulate the coding of the formative assessments. Only the two sections I directly observed throughout the semester, Sections 1 and 3, were used in this coding because without the observations of Section 2, I was unable to triangulate the coding for those students. After classifying all students into one of three levels of participation, I used relative probability to analyze the differences in performance on the lab write-up. The description of the participation groups and this analysis appear at the beginning of the relevant section in Chapter IV.

Table 7

Formative Assessment Codes

Code	Definition	Example
Problem	A concept that a student explicitly identifies as a concept they do not understand	"Now that I have completed this formative assessment, I realize that I don't really understand how the definition of the derivative and this picture of the derivative fit together" –Lacey, FA 4B
Problematic Issue	A concept that a student has mischaracterized or an error on a content question that the student does not identify as problematic	"I will use y values to approximate the derivative of the function at 5" -Brandon, FA 4A

The second analysis focused on the characterization of the Zone of Proximal Development as the interaction between students' spontaneous and scientific reasoning about a topic. After an iterative open coding process where I looked for codes in the literature, I found 16 codes that described spontaneous reasoning, scientific reasoning, or evidence of progression through the Zone of Proximal Development or were codes that provided a context to the reasoning and related this coding to the scaffolding characterization of the Zone of Proximal Development I previously described (see Figure 7). The labs were coded for this analysis in a full page version of Figure 6. Each cell of the approximation framework was classified using one of the codes in Figure 7; after the initial coding was complete, I went back through the fieldnotes and interview data to give each cell a secondary context code.

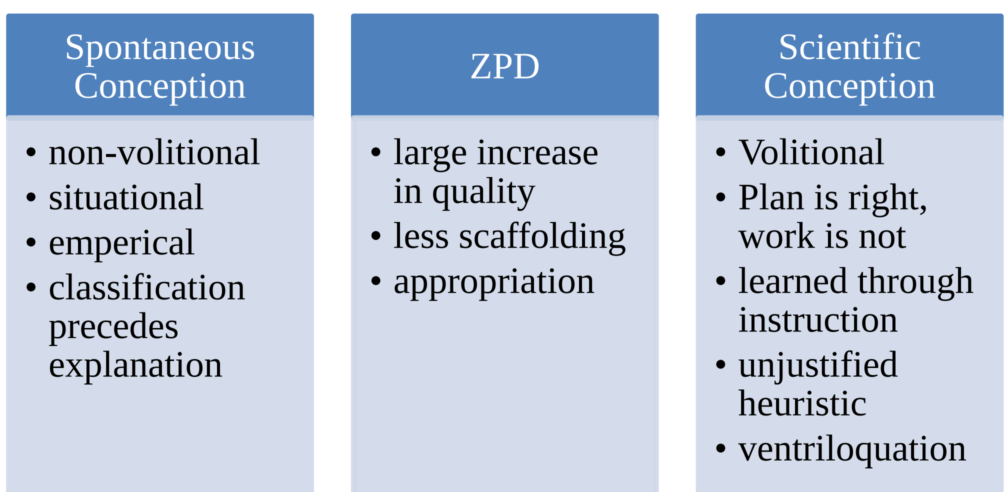


Figure 7. Spontaneous/scientific conception coding scheme.

After describing the three context codes, I discuss the definitions and standards of evidence for spontaneous conceptions, progression through the Zone of Proximal Development, and scientific conceptions in turn.

Definition taught before the lab. This code was used on personal communications with instructors. I did not observe the class before the lab so the instructors would tell me what they had covered in class the day before in order to prepare students for the lab. Since some of the more recent literature suggested the ideal pedagogy for helping students progress through their conceptual development was to bracket spontaneous activity with direction instruction and re-teaching, these were concepts we would expect students to advance the most on.

Spontaneous activity during lab. This code was used on the fieldnotes taken during the lab days. On the field notes, I made note of which portion of the lab groups was struggling with and asking for help. When they were able to start the problem, I made note of their initial ideas. The criteria for this code were the same as the spontaneous code.

Concept re-scaffolded after intervention. The code was used on the fieldnotes on the intervention days. I used this code to track which concepts were covered during the intervention and made notes of how well the intervention aligned with the list of concepts I suggested be covered in the intervention the night before after I had analyzed the post labs. These codes were used on the lab-related documents and one question on the second interview (All pre-labs, the derivatives lab “rough draft”, and the final limits, derivatives, and definite integral lab write-ups). This code was triangulated by the interview data since students were asked in the interviews to retroactively explain both their solutions and their reasoning on the aforementioned documents. Hence, the analysis for this characterization of the Zone of Proximal Development was on a much smaller subset of students than the scaffolding one ($n = 9$). All of the criteria below were adapted to the data from Vygotsky (1978).

Not volitional. Students were not consciously applying a heuristic and could not give a reason in their interview for why they gave the solution they did. For a cell to be coded in the code sheet as non-volitional, the interview participant, even after probing, was unable to give any reasoning for why they provided a particular solution. It should be noted that this code and all of the codes that followed did not presume that the student's solution was correct or incorrect.

Situationally meaningful. The reasoning the student displayed only made sense in the original context in which it was learned but not the context in which the student was applying the idea. This code was used on the documents associated with the limits lab and the derivatives lab but not the definite integral lab. For a cell to be coded as situationally meaningful, three criteria had to be met:

1. Students were able to produce the solution in that cell on a previous lab and give correct reasoning for that solution.
2. On the next lab, the student used the solution on the previous lab *unjustified heuristic* to produce the solution that was contextually inappropriate.
3. The student's reasoning for their use of an unjustified heuristic in their interview was that it had worked on the previous lab.

If all three conditions were met, the cell in the earlier lab was coded as *situationally meaningful* and the later lab was coded both as *empirical* and an *unjustified heuristic*.

Based on empirical data/personal experience. The explanation students gave or solution they provided had been previously successful in their experience. There were two different cases where this code was used.

1. Students produced a solution that was appropriate for a previous class but was not what the lab directions asked for. The most common occurrence of this was in the graphical context of labs 3 and 4. Students would produce graphs that were centered on an algebraically interesting feature of the graph, usually an asymptote, rather than the point the directions asked students to focus on. Students who produced these graphs also tended to omit scales on their axes or use integer scales.
2. The student had a *situationally meaningful* understanding of the solution on the previous lab and applied the solution to a later lab because it earned full points on the prior lab.

In either case, the interview participant had to state that previous successful experience was the reason they produced the solution they did.

Classification precedes explanation. This code was used on labs. For this code, students were able to apply definitions but could not explain why the definition applied. This code was considered evidence of spontaneous reasoning because students were applying portions of the approximation framework but not in a conscious or systematic way. The following criteria needed to be met for this code to be used:

1. Student either correctly identified what the approximations were (e.g., average rates of change) or correctly identified over- and underestimates. One of the following must also be present.

2. The explanation for why students made the classifications was missing
3. If the explanation was present, it was *non-volitional, empirical, situationally meaningful*, or *ventriliquation*.

Increase in quality in new context. This code was used in comparing the trajectory of students' reasoning over time (the derivatives lab as compared to the limits lab; the definite integrals lab as compared to the derivatives lab). When coding this, I compared interview participants' performance on the earlier lab to the later one on the major components of the approximation framework (What is unknown? How can we approximate it? What is our error? How can we find a bound for our error? What is the method to achieve the desired accuracy?). I coded an increase in quality if the following criteria were met. This increase in quality was evidence of progression through the ZPD.

1. There was a shift in code from lab to lab along this continuum:

   Blank -> Spontaneous -> ZPD -> Scientific
2. In the case of the limits lab/derivatives lab comparison, the code for the definite integrals was either the same as the derivatives lab or improved.
3. In their interview(s), participants stated that their later lab was better because they understood the concept better.

Less scaffolding needed than previous lab. This code was used for the derivatives and definite integrals labs only. For each of the interview participants, I noted in my fieldnotes how much I either (a) personally helped a participant or (b) observed someone else helping a participant on a given lab objective. Participants were also asked in the interview where and how much help they got on a lab write-up; this provided some triangulation of this code. However, the majority of help participants discussed occurred in instructor office hours or with their group outside of class, which could be a potential limitation because of the self-reported nature of the data. This code was an operationalized definition for progression through the ZPD. I used this code whenever the following criteria were met:

1. There was a previous lab.
2. A student required scaffolding on a particular area of the framework (such as what an approximation was in the context of the lab problem).
3. On the current lab, students either needed less reported/observed scaffolding ("Remember, this similar to the example you did yesterday") or no scaffolding at all to successfully write-up a solution for a lab.

There were two interesting patterns for this code: one from the limits lab to the derivatives lab and then from the derivatives lab to the definite integrals lab.

On the derivatives lab, students actually needed more scaffolding on what approximations were. None of the students started with using an average rate of change as the approximation. All 10 interview participants initially used the y-values, which was the appropriate approximation for the limits lab. On the definite integrals lab, no interview participants required additional scaffolding to calculate approximations.

On the derivatives lab, students' initial lab solutions were so terrible that the instructors returned their solutions with large amounts of comments intended to help students improve their solution. Six interview participants got comments on their graph. Four of them chose to improve the labeling, if not the size, of their graph. On the definite integrals lab, all five interview participants who were still attending class turned in a high quality graph with their definite integral pre-lab.

Appropriation. This code was used when there was sufficient evidence that a student had internalized a portion of the approximation framework as part of their schema for calculus. A cell that was appropriated was evidence that a student was no longer within their Zone of Proximal Development. This code was only used on Labs 4 and 7, though only very rarely on the derivatives lab. All of the following criteria needed to be present for a cell to be coded as appropriation:

1. The solution for the current cell had to be contextually appropriate and correct.
2. The student confirmed on the current lab that they knew how to complete this portion of the lab because it was just a contextual change of a similar strategy used on a previous lab.
3. If a cell on the derivatives lab was coded as *appropriation*, the same cell on the definite integrals lab must also meet all of the criteria for being *appropriation*.

The most likely parts of the approximation framework students appropriated were the numerical error bound cell, followed closely by numerical approximation and the graphical cells.

Volitional. This code was used on the lab-related documents and one question on the second interview (All pre-labs, the derivatives lab "rough draft", and the final limits, derivatives, and definite integrals write-ups). This code was triangulated by the interview data since students were asked in the interviews to retroactively explain both their solutions and their reasoning on the aforementioned documents. I was also able to use the interview data to determine if the student's correct solution was volitional, *ventriliquation*, or *learned through instruction*. This code was considered evidence of scientific reasoning because volitional application of strategies to solve a problem is one of the hallmarks of a scientific understanding of a concept (Vygotsky, 1978). The following criteria needed to be met to classify a solution as volitional:

1. Students did not receive instruction on this specific concept.
2. Students correctly produced the solution.
3. Student stated in the interview that they did not get any instruction on how to complete the solution of this portion of the lab.
4. If there was a later lab, the solution was still *volitional* or *appropriated* in the new context.

Definition/procedure is right, work is not. This code was used on all of the students' written documents. Students were also asked to explain their thinking on these questions in the interviews. This code was evidence that students were within their Zone of Proximal Development because they were both scientifically reasoning about how to solve the problem while using spontaneous strategies. The following conditions needed to be met for an objective to be coded this way:

1. The student's plan for finding a solution to the prompt is correct (e.g., Finding approximations in Locate the Hole by finding function values near the hole).
2. The student either did not meet the prompt (e.g., not approximating to within the specified error bound) or used an *unjustified heuristic* (e.g., approximating a derivative with function values).

3. If the procedure was correct with an arithmetic, calculator, or algebra error, this code was only used if the student indicated one of the following:
    a. In hindsight, they realized why their original reasoning was not correct and gave an explanation for why.
    b. The student gave an additional *spontaneous* explanation for why their error was unproblematic.

The most common places where this code was used were with approximations on limits lab and the derivatives lab. In the limits lab, students got this code by finding approximations correctly but not finding and approximation with sufficient accuracy to be within the error bound. On the derivatives lab (and to a lesser extent the definite integral lab), students primarily had this code for giving up on finding accurate approximations for technological reasons.

Learned through instruction. This code was used on the labs and the interviews. This code was considered evidence of scientific conception since students were explicitly taught the definition and, in some cases, the strategies for creating the solution. After something was learned through instruction, students were able to volitionally apply what they learned through instruction to similar objectives on future labs. The following criteria needed to be met for a concept being classified as being learned through instruction:

1. Students received instruction about the specific concept. This could be by
    a. The *Definition taught before the lab*
    b. Scaffolding during the lab
    c. From the intervention.

2. Students were able to correctly produce a solution after the instruction.
3. Students *volitionally* produced the solution or used an *unjustified heuristic* on the same concept on a future lab.
4. Students attributed their correct solution to the instruction documented in the first criteria.

The most common thing students claimed was learned through instruction was what approximations were (the derivatives lab and the definite integral lab). Students were highly successful at this in the derivatives lab but required explicit re-instruction on the quantities involved with approximation in the derivatives lab (average rates of change rather than function values). Students stated in their interviews that this was the reason they were able to approximate correctly. In the definite integral lab, finding approximations using Riemann sums was unproblematic for students.

Unjustified heuristic. This code was used on the lab-related documents and one question on the second interview (All pre-labs, the derivatives lab “rough draft”, and the final limits labs 4 and 7 write-ups). This code was triangulated by the interview data since students were asked in the interviews to retroactively explain both their solutions and their reasoning on the aforementioned documents. This code was considered evidence of scientific reasoning, albeit scientific reasoning that was incorrect. At least the first three criteria needed to be present for a student’s reasoning about a lab objective to be coded as an overused heuristic but all four criteria are preferred:

1. Students used a previously taught heuristic to generate a solution for the objective (e.g., if the function is increasing, the *x* value to the right will generate and overestimate).

2. The heuristic was consistently applied to the current content (“The average rate of change that uses the point on the right is the overestimate, because the function is increasing”).
3. The application of the heuristic resulted in an idiosyncratic response (see #2).
4. An interview participant identified that they applied a heuristic (or even better, misapplied) to generate their solution.

The most common overused heuristics were (in no particular order) as follows:

- Drawing a graph with a window that displayed the global behavior rather than a detailed graph around the specific point or interval for the context.
- Using y values to approximate instantaneous rates of change in all representations.
- Over/underestimates were dependent upon whether the graph was increasing or decreasing (though outside of the scope of the research question, this heuristic also appeared on Lab 5).

Ventriliquation. This code was used on the lab-related documents and one question on the second interview (All pre-labs, the derivatives lab “rough draft,” and the final limits labs 4 and 7 write-ups). This code was confirmed by the interview data since students were asked in the interviews to retroactively explain both their solutions and their reasoning on the aforementioned documents. This code could be considered evidence students were within the ZPD because they are able to at least partially provide a solution after help was provided. The following criteria needed to be met in order to code something on the current lab as ventriliquation:

1. Students received instruction about the specific concept. This could be by
    a. The *Definition taught before the lab*
    b. Scaffolding during the lab
    c. From the intervention.
2. Students produced the solution or used an *unjustified heuristic* given in instruction on the same concept on their lab solution.
3. Students attributed their solution as an attempt to mimic the instruction the received.

The codes described above form a hierarchy for spontaneous-scientific conceptual development (see Table 8). At the lowest level, students had a spontaneous level of understanding that was not organized into any formal schemas. At the next level, students were able to ventriloquate but not necessarily understand the structure of the answer. If a student could apply a previous heuristic, even if it was unjustified, to a problem, this showed a higher level of understanding than before. Next, students learned how to complete the procedure correctly through instruction but were not able to articulate why their answer was right. At the next level, students entered their Zone of Proximal Development[4]; as the solutions from lab to lab increased in quality or needed less scaffolding during instruction, there was evidence that students began to make connections between concepts. After that, a student might be able to choose the correct strategy for producing a solution independently but were unable to execute the strategy completely. If a student was able to select and execute the appropriate strategy with no external aid, their action was volitional. At the final level, students had appropriated a

[4] In this characterization of the ZPD, student were not at their ZPD until spontaneous and scientific knowledge began to be integrated.

portion of the approximation framework, which as evidenced by volitionally producing a solution in two or more contexts in a row.

Table 8

Coding Hierarchy

Level	Code(s)
0	All spontaneous codes
1	Ventriliquation
2	Unjustified Heuristic
3	Learned Through Instruction
4	Increased Quality; Less Scaffolding
5	Plan is right; work is not
6	Volitional
7	Appropriation

While this analysis plan seemed rigorous, it was important to maintain quality control throughout the data analysis process to be sure that bias was minimized. In the next section, I describe how I maintained high quality throughout the data collection and analysis.

Trustworthiness

I used Patton's (2002) framework for high quality research within a constructivist framework. Under this framework, the components of high quality research are (a) acknowledgement of subjectivity, (b) reflexivity, (c) rigorous analysis, (d) triangulation, (e) trustworthiness, (f) transferability, (g) contribution to existing literature, and (h) credibility of the researcher and methods. I have already made arguments why this dissertation project contributed to the literature (see Chapters I and II) and will not reiterate the arguments here. In this section, I argue that the dissertation study was high quality research using the remainder of this framework.

In the researcher stance in Chapter II, I disclosed my experiences and potential subjectivities that could bias the results since I was the instrument of analysis in this research. I ameliorated the subjectivities I brought to the research in two main ways. First, I kept a weekly journal about my experience in the research process; this helped me maintain an audit trail of my thought processes throughout the dissertation as well as engaging in reflexivity about how I conducted the study and evaluated if I was allowing my biases to color the results. Second, I consulted with my advisor and committee about my dissertation project; since the committee had different experiences with the research topic from me as well as more research experience than I did, they also helped keep my potential biases in check.

Rigorous qualitative researchers go beyond coding the data once or twice before moving onto writing up results or even checking with members, peers, or experts about the reasonableness of one's coding. Rigorous qualitative researchers also generate and assess rival hypotheses that could explain the results and investigate negative cases that could seem to contradict the emerging themes. The first action I took to ensure the rigor of the dissertation study was by engaging in rival hypothesis generation. I also made an effort during the data collection to investigate negative cases since formative assessment tends to raise achievement less for students earning very low and very high grades (Dibbs et al., 2013; Gallagher et al., 2006).

Triangulation has long been used to argue that a qualitative study is of high quality but there are actually many types of triangulation: of data collection, of analysts, and of theory (Patton, 2002). All three types of triangulation appeared in the dissertation project I described in the preceding sections. First, by collecting artifacts from the participants, observing participants during class, and interviewing participants, I collected data from several methods that yielded slightly different insights into the research question. Triangulation was especially apparent in the standards of evidence tables in an earlier section. By consulting with my advisor and committee on the analysis of the dissertation project, I had triangulation of analysts. Finally, by synthesizing several theories into the theoretical perspective, the study rested on stronger theoretical foundations than if a single perspective was used (Patton, 2002; Sfard, 1998).

I took two actions to increase the trustworthiness of the study. First, I maintained an audit trail with the data and journals. Second, I solicited member checks of the coding from participants in their interviews and consulted with other researchers, my advisor, and my committee for peer and expert checks. These actions also helped to increase the transferability of the findings.

Finally, the methods used for this dissertation study were credible, established methods in qualitative research that have been used in published mathematics education research (Hart, Smith, Swars, & Smith, 2009; Highfield & Goodwin, 2008). I was a credible researcher on this topic because I completed a pilot study for this project, published and presented formative assessment research in peer reviewed journals and peer reviewed conferences (Dibbs & Blasjo, 2011; Dibbs & Christopher, 2011; Dibbs, Glassmeyer, & Yacoub, in press; Dibbs & Yacoub, 2010), completed a doctoral minor in qualitative methodology, had experience with all of the methods I used, and completed all other coursework and examination prerequisites to dissertation study.

Institutional Review Board Approval

I have applied for and received Institutional Review Board (IRB) approval through the university where I conducted the pilot study. The IRB for the dissertation was submitted after the successful defense of the proposal. Both IRB approvals appear in Appendix G.

Summary

After the pilot study, several changes to the theoretical perspective and research question were made to strengthen the study. Framing the qualitative portion of the study in terms of the Zone of Proximal Development allowed for a simpler analysis. The statistical analysis of achievement on the limits, derivatives, and definite integral labs provided additional context to the qualitative findings. The results of this analysis are presented in the next chapter.

CHAPTER IV

FINDINGS

Overview

Before I could answer the research questions, I needed to make sense of the data collected during the semester. In this chapter, I conducted three analyses I will use in the final chapter to build arguments that answered the main research question. Each analysis was based upon one of the characterizations of the Zone of Proximal Development: the collaborative ZPD, ZPD as scaffolding, and ZPD as conceptual development. The first characterization was the collaborative ZPD—where students were able to solve problems as a group that members could not solve individually. The second characterization was in terms of scaffolding—where students are in their Zone of Proximal Development when they can solve a problem with help that they could not solve on their own. The major differences between these characterizations were that the first one focused on the collaboration and social aspects of Vygotskian constructivism while the latter focused on the individual learner. The final characterization of the ZPD was when students' spontaneous and scientific conceptions about a topic interacted. The last two characterizations of the ZPD were most relevant to this analysis.

This chapter is organized by the purposes outlined in Black and Wiliam's (2009) framework. In each section, the portions of each data analysis are presented in the order they are discussed in the final chapter. In order to analyze the whole class data, I needed to group the 66 students. I wanted to conduct a statistical analysis of achievement and lacked the resources to complete 66 individual case studies and develop the qualitative analysis as a grounded theory. Although I anticipated that students could be grouped into cases based on class achievement levels, I ultimately grouped the students by how many formative assessments they completed. In the next section, I argue why participation and not achievement was the appropriate mechanism for grouping students for the purpose of answering the research question.

Why Group by Participation?

When prior research has compared students completing formative assessments within a class as opposed to being compared to a control group, students have been grouped by the final grade they earned in the course to investigate differences in learning trajectories (Al Kadri et al., 2011; Minstrell & Anderson, 2011). Both studies classified students into three categories: those who earned an A in the course, those who earned either a B or a C, and those who earned a D or an F in the course. Neither study considered students who withdrew from the course. Each group appeared to have a different learning trajectory through the instruction period. Students who earned A's showed a slight improvement in mean score on every assignment; since the initial assignments for this group were very good, little absolute improvement was possible. Students who earned a B or a C generally had an initial score (either a pretest or a first assignment) near the average but had trouble applying concepts learned in earlier assignments to subsequent assignments. With additional instruction and individual formative feedback, these students showed steady improvement in subsequent assignments. Students who failed the class had a large drop in achievement between the limits and derivative labs. Unlike the students who eventually passed the course, the interventions based on the formative assessment did not appear to help students improve. As discussed later in the section, there are no significant differences in any grade-predictive variables between the students at these three participation levels so there was no reason to believe at the beginning of the semester that the students not participating in formative assessments had academic deficiencies when compared to the other two groups of students.

I began the analysis by classifying the participants into three grade bands. After scoring all of the student papers on the 20 parts of the approximation framework, I calculated the average number of correct items for each grade band on each assignment (see Figure 8). Since students received the most individual formative instruction between the two drafts of the derivatives lab, which was the first assignment where students needed to apply the approximation framework concepts learned in the limits lab, the trajectories of the students in these calculus classes appeared to be consistent with the prior literature.

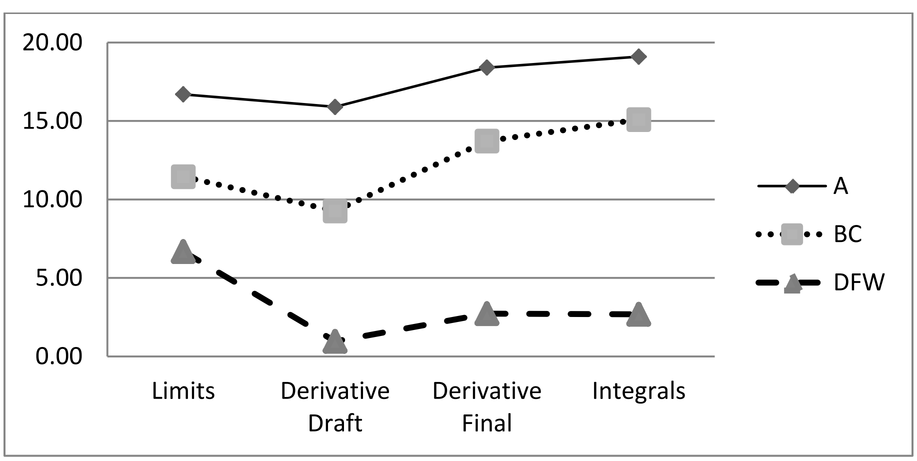


Figure 8. Average number of correct answers by final grade.

After deciding to group students by the final grade they earned in the course, I formed three case studies based on the nine students by their participation levels. Although the three students who earned A's in the course and the two students who earned F's in the course showed learning trajectories consistent with the aggregated achievement data in the previous figure, three of the other four students did not have a learning trajectory consistent with the prior trajectories in the literature or the aggregate data from this study. Two students, one who earned a B and another who earned a C, showed steady, consistent improvement from assignment to assignment with no large drop-off when asked to apply previously learned concepts. Both students had a trajectory that looked most like the students who earned an A in the course. Eva (who earned the B) and Sandra (who earned the C) appeared to improve at about the same rate as the students who earned A's, except Eva and Sandra started at a lower initial achievement level. The final two case study students, one who earned a D and one who earned a C, had nearly identical trajectories. Both students had trajectories similar to the SBC trajectory in Figure 9. They did very well on limits, struggled on derivatives, but their integral lab had more correct answers than their limits lab.

I was not able to conduct any statistical analyses on the derivatives draft; the scores were so low that reliability was adversely affected. On the other three assignments, ANOVA/ANCOVA tests showed a significant difference in mean number of items in the lab write-ups students in each participation level were able to answer correctly[1] (limits: $p = 0.011,$ derivatives: $p < 0.001,$ definite integrals $p < 0.001,$ respectively).

[1] For the derivative lab, the score on the initial draft was used as the covariate.

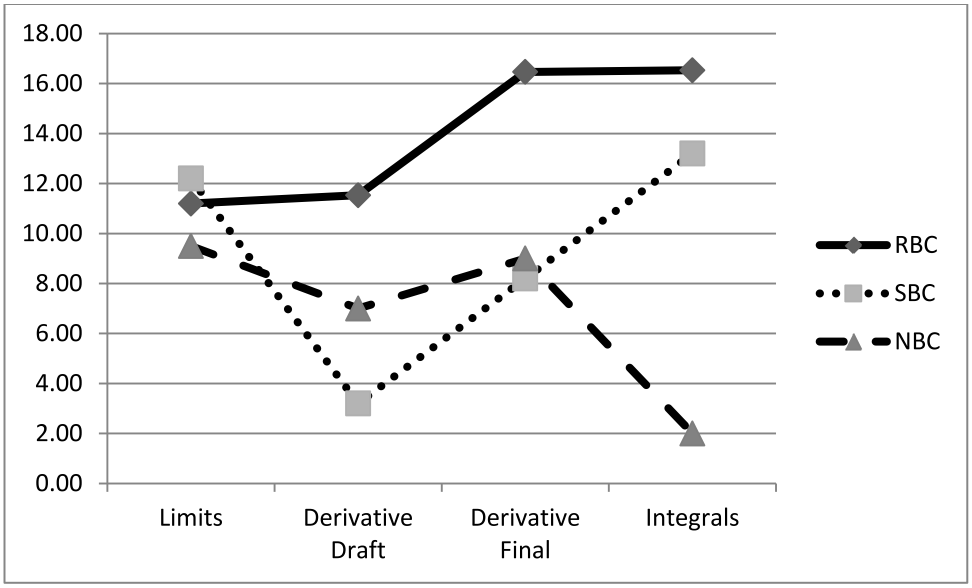


Figure 9. Average numbers of correct answers for students with a B/C grouped by participation level. The first letter of each of the codes in the legend is the participation level (regular, sporadic, nonparticipant).

I began to look for things that the students who earned A's and the two students who earned a B and a C did that the other students did not. I also looked at what the student who earned a C and the student who earned a D did that the two students who earned F's did not. The difference between these three new groups of case study students (steady improvement, large drop followed by steady improvement, large drop with no recovery) was how often they did the formative assessments. Students who showed steady improvement all earned 100% on their formative assessment grade at the end of the semester, which meant they missed no more than two formative assessments all semester. The student who earned a C completed three formative assessments all semester and the student who earned a D completed two formative assessments. Neither of the students who failed completed a formative assessment.

Although these participation levels were from self-selected groups, there was only one significant difference—gender--between the students who participated regularly, sporadically, or did not participate in formative assessments throughout the semester. Given the similarity of the learning trajectories for students at the same participation level, I argue that participation level, not final grade, was the appropriate grouping variable when looking at the effects of formative assessment within a course.

The students in the case studies were not a large enough sample to have all possible values for the numbers of formative assessment completed. In the initial analysis, I defined five participation levels (see Table 9). These categories were approximate letter grades on the formative assessment portion of the formative assessments.

Table 9

Definitions of Participation Levels

Participation Level	Number of Formative Assessments Completed
Regular	10 – 12
Frequent	7-9
Irregular	4-6
Sporadic	1-3
Nonparticipant	0

Only one frequent participant (eight assignments completed) and two irregular participants (five and four assignments completed) in the 54 students were included in the analysis. All three of these students earned C's in the course. For the analyses that follow in this chapter, the frequent participant and regular participant categories were merged and so were the sporadic and irregular participant categories. This choice of category merging was based upon the course structure: the regular and frequent participants all completed enough formative assessment to earn a 70-100% on their formative assessment grade in the course, the sporadic participants earned 10-60% on their formative assessment grade in the course, and the nonparticipants received a 0% on their formative assessment grade. When I grouped the students in each of the grade bands by their participation levels and graphed the average scores on each lab by participation, three distinct learning trajectories appeared within the B/C and the D/F grade bands. All of the students who earned A's were either regular participants or sporadic participants in the formative assessments; there appeared to be only two different trajectories. Figure 9 showed the three trajectories for the B/C grade band; each point on the line graphs was the average number of questions answered correctly on each lab write-up. For this and all of the figures that follow in the section, the first letter of each of the codes in the legend is the participation level (regular, sporadic, nonparticipant). The remainder of the code refers to the final course grade the group of students earned (A, B, or C, D/F/W).

When the students in each participation level were plotted on the same graph, the similarities in the learning trajectories became more apparent. The regular participants had a slight drop in performance on the derivative draft but seemed to perform within the achievement level of their grade band shown in Figure 10. Overall, all of the regular participants were in a trajectory that most closely resembled the “A” trajectory.

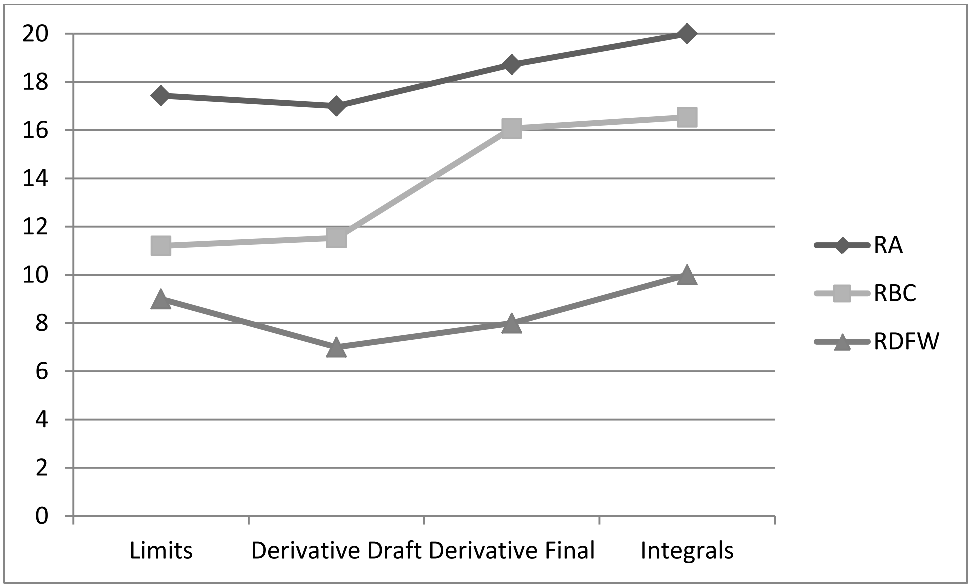


Figure 10. Average number of correct answers for regular participants grouped by final grade.

The sporadic participants all showed a drop in the number of correct answers on the derivative draft and improvement on subsequent labs. Only three students who were sporadic participants also earned A’s in the course; the small size of this group might be why this group did not appear to improve on the integral lab (see Figure 11).

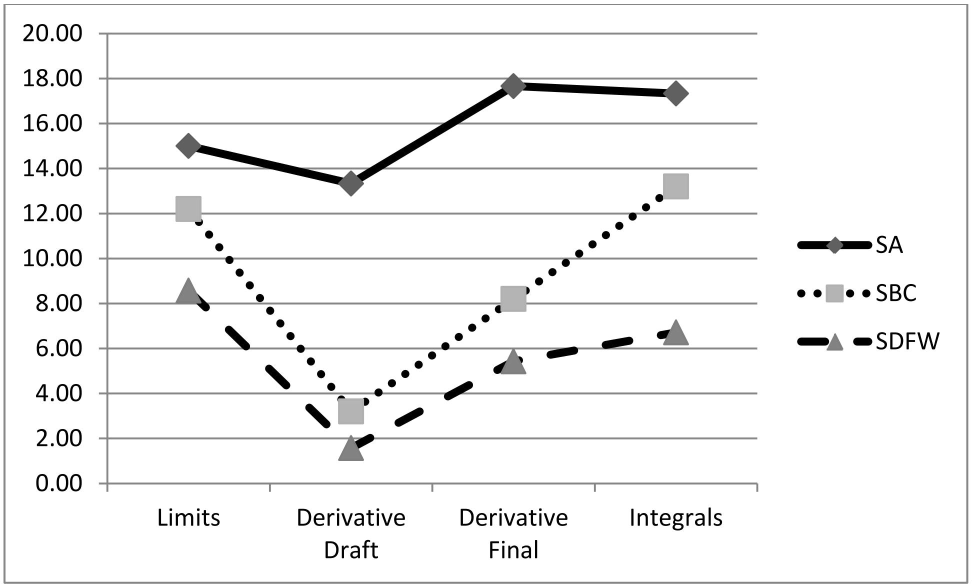


Figure 11. Average number of correct answers for sporadic participants grouped by final grade.

Students who did not complete any formative assessments had enormous difficulties with the derivative lab and never recovered (see Figure 12). Although it was mathematically possible to earn an A in the course without completing any formative assessment, a student would have needed to score an 89% or better on every other assignment in the course to earn an A.

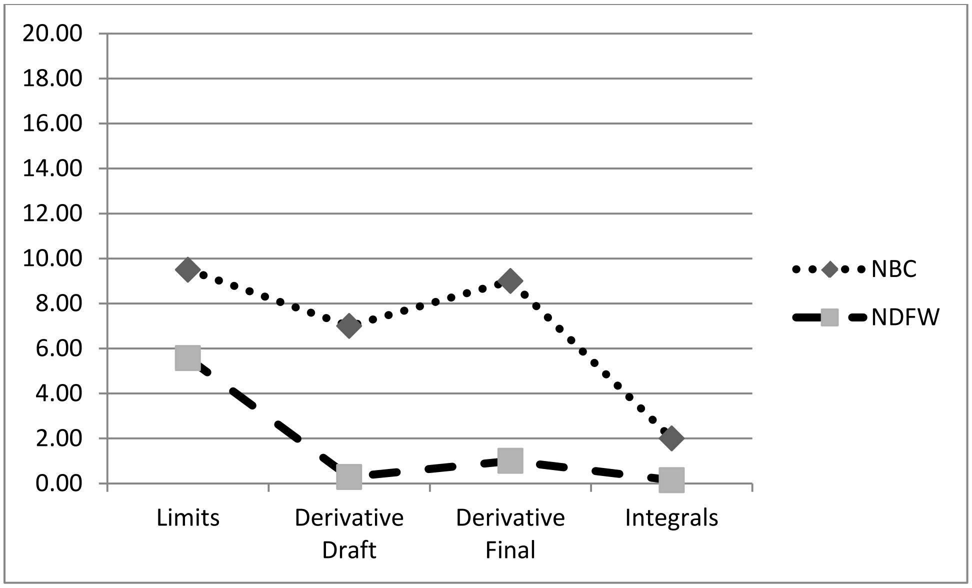


Figure 12. Average number of correct answers for nonparticipants grouped by final grade.

Before using participation level as the grouping variable in the analysis, I investigated if there were any demographic differences between the students in the different participation levels. For each demographic variable in the table, I calculated the proportion of students at each participation level who were female, non-White, non-native English speakers, and were not freshmen. I then performed four chi-squared tests for differences in proportions. Of the four chi-squared tests I performed on the demographic variables I collected from the students, there was only one significant demographic difference between the participation levels—female students were significantly more likely to be regular participants (see Table 10).

Table 10

Summary of Demographic Variable Analysis

Demographic Variable	*p* value
Gender (Male/Female)	.004
Race (White/Nonwhite)	.355
Native Language (English/Not English)	.651
Class (Freshman/Non-Freshman)	.802

I also measured four different variables known to predict student performance in introductory calculus: cumulative grade point average, ACT math score, Calculus Readiness Exam Score (CRE)[2], and the number of months between the end of the last math class a student took and the beginning of calculus. The final measure was self-reported; I obtained the other three scores from students' records. I performed ANOVA tests on each of these four quantities to see if the mean score differed across participation levels. The summary of the ANOVAs appears in Table 11. Although I used a Bonferroni correction on these and the preceding analyses, none of the *p*-values were significant even without said correction. Based on the available information, there was no reason to suspect at the beginning of the semester that students participating in formative assessment at different levels would have markedly different outcomes in the course.

[2] The Calculus Readiness Exam is a multiple choice exam all calculus students take on the second day of class.

Table 11

Summary of Analysis of Mean Grade Predictive Variable Grouped by Participation Level

Grade-Predictive Variable	*p* value
ACT Math Score	.192
Cumulative GPA	.294
CRE Score	.563
Months Between Courses	.741

After completing the analyses described above, I decided to frame all of the findings in this chapter in terms of participation levels. In Table 12, I have summarized the whole class data by participation level and final grade. Although students who earned A's and B's in the course tended to do all of the pre- and post-labs and students who failed the course tended to do no formative assignments, students who earned C's in the course did not show a consistent pattern of participation.

Table 12

Final Grade by Number of Formative Assessments Completed

	A	B	C	D/F	Total
8-13	10	6	5	1	23
1-7	2	2	5	7	16
0	0	0	5	10	15
Total	12	8	16	18	54

Students featured in the case studies were invited to interview based upon their course grades at midterms. Grouping the nine students whom I interviewed by how often they completed formative assessments resulted in the regular participant case being larger than the other two cases (see Table 13).

Table 13

Case Study Students Grouped by Participation Level

Pseudonym	Participation Level	Final Course Grade
Emily	Regular	A
Kaitlin	Regular	A
George	Regular	A-
Eva	Regular	B-
Sandra	Regular	C
Charles	Sporadic	C+
Leonard	Sporadic	D+
Lisa	Nonparticipant	F
Tre	Nonparticipant	F

To answer the research question, the data were then analyzed to see how, if at all, each of Black and Wiliam's (2009) five purposes of formative assessment applied to these undergraduate mathematics students. If there was evidence that a purpose of formative assessment was applicable to the population I studied, I investigated if students at different participation levels found that particular purpose to be equally applicable.

For the remainder of the chapter, I discuss each of Black and Wiliam's (2009) five purposes of formative assessment in the order listed in the framework: (a) clarifying learning intentions for students, (b) engineering effective class discussions, (c) providing feedback that moves learners forward, (d) activating students as resources for each other, and (e) increasing student ownership. The students in the case studies identified a sixth purpose of formative assessment, providing opportunities for peripheral participation, which was corroborated by the observational data. This is presented as the final section of the chapter. For two purposes, clarifying learning intentions and activating students as resources for each other, there was little evidence in the data that these purposes were applicable to the population of the study. In the other sections, I begin with a quantitative discussion of the whole class data followed by the qualitative analysis of how each purpose of formative assessment affected each of the case study students.

The First Purpose: Identify Learning Objectives

One of the main purposes of formative assessment in Black and Wiliam's (2009) framework was helping students identify the most important parts of an upcoming lesson. This purpose was applicable to the pre-lab so I included an interview question about how the pre-lab helped students identify the important objectives of the lab. In every interview, every student had a variation of the same response: "Of course I know that the labs are important, but the pre-labs didn't tell me that. The labs are worth 20% of the grade and there is always a question about the labs on the tests. Just looking at the syllabus is enough" (Sandra, second interview). There were no observations in class of students stating that the formative assessment helped to identify learning objectives and no student ever wrote such a statement on their pre-labs or post-labs.

The Second Purpose: Engineering Appropriate Learning Activities

The second purpose of Black and Wiliam's (2009) framework was for the instructor to use students' formative assessments to create classroom activities that addressed the issues indicated in the assignment. During the semester, these class activities took the form of an additional 10-15 minute instruction on the definitions of the approximation framework and their instantiation in the current lab.

Since students received individual written feedback on the derivatives lab instead of whole class discussion, this section is restricted to the whole class activities on the limits and integrals lab. For each of these labs, I have arranged the data chronologically. After briefly describing the students' activities in class, there is a summary of the post-labs turned in that night and a description of which items were discussed in class. After each descriptive portion, an ANOVA is presented on the performance on the questions discussed in class by participation level. The discussion of each lab's learning activities ends with the student case study data and how instruction given to students the day after the labs based on the most pressing problems students had with the labs, which will hence be known as post-lab-based instruction, helped these students in their conceptual development. Copies of the lab prompts can be found in Appendix A.

Limits

During observation of the first week's limits lab, groups were stuck on one of three questions. The first problematic question asked students to assign the unknown quantity a symbolic name. In Groups 1-5, a nonparticipant in formative assessments provided the correct solution; Groups 6 and 7 were assisted by a facilitator. The next challenge was making a plan for calculating approximations; only two groups, Groups 2 and 6, were able to complete this plan without assistance from a facilitator but all groups had successfully calculated at least one overestimate and one underestimate to the unknown value correctly before the end of class. At the end of class, all groups hit the same final stumbling block of the day-- the difference between error and error bound.

Only with explicit scaffolding from a facilitator did they move beyond this conceptual difficulty; however, even after facilitators thought they had successfully led students to the right answer, the next facilitator found the groups stuck. During the observations of three groups in each class, each facilitator would give the definition of error bound and sometimes would point out the difference on a graph. After this explanation, the group would say they understood and the facilitator would leave. As soon as the facilitator was out of earshot, the groups would call another facilitator over. These six groups spoke to every available facilitator but none of the groups moved past this obstacle.

On the post-lab that night, the regular participant group and sporadic participant group all answered the first questions correctly, although regular participants used approximately 50% more words in their responses. When asked to identify which parts of the lab or of the content covered that week they found most troublesome, the regular and sporadic participants had highly different responses (see Figure 13). None of the students in the sporadic participant group posed a question and they used no more than five words to indicate whether they needed no help or help with everything. The regular participants either asked questions about specific portions of the lab, indicated their questions were answered in class, or stated they were going to seek help the next day. All the students who stated they would seek help asked questions in office hours or before class. An email was sent to the instructors explaining that the three most pressing problems for students were the difference between error and error bounds; how to identify over- and underestimates; and what the "quality, well-labeled graph" the directions asked for would look like.

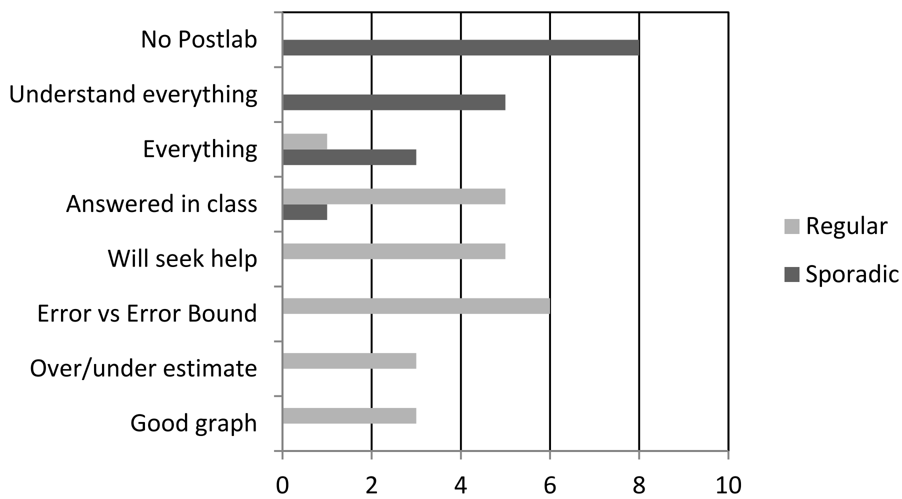


Figure 13. Limits lab, first post-lab responses.

For the instructional intervention the next day, one instructor lectured and the other did a mix of PowerPoint and short group problems. Both instructors went over one example of a graph with removable discontinuities and identified the overestimate approximation, underestimate approximation, and error and error bound. The instructors then displayed two graphs from students in a prior semester with no identification; one was a high quality graph and the other was of very poor quality. After a brief discussion, both instructors moved on to new material. Once the classes transitioned from discussing the lab to textbook material, half the students stopped taking notes. Ten minutes into the new material in both sections, six students were texting and only students in the front row or at Table 8 (see map, Chapter III) in the center of the back row were writing anything down. The disengaged students were all sporadic or nonparticipants in the formative assessments.

The second and final class day for the limits lab, like the last class day for all the labs, was a Jigsaw. Students were divided into new groups in which at least one member of each group had completed each of the different contexts. They were told to explain what they did in their groups last week and were reminded they were responsible for using another context to complete questions on the latter part of the lab. Five minutes were spent on class announcements and splitting the students into Jigsaw groups. Groups took about 10 minutes to explain the specifics of finding solutions in their context and then worked for the rest of the hour on completing the written portion of the Jigsaw assignment.

On the second post-lab, all students who submitted a post-lab answered the first two questions correctly but 7 out of 22 (32%) students explained that the overestimate was always to the right of the discontinuity because the *x* value was larger. This mistake was identified as a problematic issue because no student indicated on the final question that this was an area of difficulty for them (see Figure 14). In fact, only two students asked specific questions about any portion of the lab. The majority of students stated they understood everything or asked about a particularly tricky pair of WeBWorK questions due at the end of the week. The section not included in the analysis also had a large number of students with difficulties on the algebraic representation of errors. So the instructors were notified that identifying over- and underestimates and the difference between the algebraic error ($|(fx+h)-L|$) and error bound representations ($|f(x+h)-f(x-h)|$) were the most common student difficulties.

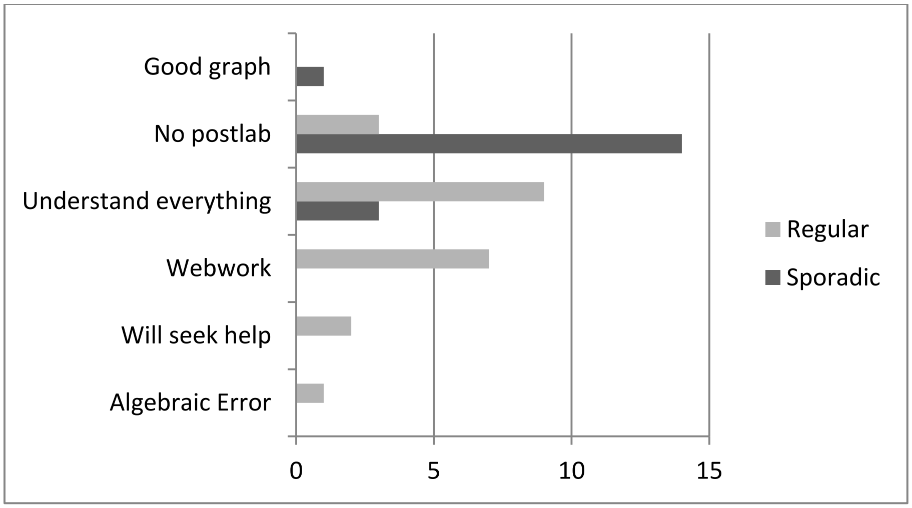


Figure 14. Limits lab, second post-lab responses.

On the final intervention day, both instructors began class by reiterating that the shape of the function determines whether an approximation will be an overestimate or an underestimate and warned that the second context the students were working with might not be the same as the original context. A contextual and algebraic review of what errors and error bounds are and how they are different followed. Emily and George, who regularly participated in formative assessment, asked clarifying questions. Both Leonard, a sporadic participant, and Tre, a nonparticipant, directed clarifying questions to their groups but not to the class as a whole. All but four students in Section 1 and five students in Section 3 took notes during the intervention; three were sporadic participants and the rest were nonparticipants. Section 1 went on to cover material from the textbook; all but seven regular participants stopped taking notes within 10 minutes of the transition and four students, all nonparticipants, texted throughout this period. Section 3 worked on the limits lab for the rest of class. Eleven of the 20 components[3] of the approximation framework were discussed in at least one of the post-lab-based instruction sessions the class after the lab, which are indicated by asterisks in Table 14.

Table 14

Questions Discussed in Post-Lab-Based Instruction

	Contextual	Graphical	Algebraic	Numerical
Unknown Value		*		
Approximation	*	*		
Error	*	*	*	*
Error Bound	*	*	*	*
Desired Accuracy		*		

An ANOVA of student performance on the items discussed in class revealed a significant difference in mean performance between at least two groups (see Table 15).

Table 15

Analysis of Variance of Items Discussed in Class, Limits Lab

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*
Between Groups	208.734	2	104.367	19.59	0.000
Within Groups	271.701	51	5.327		
Total	480.436	53			

[3] The 20 components are the four representations (context, graph, algebra, and numerical) of each of the five questions (unknown value, approximation, error, error bound, and desired accuracy. Each component is represented by one cell in Table 14 and all of the relevant tables that follow.

The Tukey Post-hoc analysis (see Table 16) showed the regular participant group had a significantly higher mean than the other two groups but the sporadic and nonparticipant groups were not significantly different from each other. Given the low mean scores of these groups, this suggested that the students who were not in the regular participant group did not benefit greatly from the post-lab-based instruction.

Table 16

Tukey Post-Hoc Analysis, Limits Lab

				Calculated *p* Value	
Groups	Count	Mean Score	Standard Deviation	Sporadic	Nonparticipant
Regular	23	7.78	0.23	$p < 0.01$	$p < 0.01$
Sporadic	16	3.67	0.47	n/a	$p > 0.05$
Nonparticipant	15	3.94	0.36	$p > 0.05$	n/a

For the total scores of the limits lab, the regular participants were significantly higher than the nonparticipants ($p < 0.01$), the sporadic participant mean score was significantly higher than the nonparticipants ($0.01 < p < 0.05$), but the mean score of the regular and sporadic participants was not significantly different ($p > 0.05$). On the items not discussed in class, the regular participants had a higher mean score than the sporadic participants ($p < 0.01$) and the sporadic participants had a significantly higher mean score than the nonparticipants ($p < 0.01$), but the nonparticipant and sporadic participant mean scores were not significantly different from each other ($p > 0.05$).

In the following section, I examine how effective the post-lab-based instruction was for the nine case study participants during the limits lab. After briefly describing each case study student and classify the coding of their conception of each item discussed in the post-lab-based instruction, I compare the students within each of the three cases--regular, sporadic, and nonparticipants.

Limits lab, regular participants. The regular participants are the most heterogeneous of the three cases in terms of their course grades. During the limits lab, Emily was earning an A, Kaitlin and George were earning B's, Eva had a D, and Sandra was failing the course. Emily and Kaitlin both asked specific questions about errors and error bounds on their post-labs. George reported on his post-lab that he did not have any questions; although much of his reasoning was spontaneous, George did correctly produce a solution for 18 of the 20 components of the approximation framework. The two post-labs Eva and Sandra missed all semester were the two post-labs associated with the limits lab.

The regular participants who asked questions on their post-labs benefited from the post-lab-based instruction but the others did not (see Table 17). Emily and Kaitlin both benefited from the post-lab-based instruction; this was unsurprising since the instruction covered exactly the items for which they asked help. George, who had figured out almost the entire lab without help, only learned how to articulate the contextual representation during the additional instruction. Eva did not take notes during the post-lab-based instruction and did not incorporate any of the ideas presented into the lab. Sandra did not benefit from the post-lab-based instruction because she visited during office hours and had all of her questions answered there. Each of the subsections following the table gives a detailed explanation of the codes below.

Table 17

Regular Participants' Limits Lab Codes

	Unknown Value (Graph)	Approximation (Context)	Approximation (Graph)	Error (Context)
Emily	Volitional	Situation Bound Reasoning (SBR)	SBR	Learned through Instruction (LTI); Intervention
Kaitlin	Plan is right; work is not	SBR	Volitional	LTI; Intervention
George	Unjustified Heuristic	LTI; UGTA	LTI; UGTA	SBR
Eva	X	LTI; UGTA	SBR	X
Sandra	Unjustified Heuristic	LTI; UGTA	LTI; UGTA	LTI; UGTA
	Error (Graph)	**Error (Algebra)**	**Error (Numerical)**	**Error Bound (Context)**
Emily	LTI; Intervention	LTI; Intervention	LTI; Intervention	LTI; UGTA
Kaitlin	LTI; Intervention	LTI; Intervention	LTI; Intervention	LTI; Intervention
George	Volitional	SBR	SBR	LTI; Intervention
Eva	X	X	SBR	X
Sandra	LTI; UGTA	LTI; Office Hours	LTI; Office Hours	LTI; UGTA
	Error Bound (Graph)	**Error Bound (Algebra)**	**Error Bound (Numerical)**	**Desired Accuracy (Graph)**
Emily	Volitional	LTI; UGTA	Volitional	Ventriloquation
Kaitlin	LTI; Intervention	LTI; Intervention	LTI; Intervention	X
George	SBR	SBR	Volitional	X
Eva	LTI; UGTA	LTI; UGTA	Volitional	X
Sandra	LTI; Office Hours	LTI; Office Hours	LTI; Office Hours	X

Note. ‘X’ denotes a question the participant left blank. LTI = Learned Through Instruction. UGTA = Undergraduate teaching assistant. SBR = Situation Based Learning.

Emily. Emily is a Secondary Mathematics major who earned an A in introductory calculus. This semester was her first exposure to calculus topics. Emily and Lisa completed the pre-calculus course the previous semester, which is why they chose to work together. Neither student had known or worked with Leonard before this semester.

Emily provided a complete and correct solution for the graphical representation of the unknown value correctly and could justify her answer in her interview. Since the pre-lab was completed outside of class, these were *volitional* acts.

Emily's verbal description of the removable singularity was in terms of polynomial rational functions with a common linear factor in the numerator and the denominator; this was coded as *empirical reasoning* because that had been her primary experience with removable singularities before the limits lab. After looking at her graph, Emily decided that *y* values near the point would work as approximations. She then graphed the initial points on the function she used to approximate the *y* value of the removable singularity. These were coded as *situation-bound reasoning* because Emily was not able to independently complete this portion of the approximation framework in the next lab.

Once Emily had approximations and moved on to finding the errors, she employed a strategy typical of regular participants. She would think about an item on which she was stuck, ask for help, and if the undergraduate teaching assistant (UGTA) made no sense, she moved on to the next question rather than seeking additional help:

> When I read the question about errors, it didn't make any sense to me, so I read the question again. It still didn't make any sense, so I asked you to come over. No offense, but you didn't make any sense either. Since there were a lot of questions left on the lab, I decided to skip that part and get more help outside of class. Then on the post-lab that night, I said that I didn't know what the difference between error and error bound was and didn't know how to do errors.

> Then [my instructor] talked about that in class the next day. That was when I got it.

Because Emily received instruction on how to complete the solutions on errors several different times, all four representations of error were clearly coded as *learned through instruction* (LTI). However, the instruction Emily found to be effective was the post-lab-based instruction so the source of the effective instruction was these discussions.

While Leonard and Lisa continued to struggle with errors during class, Emily moved on to error bounds. She received help from both of the UGTAs at different times:

> After you left my group, [the first UGTA] stopped at our table. She tried to help Leonard and Lisa with errors. I listened but it still didn't make sense. I asked her what an error bound was, and she told me it was the most we could be wrong with the approximations. Then I looked at my graph for a bit. Since the hole is between my overestimate and underestimates, I figured that the difference between the over and the under was bigger than the distance from the hole to either side. I labeled that on the graph and found this number here [points to her paper]. Then [the second UGTA] came by our table, and she tried to help Leonard and Lisa with errors some more. When she was done, I got her to help me write down what I just did as algebra.

Emily's graphical and numerical representations were *volitional* but the other two representations were completed after the UGTAs helped, so both of those representations were *learned through instruction* attributable to help from an *undergraduate teaching assistant*. On all representations of the desired accuracy portion of the approximation framework, Emily admitted that her solution was what she could remember the UGTA saying to her group at the end of the class; these representations were all coded as *ventriloquation*.

Kaitlin. Kaitlin is a pre-service Elementary Education major pursuing a concentration in mathematics; she earned an A in the course. Although she never graphed her function correctly, she did correctly explain that the unknown value was a removable singularity at $x = 2$, which is an open circle at an unknown height; this was coded as "*plan is right, work is not*." Kaitlin spent the first day of the limits lab carefully working through the four approximation representations. Her contextual representation was *situation-bound reasoning* because she was unable to complete this item in the next lab when the context was changed. She *volitionally* completed the graphical representation.

Kaitlin did not understand the difference between error and error bound with the exception of the numerical calculation of error bound; she completed that representation *volitionally*. Kaitlin spoke to all three sources of support available to her during class but none of the explanations helped her move forward, a fact she noted on her first formative assessment. Kaitlin attributed her ability to complete all of the other remaining error and error bound representations to the post-lab-based instruction in class; so these *were learned through instruction*. Kaitlin omitted the graphical representation of desired accuracy in her write-up.

George. George was a sophomore Biology major (pre-med). He chose to take calculus instead of the bio-calculus course suggested to biology majors because he thought the standard introductory calculus course would look better on his transcripts. Whereas Kaitlin's and Emily's limits lab write-ups were primarily learned through instruction, George asked for almost no help on the lab. He said he was able to complete the lab successfully because he had exposure to "functions and graphing points a lot last semester [in pre-calculus]. Once I got that was all we were doing, the rest was easy."

George's graph was centered on the asymptote of his graph instead of the removable singularity because "whenever we graphed a graph with asymptotes in pre-calculus, that was the most important part." This was coded as an *unjustified heuristic*. George was able to complete the contextual and graphical representations after instruction from the undergraduate teaching assistant and both were coded as *learned through instruction* (UGTA). His solution for the graphical representation was *volitional* but the other three representations of error were all *situation-bound reasoning*. George was able to complete three representations of the error bound *volitionally* (numerical) or using *situation-bound reasoning* (graphical; algebraic). However, he was only able to verbally describe the error bound after the post-lab-based instruction in class: "I sort of knew what I was doing, but I didn't know how to say what it [error bound] was until [my instructor] talked about in class." This item was coded as *learned through instruction* attributable to the post-lab-based instruction. George omitted the graphical representation of Desired Accuracy in his write-up.

Eva. The most remarkable thing about Eva's lab was that she did not provide an answer for 6 of the 11 portions of the approximation framework discussed in class (graphical unknown value, contextual error, graphical error, algebraic error, contextual error bound, and graphical desired accuracy). Eva was able to *volitionally* create solutions for three of the numerical representation of error bound. Two of her solutions were *situation bound reasoning*. The graphical representations of the approximations and the numerical value of error were concepts Eva claimed she learned in her pre-calculus course; all of the other items were discussed in class the next day. Eva learned to *complete through instruction* but that instruction came from an UGTA rather than the post-lab-based instruction.

Sandra. Sandra was conditionally admitted to the Chemistry master's program, which required successful completion of introductory calculus as one of the requirements. Although she had not taken a pre-calculus course and was several years removed from her last formal mathematics course, she was motivated to succeed and willing to ask for help on behalf of her group. On the first approximation lab, Sandra spent extensive time receiving help from the UGTAs and her instructor, both in class and during office hours. Her graph, like Greg's, was centered on the asymptote rather than the removable discontinuity and was coded as an *unjustified heuristic*. Graphical errors like this were not uncommon on the limits lab; most students used the graph they drew for the pre-lab, whether or not that graph was completely correct. Sandra omitted the graphical representation of desired accuracy. All of the remaining items were *learned through instruction*. Five were attributed to help from the UGTA in class (contextual and graphical approximation, contextual and graphical error, and contextual error bound). All of the remaining instruction could be attributed to instruction Sandra received during her instructor's office hours after class: "I'm older [30 at the time of the interview] than these kids, and I can't afford to mess around. What I didn't get in class I went and got help on in [my instructor's] office hours. He talked about the same stuff the next day though" (Sandra, first interview).

Overall, the helpfulness of the post-lab-based instruction depended on the student (see Table 15). For Emily and Kaitlin, who both felt that their algebra skills were rusty and did not work closely with their groups, the in-class instruction was the most common code on the items discussed in class. George did not need help to complete his write-up, and Sandra sought assistance outside of class. Eva was not able to incorporate any of the post-lab-based instruction into her write-up.

Limits lab, sporadic participants. Leonard and Charles, the two sporadic participants, were a study in extremes when it came to the effectiveness of the post-lab-based instruction. Charles earned a C+ in the course but his relatively low grade was entirely due to his lab write-ups; he earned a 96% on the portions of the course not pertaining to the labs. Charles turned in labs where the questions were either completely correct or blank; if Charles did not know how to complete a question immediately, he did not do it. All of Charles' lab codes were volitional, appropriated, or blank; there was no evidence he ever benefited from post-lab-based instruction. On the other hand, Leonard, a student who actually failed the course (D+), benefited greatly from the post-lab-based instruction. Although this appears to be an odd statement; Leonard failed the course due to a very low WeBWorK grade. It was only due to his high lab scores that he came as close to passing as he did. The 12 items discussed on the limits lab included all of Leonard's LTI codes for the limits lab. All but one of those codes could be attributed to the post-lab-based instruction (see Table 18).

Table 18

Sporadic Participants' Post-Lab-Based Discussion Codes

	Unknown Value (Graph)	Approximation (Context)	Approximation (Graph)	Error (Context)
Leonard	SBR	SBR	SBR	LTI; intervention
Charles	Volitional	X	X	X
	Error (Graph)	Error (Algebra)	Error (Numerical)	Error Bound (Context)
Leonard	LTI; intervention	LTI; intervention	LTI; intervention	LTI; UGTA
Charles	X	X	X	X
	Error Bound (Graph)	Error Bound (Algebra)	Error Bound (Numerical)	Desired Accuracy (Graph)
Leonard	SBR	LTI; intervention	Volitional	X
Charles	X	X	X	X

Note. ‘X’ denotes a question the participant left blank. LTI = Learned Through Instruction. UGTA = Undergraduate teaching assistant. SBR = Situation Based Learning.

Leonard. This pre-lab was the document Leonard, Lisa, and Emily discussed in the previous section. Leonard’s initial graph was created using an *unjustified heuristic* that a linear factor in the denominator of a function meant the presence of an asymptote.

The remaining four portions of the approximation framework were covered in the prompts in the lab activity. During the next part of the approximation framework, finding approximations to the unknown value, Leonard was almost able to reason through without the need for further instruction. “Since the unknown value is what the *y* value should be when $= 2$, we just need to plug in values close to two to get an idea what it really is,” Leonard explained in the interview. “Once I had my graph right, it was easy enough to make a chart and plot the points.” Leonard’s reasoning about approximations was inseparable from the function context in the derivatives lab so these representations were *situation-bound reasoning*.

After calculating approximations, students needed to describe the errors of their approximations in the four different representations. This was a task that Lisa, Leonard, and Emily could not complete during the allotted lab time, either individually or as a group. Leonard also had difficulty with the distinction: "I didn't really know what to do on this part [errors] until [my instructor] talked about it the next day in class. After that I was OK." All of Leonard's representations on error were coded as *learned through instruction* attributed to the post-lab-based intervention.

The concept of error bound was also difficult for Leonard. Based on the term, Leonard thought error bound was the maximum the error could be but was unsure how to find a value for error bound. I explained how the error could not be any bigger than the distance between the *y* values above and below the removable singularity. When I finished the explanation, Leonard looked at his graph and then explained to his group members that the error bound had to be the vertical distance between the two points they had already graphed (the overestimate and the underestimate). Since Leonard could not represent an error bound graphically in the next activity, his reasoning here was *situation-bound*. During the second week of the lab, Leonard learned how to algebraically represent the error bound during the second *post-lab-based instruction session*.

Charles. Charles, an Elementary Education major who needed to pass calculus as part of his elementary education mathematics concentration, was highly resistant to completing the labs throughout the semester. "I never really need to know how calculus works," he said in his first interview. "I'm just gonna teach third grade, so this has nothing to do with me." Like Leonard, there were portions of the course Charles refused to complete on a regular basis. In Leonard's case, it was the WeBWorK; Charles rarely turned in labs. Although what Charles turned in was generally correct, the number of unanswered questions far outweighed the ones he answered, particularly on the limits lab. He completed two items *volitionally*, one of which, the graphical unknown value, was discussed in class.

Limits lab, nonparticipants. The two case study students who completed no formative assessments during the semester, Lisa and Tre, attended all of the post-lab-based discussion sections while they were enrolled in the class. Tre never had a solution on any item that he learned though the post-lab-based instruction. After the post-lab-based discussions during the limits lab, Lisa was able to complete the four representations of error. However, her reasoning was almost entirely procedural on these items; she was unable to answer any of the questions discussed in class on the definite integral lab.

Tre and Lisa's codes for the items discussed in class (see Table 19) had many similarities. Both students received help from their group two times, from an UGTA once, and Lisa and Tre were able to calculate the numerical error bound volitionally. Lisa showed more spontaneous reasoning on the graphical representations related to relevant mathematical skills; Tre's only spontaneous reasoning was marginally related to mathematics. Lisa's stronger procedural knowledge of functions allowed her to take advantage of the post-lab-based instruction for errors.

Table 19

Lisa and Tre's Codes for Items Discussed in Class

	Unknown Value (Graph)	Approximation (Context)	Approximation (Graph)	Error (Context)
Lisa	Empirical	LTI; group	SBR	LTI; intervention
Tre	X	LTI; group	LTI; group	X
	Error (Graph)	Error (Algebra)	Error (Numerical)	Error Bound (Context)
Lisa	LTI; intervention	LTI; intervention	LTI; intervention	LTI; UGTA
Tre	X	X	Empirical	X
	Error Bound (Graph)	Error Bound (Algebra)	Error Bound (Numerical)	Desired Accuracy (Graph)
Lisa	X	LTI; group	Volitional	X
Tre	LTI; UGTA	X	Volitional	X

Note. 'X' denotes a question the participant left blank. LTI = Learned through instruction. UGTA = Undergraduate teaching assistant. SBR = Situation based learning.

Lisa. Like Leonard, another group member who sporadically participated in the pre-labs and post-labs, most of Lisa's situation-bound reasoning centered on the concept of functions. There were some subtle differences, however. Lisa came into the first day of this lab with only rough sketches of the graph students were going to analyze (see Figure 15). While her graph was a qualitatively accurate representation, Lisa mislabeled the axes. Her explanation for this labeling was her prior experience: "That's what you do in math. We don't know where the hole is. When you don't know something in math, you call it *x*." Since her choice of variable was based on prior experience, it was coded as *empirical* in these representations.

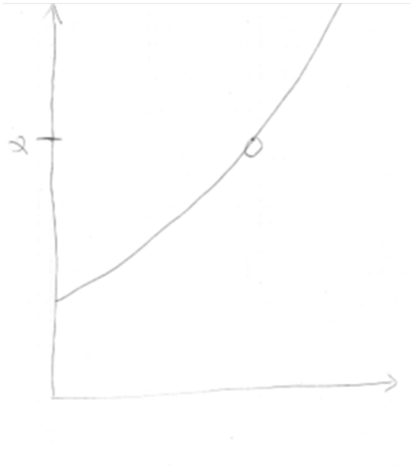


Figure 15. Lisa's limits lab graph.

Although Lisa had some idea of how to represent the unknown value, she was not sure how to approximate it:

> I didn’t know how to find where the hole is. In pre-calc last semester, when we needed to find a hole like that, we factored and cancelled. I don’t know how to factor a cube root though, so I didn’t know how to get started. Leonard showed me what to do though. He told me that if we plugged in *x* values close to 2, we could get an idea what the real value of the hole was. After I got that, the rest was easy: all I needed to do was graph points, make an *x* – *y* table, and fill it in. That’s just algebra.

Since Leonard explained the approximations to Lisa in the lab, her contextual response was coded as *learned through instruction*. Since Lisa was able to reason through how to complete the solution in the other three representations using her knowledge of functions, the rest of her approximation solution was coded *as situation-bound reasoning*.

Lisa was also unable to complete any representation of error in class even with non-peer instructional assistance. Lisa did not complete a post-lab. Error was the predominant problem in the post-labs for the regular participants and Lisa’s instructor explained the process of constructing the solution to the error portion of the lab. Lisa explained that to “do this part [errors] I just followed the notes from Wednesday [my instructor] gave. I had no idea what to do before that, and didn’t really get why I was doing what I was doing, but I got the points.”

All four representations of error were coded as *learned through instruction*. At the end of the first day of this lab, Lisa, Leonard, and Emily’s group was given instruction on what an error bound meant in the context of this lab; therefore, that representation was coded as *learned through instruction*. After hearing that explanation, Lisa looked back over her chart and subtracted the overestimate from the underestimate. Lisa explained that she could not see what was written on the paper as she is legally blind; Leonard’s graph was two feet away and drawn in pencil. Her calculation of the numerical error bound was thus concluded to be *volitional*. When asked about the algebraic representation of error bound in her first interview, Lisa explained, “I had a pretty good idea that writing the error bound down had something to do with $f - x$ [This is how Lisa always referred to function notation] notation, but I couldn’t put it together, so I got help with that part from [my instructor] in office hours.” That representation, then, was coded as *learned through instruction*. Lisa also claimed that her desired accuracy response was also a transcription from the same UGTA from whom Leonard had partially transcribed his response; Lisa’s response was thus also coded as *ventriliquation* with the exception of the numerical representation, which she completed volitionally.

Tre. Tre had a similar trajectory to Lisa except he hit his limit around midterms. Tre was a conditionally admitted Physical Therapy master’s student. He had been out of school for six years and had taken the one required math class for his bachelor’s degree four years before he graduated. After scoring a zero on the Calculus Readiness Exam on the second day of class, Tre was advised to drop calculus and take pre-calculus instead. Tre did not drop back to the prep course because his financial aid would not cover the tuition. Tre was one of three non-Caucasian students in the class and the only student with a full-time job. Like Lisa, Tre never completed a formative pre-lab or post-lab.

Every case study until this point had one point of commonality: everyone was fairly successful on Lab 3. Both of Tre's group members—Eva, who was discussed with the other regular participants, and Brandon, a sporadic participant who was not interviewed—were both very successful on the first lab. None of Tre's solutions on Lab 3 were learned from the class discussion following the labs. Of his three *learned through instruction* codes, two were attributed to instruction from a group member and the third was instruction from a teaching assistant.

Tre never answered any of the unknown value questions on the pre-lab. "Since I didn't have it done in time, it wasn't worth points, so I didn't see the point of doing it," he explained at the start of his first and only interview. When I observed that most of the correct answers on his write-up were approximation, he stated he had a significant amount of help:

> Well, I haven't had algebra in a very long time--not since high school--and I didn't have a graph, so I asked Brandon [uninterviewed group member] if I could look at his. I could see that there was a hole in the graph, but we didn't know where to start because the function didn't factor and cancel. So we called that UGTA, [name] in and she explained the approximations were points really really close to the hole. Then Brandon showed me how to graph points. After that all we needed to do was plug them in. That was the part I got.

The first two representations were coded as *learned through instruction* and he needed the context of points to be able to calculate approximations. When asked about his numerical answer, which was not correct, he explained what error meant in his experience outside of math class: “Error is how wrong you are. Like in polls, it is always plus or minus 3%. So I took my overestimate minus my underestimate and divided by 2 to get my error. That is how far the hole is from my point, so that is how big the error is.”

The last parts of the lab Tre wrote a solution for were the graphical and numerical representations of error bound. Tre explained that the undergraduate teaching assistant (UGTA) taught them how to label error bound on their graphs; once Tre received this instruction, he was able to complete the final representation he attempted without additional help:

> When I saw that [the UGTA] drew on the graph, I got that the error bound was the distance between the two points, so if I just subtracted I got the error bound. That’s why I divided by two to get the error, since the point [removable singularity] is between the two points.

In summary, Lisa was able to benefit from the post-lab based instruction when the context was familiar but Tre was not. By the end of the semester, Tre was no longer attending the course and Lisa was unable to answer any of the questions discussed in class. By the definite integral lab, the differences between the three participation groups were much more apparent than it was at the beginning of the semester.

Integrals

The definite integral lab was the last lab of the semester. At this point, the regular participants required minimal help to complete the labs. Although there were few questions that distinguished the regular participants from the sporadic participants, both groups had markedly different performances from the nonparticipants.

It was expected during the first day of the definite integral lab since students had such high quality pre-labs, only the students working on the probability context would need any help with the initial part of the labs. Other than an occasional review of Riemann sums or troubleshooting calculator entries, groups asked for very little help.

On the first post-lab, none of the 15 sporadic participants completed the assignment; neither did nine of regular participants. Of the remaining regular participants, four asked questions about the representation of error and the other 10 had no questions. There was no intervention at the beginning of the next class; instructors instead opted to use the entire class meeting for test review. The instructors were concerned students were not prepared for the final test and needed the review more than they needed to talk about the lab.

During the second week of the lab, groups came in with the easy calculation parts and the graphing (parts a-d of the definite integral lab) completed. While the first week of the lab went smoothly and the groups needed minimal help, here facilitators took a more active role. Every group needed assistance from a facilitator to construct the algebraic representation of error and to help groups iterate the Riemann sum on their calculators.

Eight regular participants and nine sporadic participants did not complete the post-lab. Of the remaining students, one regular participant had a question about a WeBWorK problem and one sporadic participant asked whether integrals could ever be applied outside a math class. None of the other students' post-labs asked questions or indicated problematic issues.

During the intervention the following day, the instructors of Section 1 and Section 3 took two slightly different approaches. In Section 1, the instructor taught index notation for the first time and then demonstrated how to iterate a large Riemann sum on the calculator. The Section 3 instructor taught index notation earlier. At the start of this class, the instructor announced a link for extra practice on Blackboard and then spent the rest of the intervention demonstrating how to use either Wolfram Alpha or calculators to find the sum. Students were more engaged and more likely to be taking notes during the intervention than they were for the rest of the class meeting.

The final day of the definite integral lab was a Jigsaw. By this point in the semester, the Jigsaw norms were firmly established and this Jigsaw looked exactly like all those before. Once students were in groups, the student who felt most confident began by explaining his/her answer; this was almost always the student who had worked in the spring context. The second most confident student would speak next and started off by saying that his/her lab was the same procedure but a different function. The probability context presenter always went last. The high-performing students were able to give a short overview explaining how their context was the same but had harder numbers. The lowest-performing students talked about how difficult the calculations were. This pattern appeared in all seven observed groups. Twenty minutes into the Jigsaw, groups had stopped presenting to each other and were either chatting off topic or working on their next write-up for the rest of the hour.

The final post-lab of the semester did not resemble any of the post-labs students had previously completed. This final post-lab could be considered a one-question interview rather than a typical post-lab. In addition to the usual final two questions on the post-lab, students were asked to explain what approximation in calculus meant to them. The analysis of this post-lab appeared in the discussion of the fifth purpose of formative assessment. Six items were discussed in the post-lab based instruction--all of the algebraic contexts and the numerical desired accuracy item (see Table 20).

Table 20

Items Discussed in Post-Lab-Based Instruction, Integrals Lab

	Contextual	Graphical	Algebraic	Numerical
Unknown Value			*	
Approximation			*	
Error			*	
Error Bound			*	
Desired Accuracy			*	*

The ANOVA of student performance on these six items that were discussed during the post-lab-based instruction revealed significant differences in mean performance between at least one pair of groups (see Table 21).

Table 21

Analysis of Variance of Items Discussed in Post-Lab-Based Instruction, Integrals Lab

Source of Variation	*SS*	*df*	*MS*	*F*	*P*-value
Between Groups	99.531	2	49.765	63.387	0.000
Within Groups	40.04	51	0.785		
Total	139.571	53			

The Tukey Post-hoc analysis (see Table 22) showed that the regular participant group had a significantly higher mean than the other two groups and the sporadic participants had a significantly higher mean than the nonparticipants. However, given the magnitudes in the means and the relatively large *p* value, there was not much difference between the regular and sporadic participation groups on the items discussed in class for this lab.

Table 22

Tukey Post-Hoc Analysis, Integrals Lab

				Calculated *p* Value	
Groups	Count	Mean Score	Standard Deviation	Sporadic	Nonparticipant
Regular	23	4.1	0.035	$p < 0.1$	$p < 0.01$
Sporadic	16	3.5	0.41	n/a	$p < 0.01$
Nonparticipant	15	1.2	0.66	$p < 0.01$	n/a

In the following section, I examine how effective the post-lab-based instruction was for the nine case study participants during the definite integrals lab. After briefly describing each case study student and classifying the coding of their conception of each item discussed in the post-lab-based instruction, I compare the students within each of the three cases: regular, sporadic, and nonparticipants.

Definite integrals, regular participants. In this lab, the final lab of the semester, none of the regular participants asked questions on their post-labs. The post-lab-based instruction was less helpful for Emily and Kaitlin because they had appropriated most of the approximation framework. George still only benefited from the post-lab-based instruction on a single item. However, Sandra and Eva benefited from the additional algebra instruction.

Emily. At this point in the semester, Emily had appropriated much of the approximation framework. Most of the non-appropriated items she was able to complete volitionally (see Table 23). Emily found approximation using Riemann sums to be more intuitive than derivatives but not quite as easy as using *y* values. The unknown value component of the approximation framework had been appropriated in all representations; Emily produced the correct solution outside of class with no outside assistance.

Table 23

Emily's Definite Integral Lab Codes

	Contextual	Graphical	Algebraic	Numerical
Unknown Value	Appropriation	Appropriation	Appropriation	Appropriation
Approximation	Volitional Increased Quality	Volitional Increased Quality	LTI (researcher)	Volitional Increased Quality
Error	Volitional Increased Quality	Appropriation	LTI (UGTA)	Volitional Increased Quality
Error Bound	Appropriation	Appropriation	LTI (intervention)	Appropriation
Desired Accuracy	Volitional	Volitional	LTI (intervention)	Appropriation

Note. ‘X’ denotes a question the participant left blank. LTI = Learned through instruction. UGTA = Undergraduate teaching assistant.

Emily was able to complete all but the algebraic representation of the approximation framework without additional instruction. These were coded as volitional or appropriated, depending on if Emily needed help on the derivatives lab. All three representations had more detail than the prior lab so these representations were also coded as increased quality. Emily asked for help on summation notation for the algebraic representation of the Riemann sum so the final approximation representation was coded as learned through instruction.

The definite integral lab was the first lab Emily did not require extensive scaffolding on the error component of the lab. She completed all but the algebraic representation without outside assistance. Using the standards of evidence outlined in Chapter III, Emily had appropriated the graphical representation and completed the other two representations volitionally. Emily sought assistance from an UGTA for help with the notation for the algebraic representation of error so this representation was coded as learned through instruction. “I really only asked for help on two things this whole lab,” Emily said. “All I needed help with was the notation on here [approximation] and here [error] and the thing with epsilon [algebraic error bound]. Other than that, I got through pretty much everything.”

Kaitlin. Kaitlin was one of the three students unable to schedule a second interview. Since she turned in a write-up, the coding scheme was altered to at least partially code her document. In Table 24, only the cells in which there were field notes of Kaitlin working on a component of the approximation framework in class were coded. Of those portions, all the cells were volitional or appropriated except for the algebraic and numerical representations of error; Kaitlin was assisted in constructing the solutions for these representations. The remaining six non-omitted cells had correct solutions but without interview or observational data, there was no way to distinguish between learned through instruction, volitional solutions, or appropriation. Kaitlin had no questions on any post-lab associated with this activity. Even considering the cells that could not be coded, it is clear that Kaitlin appropriated most of the approximation framework by the end of the semester with only minimal difficulty with errors.

Table 24

Kaitlin's Definite Integral Lab Codes

	Contextual	Graphical	Algebraic	Numerical
Unknown Value	X	Appropriation	Appropriation	Appropriation
Approximation	Volitional	Appropriation	Appropriation	Appropriation
Error	LTI/Appropriation	Appropriation	LTI (researcher)	LTI (researcher)
Error Bound	LTI/Appropriation	Appropriation	LTI/Appropriation	Appropriation
Desired Accuracy	LTI/Volitional/ Appropriation	LTI/Appropriation	LTI/Appropriation	Appropriation

Note. ‘X’ denotes a question the participant left blank. LTI = Learned through instruction.

George. All the non-coded cells of George’s definite integral lab were correct solutions. However, whether the solutions were learned through instruction, were volitional, or were appropriated could not be determined. George had no questions on his final pre-lab and his description of the approximation framework was not qualitatively different from Emily’s.

Table 25

George's Definite Integral Lab Codes (No Interview)

	Contextual	Graphical	Algebraic	Numerical
Unknown Value	Appropriation	Plan is right; work is not	Volitional	Appropriation
Approximation	LTI (UGTA)	Appropriation	X	Plan is right; work is not
Error	LTI/Volitional	Appropriation	LTI/Volitional	LTI/Volitional
Error Bound	LTI/Appropriation	Volitional	X	Appropriation
Desired Accuracy	LTI/Volitional	X	X	Appropriation

Note. 'X' denotes a question the participant left blank. LTI = Learned through instruction. UGTA = Undergraduate teaching assistant.

Although George earned a lower final grade than did Emily and Kaitlin, he showed a similar pattern of independence similar to that of Kaitlin and Emily once he was comfortable with the difference between errors and error bounds. Every one of George's labs showed progressively more scientific thinking and appropriation. George felt his instructor always said what he needed to hear after class to finish the lab but he did not make the connection between the post-labs and the content covered in class.

George omitted the portions of his write-up he was not sure were correct rather than turn in partial solutions. Although neither Emily nor Kaitlin exhibited this behavior, both of the next two students in this case study (Eva and Sandra) showed the same pattern. George's development most resembled Eva, who started with much weaker mathematical skills than George but acquired the approximation framework at the same rate.

Eva. With each lab, Eva improved the quality of reasoning in her responses and, furthermore, did not experience a large drop in quality when the context of the derivatives lab was less familiar (see Table 26). While Eva had less understanding of the first lab than did George, Kaitlin, and Emily, she improved from lab to lab in a manner more similar to these three students (who earned A's and regularly did the pre-labs and post-labs) than Leonard or Charles who earned similar grades for most of the semester and only sporadically completed the formative assessments.

Table 26

Eva's Definite Integral Lab Codes

	Contextual	Graphical	Algebraic	Numerical
Unknown Value	X	Volitional	X	Volitional
Approximation	Volitional	X	X	Appropriation
Error	Appropriation	Appropriation	LTI Increased Quality (intervention)	Volitional
Error Bound	Appropriation	LTI (UGTA)	Plan is right, work is not	Appropriation
Desired Accuracy	Ventriliquation	Ventriliquation	Ventriliquation	Appropriation

Note. 'X' denotes a question the participant left blank. LTI = Learned through instruction. UGTA = Undergraduate teaching assistant.

Sandra. Sandra's final approximation lab of the semester was her best. She needed the least amount of help to complete her write-up. The only areas that were coded as learned through instruction were the representations of error and summation notation. Most of Sandra's lab was completed volitionally and she appropriated two more portions of the approximation framework (see Table 27). In a separate educational ethnography that followed this project, Sandra—the only student in this group to continue on to the second semester of calculus—continued this pattern of steady improvement and appropriated even the error portions of the approximation framework. Sandra steadily improved her grade throughout the semester, from a mid-D to a mid-C, and continued that trajectory after the semester to eventually earn an A- in the second calculus course. For much of her work, it appeared that her lack of recent instruction over the prerequisite knowledge was the primary obstacle to her success. Sandra's development most closely resembled Eva's, but both participants have the same trajectory as the highly successful students who participated regularly in the formative assessments, albeit at a slower pace.

Table 27

Sandra's Definite Integral Lab Codes

	Contextual	Graphical	Algebraic	Numerical
Unknown Value	Volitional Increased Quality	Volitional Increased Quality	Appropriation	Plan is right, work is not
Approximation	Volitional	**X**	LTI Less Scaffolding (intervention)	Volitional
Error	LTI Less Scaffolding (UGTA)	LTI Less Scaffolding (UGTA)	LTI Less Scaffolding (intervention)	LTI Less Scaffolding
Error Bound	Volitional Increased Quality	Volitional Increased Quality	LTI (intervention)	Appropriation
Desired Accuracy	Ventriliquation	Ventriliquation	Ventriliquation	Appropriation

Note. ‘X’ denotes a question the participant left blank. LTI = Learned through instruction. UGTA = Undergraduate teaching assistant.

Definite integrals, sporadic participants. At the end of the semester, Charles had shown little improvement from his limits lab and still had no solutions attributable to post-lab-based instruction. However, Leonard was able to complete almost the entire definite integral lab without outside help except for those items covered in the post-lab-based instruction. Leonard needed this additional instruction to complete four of the six items discussed in class.

Leonard. Leonard did not complete a definite integrals lab write-up. In his second interview, he gave several reasons why he did not plan on turning one in:

> It’s due in a few days, and I have my Chem lab exam to worry about. I figured my grade, and since I didn’t do the WeBWorK all semester, I’m not going to pass

> with a C – probably a D. Since I'm retaking anyway, it's smarter for me to focus on my other classes right now.

For the first portion of the interview, I asked Leonard to answer the questions in the lab orally. After each response, I asked Leonard to explain the reasoning that led up to that answer and if anyone had helped him understand that part of the lab. Although I coded the responses based on his interview transcript rather than a written lab, I believe Leonard's interview answers and the scratch work he did provided a reasonable approximation of what he would have turned in had he been so inclined.

The first part of the lab Leonard and I talked about was the unknown value portion of the approximation framework. Here is how Leonard responded to the question, "Can you tell me how you would have completed the pre-lab?"

> Well, I didn't understand why the answer was unknown at first. In physics we just plugged numbers into that same formula. Then you brought some rubber bands out. When I pulled on it, I could tell that I wasn't using constant force. Then I knew that this was a Riemann sum problem where we are approximating an area under a curve, so I drew a quick graph like this, and a made a table of values to calculate a rough estimate. I had to ask Emily for help with the notation though.

From his comments about the pre-lab, it is clear that Leonard knew that the approximations in this context were supposed to be Riemann sums and how to calculate them. I wanted to probe further about his contextual and algebraic understanding of approximations:

> Researcher: Why is a Riemann Sum an approximation in this case?
>
> Leonard: Well, we can't use Hooke's Law directly, because the force isn't constant. So what we do is break up the interval into pieces and pretend it is constant over the piece. That gives the rectangle part of the Riemann sum. When we find the area, it makes the units work out right, so that gives us an idea for how much work was done.

Researcher: How do we represent an approximation, a Riemann sum, algebraically?

Leonard: With that weird E looking thing [sigma notation]. I got help from [my instructor]. I tried to copy what she did here [points to post-lab-based instruction notes.]

Based on this portion of the interview, I coded the contextual and graphical representations of approximations as volitional. The numerical representation of approximation was volitional. The algebraic representation of a Riemann sum was also coded as learned through instruction and attributed to the post-lab-based instruction.

Error was one part of the approximation framework Leonard struggled with throughout the semester but he was able, within this context, to reason and explain errors to me in the course of the interview:

Researcher: What are the errors in this situation?

Leonard: Well, the errors are what the amount of force we lost by pretending that the force was constant. They are these triangles here [points to graph]. There is one for the overestimate and one for the underestimate. [My instructor] taught us how to write the error in algebra the next day, but if you just want a particular error, you can replace the Riemann sum with the value of the approximation, that is the numerical error.

Researcher: Where did you learn that stuff? Error is something you've had trouble with all semester.

Leonard: Well, it's basically the same thing we did with the Iodine [derivatives lab], only the notation was different.

Based on this portion of the interview, I coded the contextual, graphical, and numerical representations of error as volitional and the algebraic representation as learned through instruction.

Error bound was the area of the approximation framework Leonard was most comfortable with and this lab was no exception:

> The error bound gives an estimate for how wrong we can be at most about the work. See, it's the overestimate minus the underestimate. [points to the graph] Or it's this rectangle here [points to the part of the rightmost rectangle that lies above the height of the first rectangle]. Algebraically, it is the integral minus the Riemann sum, but [my instructor] showed us in class how to do that.

Based on this response, the contextual and graphical representations of error bound were coded as volitional while the numerical representation of error bound was coded as appropriated based on his past performance. I coded the algebraic representation as learned through instruction because Leonard needed help to assemble the components of the error bound; however, he was able to write down the algebraic representation volitionally once he had that assistance.

For the final portion of the lab, Leonard freely admitted that on the first two labs, he just transcribed what someone else, generally a UGTA, told him to say. But this time, he seemed to have more ownership of how to get within any error bound:

> If we want to make sure we approximate the work to within some number of sigfigs, we have to first calculate how many rectangles to use. Then we can't really draw of a graph of that because there are usually too many to draw. Then you find the left and right Riemann sums. As long as you calculated your *n* right, everything should work out.

I coded this as volitional understanding of the contextual, graphical, and numerical representations of this question. Leonard, even with probing, could not articulate what the algebraic representation could be to the desired accuracy so I did not code anything in that cell (see Table 28).

Table 28

Leonard's Definite Integrals Lab Codes

	Contextual	Graphical	Algebraic	Numerical
Unknown Value	LTI; UGTA	Volitional	LTI; Group	Volitional
Approximation	Volitional	Volitional	LTI; intervention	Volitional
Error	Volitional	Volitional	LTI; intervention	Volitional
Error Bound	Volitional	Volitional	LTI; intervention	Appropriated
Desired Accuracy	Volitional	Volitional	X	LTI; intervention

Note. 'X' denotes a question the participant left blank. LTI = Learned through instruction. UGTA = Undergraduate teaching assistant.

Charles. Like Leonard, Charles put minimal effort into the final approximation lab because he felt he was locked into a grade band. "I've done enough to get a C. I'm an elementary education major. I never need to know this stuff, so I'm putting my effort into my other classes," Charles explained in his second interview. The final approximation lab Charles turned in was almost identical to the one he completed on the limits lab: he answered the algebraic representation of the unknown volitionally on the pre-lab and then turned in one volitional response on the lab write-up with no other parts answered. Charles did not complete the pre-lab or either post-lab associated with the definite integrals lab. In his interview, Charles stated that he did not know how to do any of the other questions.

Overall, Charles and Leonard were similar in their trajectories in the course. Both were reluctant to engage in formative assessment and reported that everything was fine when they did complete a post-lab. Charles and Leonard never sought for help outside of class and completed little or none of the final lab. However, Leonard benefited from the post-lab-based discussions following each lab work day, particularly when the additional instruction focused on the algebraic representations.

Definite integrals, nonparticipants. Neither Lisa nor Tre turned in a definite integral lab. Tre withdrew from the course before this lab began; despite my attempts to schedule an interview, he was not interviewed a second time. By the time the definite integral lab began, Lisa knew that it would be impossible for her to pass the class. Lisa did attend class regularly for the last month of the semester and took the final exam but she did not complete any of the online homework or the final approximation lab. Lisa's second interview was conducted as a task-based interview using the Lab 7 questions. However, even with a great deal of scaffolding, Lisa could answer only a handful of questions (see Table 29). None of the questions Lisa was able to answer were items discussed in class. Lisa maintained that even the discussions in class did not help her. "Yeah, [My instructor] talked about all of this. But there are a lotta things to write down. It all went so fast and was so hard to see I was even more lost after that," Lisa said in her second interview. There is no evidence that the class discussion helped Lisa; in fact, the discussion might have confused her more. Lisa did have a volunteer student note-taker for the course but the student providing notes to Lisa was a regular participant in the formative assessments. The note-taker (whose labs were generally very good) took sparse notes during class discussions; notes over this part of class were not always helpful for Lisa.

Table 29

Lisa's Lab 7 Codes

	Contextual	Graphical	Algebraic	Numerical
Unknown Value	X	Plan is right; work is not	X	LTI
Approximation	LTI; Group	LTI; Group	X	Plan is right; work is not
Error	X	X	X	X
Error Bound	X	X	X	Appropriation
Desired Accuracy	X	X	X	X

Note. 'X' denotes a question the participant left blank. LTI = Learned through instruction.

Overall, the post-lab-based instruction was not particularly helpful for the regular participants unless they asked specific questions on their post-lab. This was usually because the regular participants had already appropriated that portion of the approximation framework or they sought help from other channels. Of all of the case study participants, Leonard was the biggest beneficiary of the post-lab-based instruction but Charles never incorporated post-lab-based instruction into his very limited write-ups. Lisa showed some benefit from the post-lab-based instruction when the context was familiar; Tre, who was the least familiar with functions and function notation, never incorporated the post-lab-based instruction into his write-ups. The instruction given during class was intended to address the most apparent student difficulties on the post-lab but the instruction was not tailored to any particular student's understanding of a particular solution. The next purpose of formative assessment is to provide feedback tailored to individual student needs that help them progress through their ZPDs.

The Third Purpose: Providing Feedback That Moves Learners Forward

Formative feedback that moves learners forward, according to Black & Wiliam (2009), is defined as individual feedback customized to a particular learner's needs. This feedback is generally written but is not required to be so. During data collection for this project, students received individual instructor feedback from their instructor once--on the derivatives lab.

The derivatives lab had two high stakes summative assessments occur during the three weeks of the lab. On the post-labs, students asked few questions; since the previous limits labs were generally good, the discussions following each lab day were very short. However, only 8 of 54 students answered 14 or more items correctly on the lab write-up. Rather than recording those grades, the instructors gave all of the students who turned in a derivative lab individual written feedback on all of the questions they either answered incorrectly or left blank. The instructors told students that their first attempt would be considered a draft. Students were then given a week to revise and resubmit their derivative lab based upon the formative feedback; this became the final version of the derivative lab.

I first examined the derivatives lab write-ups quantitatively using an ANCOVA. In order to investigate if there were differences in student performance after feedback, I began by eliminating all of the students from the three participation levels who did not receive individual feedback. This left 21 regular participants, 10 sporadic participants, and five nonparticipants. One regular participant, the only regular participant who failed the course, was an outlier and eliminated from the sample, leaving 20 cases in the group.

The ANCOVA showed a significant difference in mean performance on the revised derivatives lab write-ups after controlling for the score on the write-up where students received initial feedback (see Table 30).

Table 30

Analysis of Covariance, Derivatives Lab

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*
Adjusted Means	252.17	2	126.08	6.38	.005
Adjusted Error	651.86	33	19.75		
Adjusted Total	904.03	35			

For the post-hoc analysis, I used simple contrasts with a Bonferroni correction to account for the multiple comparisons. The regular participants' mean performance was significantly higher than the mean of the nonparticipants. The sporadic participants' mean performance was also higher than the mean of the nonparticipants. Although the difference between the regular and sporadic participants was not significant, the relatively low *p*-value suggested that further exploration might be warranted (see Table 31).

Table 31

Post-Hoc Analysis of Co-Variance

	Initial Feedback		Final Write-up		Adjusted
Group	*M*	*SD*	*M*	*SD*	*M*
Regular	11.84	0.77	13.52	0.73	2.64a
Sporadic	2.37	1.13	9.47	1.85	7.56a
Nonparticipant	2.33	1.44	2.83	1.71	0.00

Note. Adjusted means with the same letter are not significantly different ($p < .05$) based on post-hoc analysis with a Bonferroni correction.

Since students received written feedback for every incorrect or blank response on their derivatives draft, the nonparticipants received the most instructor feedback. However, even when students' initial derivatives lab write-up scores were accounted for in the ANCOVA, students in the other participation groups were significantly outperforming the nonparticipant group on the derivative lab rewrite. This suggested that even with more extensive written feedback, the nonparticipants were not able to increase their mean scores as much as the regular and sporadic participant groups did.

In the analysis of case studies, not all of the students turned in both an initial and a revised write-up. For the case study participants who took advantage of the written feedback, there was an increase in scientific reasoning on their revised version.

Regular Participants

For the most part, the regular participants answered most of the questions correctly on their first submission of the derivatives lab and so they received minimal feedback. However, each regular participant showed improvement on the questions on which they received comments.

Emily. Emily was one of the few students who had a high quality initial derivatives lab write-up (see Table 32); this might be because she sought help immediately from lab facilitators and her instructor when she got confused on what to use for approximations. Emily had every question correct on her initial lab except for three of the desired accuracy representations. After getting the written feedback, Emily sought additional help from her instructor.

Table 32

Emily's Derivatives Lab Feedback Codes

	Desired Accuracy (Context)	Desired Accuracy (Graphical)	Desired Accuracy (Algebraic)
Lab 4 Draft	Ventriliquation	Ventriloquation	Ventriloquation
Lab 4 Rewrite	LTI (instructor)	LTI (instructor)	LTI (instructor)

LTI = Learned through instruction.

For the final component of the approximation framework, Emily explained that she still did not really understand the question: “I don’t really get what this is asking. I got full points last lab, so for this I just copied what I said last time and changed the word hole to instantaneous rate of change.” In other words, Emily ventriloquated her previous ventriliquation and the coding reflected this situation. On her derivatives lab rewrite, Emily recopied almost all of her original lab (see Table 25). The only component of the approximation framework to which she made any changes was the desired accuracy question at the end. "I hate not understanding things, "Emily explained, "Since we got a rewrite, I went to [instructor's] office hours and asked for help." Her new solution was learned through instruction during those office hours. No other codes changed from the original to the rewrite.

Kaitlin. Kaitlin, like Emily, had a high quality initial write-up and only had a few items that changed on her rewrite. Kaitlin received instructor feedback on her two incorrect responses--the algebraic and numerical representation of error.

Kaitlin received help on three parts of the lab in class: what the unknown value was in the context of the problem and the algebraic and numerical representations of error (see Table 33). On her post-labs, Kaitlin asked the difference between error and error bound on the first post-lab but otherwise had no questions. On her draft, Kaitlin made small errors in the algebraic and numerical representations of error, but the rest of her write-up was correct.

Table 33

Kaitlin's Derivatives Feedback Codes

	Unknown Value (Context)	Error (Algebraic)	Error (Numerical)
Lab 4 Draft	X	Plan is right, work is not	Plan is right, work is not
Lab 4 Rewrite	Unjustified Heuristic	LTI (formative feedback)	LTI (formative feedback)

Note. 'X' denotes a question the participant left blank. LTI = Learned through instruction.

On her derivatives lab rewrite, Kaitlin's contextual unknown value solution discussed *y* values instead of slopes and was coded as an unjustified heuristic. Kaitlin fixed her error representations based on the written comments provided by her instructor so those solutions were learned through instruction.

George. George provided a solution for 13 of the 20 (65%) questions on his first derivatives lab write-up; one of those questions was incorrect. On his rewrite, every question George improved upon could be attributed to the written formative feedback (see Table 34):

> You know, I work at a ski resort on the weekends. I didn't have time to finish this lab before I left for work. [Instructor]'s comments were really helpful-I got most of it fixed. I didn't get what [instructor] was saying here, which is why I left the questions blank.

Table 34

George's Derivatives Feedback Codes

	Lab 4 Draft	Lab 4 Rewrite
Unknown Value (Graphical)	Unjustified Heuristic	LTI (formative feedback)
Error (Context)	X	X
Error (Algebraic)	X	X
Error (Numerical)	X	LTI (formative feedback)
Error Bound (Algebraic)	X	LTI (formative feedback)
Desired Accuracy (Context)	X	LTI (formative feedback)
Desired Accuracy (Graphical)	X	X
Desired Accuracy (Algebraic)	X	LTI (formative feedback)

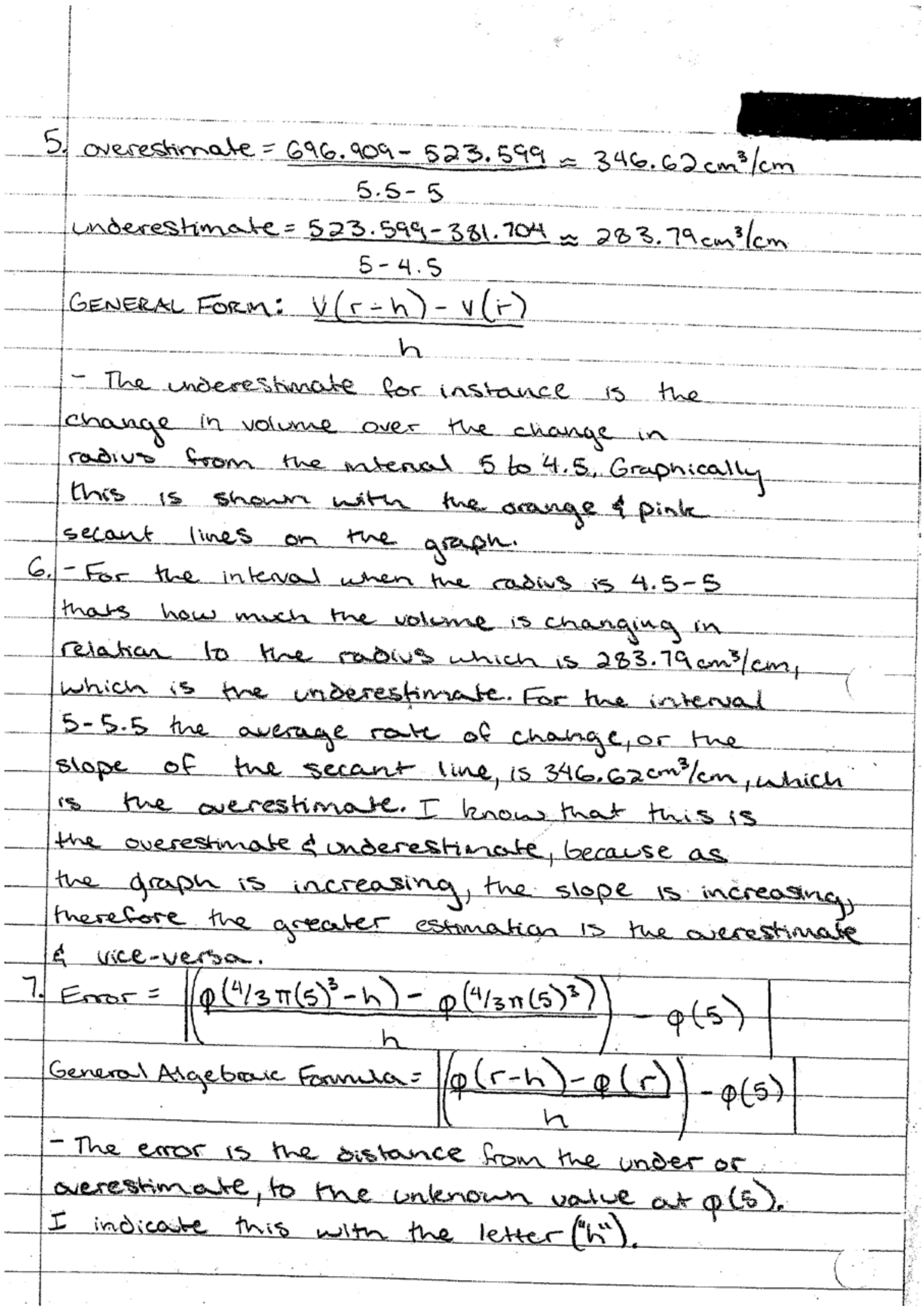
Note. 'X' denotes a question the participant left blank. LTI = Learned through instruction.

On his rewrite of the derivatives lab, George corrected six cells based on the comments provided by his instructor; the remaining portions of the lab were copied from the original assignment. All of George's comments were explicit instructions on how to correct his solution so all new solutions were coded as learned through instruction that was attributable to the formative feedback.

Eva. Eva also struggled with the Gateway exam and said she did not have a lot of time to do this lab. Eva completed the post-labs for the derivatives lab but said she understood everything she had done. Given her solution, this was an accurate statement. Her lab appears at the end of this section (see Figures 16-18). Other than the pre-lab and the final question on reaching any desired accuracy (parts of the lab that were worth minimal points of the total lab grade), Eva showed scientific reasoning or appropriation on most of the write-up (see Table 35). From the standards of evidence outlined in the previous chapter, all of the codes on the items Eva answered required the answer to be correct except for the learned through instruction code; both LTI answers were also correct. Since Eva would not have earned credit for turning in the pre-lab (unknown value row) with her revision, she decided that the 1.5 points[4] of potential improvement were not worth her time.

Eva received formative feedback on all of the blank items and the algebraic representation of error but she chose to focus on the Gateway exam rather than rewriting her best lab to date.

[4] The instructors weighted questions differently during grading than I used for the research, which accounted for the 3.5 point discrepancy.



[redacted]

5. overestimate = $\frac{696.909 - 523.599}{5.5 - 5} \approx 346.62 \text{ cm}^3/\text{cm}$

underestimate = $\frac{523.599 - 381.704}{5 - 4.5} \approx 283.79 \text{ cm}^3/\text{cm}$

GENERAL FORM: $\frac{V(r-h) - V(r)}{h}$

- The underestimate for instance is the change in volume over the change in radius from the interval 5 to 4.5. Graphically this is shown with the orange & pink secant lines on the graph.

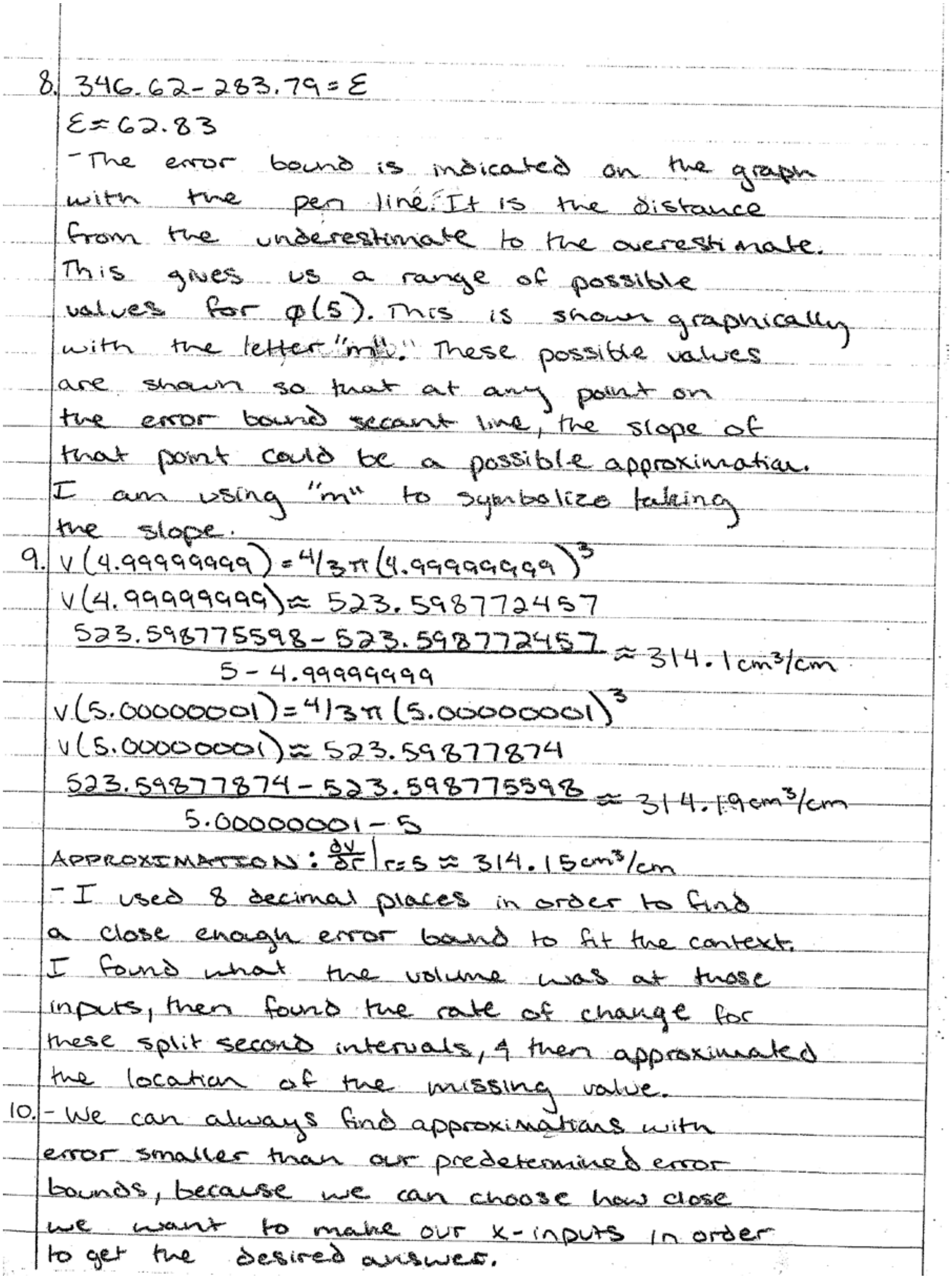
6. - For the interval when the radius is 4.5-5 thats how much the volume is changing in relation to the radius which is 283.79 cm³/cm, which is the underestimate. For the interval 5-5.5 the average rate of change, or the slope of the secant line, is 346.62 cm³/cm, which is the overestimate. I know that this is the overestimate & underestimate, because as the graph is increasing, the slope is increasing, therefore the greater estimation is the overestimate & vice-versa.

7. Error = $\left| \left(\frac{\varphi(4/3\pi(5)^3 - h) - \varphi(4/3\pi(5)^3)}{h} \right) - \varphi(5) \right|$

General Algebraic Formula = $\left| \left(\frac{\varphi(r-h) - \varphi(r)}{h} \right) - \varphi(5) \right|$

- The error is the distance from the under or overestimate, to the unknown value at φ(5). I indicate this with the letter ("h").

Figure 16. Eva's derivatives lab, page 1.



8. 346.62 − 283.79 = ε
ε ≈ 62.83
- The error bound is indicated on the graph with the pen line. It is the distance from the underestimate to the overestimate. This gives us a range of possible values for ϕ(5). This is shown graphically with the letter "m." These possible values are shown so that at any point on the error bound secant line, the slope of that point could be a possible approximation. I am using "m" to symbolize taking the slope.

9. $V(4.99999999) = \frac{4}{3}\pi(4.99999999)^3$
$V(4.99999999) \approx 523.598772457$
$\frac{523.598775598 - 523.598772457}{5 - 4.99999999} \approx 314.1\ cm^3/cm$
$V(5.00000001) = \frac{4}{3}\pi(5.00000001)^3$
$V(5.00000001) \approx 523.59877874$
$\frac{523.59877874 - 523.598775598}{5.00000001 - 5} \approx 314.19\ cm^3/cm$
APPROXIMATION: $\left.\frac{dV}{dr}\right|_{r=5} \approx 314.15\ cm^3/cm$
- I used 8 decimal places in order to find a close enough error bound to fit the context. I found what the volume was at those inputs, then found the rate of change for these split second intervals, & then approximated the location of the missing value.

10. - We can always find approximations with error smaller than our predetermined error bounds, because we can choose how close we want to make our x-inputs in order to get the desired answer.

Figure 17. Eva's derivatives lab, page 2.

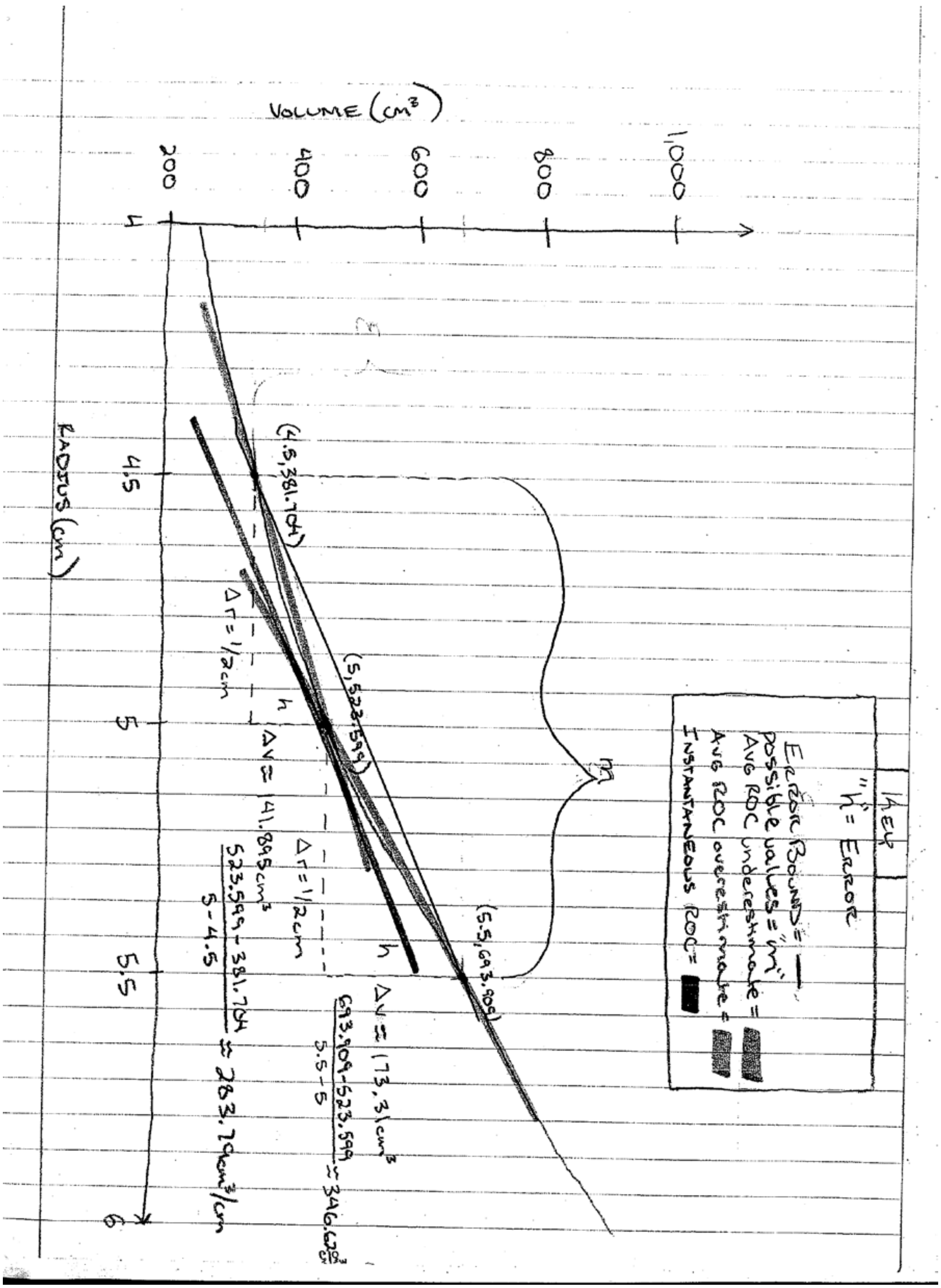


Figure 18. Eva's derivatives lab, page 3.

Table 35

Eva's Derivatives Lab Draft Codes

	Contextual	Graphical	Algebraic	Numerical
Unknown Value	**X**	**X**	**X**	**X**
Approximation	LTI (researcher)	SBR	Ventriloquation	Appropriation
Error	Volitional	Volitional	Unjustified Heuristic	LTI (intervention)
Error Bound	Volitional	SBR	Volitional	Appropriation
Desired Accuracy	X	X	X	Appropriation

Note. ‘X’ denotes a question the participant left blank. LTI = Learned through instruction.

Sandra. Like Charles, Eva, and Leonard, Sandra completed very little of the first derivatives lab write-up. In fact, she only completed 3 of the 20 items on the derivatives draft, one of which was incorrect. Sandra omitted the contextual unknown value item and had calculation errors on the numerical representation of unknown value: “After I passed the Gateway, I had time to get help. I spent a lot of time in office hours. I had started writing notes on how to do the lab, and then with [my instructor’s] comments, I was able to pretty much finish it.”

In her second attempt at this lab, for which she received extra help from her instructor both inside and outside of class, Sandra was able to complete far more of the assignment. Most of this lab was coded as learned through instruction but less scaffolding was needed to help her construct the solutions. However, some of the graphical parts were situation-bound reasoning and her contextual description of the unknown value discussed *y* values instead of slopes. In Table 36, the bold cells are the items that were correct on the pre-lab. Two items were coded as volitional because Sandra knew how to calculate the value but did not know how to calculate the approximations. Once she was able to calculate the approximations, she was able to complete those two cells without relying on any feedback.

Table 36

Sandra's Derivatives Lab Final Draft Codes

	Contextual	Graphical	Algebraic	Numerical
Unknown Value	Unjustified Heuristic	**Volitional**	**Appropriation**	LTI (instructor)
Approximation	Classification preceded explanation	SBR	LTI (formative feedback)	LTI (formative feedback)
Error	LTI Less Scaffolding (formative feedback)	LTI Less Scaffolding (formative feedback)	LTI (formative feedback)	LTI Less Scaffolding (instructor)
Error Bound	LTI (formative feedback)	LTI (formative feedback)	LTI (formative feedback)	Volitional
Desired Accuracy	LTI Increased Quality (instructor)	**X**	**X**	Volitional

Note. 'X' denotes a question the participant left blank. SBR = Situation based learning. LTI = Learned through instruction. Bold cells are items that were correct on the pre-lab.

Sandra and George both made a great deal of progress after receiving the formative feedback. Kaitlin also improved after the formative feedback on the few questions she needed to fix. Eva chose not to rewrite her lab and Emily sought help directly from her instructor. Overall, the formative feedback was helpful for those students who chose to use it. For the sporadic participants, the utility of the written formative feedback was less clear.

Sporadic Participants

For this lab, Leonard did not do a first draft of the lab and hence did not receive any formative feedback. Charles did both the draft and the rewrite. Almost all of his new correct answers on the rewrite could be attributed to his written feedback.

Leonard. For most students, this was their worst lab of the semester and Leonard was no exception. He did not turn in an initial write-up; thus, his first draft would become his final draft. In his interview, he explained why he did not turn in a write up: "I didn't do it [the derivatives write-up] because I didn't get where to start once I was on my own. In class, in the group, everything made sense, but it was gone Thursday when I went to start. I should've gotten help, but I was out of time." Leonard received no formative feedback from his instructor and all of his LTI codes on his final derivatives lab could be attributed to instruction he received from his group member, Emily.

Charles. "Since I didn't pass the Gateway right away, I figured I needed to try in case I got the penalty," Charles informed me in his second interview. This was his justification for doing an initial derivatives lab write-up and a re-write; he saw calculus as a box to check on his way to becoming a teacher. His primary goal was to do as little work as possible to pass the course. Unlike Leonard, most of Charles' reasoning on the approximation framework was spontaneous (see Table 37).

Table 37

Charles' Derivatives Lab Rough Draft Codes

	Contextual	Graphical	Algebraic	Numerical
Unknown Value	X	X	X	X
Approximation	Unjustified Heuristic	Unjustified Heuristic	Unjustified Heuristic	SBR
Error	X	X	X	X
Error Bound	X	X	Volitional	Volitional
Desired Accuracy	X	X	X	X

Note. 'X' denotes a question the participant left blank. SBR = Situation based learning.

Half of the lab was not answered and there were only two volitional responses. The items Charles answered volitionally on the limits lab were not answered in the derivatives lab in the initial write-up. He did not complete any of the derivatives lab post-labs and only completed the pre-lab as part of his rewrite. Charles explained why he rewrote the derivatives lab: "Well, I thought, that I didn't get related rates at all, and [my instructor] was gonna give us credit for the pre-lab if we did it this time. I figured I need to bank some points against the next test. Plus in the comments--it said what to do."

Charles chose to rewrite the derivatives lab because he saw it as an easy way to hedge against the poor grade he expected on the next unit test. He attributed the anticipated poor grade, which he earned, to the many attempts it took him to pass the

Gateway exam. The rewritten derivatives lab was Charles' best approximation framework lab of the semester.

Charles' overall reasoning about the approximation framework was either unjustified heuristics, learned through instruction, or ventriliquation of the comments on his rough draft. This aligned with the amount of instruction given in class and the large amounts of comments Charles received on his initial derivatives lab. Of the six new correct answers on Charles' rewrite, five of them could be attributed to the formative assessment he received (see Table 38).

Table 38

Charles' Derivatives Lab Final Draft Codes

	Contextual	Graphical	Algebraic	Numerical
Unknown Value	Volitional	Unjustified Heuristic	LTI (formative feedback)	Unjustified Heuristic
Approximation	Unjustified Heuristic	Unjustified Heuristic	LTI (formative feedback)	SBR
Error	Empirical	X	Unjustified Heuristic	X
Error Bound	LTI (formative feedback)	LTI (intervention)	Volitional	Volitional
Desired Accuracy	Vent.	X	X	LTI (formative feedback)

Note. 'X' denotes a question the participant left blank. SBR = Situation based learning. LTI = Learned through instruction.

Like Leonard, the additional scaffolding Charles received on the derivatives lab was his primary formative assessment for the semester; he also showed the greatest improvement between the two drafts of this lab. However, Leonard's additional scaffolding came from a group member and not formative feedback.

Nonparticipants

Lisa was one of the few nonparticipants who turned in a draft for the derivatives lab; she was able to improve some of her lab based upon that feedback. Tre withdrew from the course before the first write-up was due.

Lisa. Like almost all of her classmates, Lisa had a difficult time completing any of the derivatives lab solution at the end of the third week of the lab. Two things about Lisa's initial solution were in sharp contrast to Leonard and Emily. The first was the relative lack of mathematics in Lisa's solution; most of the portions of the solution Lisa provided were for the written contextual questions that did not involve calculations. The second major difference between Lisa and her two group members was the amount of spontaneous reasoning codes in her second lab (see Table 39); the majority of Leonard and Emily's labs were coded as learned through instruction. Lisa had four correct questions on the initial write-up--the approximation, error, and error bound numerical representations, and the contextual approximation. She received feedback on the other 16 questions.

Table 39

Lisa's Derivatives Lab Rough Draft Codes

	Contextual	Graphical	Algebraic	Numerical
Unknown Value	Empirical	Unjustified Heuristic	X	Unjustified Heuristic
Approximation	Vent.; Classification precedes explanation	X	X	LTI (group)
Error	Unjustified Heuristic	X	X	LTI (group)
Error Bound	Unjustified Heuristic	X	Unjustified Heuristic	Appropriation
Desired Accuracy	X	X	X	X

Note. 'X' denotes a question the participant left blank. LTI = Learned through instruction.

Lisa began the lab describing the unknown instantaneous rate of change as a hole in the graph at an unknown height, which was her empirical experience of unknown values to date. On the graph of the function, Lisa drew the general shape of the graph correctly with labeled axes and a scale given; but at the point of interest, she drew a hole. The hole was not labeled or referred to in her written work with an algebraic symbol so I did not code that representation. The graph was coded as an unjustified heuristic; the graph Lisa produced would have been a high quality graph had the unknown value been a removable singularity like the prior lab. Lisa was trying to apply the knowledge from the first lab to this one but could not separate the approximation framework from the initial context of removable singularities. Lisa explained that the unknown value was the numerical height of the removable singularity; her numerical unknown value representation was also coded as an unjustified heuristic.

During the three lab days, two different UGTAs, the instructor, and I worked with Lisa's group and explained in multiple ways that the approximations were slopes of secant lines and how to calculate those values. The numerical calculations were learned through instruction but Lisa was skeptical of our explanations:

> I didn't really understand why everyone was insisting that we weren't approximating y values. The first thing we did this time was make an $x - y$ table with x values really, really close to 5. That is exactly what we did last time with the hole. I wrote down what everyone said because they all said the same thing in class. I can tell you that this one is the overestimate because it is the bigger number, but I can't give you a better reason for why it is true.

Based on this interview response, I coded her contextual approximation representation as ventriliquation; her classification of overestimates and underestimates preceded her ability to justify her (correct) classification.

For error, one of the UGTAs showed Lisa how to write down the numerical approximation for errors using the approximations she calculated earlier on the first day of the lab, so Lisa completed that solution after instruction. However, her contextual explanation for the errors in the approximation was that error was the difference between the y values in the table and the hole, the same unjustified heuristic as before. The contextual and algebraic representations of error bound were also the product of the same unjustified heuristic; only the actual function was changed in the notation of the answers between the derivatives lab and the definite integral lab. Lisa had appropriated the numerical value for the error bound was the absolute value of the difference between the overestimate and the underestimate. This would be the first and only cell of the approximation framework Lisa would appropriate.

Lisa was more successful on the rewrite of the derivatives lab. However, two representations in her solution that were initially unjustified heuristics (graphical unknown value and algebraic error bound) were unchanged from her first write-up to her second (see Table 40). Lisa's approximation for the contextual and numerical representations and the numerical error bound were transcribed from her original solution so the coding in those cells remains unchanged. This left five cells of the approximation framework that had a new or improved solution on her derivative lab rewrite: contextual and numerical unknown value, graphical approximation, and contextual and graphical error bound.

Table 40

Lisa's Derivatives Lab Final Draft Codes

	Contextual	Graphical	Algebraic	Numerical
Unknown Value	LTI Increased Quality (formative feedback)	Unjustified Heuristic	X	LTI Increased Quality (formative feedback)
Approximation	Vent.; Classification precedes explanation	LTI (formative feedback)	X	LTI (group)
Error	X	X	X	LTI (group)
Error Bound	LTI Increased Quality (group)	LTI Increased Quality (group)	Unjustified Heuristic	Appropriation
Desired Accuracy	X	X	X	X

Note. 'X' denotes a question the participant left blank. LTI = Learned through instruction.

Lisa explained that Leonard helped her on her rewrite:

> I'm not sure I got why the unknown value was a slope and not a point until Leonard and I talked about it after class. We have physics together; when we looked at the lab that week in there, I got it…we had to draw tangent lines on the position graphs we calculated. Then we took their slope to find the speed. Other than as a starting place for the tangent line, the point didn't matter; it helped find the answer, but it wasn't the answer. That's what Leonard helped me get.

Both of the unknown value representations were coded as learned through instruction. The increased quality of Lisa's solutions indicated that she was making connections between the approximation framework and other conceptual knowledge she possessed and was in her Zone of Proximal Development. Although Lisa did not change the underlying graph from the initial solution on her rewrite, she did add secant lines to the graph. I coded these additions as learned through instruction since tangent lines were also a part of the discussion with Leonard.

Next, I asked Lisa if Leonard helped her with the rewrites she made to the error bound portion of her write-up but she explained that her help with that part of the rewrite came from another source:

> No, [my instructor] helped me with that part. On the day everyone got their labs back and she said we were gonna get rewrites she said that the part everyone had the most trouble with was error bound. So I listened really carefully to the first part of her explanation about what we doing wrong and how we should fix our answers. I used that to answer the parts I didn't the first time. I'd say the help I got came from that…I didn't change my answers for the rest of error bound because they were right.

These final two portions of the rewrite were coded as learned through instruction. Lisa's answers were more detailed and accurate than either of her prior approximation lab solutions so these responses were also coded as increased quality.

Of the five items on which Lisa improved, three were attributable to the formative feedback she received. The other two items were completed with Leonard's help. Still, Lisa did benefit from some feedback from the formative assessments.

Tre. Tre did not turn in a derivatives lab draft and stopped attending the course before the rewrite was offered.

Lisa was able to move forward on the questions from which she received formative feedback in the portions of the approximation framework where she completed multiple representations of the approximation framework. When I asked about the error items that appeared on the first draft but not the second, Lisa explained that she was not sure how to fix the context question. Overall, there was some benefit from the formative feedback for students in all three participation levels but the regular participants benefited the most.

The Fourth Purpose: Activating Students as Learning Resources for Each Other

The fourth purpose of formative assessment was to activate students as learning resources for each other and encourage collaboration. Although the lab activities were highly collaborative experiences for students, there was little direct evidence that directly related the pre-lab to student collaboration. This lack of evidence might be due to the data collection methods. During data collection, I started every lab session by collecting the pre-labs, copying them, and returning them to the students. Since the students needed the pre-lab to complete the activity and would often change their pre-labs during class, collecting and copying the pre-labs immediately allowed me to get a clean snapshot of how students were thinking before the lab began. Unfortunately, this meant that the data collection either suppressed students' discussion about the pre-labs or the students discussed the pre-labs when I was out of the room. However, three instances were recorded in the fieldnotes[5] where the last group I collected pre-labs from began a spontaneous discussion about what pre-lab solution was correct and why.

The vignette that follows was a typical example of the discussion students had about pre-labs. Emily, Leonard, and Lisa were discussing their pre-lab to the limits lab. All three of them came in with a different graphical solution and they were trying to come to a consensus for what the solution was. Although Emily had the correct solution, she did not take over the group and insist her solution was correct; the process the students took to decide that Emily's graph was correct was a collaborative one. In that sense, all of the students were working to choose the correct solution, a task none of them were able to do on their own; hence, they were within their collaborative ZPD.

The bulk of this vignette came from expanded fieldnotes. The dialogue was reconstructed from the notes I took in class and confirmed with Leonard, Emily, and Lisa during their first interviews. I have also included relevant interview quotes from each of the three participants to illustrate their thinking throughout this process.

Emily, Lisa, and Leonard

On the third Tuesday of the semester, limits lab started. The day before, students had a lecture about limits and were given the lab. They were asked to read the lab and complete the pre-lab, which was to graph the function and form a plan for finding a solution.

Collecting pre-labs for research that still allowed students to use their pre-labs as a resource was still a work in progress. I decided that the best thing to do was to collect

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[5] Lab 3 (Limits), Lab 5 (Linear Approximation, both classes).

all the pre-labs at the beginning of the class period, photocopy them, and return them to the students. It took about five minutes that the instructor would use to talk about WeBWorK or make general announcements. Later in the semester, I started making a second copy so the instructor could see students' initial thoughts. The UGTAs helped with the collection process. I took a set of identically perfect pre-labs from the all-female group in the corner, moved up to the front of the room to get the stacks from the UGTAs there, and then got the labs from the last table, which was Leonard, Emily, and Lisa.

Unlike the other groups, this group was unique in that each member was at a different participation level. At this point in the semester, all I knew about this group was Lisa was legally blind and Emily had taken pre-calculus the previous semester; I did not know Leonard at all. I had subbed for pre-calculus once and remembered Emily asked a lot of questions. At this point, I had assumed Emily was the weak student in the group since she was usually the one in class asking clarifying questions. As this first extended interaction with their group began to show, I was wrong in my initial assessment.

All three students had their graphs out on the table. Emily's was perfect; she had the shape of the graph right, the hole labeled, and the axis scale mad sense (see Figure 19).

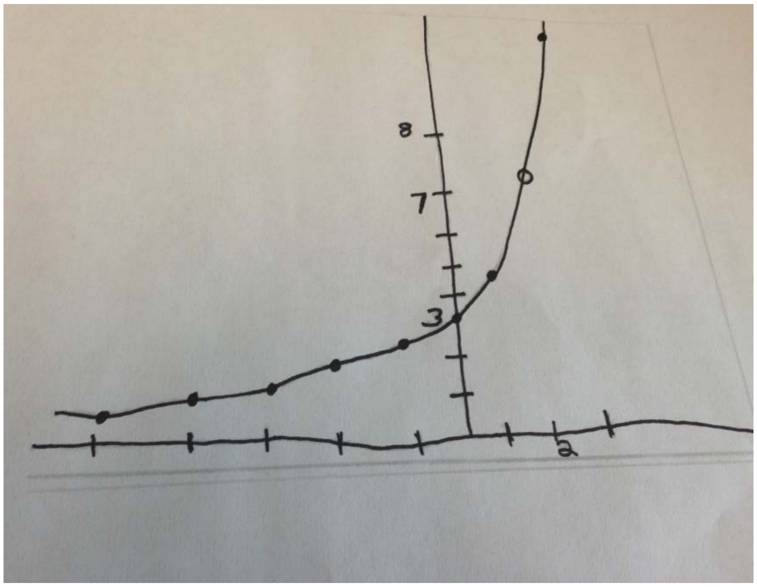
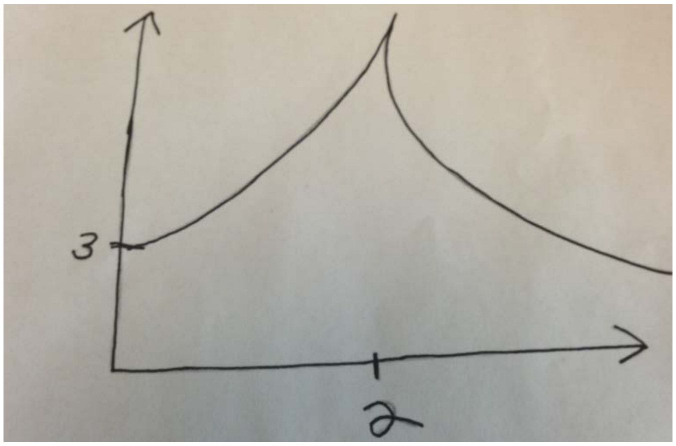


Figure 19. Emily's graph.

Lisa was finishing her pre-lab at the beginning of class; she had the general shape of the graph but had confused the labeling. Leonard had drawn a sharply pointed graph that a TI-83 calculator displayed if there was an asymptote at a particular *x* value. If he thought there was an asymptote at the point, there was no indication of that on the picture; it appeared that he had simply copied what he saw on his calculator screen onto his pre-lab (see Figure 20).

> Honestly, I just put the function into my calculator and drew what I saw. I didn't really think about if I entered anything right, because nothing before calculus was that hard to enter. Besides, every time I've seen like an *x*-2 in the denominator before this, that meant asymptote. If anything, when I saw that on my calculator, it made me less likely to check, because that was what I expected. My group set me straight right away though. (Leonard, first interview [looking at the limits pre-lab])



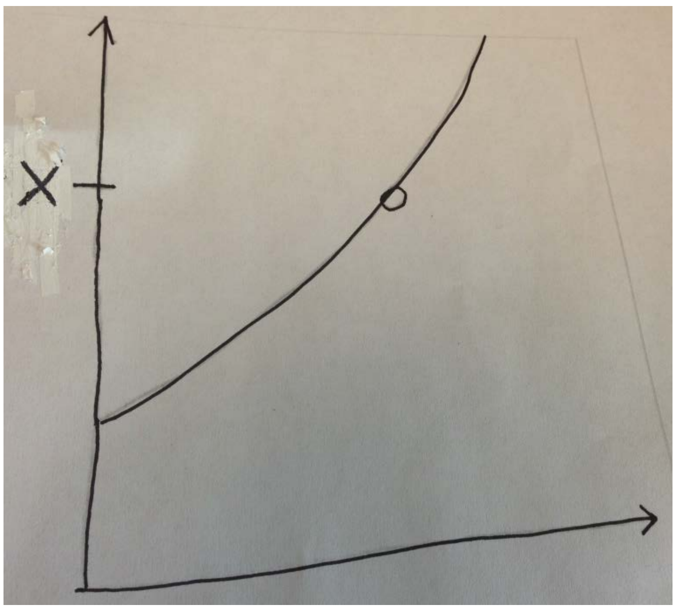


Figure 20. Leonard's graph (top) and Lisa's graph (bottom).

The three of them immediately started a discussion about Leonard’s graph; they did not initially notice that Lisa and Emily had labeled their axes differently. “How did you get your graph? It doesn’t look like mine or Lisa’s,” Emily asked Leonard. “Well, since there is an (*x*-2) in the bottom, there has to be an asymptote. So I put it in my calculator, which took a few tries, but then I drew what I saw on the screen,” Leonard replied.

"Since our answers are all different, maybe we should try re-graphing on our calculators?" Lisa suggested. They all got their calculators out and start clicking away on the buttons. They each got the same graph they drew on the pre-lab. Since the axes were not labeled on the calculator, Lisa's error was not obvious. "I thought we were right, since our two answers matched, but I let Emily do most of the explaining. She is a better explainer, so I thought it would go faster if she convinced Leonard," Lisa stated in her first interview.

After everyone looked at each of the calculator screens and saw the same disputed graphs, Emily exclaimed, "Wait, this can't be right. This is math. There is only one right answer. And the lab is called Locate the Hole, so it must be a hole. Let's look at the Y-equals screen." They all switched to their entry screens. Lisa and Emily had the same function entered. Leonard has not used enough parentheses. "I think I see the problem," Emily says slowly. "It looks like you are missing a set of parentheses here [she taps the screen]."

"Do we need those?" Leonard asked.

"Yeah, on the calculator you do, but not when writing it down. That's what my teacher in pre-calc said. The TI-83 is stupid about algebra unless you have all the parentheses in there," Emily replied.

Leonard retyped the equation into his calculator while the other two waited. "Got it!" he said, "but I don't see a hole." Emily read him her window size and Leonard finally had a graph that matched everyone else's. I moved in to take their pre-labs away so I could finally go copy the set. Leonard held his back, flipped the page over, and quickly sketched a rough drawing of the correct graph on the back (see Figure 21). Once I had that paper, I left the room to copy the pre-labs.

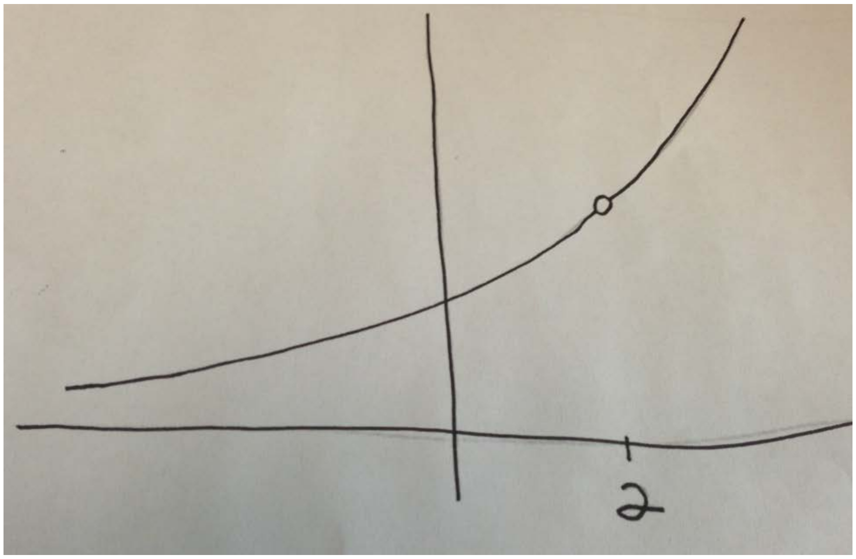


Figure 21. Leonard's second graph.

> I thought I had been careful when I did my pre-lab, but then neither of my group members had the same graph as I did. I wanted to be sure we had the right graph. Lisa had a good idea, and once we were all looking at the right thing, the rest of the Lab [3] seemed easier. Like, we could at least get something started on our own. (Emily, first interview)

This example served to illustrate that if students had retained the pre-labs for the first portion of the class, the pre-labs could have served as an artifact to begin discussions. The final purpose of formative assessment in Black and Wiliam's (2009) framework was increasing student ownership of course material. The final section details the data analysis for four of the constructs the framework defined as parts of ownership.

The Fifth Purpose: Increasing Student Ownership

In Black and Wiliam's (2009) framework, student ownership is an umbrella term that encompasses all known non-cognitive learning factors. Since the number of constructs included under this definition of ownership is too large to consider in a dissertation project, I investigated a selection of the non-cognitive learning factors: calibration, motivation, interest, and attribution. These non-cognitive learning factors were not the primary focus of the study; in the sections that follow, I present patterns in the data that warrant further exploration in another project.

Calibration

Calibration is considered to be a general metacognitive skill; it is the ability of a learner to accurately assess what they do and do not know (Hacker, Dunlosky, & Graesser, 1998). In this study, the opportunity for calibration occurred on the post-labs associates with the limits, derivative lab draft, and the definite integral lab. On those assignments, students were explicitly asked to identify which parts of the current lab they did not understand. I only considered the case study students in this analysis since I had far richer data on their perceived understandings than I did for the other participants.

In the limits, derivatives draft, and definite integral post-labs, there were sets of questions no student had asked about on their post-labs; these were the items I checked to see if students were correct in their assessment that they did not require help to complete the solution. Since none of the students asked for help on the post-lab for these items, I considered an item to be well-calibrated if the student produced the correct solution. In terms of the coding scheme used in the analysis, any item that was correct spontaneous reasoning or higher was considered to be well-calibrated--students arrived at their solution without help. I did consider “plan is right, work is not” to be well calibrated because the instances where this code was used were for digit transpositions and common calculator errors.

The qualitative case study data suggested that the three participation levels followed a similar calibration trajectory throughout the semester (see Figure 22). It is likely that familiarity with the approximation framework labs accounted for much of the improvement with calibration. However, this was one of the only indications in the data that provided any insight for why the regular participants, who were not significantly better than the other students on the grade-predictive measures at the beginning of the semester, had much higher grades by the end of the semester; the regular participants maintained high calibration levels throughout the semester. Whether this was because completing formative assessments on a regular basis helped the regular participants maintain a high calibration level or if the formative assessments helped students improve their calibration throughout the semester are areas for future research.

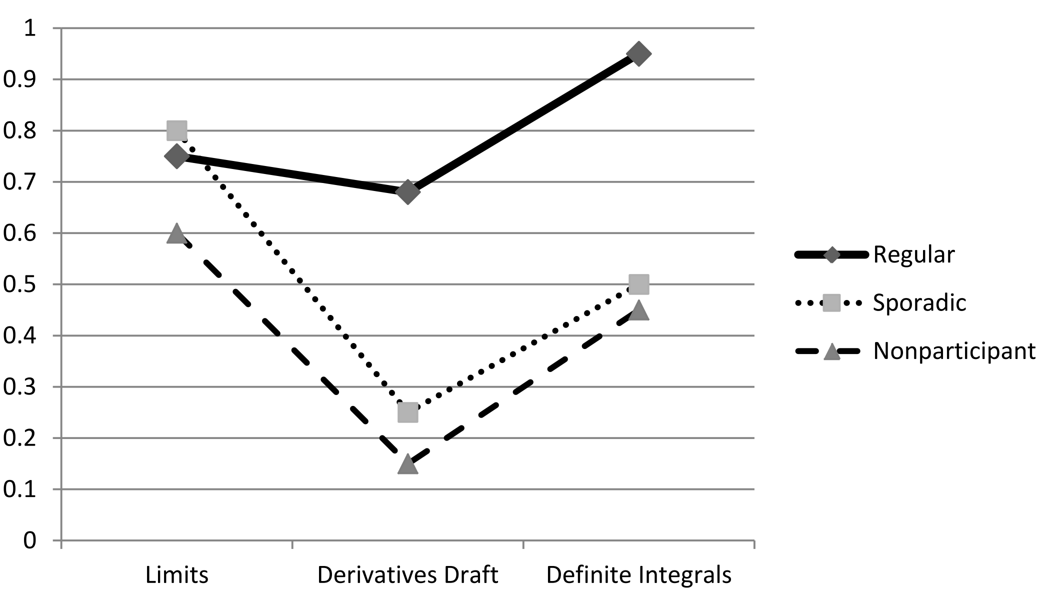


Figure 22. Percentage of well calibrated items on case study labs.

Motivation

Since motivation was not readily observable during the observations or on the documents, these data were drawn from the student interviews. Although only one participant talked about the pre-labs and post-labs being motivating for them, there was interview evidence that these assignments supported students' self-efficacy and a perception of an emotional connection with their instructor. Increases in self-efficacy and the perception by a student that an instructor cares about them could lead to increased motivation and perseverance on a challenging task (Klem & Cornell, 2004; Sakiz, Pape, & Hoy, 2012).

Charles and George were the only participants who thought the pre-labs helped their motivation; they used the difficulty of the pre-lab to decide how much effort was required to earn an acceptable grade on the lab:

> Well, I don't think the post-labs made me want to try harder-- I never did any. The pre-lab [pause] the pre-lab told me how much I had to try. If it was easy, I didn't have to work hard on Tuesday, but if it [the pre-lab] took me a long time, I started hoping [my instructor] would put me in a good group because it was going to be a lot of work. (Charles, second interview)

Leonard only completed one pre-lab before class--the one described in the previous section. "I don't think I was more motivated, but I knew I had to be a lot more careful with my calculator," Leonard explained in his first interview.

The regular participants other than George thought the pre-labs helped them feel more confident about the labs:

> I don't think the pre-labs made me try harder…if they helped me at all it was that it let me know I could do something. I'd read the questions on the lab, and other than this last one [integral lab] I had no idea where to start. At least when I looked at the pre-lab, I could always do most of it. It kinda made me think the rest might not be as hard as it looked. It made it easier the next day, knowing I couldn't get a zero on the lab. (Eva, second interview)

Although there appeared to be gender-differentiated perceptions of the pre-lab, such an analysis would be beyond the scope of the data.

Every sporadic and regular participant, when asked if the post-labs helped motivate them, they said the labs did not. However, when I asked why they thought their instructor assigned the post-labs, they all had the same response:

> I dunno about more motivated, but the post-labs make me try harder. I know [my instructor] really cares about us getting this because he reads those things every night we do them. That makes me want to try harder. Even when I don't do them, I don't want to be a disappointment. (George, first interview)

Charles, Leonard, Sandra, Emily, Kaitlin, Sandra, Eva, and George all gave similar answers--the post-lab was evidence that their instructor cared about them in all of their interviews. Although the written feedback on the derivative lab played a role, these students noticed that their questions on the post-lab were answered by their instructor in class or through Blackboard; that responsiveness was interpreted as caring.

Interest

There was no indication in the data that the formative assessments helped raise student interest. The regular participants indicated that the pre-lab might increase self-efficacy for the lab but none mentioned interest. "When I read the lab on Monday, I always get nervous. It looks so hard and I don't know what to do. Then I do the pre-lab. After I usually do that, then the rest of the lab doesn't look so scary" (Emily, first interview). The only participant who indicated that her interest in calculus increased throughout the semester was Kaitlin. However, she maintained that the labs themselves, not the formative assessments, were responsible for the change.

Attribution

According to Dweck (2006), attribution is the implicit beliefs students have about intelligence. There is a continuum of attributions--the two extreme cases being entity and incremental attribution (see Figure 23). In either case, attribution is a pattern of thoughts and behaviors that is not entirely conscious; these patterns are easiest to observe when students struggle or fail with new material. Students with entity attribution believe that intelligence is a fixed quantity. These students are focused on performance goals, e.g., grades. On material students with the entity attribution find easy, these students generally outperform students with the incremental theory of intelligence; however, students with the entity attribution tend not to persist on difficult material. Since students in this category believe that intelligence is fixed, having difficulty with material means you cannot learn the content. Students who have the incremental theory believe understanding the material is the main reason for learning. These students will show high persistence on material, regardless of the difficulty level, because effort is how learning occurs.

Theory of intelligence	Goal orientation	Confidence in present ability	Behaviour pattern
Entity Theory (Intelligence is fixed) →	Performance Goal	If high →	Seeks challenge High persistence
		If low →	Avoids challenge Low persistence
Incremental Theory (Intelligence is malleable) →	Learning Goal	If high →	Seeks challenge High persistence
		If low →	

Figure 23. Dweck's (2006) attribution model.

Most students exhibit a mixture of behaviors in the two cases. Although there are instruments for classifying students' attribution, these instruments were administered as part of this project. Attribution only emerged as a potential theme in the data after students were grouped by participation level, a grouping that only became apparent after data collection was completed. Despite the lack of quantitative data, there were patterns to student responses in the case studies and some behavior patterns during class that suggested there might be differences in attribution between participation levels.

The behaviors of the students in the nonparticipant group most closely resembled the entity attribution behaviors. In the beginning of the chapter, the nonparticipant group had the first or second highest mean score on all of the grade predictive variables. Nonparticipants led approximately 33% of the groups during the limits lab. However, after the derivatives lab, the lab scores of the students not participating in the formative assessments plummeted and never recovered. As discussed in earlier sections, after the initial derivatives lab, this group performed significantly worse than the students who were participating in the formative assessments. Neither of the case study students completed a definite integral lab. Tre did not turn in the lab because he left the course and never returned after he became frustrated on the first day of the derivatives lab and Lisa because she gave up on the class: “I was doing OK until Lab 4 [derivatives]. I didn’t get it at all. Even after rewriting it, my grade didn’t get that much better. I’m gonna fail the class anyway, so why do it? I’ve got finals for classes I will pass I ought to study for.” Lisa illustrated showed both low persistence and an orientation toward grades--both characteristics of entity attribution.

The students in the regular participation group most closely resembled the behaviors one would expect from students with an incremental orientation. The regular participants sought help throughout the semester. When they asked questions on their post-labs, the sentence started with “I don’t understand” or “I don’t get” 53% of the time. With the exception of one sporadic participant, the regular participants accounted for all of the office hour visits related to writing up the labs. The regular participants generally completed their lab write-ups; there were only nine recorded zero grades (9.7% of the assignments) for regular participants on lab write-ups. The most interesting difference between the regular participants was the difference between the wrong answers regular participants gave when compared to the sporadic and nonparticipant groups (see Figure 24). The regular participants were more likely to make computational mistakes than they were to leave a problem blank.

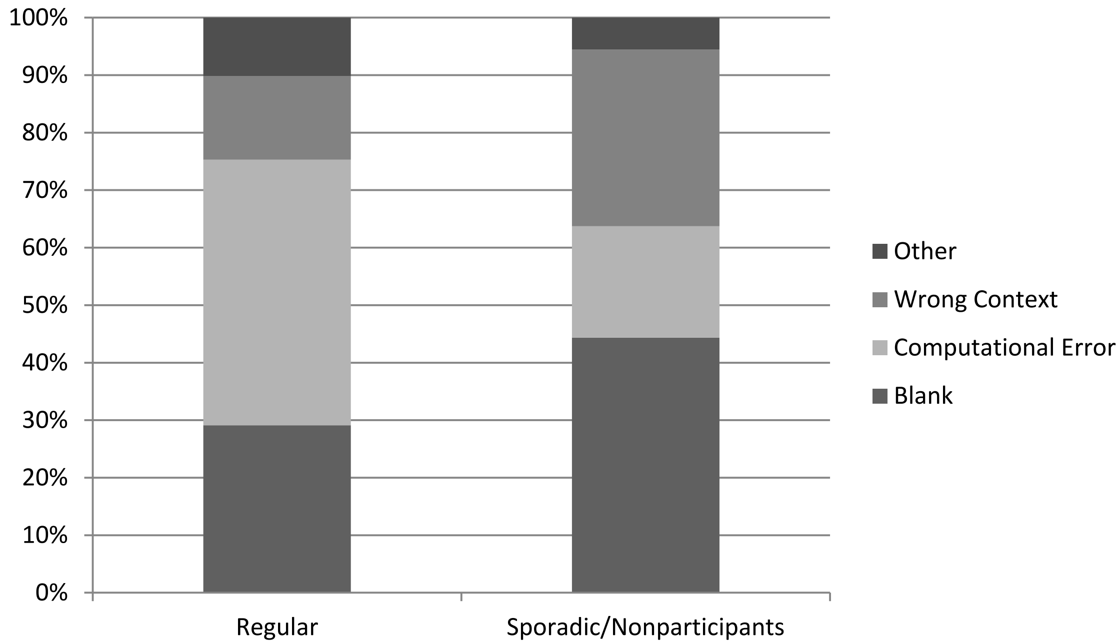


Figure 24. Types of wrong answers in all lab write-ups by participation groups.

It is difficult to say if there were any patterns as to which attribution the sporadic participants held. Charles always framed everything related to the labs in terms of the amount of effort required to ensure his desired grade of a C in the course. Leonard did not complete a final lab, for the same reason as Lisa, but he could do almost all of the questions on the final lab during his interview. Both Leonard and Charles showed some of the behaviors expected from students with entity attribution. However, four of the students classified as sporadic participants earned A's and B's. Both of the students who earned A's and one student who earned a B never asked any questions in class, emailed their instructor questions about the homework, or visited office hours. The content of the course was simply not difficult enough to speculate on the attribution of these students without quantitative measurements. The other student who was a sporadic participant who earned a B in the course led his lab group all semester, was entirely focused on understanding all of the material, emailed his instructor with homework questions, and visited office hours on a weekly basis.

Overall, there was a pattern of observations that suggested the regular participants tended to have more characteristics of incremental attribution and the nonparticipants had more characteristics of fixed attribution. Leonard and Charles seemed to have more characteristics of entity attribution. However, there were high achieving sporadic participants who found the material too easy to evaluate their attribution and one student exhibited almost all of the behaviors of a student with incremental attribution. Although the data in this project were too sparse to draw any definitive conclusions, group differences in attribution could explain the different outcomes for the three different participation levels.

A New Purpose: Participation?

Since the key grouping variable for this analysis was participation, the last thing I investigated in the data analysis was what students characterized as the benefits to participating in the formative assessments. Generally, only regular participants saw benefits to completing the formative assessments; these assignments helped students participate more fully in the class without having to admit to needing help in front of their classmates. Although this was not a major focus of the investigation, the interview and observational data suggested that the regular participants became more active participants in the course throughout the semester and eventually assumed most of the leadership roles in the lab groups.

One of the last questions I asked students in the interviews was what they thought of the purpose of completing the pre-lab and post-lab assignments. There were responses from regular participants and sporadic/nonparticipants. Charles, like the other students who did not participate regularly in the assignments, saw little motivation for the pre- and postl-abs: “Honestly, I don’t know. They are graded on completion and aren’t worth a lotta points, so they are really just busywork. I either know everything or nothing so either way it isn’t worth doing” (first interview).

The regular participants thought that the purpose of the pre-labs and post-labs was to have a chance to check their understanding and to ask questions without having to signal to their peers that they were having trouble:

> I think it’s because [my instructor] cares about us. I’m older than these kids. The pre-lab gives me a chance to review before I have to work with them, and the post-lab lets me ask questions about what I didn’t get without saying something in class. Of course, once I figured out [my instructor] cared about me, it was easier to see [my instructor] in their office, talk in class, and ask questions during the lab. (Sandra, second interview)

All of the female regular participants agreed with Sandra; they were reluctant to seek help or present their ideas at the beginning of the class. Completing the formative assessments, with opportunities to interact with their instructor and participate peripherally, helped these students have the confidence to begin to participate actively in class. George, the lone male participant, had a slightly different take on the pre-labs/post-labs and participation:

> Well, I never really had a problem talking in class. If I don’t know something, I’ll ask about it. I’ve always hated group work. I never work with my group, I work near my group. After doing the bottle lab [the first lab of the semester] and starting on Lab 3 [the limits lab], I kind figured out from the activities and the post-labs that I couldn’t get this on my own, and that was kind of the point. After that, I started really working with my group during labs. (George, first interview)

After analyzing the interview data, I turned to the field notes I created during the labs and post-lab-based instruction to see if there were any participation trends in the whole class data. In particular, I coded which student was leading each lab group and their participation level. As the semester went on, the regular participants assumed leadership of more and more groups; this was because the nonparticipants leading lab groups during the limits lab did not assume leadership roles in future labs. Sporadic participants continued to lead about the same number of groups each lab. The groups led by sporadic participants were those that consisted of sporadic and nonparticipants (see Figure 25).

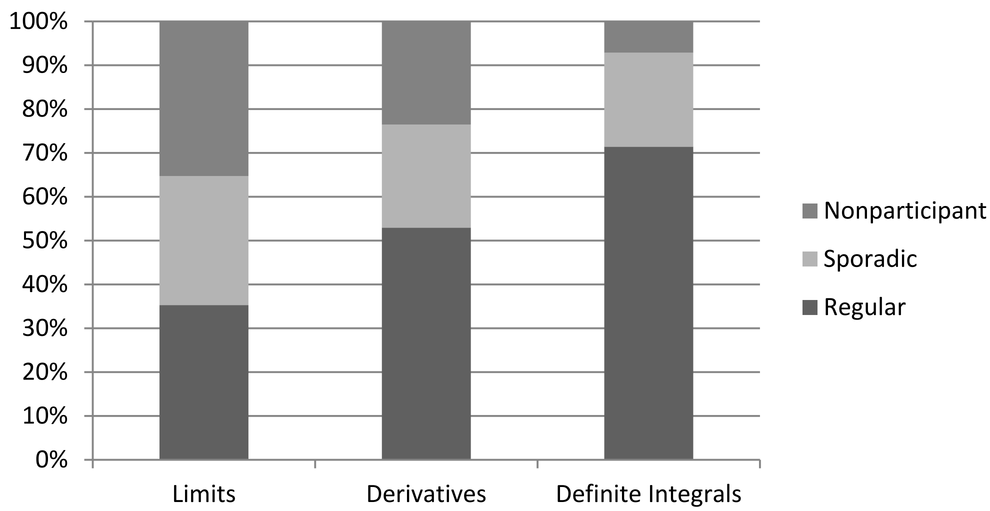


Figure 25. Group leaders by participation level.

The final type of participation I coded was during the post-lab-based instruction. Here I looked for extreme behaviors--students asking questions during the post-lab-based instruction and students who were disengaged from the instruction. Students were classified as disengaged if they were not writing notes, were not looking at their instructor, or were in engaged in non-class behaviors like texting. The students in the sporadic participation group rarely exhibited either of these behaviors but all of the other students did (see Figure 26).

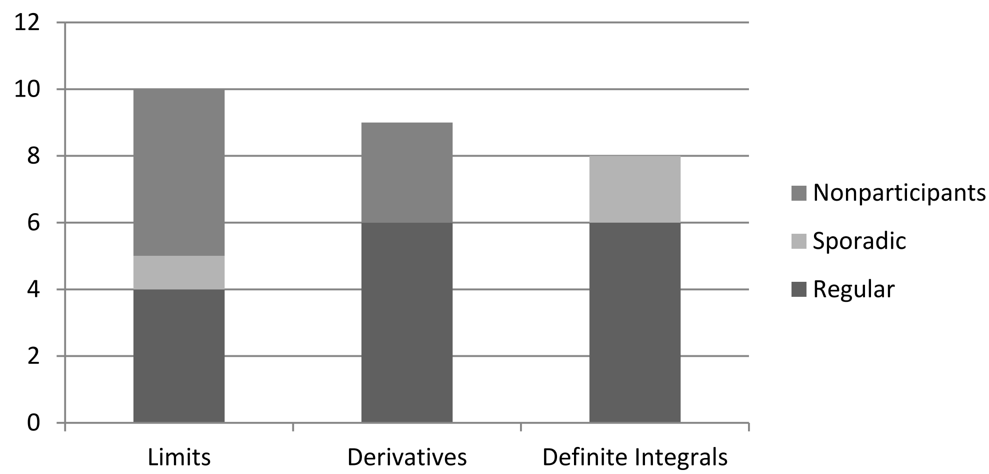


Figure 26. Questions asked in discussion by participation group.

There was a surprisingly high amount of disengaged behavior from regular and nonparticipants during the post-lab-based instruction. When asking students about their behavior after the lab, the two groups of students gave the expected responses. The regular participants were not paying attention to the parts of the discussion on parts of the labs they had already completed.

> During Lab 5 [Linear Approximation] I asked students why they did not listen (well, texted) during the post-lab-based discussion. Three of the students that do not listen, all of whom have A's and B's in the class, informed me that the reason they were not listening was that they already solved the questions being talked about in class. Checking their post-labs, they said they had no questions. None of these students are interview participants, which is unfortunate. The interview participants all listen during class. (Fieldnotes, 2/24/12)

The nonparticipants (incorrectly) believed they had a solution and were not listening for the same reason. Toward the end of the semester, nonparticipants were also not engaged because they had given up on the class.

> The other students that were not listening during the instruction I spoke to give some of the same reasons. They said that the questions being talked about in class were ones they already completed. The four students I talked to, none of whom are interview participants, have a B, C, C, and an F in the class at this time. I have to remember to check when their write-ups come in, but this seems less likely to be true. Two students also told me that class is very early in the morning and that it is hard to pay attention at that time. One student, who has gone from an A- to a C+ in the past three weeks told me that she stopped listening because nothing was making any sense. (Fieldnotes, 2/24/12)

Throughout the semester, there was an increase in disengaged behaviors during instruction. Most of this increase could be attributed to the nonparticipants (see Figure 27).

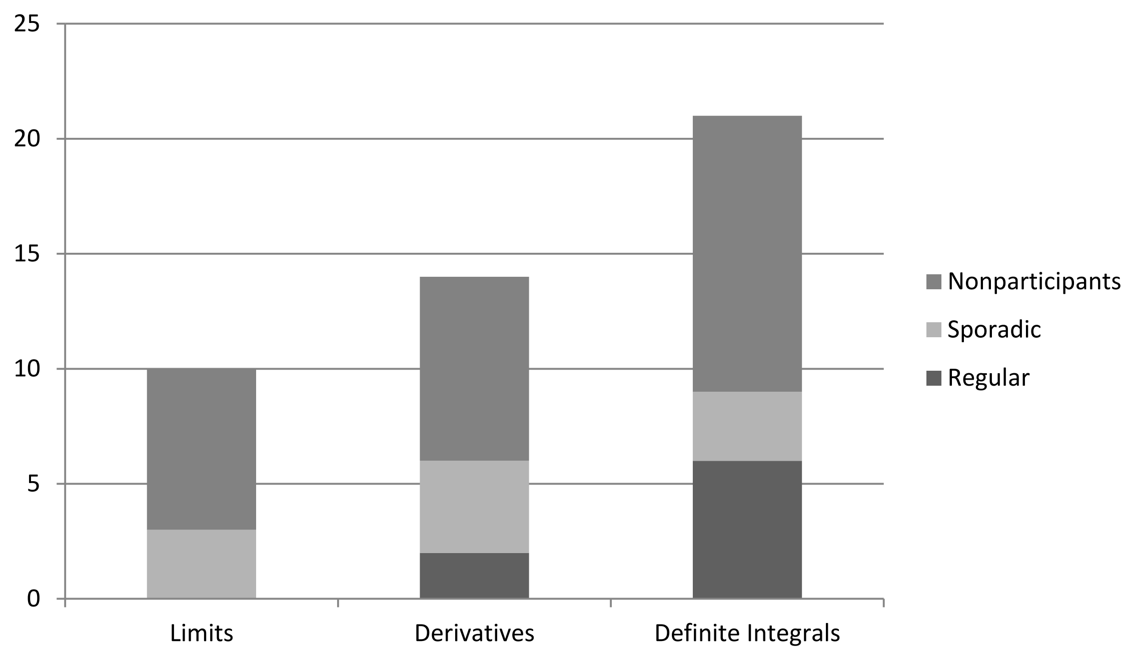


Figure 27. Number of disengaged students during post-lab-based instruction.

Although there was no measureable difference between the students at the different participation levels at the beginning of the semester, there were differences by the end of the semester in terms of achievement, classroom behavior, and conception of the approximation framework. In the final chapter, I present my argument for which functions of formative assessment were most influential on students regularly participating in the assignments, discuss the implications, and sketch several opportunities for future research based upon these findings.

CHAPTER V

DISCUSSION

Overview

After I analyzed the data and constructed the narrative in the preceding chapter, I was in a position to answer the research question that prompted this dissertation project and discuss how the answers to the research question related to the professional discourse described in Chapter II.

This final chapter consists of five major sections. In the first section, there is a brief summary and synopsis of the chapter where I have distilled the main points of the narrative for quick reference for the second section. In the second and longest section, I construct an argument for how I answered the research question. In the third major section of this chapter, I discuss the implications that could be derived from the answers to the research question in three areas: practice, theory, and methodology. The fourth section details the limitations of this dissertation project. The fifth and final section outlines several areas for further inquiry that could be built upon this dissertation study.

Summary of Findings

Students in the introductory calculus classes considered in this project participated in formative pre-labs and post-labs at one of three levels: regularly (did all formative assessments), sporadically (did no more than five formative assessments), or never. With the exception of gender (more female students were regular participants), the demographic and grade prediction variables were not significantly different between the groups. All of the groups of students had approximately identical performances on the first approximation lab. On the derivatives lab, the regular participants had significantly higher mean achievement scores than the sporadic or nonparticipants in the formative assessment. The three groups had significantly different performances on the definite integral lab and the common final exam. The regular participants outperformed the sporadic participants who in turn outperformed the nonparticipants in formative assessments. I investigated these differences by looking at students' changes in their Zone of Proximal Development throughout the semester using all three characterizations of ZPD. There was no observed evidence that the pre-labs and post-labs helped students identify the important learning objectives in the labs.

Overall, when looking at the effectiveness of the post-lab-based instruction, there seemed to be diminishing returns throughout the semester. On the items discussed in class on the limits lab, the regular participants had a higher mean score than the other two participation levels; the sporadic and nonparticipants did not have significantly different means from each other. The regular participants who asked questions on their post-labs did benefit from the post-lab-based instruction; the other regular participants already had a correct solution before the instruction occurred. Leonard and Lisa had some solutions that could be attributed to the post-lab-based instruction but Charles and Tre did not.

On the definite integrals lab, the three groups had significantly different mean scores on the items discussed in class--the regular participants had the highest mean score and the nonparticipants had the lowest score. The post-lab-based instruction helped the regular participants and Leonard complete some of the more difficult items that involved function notation; however, Lisa, Charles, and Tre (who was no longer attending class) did not show any evidence that they incorporated the post-lab-based instruction into their lab write-ups.

The students who took advantage of the written formative feedback on the derivatives lab showed considerable improvement on their revised write-up. After accounting for the initial write-up scores, the regular and sporadic participants' mean scores were not significantly different from each other; both groups had significantly higher means than the nonparticipants. Although the regular and sporadic participants' mean scores were not significantly different from each other, the relatively low p value ($p = 0.0501$) approached significance.

For the case study students, the feedback was helpful to the regular participants when they used it. However, not all of the regular participants rewrote their derivatives lab; those who chose to rewrite did not redo every question on which they received feedback. The sporadic participants and nonparticipants' case studies were less clear. Leonard did not turn in an initial lab and received no feedback. However, five of the six (83%) questions Charles rewrote correctly could be attributed to the formative feedback on his initial derivatives lab. Lisa was able to incorporate the formative feedback into three of the five (60%) questions she answered correctly on her rewrite but Tre quit coming to class before students were offered a chance to rewrite their labs. Due to the logistics of data collection, there were few instances where student collaboration could be attributed to the pre-labs.

Four aspects of ownership were investigated: calibration, motivation, interest, and attribution. Some evidence indicated that the regular participants had good calibration but there were too little data to make any suppositions about students at the other participation levels.

The sporadic and nonparticipants reported no motivational benefit inherent in completing the pre-labs and post-labs. The regular participants found that completing the pre-labs and post-labs indirectly increased their motivation. The students in the case study believed that the assignment of the pre-labs and post-labs was evidence that their instructor cared about them as students and wanted them to succeed. The regular participants reported that this perception of caring made them work harder so they did not disappoint their instructor. There was no evidence observed that the pre-labs or post-labs increased students' interest in the labs.

Although the data were not definitive, there was evidence of potential differences in attribution across the three participation groups. The regular participants showed many of the behavioral characteristics associated with incremental attribution--they sought help in office hours, missed few assignments, and sometimes took over leadership of their groups when their group members were too frustrated to continue. Also, when regular participants got items wrong on their lab write-up, these items were rarely blank. The sporadic participants were difficult to classify. Both of the case study students appeared to have entity attribution, both put in minimal effort once the course became difficult for them, and the items in their labs were either completely correct or blank. However, there was a group of very high achieving sporadic participants, none of whom consented to be interviewed; thus, it is not possible to classify these students' attributions based on observable characteristics. The nonparticipants in the case study showed many of the behaviors associated with entity attribution. As a group, this participation level appeared to be the equally prepared for calculus on the grade predictive variables. On the limits lab where the context was relatively familiar, these students performed reasonably well. However, after receiving low scores on the initial derivatives lab, most of the students not participating in the pre-labs and post-labs put very little effort into the lab assignments for the rest of the semester--a common self-protection behavior for learners with entity attribution.

There was some evidence that the pre- and post-labs helped students move from peripheral to central participation throughout the semester. The regular participants asked more questions, sought their instructor in office hours, and were leaders of their group during the lab with increasing frequency throughout the semester. In their interviews, these changes were attributed indirectly to the pre- and post-labs. The case study students reported that since these assignments were evidence of caring by the instructor, this helped the students realize their instructor was a safe person from whom to seek assistance. The nonparticipants in the pre- and post-labs showed a reverse participation pattern; they moved from central to peripheral participation throughout the semester.

With this summary of what happened this semester in the introductory calculus courses, I can argue how these data answered the research question that guided this dissertation project.

Answering the Research Question

Q1 What are the functions of formative assessment that scaffold students' peripheral participation and Zone of Proximal Development of approximation framework concepts in an introductory calculus course?

In the following sections, I discuss in turn the five components of Black and Wiliam's (2009) framework for the functions of formative assessments: clarifying learning intentions, engineering appropriate learning activities, providing feedback that moves students forward, activating students as learning resources for each other, and increasing students' ownership of their learning. Of those functions, I argue that clarifying learning intentions was not observed to be relevant for the students in this study, activating students as resources for each other was possibly applicable, and engineering classroom discussions, formative feedback, and increasing student ownership were applicable. Furthermore, the low-stakes formative assessment appeared to encourage more central participation for the regular participants and appeared to discourage such participation from the nonparticipants.

Clarifying Learning Intentions

None of the students interviewed in this project felt that the formative assessments helped them clarify the learning goals of the labs. All the students, even those who failed the course, stated they were aware the content of labs (the approximation framework) was an important part of the course. They based their reasoning on the weight given to the labs on the syllabus. This is not to say that there was no academic socialization during the semester--students gained proficiency in how to write up their labs. However, according to the case study participants, this socialization came from written comments on the lab write-ups. This written feedback could be considered formative feedback but the commented versions of the lab write-ups were not a part of this project.

Although this function of formative assessment was not identified as relevant in this study, more research is needed to determine if this phenomenon was a function of the student population at the research site. While this portion of Black and Wiliam's (2009) framework was not relevant to this study, the next function of formative assessment was one of the main purposes of the formative assessments in this study.

Engineering Appropriate Learning Activities

The learning activities engineered from the formative assessments in this study were the discussions at the beginning of the class following the lab. The pre-labs were not intended as a formal instructor planning tool; the purpose of the pre-lab was to activate students as learning resources for each other and to signal which groups needed help at the beginning of a lab. Although the regular participants showed the most consistent benefits from the post-lab-based instruction, the sporadic participants had the largest achievement gains throughout the semester on the questions discussed in class.

On the limits lab write-up, regular participants' mean score on the items discussed in class was significantly higher than the mean score of the sporadic and nonparticipant groups; the sporadic and nonparticipant group means were not significantly different from each other. Since the ANOVA for the total scores on the lab showed a significant difference between the regular and nonparticipant groups but no significant group differences on the mean score on the items that were not discussed in class, this suggested that the students participating at all in the formative assessment were benefiting from the post-lab-based instruction.

Students who worked on their lab write-ups outside of class, who could not complete the limits lab independently, and did not seek other sources of assistance benefited from the post-lab-based instruction. For four of the nine case study participants (44%), the post-lab-based instruction helped them construct solutions for at least 4 of the

10 items discussed in class. The reason why the instruction was not helpful depended on a student's participation level. Two of the regular participants in the case study attributed the source of their solution to the post-lab-based instruction. For the three regular participants with one (George) or no items (Sandra and Eva) attributable to the post-lab-based instruction, it was because they had completed the lab independently or sought additional help before the post-lab-based instruction. George had completed his lab outside of class the night of the lab, Sandra sought help in office hours after class on the lab day, and Eva got most of her help from a UGTA during the lab itself. For the sporadic participants, Leonard attributed half of the solutions of items discussed in class to that solution but Charles did not turn in a lab. All four of Lisa's representations of error were attributable to the post-lab based instruction but every item Tre did not complete during the lab day was left blank.

By the definite integral lab at the end of the semester, there are significant differences in the mean score of the participation groups on the items discussed in class: the regular participants' mean score was significantly higher than the sporadic participants' mean score, and the sporadic participants' mean score was in turn significantly higher than the nonparticipants' mean score. However, since the same pattern of significant differences was found on the items that were not discussed in class, the quantitative support for the effectiveness of class discussion at the end of the semester was much weaker than it was for the limits lab at the beginning of the semester.

Looking at the case studies, it is clear why the quantitative results did not show strong support for the effectiveness of the post-lab-based instruction. The regular participants were learning how to complete one or two of the most difficult questions on the lab through the instruction but since these students had already appropriated most of the approximation framework, there were few topic discussed in post-lab-based instruction. None of the other case study participants turned in a lab write-up; only Leonard was able to answer any of the definite integral lab questions during his final interview. Leonard, who had weak algebra skills, learned how to complete four of the five algebraic representations through the post-lab instruction.

Overall, the post-lab-based instruction was most helpful at the beginning of the semester when the students at each of the participation levels were closest in their ability to complete the approximation framework labs. By the end of the semester, the regular participants, whose post-labs set the agenda for the instruction the following day, only needed help with the most difficult algebra representations on the lab. For the 14 sporadic and nonparticipants in the formative assessment who completed the definite integral lab, the three questions most likely to be incorrect or blank were the three questions the regular participants were learning through the post-lab-based instruction; this instruction appeared to have been beyond the ZPD of those students not participating in the formative assessments.

Providing Feedback That Moves Learners Forward

Students received formative feedback once during the semester--on the initial derivatives lab. All of the students who chose to rewrite their lab benefited from the formative assessment. However, a higher portion of regular and sporadic participants rewrote the derivatives lab than the nonparticipants. In fact, the initial write-up of the derivatives lab appeared to be the point where most nonparticipants gave up on the course.

In terms of achievement of those students who turned in both an initial write-up and a revised write-up, the regular and sporadic participants benefited equally from the formative feedback. After taking the score on the initial submission of the derivatives lab into account, the sporadic participants' mean score was not significantly different from the regular participants' mean score. However; the sporadic participants' mean score was significantly higher than the nonparticipants' mean score.

In fact, on all three of the labs assigned after the derivatives lab, linear approximation, Newton's Method, and Definite Integrals, the sporadic participants had a significantly higher mean lab score than the mean score of the nonparticipant group ($p = 0.018, p = 0.002, p < 0.001,$ respectively). The sporadic participant mean score was always significantly lower than the regular participant mean score after this assignment (all p values less than 0.0001) but this was not surprising. During the post-lab-based instruction discussed in the previous section, the regular participants were always getting help on the specific questions they needed; this set of items might or might not have been the entire set of items on which the sporadic participant group needed additional instruction. When both participation groups were given the same level of customized written feedback, the sporadic and regular participant groups performed at the same level--like we would expect from groups of students with no significant differences on the grade predictive measures at the beginning of the semester.

The regular and sporadic participants in the case studies had similar reactions to formative feedback when they chose to rewrite their lab. Emily, Kaitlin, and Sandra showed increased scientific reasoning on all of the items they received formative feedback on in their initial drafts. George also showed similar improvement but he only rewrote about half of the items he got wrong in the initial draft. Charles, a sporadic participant, also showed conceptual improvement on all of his rewritten items; all but one of those improvements was attributed to the formative feedback in his interview. Eva, a regular participant, earned a B on her initial lab; since this was the highest grade she received in the course to date, she chose not to take advantage of the formative feedback. Leonard did not turn in an initial write-up and hence received no formative feedback; he showed conceptual improvement on only two items when compared to his limits lab.

The vast majority of the students in the nonparticipant group, who had never completed a pre-lab or post-lab, also did not receive any formative feedback. Only 6 of the 16 (38%) nonparticipants in formative assessment turned in both an initial and revised derivatives lab; 18 of the 23 (78%) regular participants and 11 of the 15 (73%) sporadic participants turned in both derivatives lab write-ups. Of the 10 students who failed to turn in at least one version of the derivative lab write up, eight of them were in the nonparticipant group. Two of the six (33%) nonparticipants who did turn in an initial and a rewritten derivative lab did not improve their scores on the rewrite.

Thus, of the two nonparticipant students who participated in the qualitative portion of this study, Tre, who could not separate the approximation framework from the initial function context and quit in frustration never to return to class after the derivatives lab, was more representative of a typical nonparticipant in formative assessment than Lisa. When considering the effectiveness of formative feedback, Lisa was an outlier in her participation level. Lisa completed both an initial and a revised derivative lab write-up; her score on the revised lab was the highest of any version of the derivative lab write-ups of any student at the nonparticipant level. In fact, Lisa's improvement on the revised write-up was most similar to George's; of all of the items Lisa rewrote, the improvement was attributable to the formative feedback she received. The difference between Lisa and George was that George rewrote half of his incorrect items and Lisa rewrote a third.

The conclusion about formative feedback was not surprising; students who take advantage of the formative feedback improve by similar amounts. There were two challenges with implementing individual formative feedback in undergraduate mathematics classes on a large scale. The first hurdle with formative feedback appeared to be getting all students to respond and receive the formative feedback in the first place. The second was that this type is most time intensive for teachers to implement; however, there is software developed by physics education researchers to expedite individual feedback for mathematics classrooms.

Activating Students as Resources for Each Other

Although the students were learning resources for each other, it was much less clear what role the pre-labs or post-labs played in encouraging students into those roles. The procedure I used to collect pre-labs had two unintended consequences. By removing the pre-labs from the students at the beginning of the activity, it was less likely that they discussed the pre-labs directly. Also, if students used the pre-labs as a place to start collaborating, it happened when I was not present to observe these interactions. Despite this limitation of the data collection, I still observed three instances of groups with different solutions on the pre-lab use their pre-labs to begin collaborating on what reconciled the differences. These instances of students immediately entering a collaborative ZPD to reconcile differences in their solutions suggests further study on formative assessment and this characterization of the ZPD is warranted.

The scaffolding given to the whole class based upon the responses of the post-labs was also an instance of students being activated as a learning resource for their peers; in this case, the regular participants were acting as learning resources to each other and to their classmates. As discussed two sections before, all students, even the nonparticipants, received some benefit from the regular participants' post-lab formative assessments. The final purpose of formative assessment, activating students as owners of their own learning, was primarily for the regular participants.

Activating Students of Owners of Their Own Learning

Black and Wiliam (2009) considered ownership to be a combination of four cognitive and affective constructs: (a) metacognitive self-assessment, (b) motivation, (c) interest, and (d) attribution. By activating students as owners of their own learning, we would expect to see increases on the first three characteristics and more incremental attribution behaviors. Since the Zone of Proximal Development is cold cognition[1] theory and there were no surveys to measure motivation, interest, or attribution during the semester, making claims about the influence of formative assessment on any of these ownership characteristics was beyond the limits of the theoretical perspective and the data. The purpose of the analysis for this section was to generate hypotheses for further studies in hot cognitive areas and formative assessment. Although there was no evidence in the data that the formative assessment increased students' interest in the labs, I argue that there were patterns in the data that indicated the metacognitive self-assessment and

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[1] Hot cognitive theories argue that there is an emotional component to learning and that students are not always rational actors (Lazarus, 1982).

attribution are likely to apply to undergraduates. Further targeted research is needed to investigate formative assessment on motivation and interest.

Other than some of the behaviors students displayed in the course after the derivatives lab that could indicate differences in attribution, calibration was one of the few constructs in this study where participation groups showed marked differences. The regular participants appeared to have good calibration. The students in the sporadic and nonparticipation levels had similar levels of calibration on the labs: both groups had good calibration on limits, very poor calibration on derivatives, and moderate calibration on the definitive integral lab. With the exception of the limits lab, the difference between the students in the sporadic and nonparticipant group was that the students in the sporadic participation group were answering approximately one more question not discussed correctly than the nonparticipant group. On the limits lab, the sporadic participants actually had the highest calibration of all three groups. This was likely another example of how the familiarity with the context of the limits lab tended to make students appear to understand the material better than they actually did.

There was very little direct evidence that the formative assessments in this study had any direct effect on motivation. The nonparticipant group appeared to lose motivation throughout the semester but there is no evidence in the data to suggest the formative assessments were related to it. When I asked about motivation in the interviews, all of the regular and sporadic participants said slight variations of the same statements, that formative assessments were evidence of instructor caring, and this caring caused them to try harder. The idea that formative assessments were evidence of the instructor caring about students' success, which in turn led to increased effort, was not a phenomenon I found in the literature and might be worth further exploration.

Attribution is a fundamental set of beliefs about learning (Dweck, 2006). The two types of attribution are entity and incremental. Students with entity attribution tend to outperform students with incremental attribution on tasks with familiar contexts. However, on novel tasks or tasks where knowledge must be applied in an unfamiliar context, students with incremental attribution will persevere through frustration where students with fixed attribution will quit. While specific instruments that measure attribution were not a part of this study, the behaviors of the regular participant group were consistent with incremental attribution and the behaviors of the nonparticipant group were consistent with entity attribution. To confirm the attributional difference and measure changes in attribution throughout the approximation framework of Calculus I and II courses would be a topic for further exploration. Further, the sporadic participant group did not conform to either attribution pattern; further study of students with the sporadic participation pattern using the attributional instruments is also warranted.

Participation

All of the consequences of formative documented in the previous sections were a consequence of giving students a low entry point to relay information to the instructor. By participating in the pre-lab, students received targeted class activities and specific written feedback that helped students both in their performance and their appropriation of the approximation framework. The pre-lab might have served as a jumping-off point during the labs but its primary purpose seemed to be getting groups on task quickly.

Further, evidence of calibration and attribution behaviors associated with levels of participation in formative assessment requires further investigation.

The lowest level of peripheral participation in Krummheuer's (2010) framework was overhearing. When students completed a formative assessment, they interacted with their instructor directly without the possibility another student would overhear, which reduced the social risk of asking questions. Thus, the student relayed information to the instructor, the highest level of peripheral participation, without the normal social risks of relaying information in the classroom. The instructor's response to students provided the needed information to the student.

The formative assessment combined with formative feedback seemed to encourage central participation from the regular participation group in two ways. First, by the end of the semester, students in the regular participation group had moved beyond asking questions on the formative assessments to directly and proactively interacting with their instructor for the assistance they needed. Further, the regular participants changed their role within their lab groups. During the limits lab, all of the regular participants featured in the case studies were working near their group, eavesdropping from time to time, rather than working with their group. By the derivatives lab, all of the regular participants were the acknowledged leader of their lab group. The only instances of students in the regular participant group not leading lab groups were the cases where there were multiple regular participants in the group.

For most of the students in the sporadic participation group, written feedback on the derivatives lab was the only targeted formative feedback they received all semester. Although students' write-ups were better after the feedback and all of the case study participants except for Tre showed conceptual development on the rewrite, there was no change in participation pattern after receiving feedback. Although the post-labs might have reduced the number of questions asked in class, seven of the nine case study participants said they preferred to ask a question on a post-lab than during class. The students in the sporadic participation group stayed at low levels of peripheral participation throughout the semester.

Besides scaffolding conceptual development and improved performance on the labs, the formative assessments also seemed to serve several other cognitive and affective functions for students who required further investigation. The pre-lab helped the regular participants gain self-efficacy before starting the lab and the post-labs provided students with opportunities for calibration practice. The regular participants found the opportunity to privately relay information to the instructor evidence of instructor caring, which made them try harder. There was also some observational evidence that regular participants tended to have incremental attribution and the students in the nonparticipation group tended to have fixed attribution.

This study found limited evidence that academic formative socialization was not a purpose of formative assessment for these undergraduates but might apply elsewhere. This study suggested that academic socialization might not always be a purpose of formative assessment at the undergraduate level. On the other hand, the small amount data on the pre-lab activating students as a resource for each other supported the hypothesis that the pre-lab helped students participate in their group more easily but additional investigation with alternative data collection is needed.

In the final two sections of this chapter, I discuss the implications of this study for theory, research methods, and classroom practice. In the final section, I have outlined several possible future research projects based on the findings of this dissertation.

Implications

While the answers to the research questions had intrinsic value, they also shed light on the practice, theory, and research methodology of formative assessment in the undergraduate classroom. In the sections that follow, I provide implications of the research question I answered above.

For Practice

The biggest criticism of using formative assessment, particularly at the undergraduate level, is the additional grading time (Yorke, 2003). In this section, I reflect on how the formative assessment was conducted in the classes this semester and make suggestions to minimize grading time for instructors interested in using formative assessment in their classes.

First, there needs to be transparency for the students on why they completed formative assessments, especially why the questions did not change much from week to week. Instructors should establish a classroom norm early in the semester that formative assessments serve two main functions: (a) these assignments are intended to provide a schema for students to check their understanding on the material covered in the formative assessment and (b) these assignments are a place to ask questions about the material on the formative assessment or any other material that did not made sense. I recommend the following actions by instructors to help establish this norm: (a) include these goals in the directions of the formative assessments; (b) explicitly state these goals before students complete the assignment; and (c) when using student questions to design instructional interventions, tell the students at the start of the intervention that the questions answered and topics covered are student generated.

The most difficult part of the norm to establish would be encouraging students to ask questions about what aspects of the material were most troublesome to them. I have two suggestions for instructors to incorporate should this become problematic in their class. The first modification to the formative assessments that could aid with questions would be to incorporate a meta-cognitive problem journal. In such a journal, students pick a problem they find problematic. The journaling about this problem has two parts. First, the student explains the mathematics they did to arrive at the solution they had. Second, the student explains their thinking for why they chose the strategies they did. These journals might help instructors to identify questions students did not articulate. The second suggestion is simply to require that students ask a question about something they do not understand on each formative assessment; while this will establish a norm for questions, there is also a danger that this will cause resentment among the students.

Journals such as the ones I described in the previous paragraph increase the grading burden on the instructor. However, in this dissertation project, students reported that the intervention usually answered their questions even when they had not completed the formative assessment themselves. This suggests that a sample of students completing the formative assessment each unit might be sufficient to generate all of the questions about topics students find problematic. Though this might slightly complicate grading if only a sample of students complete the formative assessments each week, the grading burden on the instructor would be considerably eased. However, I should note that using the logistics I describe in the next paragraph, I was only spending 90 minutes each week on average to grade the formative assessments for the three sections of introductory calculus that participated in this study so the grading burden is not terribly onerous with well-designed formative assessments.

In the course of this project, I tried three different formative assessment delivery methods: paper, a course management software (CMS), and email. Hard copy formative assessments, which were what I used in my own practice prior to conducting this project, seemed to be easier for students to remember for students to complete even though the assignments were posted to the CMS just like they were in the dissertation study. The hard copy assignments did not take any longer to grade than did the CMS assignments but there were a few minutes of class time lost every time a formative assessment had to be collected or returned. In larger classes, this paper shuffling could cause an unacceptable amount of lost class time. In the pilot study, the formative assessments were first emailed to the instructor; after analysis, I recorded the grades on a spreadsheet. This worked reasonably well although my inbox became cluttered very quickly; assignments continued to trickle in for a long time since there was no clear late work policy. I would not recommend this method of facilitating formative assessments, particularly for giving students feedback. Overall, posting the formative assessments as assignments on the CMS was the most efficient method of facilitation. I believe if the formative assessments were worth more points and not a larger percentage of the grade, the illusion of high impact on their grades would be enough to raise the completion rate. In very large classes where even managing CMS software becomes impractical, it might be worth exploring using additional technology, e.g., clickers, the Just in Time Teaching

(JITT) platform used in physics education, or free survey websites like Survey Monkey to complete formative assessments.

For Theory

The results of this dissertation project suggested several potential contributions to the theoretical understanding of formative assessment. In this section, I outline the hypotheses suggested by these results. Naturally, all of these hypotheses need further study before connections to any existing theoretical framework are confirmed.

Overall, the Zone of Proximal Development is not a construct explicitly linked to Black and Wiliam's (2009) formative assessment framework. However, from this dissertation project, it appeared there was a link between the formative assessment framework and the Zone of Proximal Development, particularly in terms of engineering effective classroom activities, providing effective formative feedback, and increasing some aspects of student ownership of the material. Since the population of this dissertation was significantly different than the K-6 students for which this framework was originally developed, this dissertation might be considered confirmatory of the framework for these aspects. Two functions of formative assessment, clarifying learning intentions and activating students as learning resources for each other, were not used much in the coding; further research is needed to confirm that these parts of the framework are unimportant for undergraduate mathematics students. It is possible that self-monitoring might be considered part of ownership; the literature was vague on this point. Further inquiry on self-monitoring and ownership and how these constructs appear in the undergraduate classroom is needed.

One function of formative assessment was part of the findings of this dissertation project and not explicitly identified in Black and Wiliam’s (2009) framework. The first function of formative assessment that appeared to be beneficial was providing an opportunity for peripheral participation. The peripheral participation students engaged in when completing the formative assessment, especially student behavior during the instructional intervention based on the formative assessment, suggested that students paid more attention to the instructional intervention than they did to the instruction on new material; formative assessment seemed to increase student engagement.

For Methodology

Overall, the greatest challenge of this project as a researcher was the logistics involved with translating the idea of scaffolding from a one-on-one instruction to the classroom. In this section, I reflect on the methodology of this dissertation project where I used this theoretical perspective to conduct a qualitative study in three classrooms. After describing the routines I established to conduct this research, I discuss what was successful in this endeavor and what could be improved in future iterations of these methods.

I collected and copied all student pre-labs at the beginning of class. Students would have their original copy returned to them so they had a reference during the lab. This took me about five minutes at the start of each lab; I missed no meaningful time since the instructors discussed homework problems and gave class announcements during this time. I also made a second copy for the instructors at this time so they could see what students’ original thoughts were on the pre-lab; this practice might be worth continuing, resources permitting, even without the research. All labs and tests were collected and copied after class ended since it was not important that students had their copy returned for the class activity. This data collection process was minimally disruptive in the classroom and the data storage made it easy to keep the hard copies of the data organized. The post-labs were facilitated through the research site's Course Management Software (CMS). This was easy for me to obtain access and keep the data organized; however, the particular CMS platform created some challenges in the analysis process.

Another particularly successful aspect of the methods of this dissertation study was the level of triangulation I was able to achieve both in the data collection and the analysis. This was facilitated by the relationships I was able to develop with the student participants as well as my prior relationships with the instructors of the course. I also believe that keeping my research wholly separate from the coordination and grading aspects of the instructors' role kept my participation in the classroom low key for both the students and the instructors. This level of comfort increased the quality of my data and allowed me to collect data rich enough to construct the narrative in Chapter IV. Overall, the most helpful part of the data analysis plan was the Standard of Evidence journal. S ince my pilot study used almost identical methods to the dissertation project with a smaller *n*, it was very tempting to simply use the codes developed in the pilot study in the dissertation project rather than allowing the code words to emerge organically in the analysis despite the fact that the theoretical perspective changed from the pilot study to the dissertation and those codes were no longer appropriate. By coding my journal entries on the impressions of the data and structuring those codes with the literature, I felt I was able to strike a balance between open coding the data with no direction and imposing a coding scheme on my data that did not necessarily fit.

Not everything in my research plan went smoothly. There were several things about the methods I would change if using a similar methods on a future project. While the amount of data allowed me to use documents as a primary data source to construct a narrative, which is not generally possible, the sheer amount of data collected made it difficult to analyze the data as they were collected. Without the funding for a research assistant, it was often everything I could do each week to simply keep the documents organized, expand my fieldnotes, and maintain a research journal with my impressions. Without a research assistant to organize the initial raw data, in the future I would limit the inquiry to students in a single classroom. The other difficulty with the data collection plan for this project was the unexpected levels of attrition between the first and second interviews for students who earned A's in the course. In the future, I intend to solicit far more interview participants, particularly those with poor grades, so I do not have to scramble at the end of the semester to complete interviews.

Overall, I found that interviewing groups who worked together throughout the semester, rather than individual students, yielded the richest data. Although there were both upsides and downsides to changing the composition of student groups with each activity, which were often balanced based on individual circumstances and objectives, there is no question that richer research data were obtained from the permanent lab groups in the other section; students were able to bypass the time spent on re-establishing group norms each week and delve straight into the activity. I chose groups based on group communication skills and I privileged the groups who had a large variation in grades. Although I believe this helped me understand how and when students entered the Zone of Proximal Development at various points throughout the activity and helped to illustrate the collaborative ZPD, I regret that I did not interview a homogeneous group as a foil to the other interviews. In future inquiries, I believe there should be a balance of homogeneous and heterogeneous groups interviewed. One of my biggest regrets on this project was that I did not interview either of the homogeneous permanent groups for this dissertation project.

To summarize, in addition to answering the research question, this dissertation project suggested a framework to incorporate formative assessment as a scaffolding tool in an undergraduate mathematics classroom and even in other subjects. The findings from the main research question indicated that self-monitoring and peripheral participation might be constructs that could enrich Black and Wiliam's (2009) framework when it is applied to undergraduates instead of elementary school children. Finally, the methods in this project suggested a plan for conducting document analysis in undergraduate classrooms that was efficient and minimally invasive. However, there is no such thing as a perfect study. In the next section, I discuss the major limitations of this dissertation project and the steps I took to ameliorate them.

Limitations

There was one major limitation inherent to the design of this dissertation study: the limited time scale of this investigation meant I could not investigate if the increases in self-monitoring carried forward into the second semester of calculus and beyond. By limiting the scope of the data collection to two sections of introductory calculus offered in a single semester, I was limited in the conclusions I could make about how far students moved beyond their initial Zone of Proximal Development and whether students who have had formative assessment supplemented Calculus I in their second semester course. This was an acceptable limitation because adding a longitudinal component to the dissertation project was not logistically feasible given the timeline. However, a longitudinal study on formative assessment would be an excellent follow up study when combined with some of the guided reinvention research that has been conducted on students who have completed approximation research in the past (Martin et al., 2011; Oehrtman et al., 2011; Swinyard, 2011). I discuss this potential direction of future research in the next section.

The missing second interviews due to attrition and scheduling problems were also a limitation of this dissertation project since it was possible that students who were not willing to be present in a second interview would have had significantly different answers than students who were interviewed. However, I deliberately overscheduled first interviews expecting that some students, particularly those who were not doing well in the course or whose grades declined throughout the semester, would be reluctant to interview. Given that the students who did participate in both interviews represented the achievement spectrum, I believe this limitation was minimized as much as possible due to the time and funding constraints on the project.

One could also make a reasonable argument that the standards of evidence for appropriation were too high. Given the standards of evidence used in this dissertation, it is likely that the analysis underestimated the amount of the framework students actually appropriated. I acknowledge that some of the student actions coded as volitional could be appropriation.

There were two reasons why I chose to set the bar so high. First, if all solutions solved without external help were coded as appropriation, then all of the volitional solutions on the limits lab would have been coded as appropriation. This standard would then overestimate how much of the approximation framework students actually appropriated, which was not necessarily any improvement over the current standard evidence. The second reason why a solution had to be produced volitionally in multiple contexts was to distinguish appropriation from situation bound reasoning. In both cases, determining if something was appropriated or situation bound involved looking at the subsequent lab. Counting a solution as appropriated when it could be situation-bound reasoning would again overestimate the amount of the framework students appropriated. One way to prevent this overestimation of appropriation would be to count solutions that were correctly produced in multiple contexts as appropriation, which is the current standard of evidence used.

Directions for Further Inquiry

This dissertation project was sufficient to satisfy the requirements of an initial study on the phenomenon of formative assessment at the undergraduate level. From the large, rich data set I collected in the execution of this study, I analyzed the data set for further understanding of the impact on formative assessment in this semester. Furthermore, this dissertation project suggested several additional studies that could contribute to the professional discourse on how formative assessment can affect learning in undergraduate mathematics.

Since much of the research on formative assessment has been quantitative, I believe this dissertation project was not sufficient to completely understand the qualitative effects of formative assessment in an undergraduate classroom. In the paragraphs that follow, I have briefly outlined several qualitative studies that could contribute to our understanding of this phenomenon.

Given that one of the major limitations of this study was the missing interview data and attrition, I believe an initial follow-up study to the dissertation project should use the same methods on a single undergraduate introductory calculus class with a larger number of interviews. This study should ideally be conducted incorporating the improvements to the labs and formative assessments I suggested earlier in this chapter. This would allow for more detailed analysis of these findings and might reveal additional hypotheses this project failed to reveal within the limited scope of the interviews.

After this initial study was completed, the next reasonable course of action would be to investigate the impact of formative assessment on students' Zone of Proximal Development in other undergraduate courses. First, there should be a longitudinal study conducted with similar methods to this dissertation project to see how formative assessment influenced student learning across semesters. After that, there are two large populations of students where there is a great deal of variance on where students' Zone of Proximal Development is located: pre-service teachers and students in a first proofs course. It may also be worth examining formative assessment in in-service teachers' graduate courses. Finally, formative assessment might turn out to be an invaluable tool for translating guided reinvention interviews to the classroom.

The quantitative literature on formative assessment has been primarily quasi-experimental design. Before investigating formative assessment quantitatively, I believe researchers need to design more sophisticated measurements. In this way, more methodological advances and interesting quantitative studies might eventually be conducted. The first step in this process would require a massive literature search for instruments outside of the discipline of mathematics education that could be adapted for these purposes. If adaptable instruments exist, they need to be re-piloted several times and adjusted as needed.

I feel three instrument development studies should be conducted before any quantitative research begins. Once the content and form of the formative assessments have been finalized, there should be a rigorous analysis of the formative assessments to determine their quality as measurement tools. Second, since formative assessments seem to support language acquisition of the approximation framework, a survey or a test could be developed that could measure this acquisition in a quantitative manner. I hypothesize that this language acquisition would be well described by a Rasch model. Additional qualitative research would be necessary during this instrument development process, especially if any unexplained results or new hypotheses emerged during this process.

There are two major directions to go with future quantitative research on formative assessment. The first is to use the instruments developed in the previous phase of research in quasi-experimental studies on the efficacy of formative assessment in undergraduate mathematics classrooms. There is also a need for additional studies on formative assessment and achievement in the undergraduate classroom with varying populations of undergraduates and in longitudinal settings. The final quantitative study that could stem from this dissertation project comes from the implications. When interviewing students who had not completed formative assessment, they often remarked that the intervention still met their needs; from an instructional standpoint, it is worth investigating this herd immunity phenomenon. Can a sample of students completing formative assessment derive the same benefits for the class as a whole?

After this research program is executed, including the additional qualitative, measurement, and quantitative studies that will emerge as time goes on, it will be much clearer how peripheral participation and the other theoretical implications I suggested earlier contribute to an overall framework of formative assessment. Only then will it be appropriate to publish theoretical work on formative assessment.

Overall, I achieved the goals I set out for this dissertation project. With the help of my advisor and committee, I found and defined an area for inquiry of original research. I conducted a review of the literature that contextualized the inquiry I planned. I designed a pilot study and used what I learned to formulate a stronger investigation for the larger project. I then executed the data collection plan and analyzed the data. After constructing a narrative that provided rich descriptions of what happened this semester, I answered the research question. Finally, I discussed what conclusions could be drawn from this project, considered the limitations of the study, and suggested what steps I might take next. I look forward to continuing this line of research in my next endeavor. In conclusion, Black and Wiliam’s (2009) framework appeared to apply to undergraduates in mathematics courses. Further research is required to investigate how participation levels could be used to group students and whether there is a causal link between formative assessment, calibration, or attribution. This study provides a starting point.

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APPENDIX A

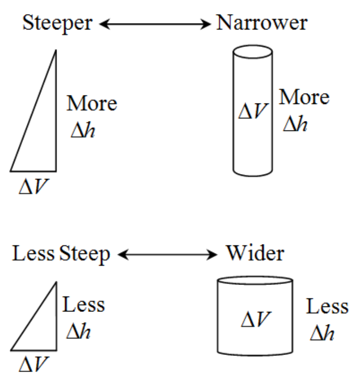
APPROXIMATION FRAMEWORK ACTIVITIES

Pilot Labs
Activity 1: Reasoning about Rates and Amounts of Change

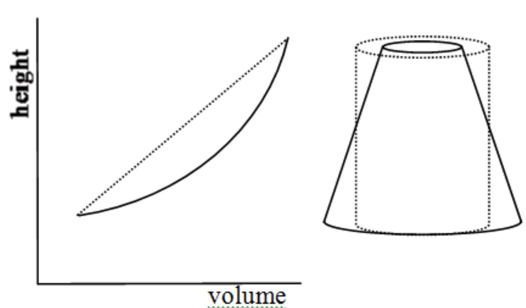
Instructions: Work with your group during class and with other students outside of class to solve these problems. Then write up the solutions *individually*. Your work must be neat and include sufficient exposition to make the solution clear to another student who has not seen the assignment (for example, a sequence of equations without explanation will most likely receive zero credit). Pay particular attention to places where explanations using multiple representations is requested, and *explicitly* discuss the connections between your explanations using different representations. Type or write all of your work *legibly* on 8½"×11" paper with at least *one-inch* margins on all sides free of writing except your name, date, and assignment number, and *staple* all pages together.

In the following, we consider plotting height of water in a bottle vs. the volume of the water in the bottle. That is, height is on the vertical axis (dependent variable) and volume is on the horizontal axis (independent variable).

1. Recall that the definition of an increasing function *f* is that $f(x_1) < f(x_2)$ whenever $x_1 < x_2$. The graph must always be increasing (going up as we move from left to right) since more volume has to correspond to more height. Rewrite the definition of an increasing function using $h(V)$ instead of $f(x)$. Then explain the ***meaning*** of this definition in terms of the bottle.
2. Steepness of the graph is related to the cross-sectional area of the bottle. Explain why a steeper graph corresponds to a narrower bottle and a less steep graph corresponds to a wider bottle, as shown to the right. Make sure that you are talking meaningfully about the ***rate of change*** of height with respect to volume by breaking down your explanation in terms of ***amounts of change*** in height and ***amounts of change*** in volume.



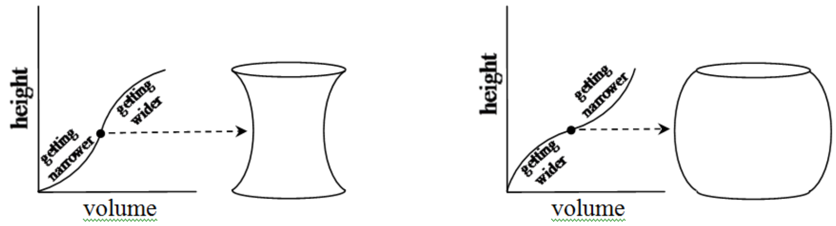
3. Translate the ideas from Problem 2 to the context of motion, for which the rate of change of distance with respect to time (velocity) is certainly an important idea. Explain why a steeper graph corresponds to faster motion and a less steep graph corresponds to slower motion. Be sure to frame your argument in terms of the ***amounts of change*** in distance and time.



4. The diagram to the right depicts a bottle that is wide at the bottom and narrow at the top (drawn with a solid line). The solid line in the graph shows the relationship of height vs. volume for this bottle. Recall that in order to think about the meaning of an average rate of change it is often helpful to

introduce an auxiliary situation where the rate is constant. In this case, we can imagine a cylindrical bottle (as drawn with a dotted line) and corresponding linear graph.
Use the auxiliary cylindrical bottle and graph to explain the meaning of the average rate of change of height with respect to volume for the original bottle that is wide at the bottom and narrow at the top.

5. Inflection points correspond to points where the bottle changes from getting narrower to getting wider (or vice-versa). This is because an inflection point on the graph occurs when the graph changes from getting steeper to becoming less steep (or vice-versa)



   a. Explain what is happening at the inflection points for the two bottles shown above using language about ***amounts of change***.

Describe what an inflection point means in a graph of distance traveled as a function of time. Provide explanations both in terms of the ***rate of change*** and the ***amount***

Activity 2: Locate the Hole

The graph of $f(x) = \dfrac{\sqrt[3]{x+7}-2}{x-1}$ has a hole. Your task is to determine the location of this hole using approximation techniques (no fancy limit computations allowed).

1. Identify what **unknown numerical value** you will need to approximate. Give it an appropriate shorthand name (that is, a variable).
2. Determine what you will use for **approximations**. Write a description of your answer using algebraic notation (for example, function notation, variables, formulas, etc.)
3. Find an approximation and give its numerical value to 4 decimal places. Is this an **underestimate** or **overestimate**? Explain how you know. Find both an underestimate and an overestimate.
4. Draw the graph using an entire sheet of paper. Depict your answers to #1, #2 and #3 on the graph with labels for each part of your answers.
5. Illustrate the **error** for your two approximations on your graph. Explain why you can't determine the numerical values of these errors. What is an algebraic representation for the error in your approximations?
6. Use your underestimate and overestimate to find a **bound on the error** for these two approximations. Explain your work.
7. List three fairly decent pairs of underestimates and overestimates (you can include the one you computed above). For each pair, give a bound for the error and use this to determine a range of possible values for the actual *y*-value of the hole. Add one of these underestimate-overestimate pairs to your graph and depict both the error bound and the **range of possible values**. Don't forget to label everything!

Underestimate	Overestimate	Error Bound	Range of Possible Values

8. Find an approximation with error smaller than 0.0001. Then describe ***all*** of the *x*-values you could use to get approximations that would have an error smaller than 0.0001. Add this to your picture.
For any pre-determined error bound, can you find an approximation with error smaller than that bound? Explain in detail how you know.

Activity 3: At this Rate

Instructions: You will approximate the instantaneous rate of change for one of the situations below using appropriately chosen average rates. Answer the following questions algebraically, numerically, graphically, and by representing each quantity in your diagram:

1. Imagine how things are changing in this situation. What quantities are changing and what quantities are constant? What are the important quantities for finding the requested approximations? Describe ***how*** these are changing in relation to each other.
2. Draw a **large** picture of the physical situation for the context given. Include several "snapshots" showing i) the system at the moment for which the instantaneous rate is requested and ii) configurations that clearly illustrate your description from Question 1. Illustrate and label the changes in the relevant quantities to support your answer to Question 1. You will return to this diagram to include additional information.
3. Draw a graph showing the relationship between the two quantities involved in the instantaneous rate that you are asked to approximate. Label several points on your graph corresponding to i) the moment for which the instantaneous rate is requested and ii) configurations that clearly illustrate your description from Question 1. Illustrate and label the changes in the relevant quantities to support your answer to Question 1. You will return to this graph to include additional information.
4. Describe in more detail what you have been asked to approximate using language from your given context. Give this quantity an appropriate algebraic representation and explain how it can be represented graphically. Add and label this on your graph.
5. Compute average rates of change that approximate the requested instantaneous rate. Explain the meaning of one of your average rates of change in terms of your context. Give an algebraic expression showing how to compute these average rates in general. Explain how these average rates can be represented graphically and add and label them on your graph.
6. Find both underestimates and overestimates for the requested instantaneous rate. Justify your answer in terms of your context. Explain how this can be seen on both the diagram of the situation and on the graph.
7. What are the errors? Give an algebraic representation of the errors for both an underestimate and an overestimate. Give the general form of this algebraic expression. Explain how these errors are represented graphically. Add and label the errors on your graph.
8. Find an error bound for one of your approximations. Justify your answer. Explain how this error bound is represented graphically. Add and label the error bound on your graph. What is the resulting range of possible values for your instantaneous rate? Explain how this range is represented graphically.
9. Find an approximation accurate to within the error bound given in your problem. Show and explain all of your work.
10. How can you find an approximation with error smaller than any predetermined error bound? Describe the process in detail.

In-Class Context: A bolt is fired from a crossbow straight up into the air with an initial velocity of 49 m/s. Accounting for wind resistance proportional to the speed of the bolt, its height above the ground is given by the equation $h(t) = 7350 - 245t - 7350e^{-t/25}$ meters (with *t* measured in seconds). Approximate the speed when $t = 2$ seconds accurate to within 0.1 m/s.

Group 1: Approximate the instantaneous rate of change of the volume of a sphere with respect to its radius when the radius is 5 cm accurate to within 0.1 cm^3/cm.

Group 2: NASA has determined that asteroid 1999 RQ36 has a 1 in 1000 chance of colliding with Earth on September 24, 2182*. The force of gravity in Newtons (N) between two objects is inversely proportional to the square of the distance separating them. The constant of proportionality is *GMm* where *G* is the "universal constant of gravity" $6.67´\ 10^{-11}$ Nm^2/kg^2 and $M = 5.97´\ 10^{24}$ kg and $m = 1.4´\ 10^{11}$ kg are the masses of the earth and the asteroid, respectively. Approximate the instantaneous rate of change of the gravitational force between the Earth and 1999 RQ36 with respect to distance when the two objects are 10,000 km apart accurate to within 0.1 N/m.

Group 3: The half-life of Iodine-123, used in medical radiation treatments, is about 13.2 hours. Approximate the instantaneous rate at which the Iodine-123 is decaying 5 hours after a dose of 6.4 μg is administered accurate to within 0.0001 μg/hr. *Class is canceled on September 24, 2182

Activity 4: Linear Approximation

The NASA Q36 Robotic Lunar Rover can travel up to 3 hours on a single charge and has a range of 1.6 miles. After *t* hours of traveling, its speed is $v(t)$ miles per hour given by the function $v(t)=\sin\sqrt{9-t^2}$. One hour into a trip, the Q36 will have traveled 0.19655 miles. Two hours into a trip, the Q36 will have traveled 0.72421 miles.

1. Use your calculator to graph $v(t)$. Explain in words what the graph says about how the Q36 moves during a 3-hour trip starting with a full charge.
2. What is the fastest speed achieved by the Q36? When does this happen?
3. Using a full sheet of paper, carefully sketch a graph of the distance $x(t)$ traveled by the Q36 measured in miles during this trip as a function of time in hours. Explain precisely why you drew the graph as you did.
4. Find the function $a(t)$ that gives the acceleration of the Q36 measured in miles per hour^2. Find $a(1)$ and explain the meaning of this value in terms of the motion of the Q36.
5. When does $a(t)=0$? Explain what this means on the graphs of $v(t)$ and $x(t)$. What does it mean in terms of the motion of the Q36?
6. Draw tangent lines to the graph of $x(t)$ at times $t=0$, $t=1$, $t=2$, and $t=3$. Label each tangent line with its equation. Use the variables *x* and *t* in these equations.
7. Approximate how far the Q36 traveled in the 10 minutes immediately following the $t=1$ hour mark. Is this an underestimate or overestimate? Explain. Use the speed at the end of this time interval to find a different approximation for the distance traveled during this 10 minutes. Is this an underestimate or overestimate? Explain. Draw a large graph emphasizing this 10-minute time interval and include the tangent lines used for both of your linear approximations. Label the changes Δt and Δx and the linearized differentials *dt* and *dx* corresponding to both of your approximations in the question.
8. Controllers want to turn the Q36 around to head back to its base after traveling 0.75 miles. Approximately what time will this happen? Will the actual time be a little earlier or a little later than your estimate? Explain. Draw a large graph emphasizing the portion of the trip starting at $t=2$ hours until the 0.75 mile mark. Include the tangent line used for your linear approximation. Label the changes Δt and Δx and the linearized differentials *dt* and *dx* corresponding to your approximation.

Quadratic Approximation

So far you have used tangent lines or "best fit lines" to approximate values of $x(t)$. Lines with slope *m* through the point (t_0, x_0) can be written in point-slope form as $x = x_0 + m(t-t_0)$. You then used the derivative $v(t)=x'(t)$ to find the slope at t_0.

We could improve our approximations by using "best fit parabolas." For the following problems, note that $x = x_0 + a(t-t_0)+b(t-t_0)^2$ is the equation of a parabola that passes through the point (t_0, x_0). Changing the parameters *a* and *b* will change the shape of the parabola without changing the fact that it passes through that point.

1. Sketch a parabola on your large graph of $x(t)$ that you think represents the best fit parabola at time $t=2$.
2. Find the slope at t_0 of the parabola with equation $x = x_0 + a(t-t_0)+b(t-t_0)^2$. Also find the second derivative at t_0. Your answers will involve the parameters *a* and *b* that control the shape of the parabola.
3. Determine the equation of the parabola $x = x_0 + a(t-t_0)+b(t-t_0)^2$ that passes through the point $(t_0, x_0)=(1, 0.19655)$ and has the same first and second derivatives as the actual function $x(t)$ at

$t_0 = 1$, that is $x'(1) = v(1)$ and $x''(1) = a(1)$. Use this equation to find a more accurate approximation to your answer to Question 7.

4. Determine the equation of the parabola $x = x_0 + a(t - t_0) + b(t - t_0)^2$ that passes through the point $(t_0, x_0) = (2, 0.72421)$ and has the same first and second derivatives as the actual function $x(t)$ at $t_0 = 2$, that is $x'(2) = v(2)$ and $x''(2) = a(2)$. Use this equation to find a more accurate approximation to your answer to Question 8.
5. We could continue to improve these approximations by finding higher degree polynomials with derivatives that match at a specified point. Consider approximating the distance traveled after one hour and 10 minutes, that is $x(1.16667)$. Review the approximation framework and determine
   a. What is being approximated?
   b. What are the approximations?
   c. What is the controlling variable?
   d. What is the singularity for the controlling variable (a value we can't actually plug in)?

   Then write the value of $x(1.16667)$ as an appropriate limit, using your answers above.

Activity 5: Newton's Method and Orbital Mechanics

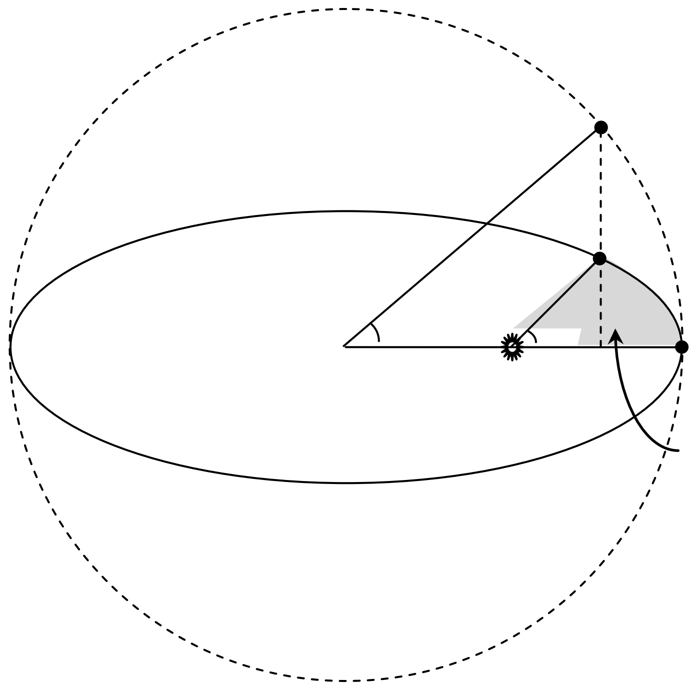
Kepler's first law of planetary motion states that the orbit of a planet is an ellipse with the star at one focus. His second law states that the line joining the planet and the star sweeps out equal areas during equal intervals of time. If we know the period (length of time for one complete orbit), the shape of the ellipse, and the time of the pericenter (when the planet is closest to the star), then Kepler's first and second laws are sufficient to determine the location of the planet in orbit at any other time.

There are three important angles about an orbit that you will need:

The *True Anomaly* (*TA*) is the angle measured at the star between the pericenter and the planet. This gives the planet's position relative to the star, so it is the angle we will be trying to determine.

The *Eccentric Anomaly* (*EA*) is the angle measured at the center of the ellipse from the pericenter to the projection of the planet on the auxiliary circle. This is an intermediate angle that we will need in our computations.

The *Mean Anomaly* (*MA*) is the angle from the pericenter that would have been swept out by the planet if it were moving at a constant angular velocity. This is a useful angle because it changes at a constant rate in time and can easily be converted into time units by a simple proportion.



Finally, the shape of the ellipse is measured by its *eccentricity* I which is the fraction of the distance along the semimajor axis at which the focus lies. The distance from the star to the planet I can then be determined from

$$r = a(1 - e\cos EA) \qquad (0)$$

Where a is the length of the semimajor axis.

Basic geometry allows one to determine the following relations among the angles defined on the previous page:

$$MA = EA - e\sin EA \qquad (1)$$

$$\tan\frac{TA}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{EA}{2} \qquad (2)$$

The general strategy to determine the position of a planet at any time is to determine *MA* based on the fraction of one orbit that has elapsed. Then use the first equation above to determine *EA*. Finally use the second equation above to determine *TA*. The only difficulty is that the first equation cannot be explicitly solved for *EA*! Thus we will have to apply Newton's method to this equation.

The Idea of Newton's Method – Lecture

1. First we will get a sense for how Newton's method works. Use your calculator to draw the graph of $y = x^4 - 10x^3 - x^2 + 3x - 50$. Draw the graph on your paper and estimate where the zeros are.
2. Now draw the graph on a one-unit interval in the domain containing one of the zeros. For example, if you think a zero is somewhere near $x = 3.5$ you would graph the function over the interval [3,4]. Use this new graph to better estimate the *x*-value of this zero, then evaluate f at that value. You should get something near 0. What is your guessed *x*-value for this zero and what does evaluating f at this point tell you?
3. Now graph the same function on a one-unit interval in the domain containing the other zero of f. Use this new graph to better estimate the *x*-value of this zero, then evaluate f at that value. What is your guessed *x*-value for this zero and what does evaluating f at this point tell you?

4. Find the derivative of *f*. Reproduce the graph of *f* near the smallest zero you found, but this time include the graph of the tangent line at the point where you guessed the zero should be located. Solve for the location where your tangent line crosses the *x*-axis. This should be near your guessed zero for *f* but not exactly the same. Why was this possible even though it is not possible to solve for when $f(x) = 0$?
5. Evaluate *f* at this new *x*-value. Does it produce an output closer to zero than your first guess did? Explain. Did you get an output smaller than 0.01? If not, repeat the procedure above until you get an output smaller than 0.01.
6. Reproduce the graph of *f* near the larger zero you found, but this time include the graph of the tangent line at the point where you guessed this zero should be located. Solve for the location where your tangent line crosses the *x*-axis. This should be near your guessed zero for *f* but not exactly the same.
7. Evaluate *f* at this new *x*-value. Does it produce an output closer to zero than your first guess did? Explain. Did you get an output smaller than 0.01? If not, repeat the procedure above until you get an output smaller than 0.01.

MATLAB Version:
Your program must give the true anomaly, *T*, and distance from the star, *r*, at regular time intervals for an entire orbit. The program should start by defining the following variables so that they can later be changed by the user:

- The semimajor axis of the orbit
- The eccentricity of the orbit
- The time required for one orbit (period)
- The time change between each calculation to be made

Your program should be structured as a loop that runs through an entire orbit at the specified time intervals. The easiest way to do this is to use a "for" loop. See us if you do not remember the syntax. Inside this loop, you should compute, *M*, then iterate Newton's method 3 times with equation (1) to get *E*. This is also best done with a "for" loop. Then, still inside the loop, use equation (2) to determine *T*. You can use equation (0) to determine *r*. Then you can give the values for *T* and *r* on the same line with the simple statement disp([T,r])

You can use the following information to run your program for both Earth and Mars using increments of one day on that planet:

Planet	Eccentricity	Semimajor Axis (Astronomical Units)	Heliocentric Period (terrestrial days)	Equatorial Period (hours)
Earth	0.0167175	1.000	365.256	23.9345
Mars	0.0933865	1.489	686.980	24.6229

Instead of having the program print out pairs of values for true anomaly (*T*) and distance I, have the program display a plot of the planet's location at regular time intervals. To do this, place the following code after your assignment of parameters:

```
set(gca,'DataAspectRatioMode','manual')
hold on
plot(0,0,'*')
```

The first line prevents Matlab from rescaling the plot which would make all ellipses look like circles – a bad thing if you want to see what an orbit looks like. The second line keeps a single plot so that multiple points can be put on it. The third line plots a * at the origin to represent the sun.

Then replace the "disp" command with the following:

```
plot(r*cos(T),r*sin(T),'.g')
```

This plots the planet at the appropriate *x* and *y* coordinates from the sun. The period in '.g' makes the dots big enough to see, and the g makes it green – use '.r' for red dots, '.b' for blue dots, etc.

By running the program again before closing the figure window, you can plot orbits for multiple planets at the same time. Below are some additional data. You can go online to find data for other planets and orbiting objects.

Planet	Eccentricity	Semimajor Axis (Astronomical Units)	Heliocentric Period (Terrestrial days / yrs)
Earth	0.0167175	1.000	365.256 days
Mars	0.0933865	1.489	686.980 days
2003 UB_{313} 10^{th} planet?	0.4378	67.89	557 years
Comet Halley	0.967	17.2	76 years

Print out at least three plots that show two orbiting objects each. Label the objects and the time intervals between locations for each object. Then, for each plot, write a sentence or two about something interesting that the plot tells you about the motion of these objects.

When you model the orbit of 2003 UB_{313}, you should notice something funny about the orbit. Identify and fix the problem in your program.

Copy and paste your output for each of these runs into an email, attach your m-file program with your submission.

Remember that the first line of the program should be a comment with your name. The second line should be the comment "% Orbital Mechanics" to indicate the assignment.

Activity 6: Optimization Problems

Instructions: Write up all parts of your group's numbered problem. This should include

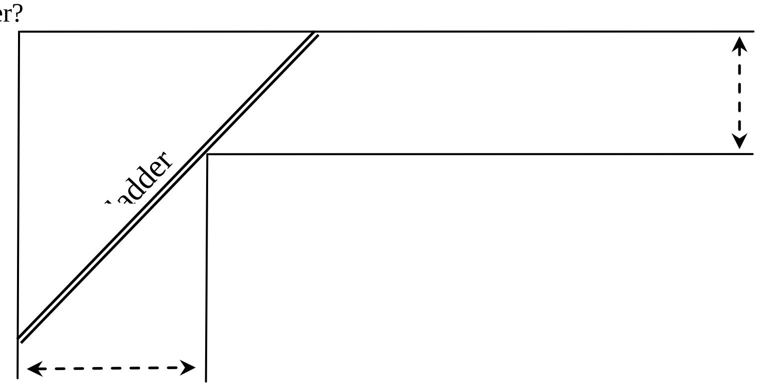
1. a diagram with all relevant constants and variables labeled,
2. a function expressing the quantity to be minimized/maximized as a function of ***one*** other variable (and the work required to create the function),
3. the domain of the function covering all possible configurations for your problem,
4. the derivative and critical points of the function, and
5. an analysis of all critical points and endpoints to determine the minimum/maximum.

1. In an extremely unlikely coincidence, several students in our class have independently opened three different soft-drink companies.
   a) Alex and Alex have opened a mathematically-correct soft-drink business. Find the dimensions of a can made of aluminum, holding 12 ounces of their Calcu-Cola, using the least amount of aluminum (1 oz. is 29.57 cm^3).
   b) Jonathan and Ryan open a competing soft-drink business and have developed a new technology that allows them to use a thinner aluminum in the side walls of the can. Their 12-ounce can of Drink-and-Derive is constructed so that the top and bottom of the can are *k* times as thick as the side (the exact value of the factor *k* is a company secret). What dimensions should Jonathan and Ryan make their can to minimize the aluminum in the can (as a function of *k*)?
   c) Tausha and Elizabeth figure out a way to manufacture a can using thinner aluminum for the ends as well as the sides for their drink Tasty-Tangent. Unfortunately, the pieces must be cut out of a rectangular sheet of aluminum. There is no waste involved in cutting the metal that makes the vertical sides of the can because that can be a rolled-up rectangle. But each circular end piece is cut

from a square of metal and the corners of the square are wasted. Find the dimensions of the most economical 12-ounce can they can make.

2. In another striking coincidence, all of the students at one table are having real-life issues involving circular sectors.
   a) To relieve stress from doing WeBWorK problems, Molly has taken up gardening. She has designed her garden plot in the shape of a circular sector with radius R and angle θ. Based on the amount of vegetables she wants to grow, she has determined the garden should have an area of A square feet. She needs to build an electric fence around the perimeter to keep Donovan and Jeremy out of the tomatoes. Find the dimensions (R,θ) which minimize the length of fence Molly will need to build.
   b) Elizabeth is fed up with Hannah's jokes and decides to make her wear a dunce cap in calculus class. She starts out with a paper circle of radius 12 inches and wants to make the hat as large as possible for optimal humiliation effect. To do this, she cuts out a sector with angle φ and tapes together the resulting edges to form the cone. Find the magnitude of φ so that the volume of the dunce hat is maximized.
   c) Keith and Graham finished their Mathematica Lab early and are enjoying a day at Horsetooth Reservoir, but soon get into an argument. Keith pushes Graham off of their "Little Mermaid" floaty 200 feet from shore and paddles off. The icy cold water has momentarily made Graham forget he is a really good swimmer. Allie is at a point 200 feet down the shore from the point closest to Graham. She can run 18 ft/s and can swim at a rate of 5 ft/s. To what point on the shore should she run before diving into the lake if she wants to reach Graham as quickly as possible? Once Graham falls into the water, he can manage to thrash about for exactly 51 seconds. Can Allie reach him in time?

3. Danielle and Alison went to a fortune-teller at the Colorado state fair who tipped them off that their next calculus assignment would have a problem involving ladders sliding down a wall. Steve suggested they should replicate this in real life to gain an advantage on the rest of the class, so they all decided to sneak a long ladder into ROSS Hall. Brandon further added that they should use as long of a ladder as possible for an optimal experience. Steve points out the problem that they must maneuver the ladder through a turn where the hallway constricts from 8 feet down to 5 feet as indicated in the picture. What is the longest ladder Steve, Brandon, Danielle and Alison can carry horizontally around the corner?

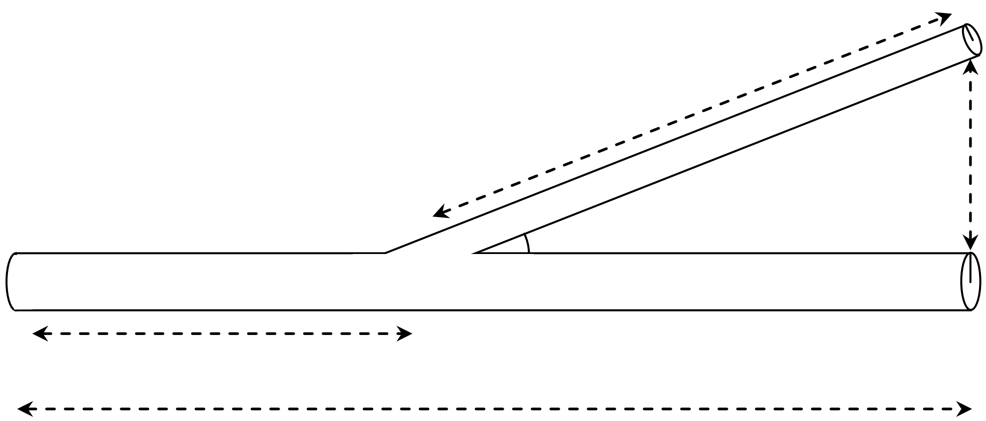


4. A billboard advertising "Have your picture taken with the buffalo at Kaci's Pottery Mega-Warehouse" is k feet wide, perpendicular to a straight road, and s feet from the road. At what point on the road would Corinne have the best view of the billboard as she drives by, thus tempting her to take the 30-mile detour for the photo-op. That is, at what point on the road is the angle subtended by the billboard a maximum?

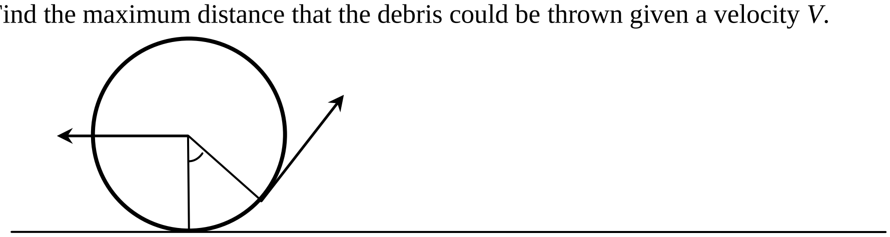
5. Jose and Levi have decided to try to get on Dr. Oehrtman's good side by building a life-size origami statue of him as a gift. They begin with a sheet of paper that is 40' long and 60' wide. Janelle suggests that the first fold should be particularly significant. Shea gets really excited and yells "We should fold the bottom right corner to a point on the left side so that the length of the crease is a minimum!" To what point should Shea, Janelle, Jose and Levi fold the corner to achieve this incredibly symbolic feat?

6. Julie and Stephanie are studying for a biology exam, but wishing they could get back to their favorite homework – calculus. This chapter is on the system of blood vessels in the body, which is made up of arteries, arterioles, capillaries, and veins. Julie wonders out loud if there is a reason for the branching

patterns she sees in the textbook diagrams. Ben happens to walk by and overhear the conversation. He suggests that the reason might be that the transport of blood from the heart through all organs of the body and back to the heart should be as efficient as possible. He suggests to Julie and Stephanie that one way this can be done is by having large enough blood vessels to avoid turbulence and small enough blood cells to minimize viscosity. Then Connor wanders by and suggests they use calculus to derive the angle θ for branches in blood vessels such that total resistance to the flow of blood is minimized. He draws the picture below and says they could assume that a main vessel of radius r_1 runs along the horizontal line from *A* to *B.* A side artery, of radius r_2, heads for a point *C* that is *s* units away from the main vessel. Connor then labels *D* as the point where the branching vessel leaves the main vessel. Julie points out that in order to solve the problem, they would also have to know how the resistance of blood flow is related to the size of the vessel. Fortunately, Jessica happens to walk by and says that they can use Poiseuille's law for that. Specifically, the resistance *R* in the system is proportional to the length *L* of the vessel and inversely proportional to the fourth power of the radius *r*. That is, $R = k \cdot \frac{L}{r^4}$, where *k* is a constant determined by the viscosity of the blood. Labeling lengths $AB = L_0$, $AD = L_1$, and $DC = L_2$, Nefty notes that the total resistance from *A* to *C* is the sum of the resistance on *AD* and the resistance on *DC*. What angle θ minimizes this resistance?



7. Richard and Caleb are driving a large truck down a road with lots of gravel. The tread in his rear tires occasionally pick up small pieces of gravel and fling them into the air as they work loose from the tire. Catrina and Melissa are following behind but don't want their windshield to get cracked from these flying rocks. They decide to use calculus to determine a safe distance to travel behind Richard's truck. The distance traveled by the gravel will vary depending on the angle, α, at which the gravel is thrown from the tire. Find the maximum distance that the debris could be thrown given a velocity *V*.



7. In yet another stroke of great coincidence, several students in our class have landed summer internships conducting wildlife research in the area.
   a) Clayton and Zach are studying the population of brook trout in Sprague Lake in Rocky Mountain National Park. Clayton finds a population model on mathworld.com to predict the population of a species one year depending on the previous year's population. His model determines the

population of trout next year by the function $f(T) = Te^{r(1-T/P)}$, where *T* is the current year's trout population, *P* is the natural equilibrium population, and *r* is a constant that depends upon how fast the population grows. For different population sizes *T*, different amounts of trout can be fished (harvested) in a year so that the population remains the same size, in which case such a harvest is sustainable over time. Zach computes a derivative in his head and realizes that they won't be able to solve for the trout population T_0 which will support the maximum sustainable harvest. After a little mental arithmetic of his own, Clayton quickly sees Zach's point, but suggests that with a little work they might be able to express the size of the maximum sustainable harvest in terms of T_0, *r*, and *P* without using an exponential. What is this expression?

b) Andrew and Kyle are studying bird migration patterns for the Northern Prairie Wildlife Research Center. Andrew reasons that the length of time a migrating bird can fly depends on how fast it flies, that is, the flight time is some function $T(v)$. Assuming *E* is the bird's initial energy, Kyle derives that the bird's effective power is given by kE/T, where *k* is the fraction of the power that can be converted into mechanical energy. Andrew looks up the Beginner's Guide to Aerodynamics on NASA's website where he finds that according to the principles of aerodynamics, this effective power is also related to the wind speed *S* and the induced power *I* (or rate of working against gravity) . Specifically the effective power is equal to $aSv^3 + I$ for some positive constant *a*. Find the velocity that a bird would need to fly to migrate a maximum distance. (Kyle notes that this will depend on some, but not all, of the parameters listed above).

Activity 7: Related Rates Problems

Instructions: Do not immediately answer the questions asked in problems 1-8 below. Instead, follow these instructions:

1. Draw ***and label*** a picture of the situation.
2. The rate(s) you know and the rate you are seeking should be the time derivatives of quantities you have labeled. State what those quantities are.
3. Determine an algebraic relationship involving only the varying quantities you identified in Part b. (These are the variables for which we either know or want to find a derivative value.)
4. Finally, venture a guess as to what type of answer you would get. Will it be positive or negative? How would the rate depend on the variables in the problem?
5. Differentiate the expression you found in Part d) with respect to time, *t*. Plug in the appropriate values given for any variables or rates then solve for the requested rate and answer the question.

PROBLEMS

1. Graham and Keith got a new sand box over the weekend and took their "friends," He-Man and Skeletor, out to play make-believe.
   a. Skeletor tied He-Man to a pole and began dumping sand on top of him at a rate of 4 cubic inches per second. He-Man is six inches tall. At the moment he is half buried, He-Man notices that the sand is rising at a rate of ½ inch per second. How much longer does Grah... I mean He-Man have to come up with a way to escape before he is completely buried?
   b. Allie and Kaci show up with Bat-Girl at the new sand box. Bat-Girl says that burying He-Man is boring and she would rather do Calculus! Help Bat-Girl and Skeletor do the following problems from one of Dr. Oehrtman's old exams:
   In the following equations, suppose that each variable is actually a function of time, *t*, unless otherwise specified, and differentiate the expression with respect to *t*.
      i. $x^2 + y^2 = 100$
      ii. $\frac{x+s}{5} = \frac{s}{1.5}$
      iii. $40y - xy = 80$
      iv. $(x+7)(7-gt^2) = 9x$ where *g* is a constant

v. $V = kh^3$ where *k* is a constant

2. a. Elizabeth and Jonathan have invented a "magic triangle." Its base is on a horizontal surface and no matter what you do to its height, the triangle always has area 10. If Alex pushes down on the top of the triangle so that it becomes shorter at a rate of 3 cm/sec, how fast will the length of the base be changing when the triangle is 5cm tall?
   b. The speed limit on a straight stretch of highway is 55 mph. Tausha, a highway patrol officer, stations herself at a point out of view of the motorists 50 feet off the highway. She is equipped with a radar gun which measures the speed at which a car approaches ***her position***. She takes a reading of suspected speeders by pointing her radar gun at a point on the highway 120 feet from the point on the highway closest to her. The radar gun picks up a reading of 48 feet/sec for a green Chevy driven by Ryan and Alex. How fast are they traveling?
3. a. Hannah and Elizabeth are on a Ferris wheel relaxing from a long morning of bull fighting at the Pontotoc County Fair in Ada, Oklahoma (ask them). The plane of the Ferris wheel lines up sun with the sun which is at a 60° angle with the ground (in-line with the one-o'clock position of the Ferris wheel). The diameter of the wheel is 50 feet, and it is rotating at a rate of 0.1 revolution per second. (i) What is the speed of Hannah and Elizabeth's shadow on the ground when they are at a two-o'clock position? (ii) a one-o'clock position?
   b. Inspired by recently renting "Saturday Night Fever" Donovan and Jeremy are redecorating their 16'x12' living room in a disco theme. Molly stops by to bring them some tomatoes from her garden, but when Jeremy opens the door, she is shocked by the sight of a disco ball rotating once every 2 seconds from the center of the ceiling. Her horror is replaced by a trance-like state as she is hypnotized from tracking one of the spots of light spinning around the room. As this spot of light enters a corner going from a long wall to a short wall, how fast is it moving?
4. a. Assume that Richard is perfectly spherical and that he melts at a rate proportional to his surface area, *A* (i.e., $\frac{dV}{dt} = kA$ for some negative constant *k*.) how fast is Richard's radius changing when his radius passes the 3 cm mark? When his radius is 5 cm? when his radius is *r* cm? (Your answers might involve the constant *k*.)
   b. Catrina and Melissa have made themselves two dimensional! Catrina moves right along the positive *x*-axis, and Melissa moves right on the graph of $f(x) = -\sqrt{3x}$. At a certain time, Catrina is at the point (5,0) and moving at 3 units/sec, and Melissa is at a distance of 2 units from the origin moving with speed 4 units/sec. At what rate is the distance between Catrina and Melissa changing?
   c. Caleb is an expanding 4-dimensional sphere! Specifically, when his radius is *r* meters, his 4-dimensional volume is $\frac{\pi^2}{2}r^4$ m^4. How fast is Caleb's radius changing when his 4-dimensional volume is 37 m^4 and increasing at a rate of 1.2 m^4/s?
5. a. A light is on the ground 40 meters away from a building. Kyle walks from the light toward the building at 2 meters/second. Zach is standing at the wall looking at the shadow. At what rate does Zach observe Kyle's shadow on the building shrinking when Kyle is 20 meters away?
   b. Andrew and Clayton are on their annual hunting trip. This time, however, ***they*** plan on outsmarting the deer! Andrew is sitting to Clayton's right (east) when the perfect buck appears 40 meters to the north. Clayton aims, but Andrew sneezes. The deer startles and takes off straight southeast at 13 meters per second. Clayton turns to keep the deer centered in his sight, but can't get a clean shot. At the instant Clayton smacks Andrew in the head with the barrel of the rifle, how fast was he rotating?
6. a. Ben is painting the walls of his room Bear Navy and Gold and is standing on top of a two-piece extension ladder leaning against the wall. Nefty walks by and is upset that Ben is covering up the orange and yellow paisley wallpaper, so he kicks the base of the ladder. Suddenly, the ladder starts collapsing at the rate of 2 feet per second AND, at the same time, its base starts sliding away from the wall at the rate of 3 feet per second. How fast is Ben falling (the top of the

ladder moving down the wall) when he is 8 feet from the ground and the base is 6 feet from the wall?

b. Stephanie and Julie live next to Connor who has a very loud stereo. The volume knob goes to 11, turning half a circle (angles θ between $0°$ and $180°$). The volume of the music, usually , is given by the function $V(\theta) = 110\sin\left(\frac{\theta}{2}\right)$ decibels (dB). One night at 3:30 in the morning they hear the lyrics "That's when I saw her, ooh, I saw her. She walked in through the out door, out door..." increasing from a volume of 88 dB at a rate of 1 decibel per second! At what rate can Stephanie and Julie deduce that Connor is turning the volume knob?

7. A streetlight hangs 5 meters above the ground.
   a. Levi walks away from the point under the light at a rate of 1.5 meters per second. How fast is his shadow lengthening when he is 7 meters away from the point under the light?
   b. Jose has the ability to magically shrink himself. At what rate must he do this to keep his shadow a constant length of 3 meters while walking away from the light at a speed of 2 m/s?
   c. Shea is running in circles around the streetlight at a distance of 10 meters and a speed of 4.5 m/s. How fast is the head of his shadow moving?
   d. Janelle has the ability to walk on vertical surfaces, and is walking up and down the lamp post. How fast is her shadow shrinking when she is on her way down and reaches the ground?
8. One sweltering 105^{o} day last August, Steve and Danielle were cleaning the gutters on Corinne's house in repentance for having earlier hit a baseball through her dining room window. While Steve was perched atop a 10 foot ladder, he made the mistake of angering Danielle by not laughing at her puns. In retaliation, Danielle began to pull the base of the ladder away from the wall at a rate of ½ ft/sec. Steve's balance is very good, and the ladder was originally flat against the wall.
   a. Alison and Brandon are watching in amusement from across the street and decide to see if Steve is falling faster and faster or slower and slower. How far does Steve fall during his first four seconds of motion? The next four? The next four? The next four? The last four? (Use a calculator.)
   b. How fast is Steve approaching the ground when Danielle has pulled the bottom of the ladder 6 feet from the wall.
   c. How fast is Steve moving when he hits the ground?
   d. Corinne notices that the triangle formed by the ladder, the wall and the ground first gets bigger then gets smaller. How fast is the area of the triangle changing when Steve is 8 feet from the ground? Is the triangle getting larger at this time, or smaller? How fast is the area of the triangle changing when Danielle has pulled the base 8 feet from the wall? Is the triangle getting larger at this time, or smaller?

Activity 8: Definite Integrals

Context 1: For a constant force* F to move an object a distance d requires an amount of energy** equal to $E = Fd$. Hooke's Law says that the force exerted by a spring displaced by a distance x from its resting length is equal to $F = kx$, where k is a constant that depends on the particular spring. In this activity you will approximate the energy required to stretch the spring with $k = 0.155$ N/cm from 5 cm past its natural length to 10 cm.

*The standard unit of force is Newtons (N), where 1 N = 1 kg·m/s^2 or the force required to accelerate a 1 kg mass at 1 m/s^2. Increasing either the mass or the acceleration rate therefore requires a proportional increase in force.

**The standard unit of energy is Joules (J), where 1 J = 1 N·m or the energy required to move an object with a constant force of 1 N a distance of 1 m. Increasing either the force or the distance requires a proportional increase in energy.

1. Draw and label a large picture of a spring initially displaced 5 cm from its natural length then stretched to a displacement of 10 cm.
2. Explain why we cannot just multiply a force times a distance to compute the energy.
3. Use Riemann sums with 10 terms to find both an underestimate and overestimate for the energy required to stretch the spring from 5 cm to 10 cm. Write out your sums numerically and with summation notation. Illustrate the terms of your sum on your picture.

4. Write an algebraic expression for your error. What is the bound on the error for your approximations? What is the range of possible values for the energy required to stretch the spring from 5 cm to 10 cm?
5. Find an approximation accurate to within $0.000005 = 5\times10^{-6}$ Joules.
6. Write a formula indicating how to find an approximation accurate to within any pre-determined error bound, ε.
7. Illustrate your answers to b) and c) in terms of area under an appropriate graph. Label your axes, underestimate, overestimate, actual value, error, and error bound in your diagram.
8. Write a definite integral expressing the exact amount of energy required to stretch the spring.

Context 2: A uniform pressure P^{**} applied across a surface area *A* creates a total force* of $F = PA$. The density of water is 1000 kg per cubic meter, so that under water the pressure varies according to depth, *d*, as $P = 1000d$. In this activity you will approximate the total force of the water exerted on a dam 62 meters wide and extending 25 meters under water.

> *The standard unit of force is Newtons (N), where 1 N = 1 kg·m/s^2 or the force required to accelerate a 1 kg mass at 1 m/s^2. Increasing either the mass or the acceleration rate therefore requires a proportional increase in force.
> **Pressure is the force per unit area, $P = F/A$, so for example a force of 6 N applied over a 2 m^2 area would generate a pressure of 3 N/m^2. Increasing the force would increase the pressure proportionally. Increasing the area would decrease the pressure proportionally (an inverse proportion).

1. Draw and label a large picture of a dam 62 m wide and extending 25 m under water.
2. Explain why we cannot just multiply a pressure times an area to compute the force.
3. Use a Riemann sum with 5 terms to find both an underestimate and overestimate for the total force of the water exerted on this dam. Write out your sums numerically and with summation notation. Illustrate the terms of your sum on your picture.
4. What is the error bound for each of these approximations?
5. Find an approximation accurate to within 5000 N.
6. Write a formula indicating how to find an approximation with any pre-determined tolerance, ε.
7. Illustrate your answers to c) and d) in terms of area under an appropriate graph. Label your axes, approximation, actual value, error, and error bound in your diagram.
8. Write a definite integral expressing the exact force of the water on the dam.

Context 3: The mass *M* of an object with constant density *d* and volume *v* is $M = dv$. A 10-meter long, 10-cm diameter pole is constructed of varying metal composition so that its density increases at a constant rate from 4.2 grams per cubic centimeter at one end to 33.8 grams per cubic centimeter at the other. In this activity you will approximate the mass of this pole.

1. Draw a large picture of the pole labeling all dimensions and representing the variable density.
2. Explain why we cannot just multiply a density times a volume to compute the mass.
3. Use a Riemann sum with 4 terms to find both an underestimate and overestimate for the mass of the pole. Write out your sums numerically and with summation notation. Illustrate the terms of your sum on your picture.
4. What is the error bound for each of these approximations?
5. Find an approximation accurate to within 300 grams.
6. Write a formula indicating how to find an approximation with any pre-determined tolerance, ε.
7. Illustrate your answers to c) and d) in terms of area under an appropriate graph. Label your axes, approximation, actual value, error, and error bound in your diagram.
8. Write a definite integral expressing the exact mass of the pole

Context 4: The volume *V* of an object with constant cross-sectional surface area, *A*, and height, *h*, is $V = Ah$. In this activity you will approximate the volume of water in a large spherical bottle of radius 1 foot that is filled to height of 21.7 inches*.

*Since you can easily compute the volume of the bottom half of the sphere, you will focus on approximating the volume contained in the remaining 9.7 inches.

1. Draw a large picture of the spherical bottle labeling all dimensions and representing the variable cross-sectional area at different heights.
2. Explain why we cannot just multiply an area times a height to compute the volume.
3. Use a Riemann sum with 4 terms to find both an underestimate and overestimate for the volume of water in the bottle. Write out your sums numerically and with summation notation. Illustrate the terms of your sum on your picture.
4. What is the error bound for each of these approximations?
5. Find an approximation accurate to within 0.37 in^3.
6. Write a formula indicating how to find an approximation with any pre-determined tolerance, ε.
7. Illustrate your answers to c) and d) in terms of area under an appropriate graph. Label your axes, approximation, actual value, error, and tolerance in your diagram.
8. Write a definite integral expressing the exact volume of water in the bottle.

Context 5: The average annual household income in the U.S. is $49,443 with standard deviation $23,470. Assuming a normal distribution of household incomes, the probability density would be

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where $\mu = 49,443$ and $\sigma = 23,470$. In situations where the probability density is a constant, *p*, the proportion of cases falling within a range $a < x < b$ is $(b-a)p$. In this activity, you will approximate the proportion of households earning more than the mean annual income but less than $100,000 annually.

1. Draw a large graph of *f*. Show what area corresponds to the proportion of households earning more than the mean annual income but less than $100,000 annually
2. Explain why we cannot just multiply a probability density by the size of the income range to determine the proportion of households in that range.
3. Use a Riemann sum with 4 terms to find both an underestimate and overestimate for the overall proportion. Write out your sums numerically and with summation notation. Illustrate the terms of your sum on your picture.
4. What is the error bound for each of these approximations?
5. Find an approximation accurate to within 0.01% (that is the proportion should be within 0.0001).
6. Write a formula indicating how to find an approximation with any pre-determined tolerance, ε.
7. Illustrate your answers to c) and d) on your graph. Label your axes, approximation, actual value, error, and error bound in your diagram.
8. Write a definite integral expressing the exact proportion of households earning more than the mean, but less than $100,000 annually.

Activity 9: Modeling with Definite Integrals

1. The kinetic energy of an object with mass m and constant speed v is , at least in the case where the entire object is moving at the same speed. Suppose a 10 cm long rod weighing 30 grams is rotating around one of its ends at a rate of one revolution per minute, much like the second hand of a clock.
   a. Write a definite integral that gives the kinetic energy of the rod in Joules (kg•m2/s2), and evaluate the integral.
   b. Describe the meaning of each factor of your integral and give the units it is measured in.
   c. If the rod is only half as long but moves twice as fast, does the kinetic energy increase or decrease?
2. The density of oil in a circular oil slick on the surface of the ocean at a distance r meters from the center of the slick is given by kg/m2.
   a. If the slick extends from to m, write a definite integral that gives the total mass of oil in the slick, and evaluate the integral.
   b. Describe the meaning of each factor of your integral and give the units it is measured in.
   c. Within what distance r is half the oil of the slick contained?
3. The force of gravity that the earth exerts on an object diminishes as the object gets further away from the earth. The energy required to lift an object 1 foot at sea level is greater than the energy required to lift the same object the same distance at the top of Mt. Everest. However, the difference in altitudes is so small in comparison to the radius of the earth that the difference in work is negligible. On the other

hand, when an object is rocketed into space, the fact that the force of gravity diminishes with distance from the center of the earth is critical. According to Newton, the force of gravity on a given mass is proportional to the reciprocal of the square of the distance of that mass from the center of the earth. That is, there is a constant k such that the gravitational force at distance r from the center of the earth is given by the energy required to move an object a distance d is , if the force is constant over the distance d. How much

4. Write a definite integral that gives the energy in Joules (1J = 1 Nm) required to lift a 1-kg payload from the surface of the earth to the moon, which is about 362,570 km away at its closest point. (Hint: The earth's surface is at a distance of 6,371 km from its center. At this value of r, the force of gravity on the 1 kg object is 1 N. Use this to determine the constant k.)
   b. Describe the meaning of each factor of your integral and give the units it is measured in.
   c. How much energy is required to lift the 1-kg payload half-way to the moon?
5. The energy required to move an object a distance x while exerting a constant force F is E = Fx. Suppose that you have two magnets and a wire. One magnet is attached to the end of the wire and the other can slide along the wire. If the magnets are arranged so that they repel each other, then it will require force to push the movable magnet toward the fixed magnet. The amount of force needed to move the magnet increases as the two get closer together. In fact, the force at a distance d is proportional to .
   a. Using a constant of proportionality k between the force and distance, write a definite integral that gives the energy required to move the magnet from 5 cm away to 3 cm away in Joules (kg•m2/s2), and evaluate the integral (your answer will depend on k).
   b. Describe the meaning of each factor of your integral and give the units it is measured in.
   c. Which will require more energy, to move the magnet from 5 cm away to 3 cm away, or from 3 cm away to 2 cm away?
6. An exponential model for the density of the earth's atmosphere says that if the temperature of the atmosphere were constant, then the density of the atmosphere as a function of height, h (in meters), above the surface of the earth would be given by kg/m3.
   a. Write a definite integral that gives the mass of the portion of the atmosphere from to m (i.e., the first 100 meters above sea level). Assume the radius of the earth is 6400 km. Then evaluate the integral.
   b. Describe the meaning of each factor of your integral and give the units it is measured in.
   c. Estimate the total mass of the earth's atmosphere.
7. The gravitational attraction between two particles of mass and at a distance r apart is
   .a. Write a definite integral that that gives the gravitational attraction between a thin uniform rod of mass M and length l and a particle of mass m lying in the same line as the rod at a distance a from one end.
   b. Describe the meaning of each factor of your integral and give the units it is measured in.
   c. Two long, thin, uniform rods of length and lie on a straight line with a gap between them of length a. Suppose their masses are and , respectively. What is the force of attraction between the rods? (Use the result of Part a.)

Activity 10: Modeling with Differential Equations

1. Often scientists use rate of change equations in their study of population growth for one or more species. In this problem we study systems of rate of change equations designed to inform us about the future populations for two species that are either competitive (that is both species are harmed by interaction) or cooperative (that is both species benefit from interaction).
   Which system of rate of change equations below describes a situation where the two species compete and which system describes cooperative species? Explain your reasoning.
   System A System B
2. A group of scientists is studying a fish population and modified this equation to get , where P represents thousands of fish in Lake Minnetonka and t is in years.
   a. Plot by hand a tangent vector field for this rate of change equation that you believe shows important features are describe what those important features are.

b. What does this rate of change equation predict about the long-term outcome of the fish population if the initial population is 2 (i.e., P = 2 at t = 0)? How about if P = 6 at t = 0?
c. Why are the predictions you made in part (b) be reasonable (or not) for a fish population? Explain.

3. Consider the following systems of rate of change equations:
System A System B
In both of these systems, x and y refer to the number of two different species at time t. In particular, in one of these systems the prey are large animals and the predators are small animals, such as piranhas and humans. Thus it takes many predators to eat one prey, but each prey eaten is a tremendous benefit for the predator population. The other system has very large predators and very small prey.
Figure out which system is which and explain the reasoning behind your decision.
4. In System A from Question 3, assume x and y are measured in thousands of animals.
   a. What are and if . Explain in practical what will happen to x and y in this circumstance. That is what happens to the populations?
   b. What are and if . Explain in practical what will happen to x and y in this circumstance. That is what happens to the populations?
   c. What nonzero populations of the predator and prey result in and . Explain in practical what will happen to x and y in this circumstance. That is what happens to the populations?
   d. Draw the xy-plane on a region that comfortably shows all of the features you discussed above. Label and number your axes and sketch several arrows showing the direction of change for various x,y-pairs in the plane.
5. Apply five steps of the Euler method to the initial value problem with to estimate at showing all of your work. Is your result an underestimate or overestimate? Explain how you know.

Lab:

1. Determine the equation of the parabola $x = x_0 + a(t-t_0) + b(t-t_0)^2$ that passes through the point $(t_0, x_0) = (1, 0.19655)$ and has the same first and second derivatives as the actual function $x(t)$ at $t_0 = 1$, that is $x'(1) = v(1)$ and $x''(1) = a(1)$. Use this equation to find a more accurate approximation to your answer to Question 2 in Part 1 of this lab.
2. Determine the equation of the parabola $x = x_0 + a(t-t_0) + b(t-t_0)^2$ that passes through the point $(t_0, x_0) = (2, 0.72421)$ and has the same first and second derivatives as the actual function $x(t)$ at $t_0 = 2$, that is $x'(2) = v(2)$ and $x''(2) = a(2)$. Use this equation to find a more accurate approximation to your answer to Question 3 in Part 1 of this lab.
3. We could continue to improve these approximations by finding higher degree polynomials with derivatives that match at a specified point. Consider approximating the distance traveled after one hour and 10 minutes, that is $x(1.16667)$ with such an n^{th}-degree polynomial $P_n(x)$. Then in the situation in this lab, as well as many others, it turns out that
$$x(1.16667) = \lim_{n\to\infty} P_n(1.16667).$$
Review the approximation framework and determine
   a. What is being approximated?
   b. What are the approximations?
   c. What are the errors?
   d. What is the controlling variable (i.e., what makes the approximation more accurate)?
   e. What is the singularity for the controlling variable (a value we can't actually plug in)?

Lab 6: Newton's Method (Unchanged from Pilot)
Lab 7 Definite Integrals

Context 1: For a constant force* F to move an object a distance d requires an amount of energy** equal to $E = Fd$. Hooke's Law says that the force exerted by a spring displaced by a distance x from its resting length is equal to $F = kx$, where *k* is a constant that depends on the particular spring. In this activity you will approximate the energy required to stretch the spring with $k = 0.155$ N/cm from 5 cm past its natural length to 10 cm.

*The standard unit of force is Newtons (N), where 1 N = 1 kg·m/s^2 or the force required to accelerate a 1 kg mass at 1 m/s^2. Increasing either the mass or the acceleration rate therefore requires a proportional increase in force.

**The standard unit of energy is Joules (J), where 1 J = 1 N·m or the energy required to move an object with a constant force of 1 N a distance of 1 m. Increasing either the force or the distance requires a proportional increase in energy.

1. Draw and label a large picture of a spring initially displaced 5 cm from its natural length then stretched to a displacement of 10 cm.
2. Explain why we cannot just multiply a force times a distance to compute the energy.
3. Use Riemann sums with 10 terms to find both an underestimate and overestimate for the energy required to stretch the spring from 5 cm to 10 cm. Write out your sums numerically and with summation notation. Illustrate the terms of your sum on your picture.
4. Write an algebraic expression for your error. What is the bound on the error for your approximations? What is the range of possible values for the energy (in N·m) required to stretch the spring from 5 cm to 10 cm?
5. Find an approximation accurate to within $0.000005 = 5\times10^{-6}$ Joules.
6. Write a formula indicating how to find an approximation accurate to within any pre-determined error bound, ε.
7. Illustrate your answers to b) and c) in terms of area under an appropriate graph. Label your axes, underestimate, overestimate, actual value, error, and error bound in your diagram.
8. Write a definite integral expressing the exact amount of energy required to stretch the spring.
9. Let *n* be the last three digits of your Bear Id†. For Contexts 1-5, write
   a. the Riemann sums for the underestimate and overestimate with *n* terms using summation notation,
   b. the Riemann sums for the underestimate and overestimate with *n* terms using calculator notation, i.e., sum(seq(…)), and their numerical results, and
   c. the definite integral representing the exact answer.

†If the last three digits of your Bear Id is greater than 800, subtract 400 to get your value for *n*.

Lab 7 Definite Integrals

Context 2: A uniform pressure *P*** applied across a surface area *A* creates a total force* of $F = PA$. The density of water is 1000 kg per cubic meter, so that under water the pressure varies according to depth, *d*, as $P = 1000d$. In this activity you will approximate the total force of the water exerted on a dam 62 meters wide and extending 25 meters under water.

*The standard unit of force is Newtons (N), where 1 N = 1 kg·m/s^2 or the force required to accelerate a 1 kg mass at 1 m/s^2. Increasing either the mass or the acceleration rate therefore requires a proportional increase in force.

**Pressure is the force per unit area, $P = F/A$, so for example a force of 6 N applied over a 2 m^2 area would generate a pressure of 3 N/m^2. Increasing the force would increase the pressure proportionally. Increasing the area would decrease the pressure proportionally (an inverse proportion).

1. Draw and label a large picture of a dam 62 m wide and extending 25 m under water.
2. Explain why we cannot just multiply a pressure times an area to compute the force.
3. Use a Riemann sum with 5 terms to find both an underestimate and overestimate for the total force of the water exerted on this dam. Write out your sums numerically and with summation notation. Illustrate the terms of your sum on your picture.
4. What is the error bound for each of these approximations?
5. Find an approximation accurate to within 5000 N.
6. Write a formula indicating how to find an approximation with any pre-determined tolerance, ε.

7. Illustrate your answers to c) and d) in terms of area under an appropriate graph. Label your axes, approximation, actual value, error, and error bound in your diagram.
8. Write a definite integral expressing the exact force of the water on the dam.
9. Let n be the last three digits of your Bear Id. For Contexts 1-5, write
   a. the Riemann sums for the underestimate and overestimate with n terms using summation notation,
   b. the Riemann sums for the underestimate and overestimate with n terms using calculator notation, i.e., sum(seq(…)), and their numerical results, and
   c. the definite integral representing the exact answer.

†If the last three digits of your Bear Id is greater than 800, subtract 400 to get your value for *n*.

Definite Integrals

Context 3: The mass *M* of an object with constant density *d* and volume *v* is $M = dv$. A 10-meter long, 10-cm diameter pole is constructed of varying metal composition so that its density increases at a constant rate from 4.2 grams per cubic centimeter at one end to 33.8 grams per cubic centimeter at the other. In this activity you will approximate the mass of this pole.

1. Draw a large picture of the pole labeling all dimensions and representing the variable density.
2. Explain why we cannot just multiply a density times a volume to compute the mass.
3. Use a Riemann sum with 4 terms to find both an underestimate and overestimate for the mass of the pole. Write out your sums numerically and with summation notation. Illustrate the terms of your sum on your picture.
4. What is the error bound for each of these approximations?
5. Find an approximation accurate to within 300 grams.
6. Write a formula indicating how to find an approximation with any pre-determined tolerance, ε.
7. Illustrate your answers to c) and d) in terms of area under an appropriate graph. Label your axes, approximation, actual value, error, and error bound in your diagram.
8. Write a definite integral expressing the exact mass of the pole.
9. Let n be the last three digits of your Bear Id. For Contexts 1-5, write
   a. the Riemann sums for the underestimate and overestimate with n terms using summation notation,
   b. the Riemann sums for the underestimate and overestimate with n terms using calculator notation, i.e., sum(seq(…)), and their numerical results, and
   c. the definite integral representing the exact answer.

†If the last three digits of your Bear Id is greater than 800, subtract 400 to get your value for *n*.

Definite Integrals

Context 4: The volume *V* of an object with constant cross-sectional surface area, *A*, and height, *h*, is $V = Ah$. In this activity you will approximate the volume of water in a large spherical bottle of radius 1 foot that is filled to height of 21.7 inches*.

*Since you can easily compute the volume of the bottom half of the sphere, you will focus on approximating the volume contained in the remaining 9.7 inches.

1. Draw a large picture of the spherical bottle labeling all dimensions and representing the variable cross-sectional area at different heights.
2. Explain why we cannot just multiply an area times a height to compute the volume.
3. Use a Riemann sum with 4 terms to find both an underestimate and overestimate for the volume of water in the bottle. Write out your sums numerically and with summation notation. Illustrate the terms of your sum on your picture.
4. What is the error bound for each of these approximations?
5. Find an approximation accurate to within 0.37 in^3.
6. Write a formula indicating how to find an approximation with any pre-determined tolerance, ε.
7. Illustrate your answers to c) and d) in terms of area under an appropriate graph. Label your axes, approximation, actual value, error, and tolerance in your diagram.

8. Write a definite integral expressing the exact volume of water in the bottle.
9. Let *n* be the last three digits of your Bear Id. For Contexts 1-5, write
   a. the Riemann sums for the underestimate and overestimate with *n* terms using summation notation,
   b. the Riemann sums for the underestimate and overestimate with *n* terms using calculator notation, i.e., sum(seq(…)), and their numerical results, and
   c. the definite integral representing the exact answer.

†If the last three digits of your Bear Id is greater than 800, subtract 400 to get your value for *n*.

Definite Integrals

Context 5: The average annual household income in the U.S. is $49,443 with standard deviation $23,470. Assuming a normal distribution of household incomes, the probability density would be

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where $\mu = 49,443$ and $\sigma = 23,470$. In situations where the probability density is a constant, *p*, the proportion of cases falling within a range $a < x < b$ is $(b-a)p$. In this activity, you will approximate the proportion of households earning more than the mean annual income but less than $100,000 annually.

1. Draw a large graph of *f*. Show what area corresponds to the proportion of households earning more than the mean annual income but less than $100,000 annually
2. Explain why we cannot just multiply a probability density by the size of the income range to determine the proportion of households in that range.
3. Use a Riemann sum with 4 terms to find both an underestimate and overestimate for the overall proportion. Write out your sums numerically and with summation notation. Illustrate the terms of your sum on your picture.
4. What is the error bound for each of these approximations?
5. Find an approximation accurate to within 0.01% (that is the proportion should be within 0.0001).
6. Write a formula indicating how to find an approximation with any pre-determined tolerance, ε.
7. Illustrate your answers to c) and d) on your graph. Label your axes, approximation, actual value, error, and error bound in your diagram.
8. Write a definite integral expressing the exact proportion of households earning more than the mean, but less than $100,000 annually.
9. Let *n* be the last three digits of your Bear Id. For Contexts 1-5, write
10. the Riemann sums for the underestimate and overestimate with *n* terms using summation notation,
    a. the Riemann sums for the underestimate and overestimate with *n* terms using calculator notation, i.e., sum(seq(…)), and their numerical results, and
    b. the definite integral representing the exact answer.

†If the last three digits of your Bear Id is greater than 800, subtract 400 to get your value for *n*.

Definite Integrals

Context 6: Fluid traveling at a velocity *v* across a surface area *A* produces a flow rate of $F = vA$. Poiseuille's law says that in a pipe of radius *R*, the viscosity of a fluid causes the velocity to decrease from a maximum at the center ($r = 0$) to zero at the sides ($r = R$) according to the function $v = v_{\max}\left(1 - \frac{r^2}{R^2}\right)$. In this activity you will approximate the rate that water flows in a 4-inch diameter pipe if $v_{\max} = 4.44$ ft/s.

1. Draw a large picture of a cross-section of the pipe labeling all dimensions and representing the variable flow rate at different places.
2. Explain why we cannot just multiply a velocity times an area to compute the flow rate.

3. Use a Riemann sum with 4 terms to find both an underestimate and overestimate for the overall flow rate in the pipe. Write out your sums numerically and with summation notation. Illustrate the terms of your sum on your picture.
4. What is the error bound for each of these approximations?
5. Find an approximation accurate to within 0.0001 cfs.
6. Write a formula indicating how to find an approximation with any pre-determined tolerance, ε .
7. Illustrate your answers to c) and d) in terms of area under an appropriate graph. Label your axes, approximation, actual value, error, and error bound in your diagram.
8. Write a definite integral expressing the exact flow rate in the pipe.

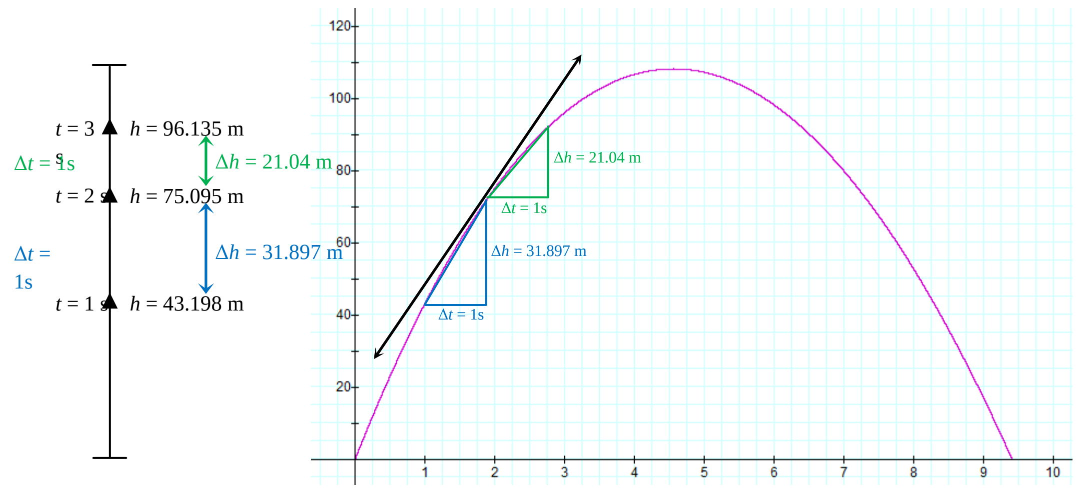
APPENDIX B

PILOT FORMATIVE ASSESSMENTS

Formative Assessment #1

Directions: Answer the following questions to the best of your ability. Responses need not be lengthy, but should answer all parts of the question. Please type your answers into this word document and email it back to [Me] at [Your.instructor@unco.edu] by [9 pm tonight].

1. A bolt is fired from a crossbow straight up into the air with an initial velocity of 49 m/s. Accounting for wind resistance proportional to the speed of the bolt, its height above the ground is given by the equation $h(t) = 7350 - 245t - 7350e^{-t/25}$ meters (with t measured in seconds). Approximate the speed when $t = 2$ seconds accurate to within 0.1 m/s. Use the graphs and calculations below to answer the following questions.



The average rate of change during the second and third seconds are

$$\frac{\Delta h}{\Delta t} = \frac{75.095 \text{ m} - 43.198 \text{ m}}{2 \text{ s} - 1 \text{ s}} = \frac{31.897 \text{ m}}{1 \text{ s}} = 31.897 \text{ m/s}$$

and

$$\frac{\Delta h}{\Delta t} = \frac{96.135 \text{ m} - 75.095 \text{ m}}{3 \text{ s} - 2 \text{ s}} = \frac{21.04 \text{ m}}{1 \text{ s}} = 21.04 \text{ m/s}$$

   a. hat is the unknown value we are trying to approximate?
   b. In the context of this problem, what does the value mean?
   c. What are the approximations?
   d. Identify an approximation that is an overestimate. How do you know it is an overestimate?
   e. Write down a formula for the error (in words or math symbols)
   f. What is a bound on the error?
2. Now consider situation your group worked on today.
   a. What is the unknown value we are trying to approximate?
   b. In the context of this problem, what does the value mean?
   c. What are the approximations?
   d. Identify an approximation that is an overestimate. How do you know it is an overestimate?
   e. Write down a formula for the error
   f. What is a bound on the error?
3. Write a short paragraph that answers the following two questions. What mathematical concepts or phrases used so far this week do you recognize from calculus? From other mathematics courses?
4. What questions do you have about the material we have covered so far in class?

Formative Assessment #2

Directions: Answer the following questions to the best of your ability. Responses need not be lengthy, but should answer all parts of the question. Please type your answers into this word document and email it back to [Me] at [Your.instructor@email.edu] by [9 pm tonight].

1. Fill in blanks with the letter(s) from the definition of the derivative to label the quantities marked on the graph of $y = f(x)$ as illustrated below.

   Error Bound = ________________

   Average Rate of Change = ________________

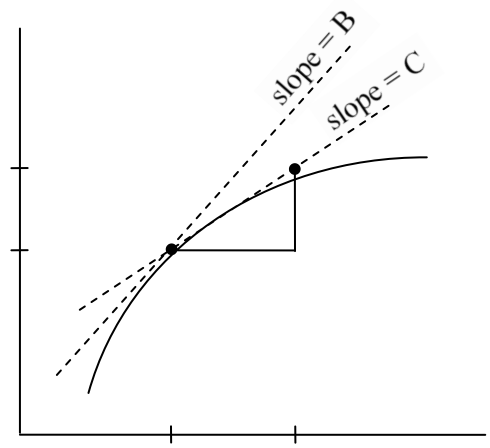
   Instantaneous Rate of Change= ________________

   Δy = ________________

   Δx = ________________

   x = ________________

   $x + h$ = ________________



2. Write a short paragraph that answers the following two questions. What mathematical concepts or phrases used so far this week do you recognize from calculus? From other mathematics courses?
3. What questions do you have about the material we have covered so far in class?

Formative Assessment #3 appeared as a question on Test #2

4. Fill in blanks with the appropriate expressions from the definition of the derivative to label the quantities marked on the graph of $y = f(x)$ as illustrated below.

   A = ________________

   B = ________________

   C = ________________

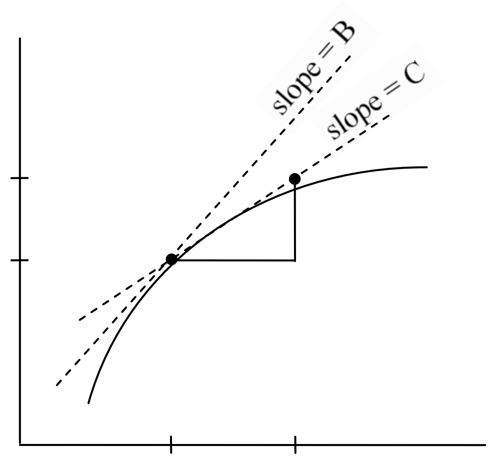
   D = ________________

   E = ________________

   F = ________________

   G = ________________

   H = ________________



Formative Assessment #4

Directions: Answer the following questions to the best of your ability. Responses need not be lengthy, but should answer all parts of the question. Please type your answers into this word document and email it back to Dr. Oehrtman by 9 pm tonight.

1. What are you approximating when you use a linear approximation?
2. How do we calculate approximations?
3. How can we tell if our approximation is an overestimate or an underestimate?
4. How can we make our approximations more accurate?
5. Write a short paragraph that answers the following two questions. What mathematical concepts or phrases used so far this week do you recognize from calculus? From other mathematics courses?
6. What questions do you have about the material we have covered so far in class?

Formative Assessment #5

Directions: Answer the following questions to the best of your ability. Responses need not be lengthy, but should answer all parts of the question. Please type your answers into this word document and email it back to [Me] at [Your.instructor@unco.edu] by [9 pm tonight].

In Activity 4, we were given information about the NASA Q36 Robotic Lunar Rover. Specifically, it can travel up to 3 hours on a single charge and has a range of 1.6 miles. After *t* hours of traveling, its speed is $v(t)$ miles per hour given by the function $v(t) = \sin\sqrt{9-t^2}$. One hour into a trip, the Q36 will have traveled 0.19655 miles. Two hours into a trip, the Q36 will have traveled 0.72421 miles.

Consider the following table of velocities:

Time *t* in hours	0	.5	1	1.5	2
Velocity *v*(*t*) in mph	0.14112	0.18252	0.30807	0.51715	0.78675

Assuming the speed at the beginning of each half hour, we would determine the Q36 traveled

$$\frac{1}{2}(0.14112)+\frac{1}{2}(0.18252)+\frac{1}{2}(0.30807)+\frac{1}{2}(0.51715)=0.57443 \text{ miles.}$$

Assuming the speed at the end of each half hour, we would determine the Q36 traveled

$$\frac{1}{2}(0.18252)+\frac{1}{2}(0.30807)+\frac{1}{2}(0.51715)+\frac{1}{2}(0.78675)=0.89725 \text{ miles.}$$

a. What is the unknown value we are trying to approximate?
b. In the context of this problem, what does the value mean?
c. What are the approximations?
d. Identify an approximation that is an overestimate. How do you know it is an overestimate?
e. Write down a formula for the error (in words or math symbols)
f. What is a bound on the error?

2. Now consider situation your group worked on today.
    a. What is the unknown value we are trying to approximate?
    b. In the context of this problem, what does the value mean?
    c. What are the approximations?
    d. Identify an approximation that is an overestimate. How do you know it is an overestimate?
    e. Write down a formula for the error
    f. What is a bound on the error?
3. Write a short paragraph that answers the following two questions. What mathematical concepts or phrases used so far this week do you recognize from calculus? From other mathematics courses?
4. What do you understand about approximating distance traveled? About your group's context? What questions do you still have?

Formative Assessment #6

Directions: Answer the following questions to the best of your ability. Responses need not be lengthy, but should answer all parts of the question. Please type your answers into this word document and email it back to [Me] at [Your.instructor@unco.edu] by [9 pm tonight].

1. How can you approximate what the definite integral of a function is on an interval?
2. How could I get a bound on my error?
3. How could I get a better error bound?
4. Write a short paragraph that answers the following questions. How is approximating a definite integral like this similar to earlier in the semester when we were working on differentiation? How is it different?
5. Write a short paragraph that answers the following two questions. What mathematical concepts or phrases used so far this week do you recognize from calculus? From other mathematics courses?

APPENDIX C

DISSERTATION FORMATIVE ASSESSMENTS

Post-Lab 1a

Directions: Answer the following questions to the best of your ability. Responses need not be lengthy, but should answer all parts of the question.

1. Which question is your group working on?
2. What have you figured out about the answer so far?
3. Write a short paragraph that answers the following two questions. What mathematical concepts or phrases used so far this week do you recognize from calculus? From other mathematics courses?
4. What questions do you have about the material we have covered so far in class?

Post-Lab 1b: Bottle Jigsaw

1. How were the other groups' problems similar to the problem you worked on? Different?
2. Write a short paragraph that answers the following two questions. What mathematical concepts or phrases used so far this week do you recognize from calculus? From other mathematics courses?
3. What parts of the material in the lab did this week did you understand? What questions do you have about the material we have covered so far in class, either in the lab or in lecture?

Post-Lab 3a: Locate the Hole

Directions: Answer the following questions to the best of your ability. Responses need not be lengthy, but should answer all parts of the question.

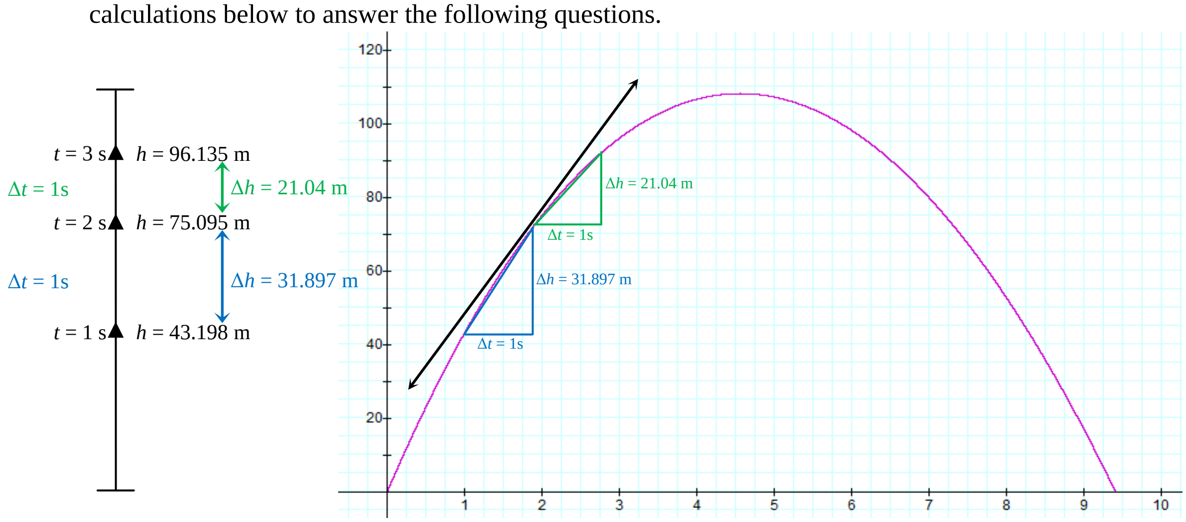
1. Which question is your group working on?
2. What have you figured out about the answer so far?
3. Write a short paragraph that answers the following two questions. What mathematical concepts or phrases used so far this week do you recognize from calculus? From other mathematics courses?
4. What questions do you have about the material we have covered so far in class?

Post-Lab 3b: Locate the Hole Jigsaw

1. Choose one of the problems you heard about in the Jigsaw and explain how the other group arrived at their solution.
2. How were the other groups' problems similar to the problem you worked on? Different?
3. Write a short paragraph that answers the following two questions. What mathematical concepts or phrases used so far this week do you recognize from calculus? From other mathematics courses?
4. What parts of the material in the lab did this week did you understand? What questions do you have about the material we have covered so far in class, either in the lab or in lecture?

Post-Lab 4a: At this Rate, Week 1

A bolt is fired from a crossbow straight up into the air with an initial velocity of 49 m/s. Accounting for wind resistance proportional to the speed of the bolt, its height above the ground is given by the equation $h(t) = 7350 - 245t - 7350e^{-t/25}$ meters (with *t* measured in seconds). Approximate the speed when $t = 2$ seconds accurate to within 0.1 m/s. Use the graphs and calculations below to answer the following questions.



The average rate of change during the second and third seconds are

$$\frac{\Delta h}{\Delta t} = \frac{75.095\text{ m} - 43.198\text{ m}}{2\text{ s} - 1\text{ s}} = \frac{31.897\text{ m}}{1\text{ s}} = 31.897\ \text{m/s}$$

and

$$\frac{\Delta h}{\Delta t} = \frac{96.135\text{ m} - 75.095\text{ m}}{3\text{ s} - 2\text{ s}} = \frac{21.04\text{ m}}{1\text{ s}} = 21.04\ \text{m/s}$$

a. What is the unknown value we are trying to approximate?
b. In the context of this problem, what does the value mean?
c. What are the approximations?
d. Identify an approximation that is an overestimate. How do you know it is an overestimate?
e. Write down a formula for the error (in words or math symbols)
f. What is a bound on the error?

5. Now consider situation your group worked on today.
   a. What is the unknown value we are trying to approximate?
   b. In the context of this problem, what does the value mean?
   c. What are the approximations?
   d. Identify an approximation that is an overestimate. How do you know it is an overestimate?
   e. Write down a formula for the error
   f. What is a bound on the error?
6. Write a short paragraph that answers the following two questions. What mathematical concepts or phrases used so far this week do you recognize from calculus? From other mathematics courses?

Post-Lab 4b At this Rate, Week 2

1. Fill in blanks with the letter(s) from the definition of the derivative to label the quantities marked on the graph of $y = f(x)$ as illustrated below.

Error Bound = ________________

Average Rate of Change = ________________

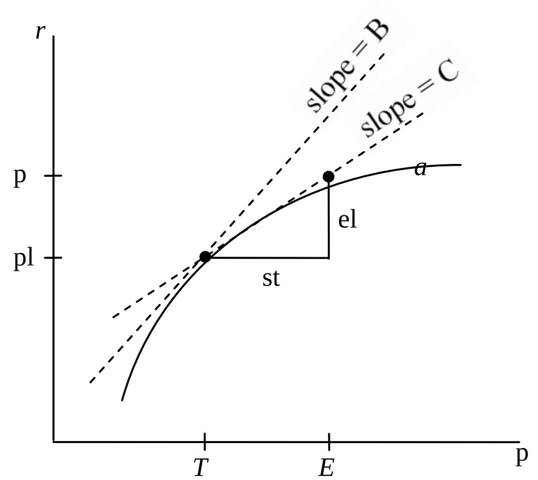
Instantaneous Rate of Change= ________________

Δy = ________________

Δx = ________________

x = ________________

$x + h$ = ________________



2. Write a short paragraph that answers the following two questions. What mathematical concepts or phrases used so far this week do you recognize from calculus? From other mathematics courses?
3. What parts of the material in the lab did this week did you understand? What questions do you have about the material we have covered so far in class, either in the lab or in lecture?

Post-Lab 4c (common test question)

Fill in blanks with the appropriate expressions from the definition of the derivative to label the quantities marked on the graph of $y = f(x)$ as illustrated below.

A = ________________

B = ________________

C = ________________

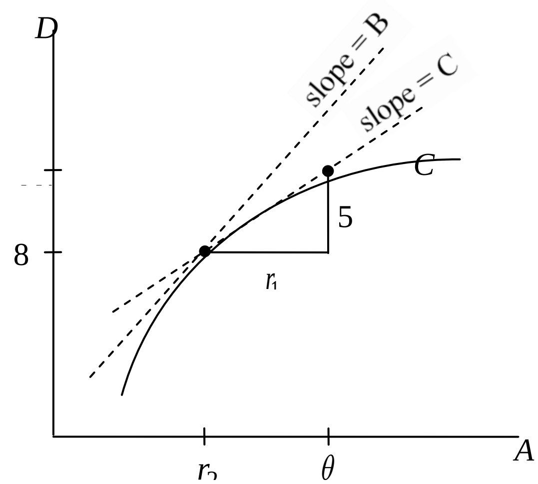
D = ________________

E = ________________

F = ________________

G = ________________

H = ________________



Post-Lab 5a: Linear Approximation

There will not be a post-lab after the first week of linear approximation; students should study for test 2 instead.

Post Lab 5b: Linear Approximation, Week 2

1. What are you approximating when you use a linear approximation?
2. How do we calculate approximations?
3. How can we tell if our approximation is an overestimate or an underestimate?
4. How can we make our approximations more accurate?
5. Write a short paragraph that answers the following two questions. What mathematical concepts or phrases used so far this week do you recognize from calculus? From other mathematics courses?

Post Lab 6a: Newton's Method

1. What is Newton's Method?
2. How did we use Newton's Method in this lab?
3. Write a short paragraph that answers the following two questions. What mathematical concepts or phrases used so far this week do you recognize from calculus? From other mathematics courses?
4. What have we covered in this activity that makes sense to you? What questions do you have about the material we have covered so far in class?

Post-Lab 6B

1. How does this picture relate to Newton's Method?
2. How does this lab relate to the calculus concepts we have covered in this course?
3. Write a short paragraph that answers the following two questions. What mathematical concepts or phrases used so far this week do you recognize from calculus? From other mathematics courses?
4. What have we covered in this activity that makes sense to you? What questions do you have about the material we have covered so far in class?

Post Lab 7a

In Activity 4, we were given information about the NASA Q36 Robotic Lunar Rover. Specifically, it can travel up to 3 hours on a single charge and has a range of 1.6 miles. After *t* hours of traveling, its speed is $v(t)$ miles per hour given by the function $v(t) = \sin\sqrt{9-t^2}$. One hour into a trip, the Q36 will have traveled 0.19655 miles. Two hours into a trip, the Q36 will have traveled 0.72421 miles.

Consider the following table of velocities:

Time *t* in hours	0	.5	1	1.5	2
Velocity *v*(*t*) in mph	0.14112	0.18252	0.30807	0.51715	0.78675

Assuming the speed at the beginning of each half hour, we would determine the Q36 traveled

$$\frac{1}{2}(0.14112)+\frac{1}{2}(0.18252)+\frac{1}{2}(0.30807)+\frac{1}{2}(0.51715)=0.57443 \text{ miles.}$$

Assuming the speed at the end of each half hour, we would determine the Q36 traveled

$$\frac{1}{2}(0.18252)+\frac{1}{2}(0.30807)+\frac{1}{2}(0.51715)+\frac{1}{2}(0.78675)=0.89725 \text{ miles.}$$

1.
   a. What is the unknown value we are trying to approximate?
   b. In the context of this problem, what does the value mean?
   c. What are the approximations?
   d. Identify an approximation that is an overestimate. How do you know it is an overestimate?
   e. Write down a formula for the error (in words or math symbols)
   f. What is a bound on the error?

2. Now consider situation your group worked on today.
   a. What is the unknown value we are trying to approximate?
   b. In the context of this problem, what does the value mean?
   c. What are the approximations?
   d. Identify an approximation that is an overestimate. How do you know it is an overestimate?
   e. Write down a formula for the error
   f. What is a bound on the error?
3. Write a short paragraph that answers the following two questions. What mathematical concepts or phrases used so far this week do you recognize from calculus? From other mathematics courses?
4. What do you understand about approximating distance traveled? About your group's context? What questions do you still have?

Post Lab 7b

1. How can you approximate what the definite integral of a function is on an interval?
2. How could I get a bound on my error?
3. How could I get a better error bound?
4. Write a short paragraph that answers the following questions. How is approximating a definite integral like this similar to earlier in the semester when we were working on differentiation? How is it different?
5. Write a short paragraph that answers the following two questions. What mathematical concepts or phrases used so far this week do you recognize from calculus? From other mathematics courses?

Post-Lab 7c

1. Throughout this semester, we have used the idea of approximation in these activities. In a short paragraph, define what the terms approximation, error, error bound mean to you, and how they relate to the idea of limits.
2. Write a short paragraph that answers the following two questions. What mathematical concepts or phrases used so far this week do you recognize from calculus? From other mathematics courses? What do you understand

APPENDIX D

COURSE SYLLABUS AND SCHEDULE

Pilot

CALCULUS I
MATH 131, Fall 2011

INSTRUCTOR: Dr. Michael Oehrtman
OFFICE: ROSS 2239F
PHONE: (970) 351-2380
OFFICE HOURS: MTWF: 9:00 am – 10:00 am, and by appointment
EMAIL: 268ebecca268@gmail.com

CREDITS: 4 semester credits

PREREQUISITES: Strong algebra and trigonometry background, an understanding of basic functions (polynomial, exponential, logarithmic, etc.), and a willingness to work hard.

REQUIRED TEXT: Hughes-Hallett, Gleason, McCallum, et al., *Calculus, 5th ed*, Wiley, 2009.

TECHNOLOGY: You will need a graphing calculator for this course. I strongly recommend a TI-83, TI-83 Plus or TI-84. We will also use a computer algebra system called Mathematica® for lab activities. UNC has a site license for the software, and it is available in ROSS and UC labs. WeBWorK will be used for homework assignments, and course materials will be available on Blackboard (unco.blackboard.com).

COURSE DESCRIPTION: Inspired by problems in astronomy, Isaac Newton and Wilhelm Gottfried Leibnitz developed the ideas of calculus roughly 300 years ago. Since then, calculus has provided the foundation for advances in many other fields, even those which seem far removed from mathematics. You will find applications in chemistry, physics, economics, biology, medicine, business, psychology, and of course mathematics. Calculus is so important that it is often considered the gateway to many of the disciplines in which it is used.

The power of calculus lies in its power to reduce complicated problems to simple rules and procedures. While these procedures can be (and often are) taught with little regard to the underlying mathematical concepts or their practical uses, our emphasis will be on understanding all of these: concepts, procedures and uses. We will engage in the full mathematics process, which includes searching for patterns, order and reason; creating models of real world situations to clarify and predict better what happens around us; understanding and explaining ideas clearly; and applying the mathematics we know to solve unfamiliar problems. Participation in this variety of mathematical activities is challenging, and for many students, the experience will be vastly different from experiences in more traditional mathematics course.

So what is calculus? Very briefly, calculus is the study of changing quantities. It has two main themes: differentiation, which studies rates of change and is the focus of this course; and integration, which studies accumulating quantities and will be introduced this semester but is more fully developed in Calc II. Calculus I is an introduction to the tools, methods, and applications of single-variable differential calculus. Central concepts of the course are that of a function and its derivative. We begin by a review of basic functions and their properties. Next we'll discuss a concept of a limit that is necessary to give the definition of a derivative. After mastering limits and their use in defining derivatives of basic functions, we'll study a collection of simple rules that allows us to easily compute the derivative of any function expressible in terms of elementary functions. We'll discuss various applications of differential calculus to real-life problems. In particular, we'll talk about differential equations, their (numerical) solutions. Finally we will introduce the idea of the definite integral to model aspects of accumulation.

GOALS. Our course is one of the General Education courses and it aims to satisfy the following outcome objectives in the area of Mathematics:

- Students will demonstrate proficiency in the use of mathematics to structure their understanding of and investigate questions in the world around them.
- Students will demonstrate proficiency in treating mathematical content at an appropriate level.
- Students will demonstrate competence in the use of numerical, graphical, and algebraic representations.
- Students will demonstrate the ability to interpret data, analyze graphical information, and communicate solutions in written and oral form.
- Students will demonstrate proficiency in the use of mathematics to formulate and solve problems.
- Students will demonstrate proficiency in using technology such as handheld calculators and computers to support their use of mathematics.

ACTIVITIES: On Tuesdays we will work in small groups on activities that develop the central concepts in the course. Attendance and participation is especially crucial on these days. You will turn in individual write-ups of these class activities and make presentations of your work to the other groups in the class. It is also important to ask questions of the other groups (who will generally work on related but slightly different problems than your own group) when they present as you will be responsible for all the problems on exams.

ATTENDANCE: There may be topics covered in class that are not in the text. You are responsible for all material covered. I don't take attendance, but there is a strong correlation between attendance and final grades. Missing class more than once or twice during the semester is likely to affect your grade, either directly or indirectly. If you do miss class, you should get notes and/or handouts from your classmates and/or see me during office hours.

HOMEWORK: There are three types of homework assignments in this class:

- **Written homework** will usually consist of a small number of relatively comprehensive problems mostly drawn from writing up the Tuesday group activities. It will be posted on Blackboard, about once a week. The emphasis for these assignments is on presentation and explanation. You will turn in these problems and they will be graded. Your lowest written homework score will be dropped in computing your homework grade.
- **On-line homework** will usually consist of two WeBWorK-based assignments each week. You are allowed six attempts for each question and you can get partial credit if you only get part of a problem right. These problems will be similar to those in the book and are graded immediately.
- **Suggested practice problems from the text**. The answers to most of these problems are in the text, so I will not collect them. However, you will see some of these problems (verbatim or with slight variations) on tests, so completing the problems is strongly encouraged!

The key to success in this course is regularly working with other students in the class, doing the homework early and asking questions when you have them!!! We will discuss homework problems in class, but there will often not be enough time to discuss all of them. Please come to office hours or visit the math tutoring lab if you have additional questions about the homework.

LATE POLICY: WeBWorK assignments will have a closing date and time and will not be accepted late. All other work is due *at the beginning of class* on the announced due date. I may accept late written homework for reduced credit, until I have graded an assignment or project. After I have graded the pile, I will no longer accept late work and you will receive a 0. I generally grade materials within a couple days of collecting them, and sometimes grade them the same day they are collected. Expect to lose approximately 10% for each day an assignment is late.

EXAMS: We will have four in-class exams (roughly covering Chapters 1, 2, 4 and 5), and a comprehensive final exam. The final exam will be Monday, December 5th, from 4:15 to 6:45 pm. Make-up exams are possible only if there is a *documented* emergency.

GATEWAY TEST: There will be a WeBWorK-based test on differentiation after we have covered the short-cut rules for taking derivatives in Chapter 3. You will be able to take the test as many times as you like during the 2 week period that it is open. A passing grade is 12 out of 15, and each problem is graded as correct or incorrect (no partial credit). Your grade on the Gateway Test is not figured into your weighted average at the end of the course. However, failure to pass the Gateway Test during the window it is available will result in your final course grade being lowered by 2/3 of a letter grade.

WORKLOAD AND ASSISTANCE: You should expect to spend **8 to 12 hours each week**, outside of class, on the course material. This includes reading, homework, and studying for quizzes and exams. Some weeks (those in which an exam is scheduled, for instance) may require more of your time, other weeks may require less, but *on average*, budget 8 to 12 hours each week. **I can't stress enough that in order to be successful in this class you should spend much of this time working with other students in the class!** Please ask questions and seek assistance as needed. You may email me at any time, and I encourage you to make use of my office hours and the Thursday group study room. In addition there are two tutoring centers (see http://www.unco.edu/tutoring.htm for hours and more information):

- We will also have the rooms listed below reserved just for studying calculus in groups from 9:00 am – 5:00 pm on Thursdays. We strongly encourage you to drop in or organize group studying at these times and will have a calculus instructor staffing this room at most times to assist you.

9:00 am – 10:50 am	Ross 1080
11:00 am – 1:00 pm	Ross 1090
1:00 pm – 5:00 pm	Ross 2090

- The math tutoring lab is located in Ross 1250 and will open the second week of classes. It is a great place to go if you have a quick question or get stuck on a particular problem. No appointment is necessary.
- The university tutoring Center is located in Michener L120. It provides more personalized one-on-one tutoring in many areas (including Mathematics). An appointment is necessary.

COLLABORATION AND ACADEMIC INTEGRITY: I assume that you are here to learn. If you talk to each other, you will learn from each other, perhaps more than you will learn from me. I encourage you to form study groups. Try the homework yourself, and then get together with a study group to go over questions, and to study for tests. You will learn a great deal from articulating your questions and explaining material to your peers. Discussion of assigned homework is encouraged, but you should be sure you fully understand the material by writing your solutions on your own. Evidence of any cheating or collaboration on work assigned to be completed individually will result in a 0 for that work, at minimum.

HONOR CODE: All members of the University of Northern Colorado community are entrusted with the responsibility to uphold and promote five fundamental values: Honesty, Trust, Respect, Fairness, and Responsibility. These core elements foster an atmosphere, inside and outside of the classroom, which serves as a foundation and guides the UNC community's academic, professional, and personal growth. Endorsement of these core elements by students, faculty, staff, administration, and trustees strengthens the integrity and value of our academic climate. UNC's policies and recommendations for academic misconduct will be followed. For additional information, please see the Dean of Student's website, Student Handbook link **http://www.unco.edu/dos/handbook/index.html**

PORTABLE ELECTRONIC DEVICES: Please extend courtesy to your instructor and fellow students by turning off your portable electronic devices, and putting them away in your bag, during class. If you know that you may need to accept an emergency phone call during class or if you have children in childcare or school, please let the instructor know. If you need to take a phone call during class, please step out of the classroom while you complete your call.

STUDENTS WITH DISABILITIES: Students who require special accommodations due to a disability should contact Disabilities Support Services (351-2289) as soon as possible to better ensure that accommodations are implemented in a timely fashion.

GRADING:

Written homework sets	10 %
Formative Assessments	5%
Online WeBWorK assignments	15 %
Chapter 1 Exam	10 %
Chapter 2 Exam	15 %
Chapter 4 Exam	15 %
Chapter 5 Exam	15 %
Final Exam	15 %

An overall score of 93% or above will receive at least an A, 90% or above at least an A-, 87% or above at least a B+, 83% or above at least a B, 80% or above at least a B-, and so on.

M	T	W	F
Welcome, 1.1 – Rate	Bottle Activity	**WeBWorK Introduction** **Diagnostic Quiz**	**WeBWorK 1 (1.1 & 1.2)** Bottle Activity Review 1.2 – Exponentials & Rate Petrie Dish Activity
Bottle Activity HW Locate the Hole Activity	1.8 – Limits	**WeBWorK 4 (1.7* & 1.8)** Review 1.1, 1.2, 1.8, & Activities	**WeBWorK Review 1** Exam 1
Labor Day	2.1 – Measuring Speed (Bolt) At this Rate Activity	**WeBWorK 2 (1.3* & 1.4*)** **WeBWorK 3 (1.5* & 1.6*)** At this Rate Review	**WeBWorK 5 (2.1 & 2.2)** 2.3 – Derivative Function 2.4 – Interpretations of Deriv
Locate the Hole HW 2.5 – Second Derivative 2.6 – Differentiability	(Finish Bolt Discussion) At this Rate Activity	**WeBWorK 6 (2.3 & 2.4)** Derivative Review At this Rate Review	**WeBWorK 7 (2.5 & 2.6)** 3.1 – Powers & Polynomials 3.2 – Exponential
At this Rate HW 3.3 – Product & Quotient	At this Rate Presentations	**WeBWorK 8 (3.1 & 3.2)**	**WeBWorK Review 2** Exam 2
3.5 – Trig Functions	3.9 – Linear Approximation Activity	**WeBWorK 9 (3.3 & 3.4)** **WeBWorK 10 (3.4 & 3.5)** Chapter 3 Review	**WeBWorK 11 (3.6 & 3.7*)** **WeBWorK 12 (3.7* & 3.9)** Start Derivative Mastery
Linear Approximation HW	Newton's Method Activity	**WeBWorK Review 3** 4.2 – Optimization 1	**WeBWorK 13 (4.1 & 4.3*)** 4.4 – Optimization 2
Newton's Method HW 4.5 – Marginality (or other additional linearization)	Optimization Activity	4.6 – Related Rates	**WeBWorK 14 (4.4 & 4.5*)** Derivative Mastery Deadline Withdrawal Deadline 4.6 – Related Rates
Optimization HW Chapter 4 Review	Related Rates Activity	Optimization & Related Rates Review	**WeBWorK 15 (4.6 & 4.7*)** Exam 4
5.1 – Measure distance traveled (toy car)	Definite Integral Activity	Distance: error, error bound	Modeling with the Definite Integral
Related Rates HW 5.2 – Definite Integral & Area, graphical interpretation	Definite Integral Activity	5.3 – FTC	**WeBWorK 16 (5.1 & 5.2)** 5.3 – FTC 5.4
Definite Integral HW	Definite Integral	6.1	**WeBWorK 17 (5.3 & 5.4)**

Ch. 5 Review	Presentations		Exam 5
6.2	DE Modeling Activity	6.3	**WeBWorK 18 (6.1 & 6.2)** 6.4
Integral Modeling HW 6.5	DE Modeling Activity	*Thanksgiving*	*Break*
DE Modeling HW Ch. 1 & 2 Review	Review Activity	**WeBWorK 19 (6.3*, 11.2* & 11.3*)** Ch. 3 & 4 Review	**WeBWorK 20 (6.4 & 6.5)** Ch. 5 & 6 Review

Dissertation

CALCULUS I
MATH 131, Spring 2012

INSTRUCTOR: ------
OFFICE: ROSS -----
PHONE: -----
OFFICE HOURS: -----, and by appointment
EMAIL: -----

CREDITS: 4 semester credits

PREREQUISITES: Strong algebra and trigonometry background, an understanding of basic functions (polynomial, exponential, logarithmic, etc.), and a willingness to work hard.

REQUIRED TEXT: Hughes-Hallett, Gleason, McCallum, et al., *Calculus, 5th ed*, Wiley, 2009.

TECHNOLOGY: You will need a graphing calculator for this course. I strongly recommend a TI-83, TI-83 Plus or TI-84. We will also use a computer algebra system called Mathematica® for lab activities. UNC has a site license for the software, and it is available in ROSS and UC labs. WeBWorK will be used for homework assignments, and course materials will be available on Blackboard (unco.blackboard.com).

COURSE DESCRIPTION: Inspired by problems in astronomy, Isaac Newton and Wilhelm Gottfried Leibnitz developed the ideas of calculus roughly 300 years ago. Since then, calculus has provided the foundation for advances in many other fields, even those which seem far removed from mathematics. You will find applications in chemistry, physics, economics, biology, medicine, business, psychology, and of course mathematics. Calculus is so important that it is often considered the gateway to many of the disciplines in which it is used.

The power of calculus lies in its power to reduce complicated problems to simple rules and procedures. While these procedures can be (and often are) taught with little regard to the underlying mathematical concepts or their practical uses, our emphasis will be on understanding all of these: concepts, procedures and uses. We will engage in the full mathematics process, which includes searching for patterns, order and reason; creating models of real world situations to clarify and predict better what happens around us; understanding and explaining ideas clearly; and applying the mathematics we know to solve unfamiliar problems. Participation in this variety of mathematical activities is challenging, and for many students, the experience will be vastly different from experiences in more traditional mathematics course.

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LABS: On Tuesdays we will work in small groups on labs that develop the central concepts in the course. Attendance and participation is especially crucial on these days. You will turn in individual write-ups of these class activities and make presentations of your work to the other groups in the class. It is also important to ask questions of the other groups (who will generally work on related but slightly different problems than your own group) when they present as you will be responsible for all the problems on exams.

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EXAMS: We will have four in-class exams (roughly covering Chapters 1, 2, and 4), and a comprehensive final exam. The final exam will be Monday, April 30th, from 4:15 to 6:45 pm. Make-up exams are possible only if there is a *documented* emergency.

GATEWAY TEST: There will be a WeBWorK-based test on differentiation after we have covered the short-cut rules for taking derivatives in Chapter 3. You will be able to take the test as many times as you like during the 2 week period that it is open. A passing grade is 6 out of 7, and each problem is graded as correct or incorrect (no partial credit). Your grade on the Gateway Test is not figured into your weighted average at the end of the course. However, failure to pass the Gateway Test during the window it is available will result in your final course grade being lowered by 2/3 of a letter grade.

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COLLABORATION AND ACADEMIC INTEGRITY: I assume that you are here to learn. If you talk to each other, you will learn from each other, perhaps more than you will learn from me. I encourage you to form study groups. Try the homework yourself, and then get together with a study group to go over questions, and to study for tests. You will learn a great deal from articulating your questions and explaining material to your peers. Discussion of assigned homework is encouraged, but you should be sure you fully understand the material by writing your solutions on your own. Evidence of any cheating or collaboration on work assigned to be completed individually will result in a 0 for that work, at minimum.

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PORTABLE ELECTRONIC DEVICES: Please extend courtesy to your instructor and fellow students by turning off your portable electronic devices, and putting them away in your bag, during class. If you know that you may need to accept an emergency phone call during class or if you have children in childcare or school, please let the instructor know. If you need to take a phone call during class, please step out of the classroom while you complete your call.

STUDENTS WITH DISABILITIES: Students who require special accommodations due to a disability should contact Disabilities Support Services (351-2289) as soon as possible to better ensure that accommodations are implemented in a timely fashion.

GRADING:

Written homework sets	25 %
Formative Assessments	5%
Online WeBWorK assignments	25 %
Chapter 1 Exam	10 %
Chapter 2 Exam	10 %
Chapter 4 Exam	10 %
Final Exam	15 %

An overall score of 93% or above will receive at least an A, 90% or above at least an A-, 87% or above at least a B+, 83% or above at least a B, 80% or above at least a B-, and so on.

Week	M	T	W	F
1 1/9 – 1/13	Welcome, 1.1 – Rate Meaning of constant rate and average rate in velocity-distance-time context (This sets up Lab 1)	(one question per group – focus on correct language and practice precision)	**WeBWorK Introduction** **Diagnostic Quiz**	Lab 1, Part 1 Due Bottle Activity Review
2 1/16 – 1/20	*MLK Day*		1.2 – Exponentials & Rate (Emphasize rate is proportional to amount, e.g., students fill out Lab 2 during interactive lecture)	**WeBWorK 1 (1.1 & 1.2)** 1.2 – Exponentials & Rate Petrie Dish Activity Review
3 1/23 – 1/27	Lab 1, Part 2 Due 1.4 – Logarithmic Functions Lab 3 Prep – Part 1	3. Locate the Hole Lab	1.5 – Trigonometric Functions	**WeBWorK 2 (1.3 & 1.4)** 1.6 – Powers, Polynomials, and Rational Functions
4 1/30 – 2/3	Lab 3, Part 1 Due 1.7 – Introduction to Continuity Lab 3 Prep – Part 2	3. Locate the Hole Lab	**WeBWorK 3 (1.5 & 1.6)** 1.8 – Limits Locate the Hole Lab Review	**WeBWorK 4 (1.7 & 1.8)** (Serves as example for Lab 4)
5 2/6 – 2/10	Lab 3, Part 2 Due Lab 4 Prep – Part 1	4. At this Rate Lab	**WeBWorK Review 1** 2.3 – Derivative Function 2.4 – Interpretations of Deriv	**WeBWorK 5 (2.1 & 2.2)** CHAPTER 1 EXAM
6 2/13 – 2/17	Lab 4, Number Check Due Lab 4 Prep – Part 2	4. At this Rate Lab	**WeBWorK 6 (2.3 & 2.4)** 2.5 – Second Derivative 2.6 – Differentiability	**WeBWorK 7 (2.5 & 2.6)** 3.1 – Powers & Polynomials 3.2 – Exponential
7 2/20 – 2/24	Lab 4, Part 1 Due 3.3 – Product & Quotient	4. At this Rate Lab	**WeBWorK 8 (3.1 & 3.2)**	**WeBWorK Review 2** 3.5 – Trig Functions
8 2/27 – 3/2	Lab 4, Part 2 Due Lab 5 Prep – Part 1	5. Linear Approxima tion Lab	**WeBWorK 9 (3.3 & 3.4)** CHAPTER 2 EXAM	**WeBWorK 10 (3.4 & 3.5)** START DERIVATIVE MASTERY 4.1 – Graphing
9 3/5 – 3/9	Lab 5, Part 1 Due Lab 5 Prep – Part 2	5. Linear Approxima tion Lab	**WeBWorK 11 (3.6 & 3.7*)** 4.2 – Optimization 1	**WeBWorK 12 (3.7* & 3.9)** 4.3 – Families of Functions
3/12 – 3/16				
10 3/19 – 3/23	Lab 5, Part 2 Due Lab 6 Prep – Part 1	6. Newton's Method Lab	**WeBWorK Review 3** 4.4 – Optimization 2	END DERIVATIVE MASTERY 4.5 – Marginality
11 3/26 – 3/30	Lab 6, Part 1 Due Lab 6 Prep – Part 2	6. Newton's Method Lab	**WeBWorK 13 (4.1 & 4.3)** 1.6 – Related Rates	**WeBWorK 14 (4.4 & 4.5)** (Serves as example for Lab 7)
12 4/2 – 4/6	Lab 6, Part 2 Due Lab 7 Prep – Part 1	7. Definite Integral Lab	**WeBWorK 15 (4.6 & 4.7*)** Chapter 4 Review	CHAPTER 4 EXAM

13 4/9 – 4/13	Lab 7, Number Check Du Lab 7 Prep – Part 2	7. Definite Integral Lab	5.3 – FTC	**WeBWorK 16 (5.1 & 5.2)** 5.4 – Theorems about Integrals
14 4/16 – 4/20	Lab 7, Part 1 Due 6.1 – Antiderivatives Graphically and Numerically Lab 7 Prep – Part 3	7. Definite Integral Lab	**WeBWorK 17 (5.3 & 5.4)** 6.2 – Antiderivatives Analytically	**WeBWorK 18 (6.1 & 6.2)** 6.3 – Differential Equations
15 4/23 – 4/27	Lab 7, Part 2 Due 6.4 – Second FTC	6.5 – Equations of Motion	**WeBWorK 19 (6.3*, 11.2* & 11.3*)** Review	**WeBWorK 20 (6.4 & 6.5)** Review

APPENDIX E

OBSERVATION PROTOCOLS

Group Work Observation Protocol

Date: ___________ Page: __________

Time	Code (OTB, FA, C, PP, CP, MSC)	Group	Description	Classification
				Present/Absent Appropriate/Inappropriate
				Present/Absent Appropriate/Inappropriate
				Present/Absent Appropriate/Inappropriate
				Present/Absent Appropriate/Inappropriate
				Present/Absent Appropriate/Inappropriate
				Present/Absent Appropriate/Inappropriate
				Present/Absent Appropriate/Inappropriate
				Present/Absent Appropriate/Inappropriate

Class Intervention Observation Protocol

Date: ___________

Time	Code	Group	Notes
Initial Reaction			
Time	Code	Group	Notes
Initial Reaction			

APPENDIX F

INTERVIEW PROTOCOLS

Interview Invitation Letter

Dear <Name>,

I would like to invite you to participate in a short interview as part of my dissertation research.

We are interested in hearing about your experience to help me identify ways to improve the introductory calculus course. This research is about the assignments you complete in the course, and how, if at all, they help you. You are one of a group of people I would like to talk to, and the information you share with us will help me take action to improve this course for future calculus students. I expect the interview to last about twenty minutes.

I would especially like to talk with you about this course because [1) agreed to participate in interviews in your initial consent letter and 2) something personal I pull from fieldnotes].

I am interested in your experiences in this course, both individually and in your group. Any information you provide to me will be kept confidential; your instructor will not be informed of anything you choose to share.

I have the following times available next week to schedule your interview:

[Insert times here]

I have included an interview information sheet in this envelope. If you are willing to be interviewed, please fill out this sheet and bring it you your interview. This will give us a place to start talking.

Let’s talk soon about what interview time will work best for you.

Yours sincerely,

Becky-Anne Dibbs
Doctoral Candidate
283ebecca.dibbs@unco.edu

First Interview Protocol

Verbal Script*: Thank you for taking time out of your day to help me with my research. I expect this interview to last about thirty minutes. From the consent form you signed to participate in this study, you said it was ok for me to record this interview. I want to remind you that no one except me will listen to your recording, and I will always use your pseudonym if I quote anything from this interview. Your instructor will not be informed of anything you say during this interview. Knowing that, are you ok with me recording this interview? Thank you. If at any time you become uncomfortable and want me to stop recording let me know*.

I'd like to start with a few short questions to get to know you a little better:

1) What is your major?
2) What year of college are you in?
3) Why are you taking calculus?
4) Have you had calculus before?
    a. If so when?
    b. Was it at UNC or somewhere else?
5) Is there anything else about you that you would like share with me?

Thanks. Now my research is about how calculus is taught, so we are going to talk today about some of the assignments you have done so far in the semester. First, I'm going to talk about the formative assessments you have been doing after the group work.

6) What do you think about these assignments?
7) Do these formative assignments help you?
8) What do you think about the classes when there is not a group work activity?
9) Now, I want you to think about the class the day after the group activity. Is this class better than the other two classes, worse than the other classes, or about the same?
    a. Why or why not?
10) After you work on the formative assessment, how - if at all - does working on the formative assessment change
    a. How you approach class the next day? (Paying more attention etc.)
    b. How you write up group work?
    c. How you answered the similar question on the test?

I have copies of some of your formative assessments, At this Rate write up, and your test here. I'd like to ask you a few questions about some of the answers you put down. Now, just because I'm asking a question about a particular problem doesn't mean that your answer is wrong; what I am interested in knowing is how you thought about the question and why you put that answer down.

Note: I will mostly ask students about the parts of the formative assessments, homework, and exams that I coded as idiosyncratic thinking.

11) (Content question FA 3) Can you walk me through how you went about answering this?
12) (Transfer question FA 3) Knowing what you know now, is there anything you would change about this answer?
13) How do you decide if you understand something?
14) Do you feel like the questions you asked got answered in the next class?
    a. Why or why not?
15) - 18) Re-ask questions 11-14 as necessary for the other formative assessment
19) (Test Question) Can you walk me through about how you went about answering this?
    a. How, if at all, did you use the formative assessment to prepare for the test?
20) How could these formative assessments be changed to make them more helpful for future calculus students?
21) What could we change to make the lectures better for future calculus students?

Now I'd like to switch gears and ask you a few questions about some of the other parts of the class.

22) Can you explain to me how I could approximate what the derivative of a function is at a point?
    a. In the other contexts?
23) How could I get a bound on my error?
24) How could I get a better error bound?

25) What else should I have asked you about
    a. The formative assessments?
    b. The group activities?
    c. The test?
26) What do you think you will remember the most about this class so far?
27) Is there anything else you would like to tell me?

Second Interview Protocol

Introduction*: Thank you for taking time out of your day to help me with my research. I expect this interview to last about thirty minutes. From the consent form you signed to participate in this study, you said it was ok for me to record this interview. I want to remind you that no one except me will listen to your recording, and I will always use your pseudonym if I quote anything from this interview. Your instructor will not be informed of anything you say during this interview. Knowing that, are you ok with me recording this interview? Thank you. If at any time you become uncomfortable and want me to stop recording let me know*.

1) How have you been since the last time we talked?

Thanks. Now my research is about how calculus is taught, so we are going to talk today about some of the assignments you have done so far in the semester. First, I'm going to talk about the formative assessments you have been doing after the group work. I asked these questions last time too, so I want you to think about knowing what you know now, if any of your answers are different now.

2) What do you think about these assignments?
3) Do these formative assignments help you?
4) What do you think about the classes when there is not a group work activity?
5) Now, I want you to think about the class the day after the group activity. Is this class better than the other two classes, worse than the other classes, or about the same?
    a. Why or why not?
6) After you work on the formative assessment, how - if at all - does working on the formative assessment change
    a. How you approach class the next day? (Paying more attention etc.)
    b. How you write up group work?
    c. How you answered the similar question on the test?

I have copies of some of your formative assessments, At this Rate write up, and your test here. I'd like to ask you a few questions about some of the answers you put down. Now, just because I'm asking a question about a particular problem doesn't mean that your answer is wrong; what I am interested in knowing is how you thought about the question and why you put that answer down.

Note: I will mostly ask students about the parts of the formative assessments, homework, and exams that I coded as idiosyncratic thinking.

7) (Content question FA #) Can you walk me through how you went about answering this?
8) (Transfer question FA #) Knowing what you know now, is there anything you would change about this answer?
9) How do you decide if you understand something?

10) Do you feel like the questions you asked got answered in the next class?

    a. Why or why not?

11) - 18) Re-ask questions 11-14 as necessary for the other formative assessment

1. (Test Question) Can you walk me through about how you went about answering this?
    a. How, if at all, did you use the formative assessment to prepare for the test?
2. How could these formative assessments be changed to make them more helpful for future calculus students?
3. What could we change to make the lectures better for future calculus students?

Now I'd like to switch gears and ask you a few questions about some of the other parts of the class.

4. Can you explain to me how I could approximate what the definite integral of a function is on an interval?
    b. In the other contexts?
5. How could I get a bound on my error?
6. How could I get a better error bound?
7. How is approximating a function like this similar to earlier in the semester when we were working on differentiation?
    c. How is it different?
8. What else should I have asked you about
    d. The formative assessments?
    e. The group activities?
    f. The test?
9. What do you think you will remember the most about this class so far?
10. Is there anything else you would like to tell me?

APPENDIX G

INSTITUTIONAL REVIEW BOARD APPROVALS

UNIVERSITY of
NORTHERN COLORADO
Institutional Review Board (IRB)

August 31, 2011

TO: Megan Babkes Stellino
School of Sport and Exercise Science

FROM: The Office of Sponsored Programs

RE: Exempt Review of *Impact of Formative Assessment on Learning Trajectories and Transfer Facilitation in Introductory Calculus,* submitted by Rebecca-Ann Dibbs (Research Advisor: Michael Oehrtman)

The above proposal is being submitted to you for exemption review. When approved, return the proposal to Sherry May in the Office of Sponsored Programs.

I recommend approval.

Megan B Stellino 9/13/11
Signature of Co-Chair Date

The above referenced prospectus has been reviewed for compliance with HHS guidelines for ethical principles in human subjects research. The decision of the Institutional Review Board is that the project is exempt from further review.

IT IS THE ADVISOR'S RESPONSIBILITY TO NOTIFY THE STUDENT OF THIS STATUS.

Comments:
- email 9/12
- revise consent, clarify participant procedure
- questionnaire?

25 Kepner Hall ~ Campus Box #143
Greeley, Colorado 80639
Ph: 970.351.1907 ~ Fax: 970.351.1934

UNIVERSITY of NORTHERN COLORADO
Institutional Review Board (IRB)

December 11, 2011

TO: Megan Babkes Stellino
School of Sport and Exercise Science

FROM: The Office of Sponsored Programs

RE: Exempt Review of *The Effects of Formative Assessment on Students' Zone of Proximal Development in Introductory Calculus* submitted by Rebecca-Anne Dibbs (Research Advisor: Michael Oehrtman)

The above proposal is being submitted to you for exemption review. When approved, return the proposal to Sherry May in the Office of Sponsored Programs.

I recommend approval. ✓

pending permission submission

Megan B Stellino 1/3/12
Signature of Co-Chair Date

The above referenced prospectus has been reviewed for compliance with HHS guidelines for ethical principles in human subjects research. The decision of the Institutional Review Board is that the project is exempt from further review.

IT IS THE ADVISOR'S RESPONSIBILITY TO NOTIFY THE STUDENT OF THIS STATUS.

Comments:
- email 12/28
- instructor permission? or researcher = instructor? - will submit when obtained
- small consent typo ✓
- no consent copies necessary ✓

25 Kepner Hall ~ Campus Box #143
Greeley, Colorado 80639
Ph: 970.351.1907 ~ Fax: 970.351.1934

APPENDIX H

CONSENT FORM FOR HUMAN PARTICIPANTS IN RESEARCH

UNIVERSITY of NORTHERN COLORADO

CONSENT FORM FOR HUMAN PARTICIPANTS IN RESEARCH
UNIVERSITY OF NORTHERN COLORADO
Project Title: The effects of formative assessment on students' zone of proximal development in introductory calculus

Researcher: Rebecca-Anne Dibbs, School of Mathematical Sciences, 970-351-2229
Research Supervisor: Dr. Michael Oehrtman 970-351-2344 michael.oehrtman@unco.edu

My research will help to determine in which ways the formative assessments you complete in class help students learn calculus more effectively. This will allow me to make suggestions for improvement to future calculus courses, if necessary.

If you choose to participate in this study, I will keep a copy of all of your formative assessments and tests. I will also observe your class on Tuesdays and Wednesdays. I will also interview a few people to understand how students think about formative assessments. If you agree to participate, please fill out the back of this form, including a signature and date. Check the box if you agree to be contacted for interviews. If you are selected to be interviewed, you will be interviewed twice this semester; I expect the interviews to last about thirty minutes each time.

To help maintain confidentiality, I will give each participant a pseudonym. You may give yourself a pseudonym if you wish; otherwise I will choose one for you. All data collected from you, including copies of exams and interview transcripts, will be identified with this name. The key with the participants' names and identifiers will be available to me alone, and I will discard upon completion of this study. If you do choose to participate in this study, you will not be identifiable in the final report.

I foresee no risks to you beyond those that are normally encountered in a classroom setting. Participation is voluntary. You may decide not to participate in the study and if you begin participating, you may still decide to stop and withdraw at any time. Nonparticipation or withdrawal from the study will not affect your grade in Math 131. Your instructor will not know who in the class is participating. Your decision will be respected and will not result in loss of benefits to which you are otherwise entitled.

Please feel free to contact me if you have any questions or concerns about this research. I appreciate your willingness to help me with my research.

 Initials: _____________

I _____________________ (Print Name), having read the front page of this letter and having an opportunity to ask any questions, would like to participate in this research and my signature below indicates my informed consent to participate. A copy of this form will be given to you to retain for future reference. If you have any questions or concerns about your selection or treatment as a participant, please contact the Office of Sponsored Programs, Kepner Hall, University of Northern Colorado Greeley, Co 80639; 970-351-2161

I agree to be contacted for an interview: YES NO (Circle one)

If you circled YES above, please give the email address you prefer to be contacted at:

____
Participant email

_______________________________ ______________
Participant Signature Date (month/day/year)

_______________________________ ______________
Participant's Name (please print) Pseudonym

_______________________________ ______________
Researcher's Signature Date(month/day/year)

_______________________________ ______________
Researcher Supervisor Signature Date(month/day/year)