Journal of Educational Research and Innovation

Volume 9
Number 1 *Revisiting Contexts and Conditions for Teaching and Learning*

Article 4

2021

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Journal of Educational Research and Innovation
2021, Vol. 9, No. 1

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Introduction

Though definitions are considered to be fundamental in mathematics, a mathematical concept is defined differently based on the logical relationship between different mathematical statements related to the concept (Winicki-Landman & Leikin, 2000). There is disagreement in the field of mathematics education regarding whether a definition should be as minimal as possible with some scholars insisting on a full reduction of extraneous properties and others honoring the role of context, allowing for more redundancy (Zaslavsky & Shir, 2005). Which properties and how many to include in a definition is somewhat arbitrary, and the value of a definition depends on the perspective of its author. That many different definitions can be written for the same concept is difficult for preservice teachers to understand (Linchevski et al., 1992).

Few studies investigate student conceptions of a mathematical definition (e.g., Zaslavsky & Shir, 2005). Using written responses and recordings of small group discussions, Zaslavsky and Shir (2005) investigated four students' conceptions of definitions for square and isosceles triangle (among other non-geometry concepts), and how these conceptions were reflected in and developed through activities that elicited consideration of alternative ways to define a mathematical concept. They found that asking students to consider a variety of definitions is a powerful learning environment wherein concept definitions could be gradually refined along with conceptions of definition in general.

In this study, we sought to better understand preservice elementary mathematics teachers' beliefs about the process of writing mathematical definitions and the definitions themselves. We refer to the process of writing a definition as the act of defining (de Villiers, 1998; Kobiela & Lehrer, 2015; Zandieh & Rasmussen, 2010). We wondered how beliefs about definition and mathematics itself might be exposed by the experience of authoring a mathematical definition for consideration, validation, and use by others. We specifically asked, what beliefs about mathematical definitions and the act of defining are exposed when reflecting on a classroom episode focusing on writing definitions for special quadrilaterals?

Related Research

We draw on two areas of related research. First, we will establish a perspective in the literature that parses mathematical definitions from the act of defining. Second, we will establish what it means for individuals to participate in the act of defining in the mathematics classroom.

Definition versus Defining

Tall and Vinner's (1981) terms, concept definition and concept image, are often used to frame the nature of mathematical definitions. Tall and Vinner (1981) describe a concept image as "the total cognitive structure that is associated with the concept which includes all the mental pictures and associated properties and processes" (p. 152). These authors distinguish this from a concept definition or "a form of words used to specify that concept" (Tall & Vinner 1981, p. 152). Rather than funneling individuals toward a single and specific verbalized definition, the authors indicate that individuals may hold concept definitions as independent and different from a formal concept definition accepted by the mathematical community. This puts emphasis on constructed learning and enables the perspective that mathematical definitions can be simultaneously individual and socially constructed. This is not a typical perspective; mathematics students rarely experience the autonomy of writing a definition and usually encounter just one external definition, generally attributed to a textbook (Zaslavsky & Shir, 2005).

In contrast to Zaslavsky and Shir (2005), many in the mathematics education community discourage introducing students to finished products and support the act of defining as a mathematical process (de Villiers, 1998; Kobiela & Lehrer, 2015; Zandieh & Rasmussen, 2010).

> The construction of definitions (defining) is a mathematical activity of no less importance than other processes such as solving problems, making conjectures, generalizing, specializing, proving, etc., and it is therefore strange that it has been neglected in most mathematics teaching. (de Villiers, 1998, p. 294)

Extending this beyond the mathematical benefits and illustrating the pedagogical power, Jansen (2020) wonders how to foster a mathematics classroom culture where

> … participating during mathematics class is an opportunity to continue learning, not an obligation to perform what we already know [e.g., provided definitions]. As we communicate, our ideas are used to reflect, to hear ourselves think, to get feedback from others, and to make sense of our ideas through reflecting on what we heard ourselves say or write. We share our thinking and grow our ideas through communicating. (p. 2)

This sentiment echoes other recommendations to emphasize the act of defining over the learning of definitions. De Villiers (1998) recommends that students formulate their own definitions, and then collaboratively discuss and compare them in order to help students see the benefits of different defining systems for equivalent meanings. For example, defining quadrilaterals by properties of their diagonals rather than focusing on their sides and angles can be an enlightening discussion that highlights unfamiliar properties.

There are also advantages of using inclusive definitions (e.g., squares are also rectangles) over those that partition shapes (each shape has at most one category) (de Villiers, 1998). Though geometers, mathematicians and authors of college-level texts prefer the use of inclusive definitions (Usiskin, 2008) because of the mathematical advantages they provide (i.e., simplification of the wording of theorems), discussions that result in classrooms from "act of defining activities" can also be thought provoking and challenging when inclusive definitions are the end goal. Keiser (2004) reports great value in, at the very least, delaying the presentation of any formal definitions in favor of spending time with informal exploration and description similar to recommendations by Battista (2008).

Learning to Define

Povey and Burton (1999) challenge the idea that authorship is vested in mathematicians and the texts in which mathematics is conveyed. They posit that this emphasizes a cultural transmission view of learning as opposed to one of interpretation and meaning making. However, what does it mean for learners to become authors of mathematics? When it comes to the act of defining, sociomathematical norms of undergraduate students are not the same as mathematicians (cf. Sánchez & García, 2014; Fernández-León et al., 2021), and there can be a conflict about issues like whether to select the most complete definition (i.e., most descriptive) or to remain minimal. Fernández-León et al. (2021) found that even adult learners are not sure how to apply consistent criteria when authoring "good" definitions, and can conflate description with the act of defining.

When we take a view of mathematics as a humanistic discipline where mathematics is socially constructed and personal values influence our evaluation of results, then it is important for instruction to be participatory. Here, definitions are not just mathematical tools to be internalized, but teaching tools that help convey perceived meaning to others. "By constructing and negotiating their own definitions, students can acquire more robust understandings of specific mathematics concepts" (Harel et al., 2006, p. 151). In order for that to happen students should have agency and voice in the classroom (White, 1993). If we understand meaning as negotiated, then authority belongs to the knower, even as external sources are considered and evaluated critically (Langer-Osuna, 2017).

For the teacher, recognizing that the choice of which definition to write or use in mathematics classrooms is also based on the pedagogical context. This might include curricular approaches, learning trajectories, the students in the classroom, and a desire for clarity or elegance (Winicki-Landman & Leikin, 2000). In order to develop a classroom culture where this is possible, we posit that teachers need to be aware of and able to comprehend the perspectives and mathematical thinking of their students.

Araki (2015) refers to this ability as mathematical empathy. Building on the notion of empathy as seeking to understand another through their frame of reference, Araki defines Mathematical Empathy as "the ability to comprehend another person's ideas and the true meaning or purpose behind them, seeking to utilize the other person's frame of reference" (p. 118). If teachers are intended to elevate student thinking within mathematics instruction, then mathematical empathy is required. Mathematical empathy is what allows us to play the believing game (Harkness, 2009) and find the truth in the mathematics that students share.

It is important to this study that we are viewing the work of the participants as grounded in these ideas. Our classroom episode was intended to provide opportunities for students to author definitions with agency and voice. We continue to center their voices in our reflection on that episode and the potential of this type of curricular experience. By choosing to focus on participant reflections, we seek not to establish a measure of effectiveness in learning about quadrilaterals or even construct a measure of ability to write a high-quality definition. Rather, to understand their beliefs about mathematical authorship and beliefs about mathematics and teaching through the lens of that experience.

Methodology

This study examines preservice elementary mathematics teachers' (PSETs) beliefs about mathematical definitions and the process of writing them. Participants were recruited from two sections of a course on geometry for elementary (PK-3) teachers with a total of 71 preservice elementary teachers. Samples of reflective writing were analyzed with grounded theory (Vollstedt & Rezat, 2019) to build and then apply a framework by which to give nuance to what we know about what PSETs believe about the purpose, nature, and origin of mathematical definitions. In this section, we will first give an overview of the defining activities on which the PSETs reflect followed by the methods of data collection and analysis. We will conclude with an overview of the framework that emerged through grounded theory.

Defining Activities

Defining in a Collaborative Space. In a face-to-face environment, we asked PSETs to explore dynamic quadrilaterals constructed with interactive geometry software (IGS). Dynamic quadrilaterals are on-screen manipulable shapes, where the geometric properties of the specific quadrilateral are maintained (e.g., congruent side lengths, opposite parallel sides). The PSETs engaged in activities using the dynamic quadrilaterals to promote their development of a concept image for specific types of quadrilaterals, which they could then use for defining each quadrilateral. The PSETs worked in small groups for the first class session, and then returned to the material as a whole class to collaboratively create a series of definitions for quadrilateral, kite, parallelogram, rectangle, rhombus, square, and trapezoid. Each of the two sessions lasted approximately 60 minutes.

IGS was selected as a foundational experience for three reasons. First, it is an opportunity to explore a digital world that goes beyond the tutorial and practice models that are so prevalent among online applications. Secondly, the IGS activities provided a common experience for all of the PSETs to draw upon. Lastly, the dynamic shapes were designed in such a way as to help students explore quadrilaterals both holistically and analytically. PSETs were able to generate many examples using the dynamic quadrilaterals. We, like de Villiers (1998), hypothesize that exposure to dynamic figures constructed using IGS may make it easier for PSETs to accept a more inclusive hierarchical classification of quadrilaterals.

Reading *The Role of Definition*. We wanted to understand the PSETs' experiences with the process of creating these definitions from a first-hand perspective, but also wanted to understand how this experience might influence the way they perceived the role of definition in the primary setting. In order to help them frame their comments as both learners of mathematics and future teachers of mathematics, we first asked them to individually read The Role of Definition (Keiser, 2000).

The article was chosen as a catalyst for reflection on this experience because it suggested that early presentation of formal definitions can curtail thinking in middle grades classrooms and argued for student-generated fluid definitions based on concept imagery (Tall & Vinner, 1981) relevant to classroom learning. Even though our course focuses on the mathematics of early elementary classrooms, we did not feel that the context of middle grades would interfere with our PSETs reading of the article. We felt that it might give their recent emotionally and intellectually challenging classroom activity some legitimacy as they empathized with the learners in the article. We also hoped it would help PSETs position themselves as future teachers of children when imagining the role of definition in mathematical learning.

Role as Researchers

The three researchers are coming from a non-positivist paradigm, specifically a constructivist paradigm. We believe that learning occurs as learners are actively involved in a process of meaning and knowledge construction. We also believe that social interaction plays a fundamental role in the process of cognitive development. The first author played a dual role in this research, as an instructor of the PSET courses and as a researcher, while the other authors were not involved in the instructional activity.

Data Collection

After they had completed the IGS in-class activity, the defining discussion and the assigned reading, we asked our PSETs to write a written reflection in an online setting. Specifically, we prompted, "After reading the article, The Role of Definition, what new thoughts do you have about the conversations we had in class about defining quadrilaterals? How about using definitions with children?" PSETs were aware that their reflections would eventually be read by both their classmates and their instructor and that their instructor was participating in the discussion. There was a grade associated with the assignment based solely on completion. PSETs could not read the reflections posted by their classmates until they had uploaded their own. Once they had responded to the prompt, they were given access to their classmates' reflections and were asked to participate in an online discussion of what had been shared. The data in this study comes from only the initial posted reflections (n=71).

We could certainly learn a great deal about the existing mathematical content knowledge of our teachers (Ball et al., 2008) if we analyzed the definitions that were written during the lesson. We could also have conducted a survey to compare our PSETs' beliefs about the features or roles of definitions with the results shared by Zaslavsky and Shir (2005). However, we chose to focus on our PSETs beliefs about definitions and the process of writing them. Thus, we chose to use reflective writing samples written in response to these defining activities.

Data Analysis Procedures

We collected data during a single semester and analyzed the data systematically using grounded theory approaches (Vollstedt & Rezat, 2019), analyzing the data for recurring themes. In the analytic process, we made initial codes from the existing data and then continually revisited and revised those codes in subsequent analyses. Data analysis involved a constant comparison of description of codes to accurately reflect the evidence leading to codes. The data were searched, looking for both confirming and disconfirming evidence that either supported or challenged a particular code description. When disconfirming evidence was found, the data were searched for additional instances and the results presented here include consideration of all such evidence.

To begin analysis, all three researchers read the 71 original reflection posts and wrote memos about emergent themes across the data (Multiplicity, Authorship, Authority, Audience, Empathetic Awareness and Empathetic Comprehension). Through iterative discussions by the researchers, we developed, applied and refined this framework.

Our second round of analysis was intended to refine those emergent themes. Descriptions of these themes can be found in the following section. In this round, 14 reflection posts were randomly selected from the entire set, and coded with the emergent framework by all three researchers. The researchers compared their coding results, and then refined the framework to establish a more coherent description for each of the six themes. No data on inter-rater reliability (IRR) was kept and the initial 14 posts were included in two subsequent rounds of coding.

Table 1

Inter-Rater Reliability for Each Coding Category in Round 3, Round 4, and Overall

Themes	Round 3	Round 4	Overall IRR
Multiplicity	0.69	0.83	0.76
Authorship	0.72	0.71	0.72
Authority	0.53	0.80	0.66
Audience	0.72	0.80	0.76
Empathetic Awareness	0.81	0.83	0.82
Empathetic Comprehension	0.92	1.00	0.96

Two additional rounds of analysis were conducted with documented IRR (Table 1). The data were split into two halves. The first half (n=36) was coded in the third round, which led to further refinement of the codes and discussion prior to the analysis of the second half of the data. In both cycles, each post was coded independently by two randomly assigned researchers. Researchers met after round 3 to discuss, compare, and come to a consensus for each reflection post after documenting their IRR on the initial themes; and found that a low initial agreement about coding Authority (.53) indicated a need to refine the description of the code for round 4.

In round 4 the remaining 35 reflection posts were coded. With the exception of the code Authorship (where there was only a small difference), all of the IRR scores increased from Round 3 to Round 4.

The overall IRR scores for the initial coding themes used in this paper were 0.76 for Multiplicity, 0.72 for Authorship, 0.66 for Authority, 0.76 for Audience, 0.82 for Empathetic Awareness and 0.96 for Empathetic Comprehension. The overall IRR for all posts was 0.77. After the IRR was recorded, pairs of coders discussed disagreements until a final consensus was reached. In a few instances, the opinion of the third researcher was used to help reach consensus.

As we shared the analysis with other colleagues in informal sessions, it was brought to our attention that some of the data indicated not only a lack of Multiplicity, but the presence of the opposite. On their advice, the research team revisited the entire corpus of data to look for evidence of what we referred to as Singularity.

Emergent Framework

Our resulting framework had seven different themes. In this section, we will define and illustrate each of the themes in the emergent framework. The first five (Multiplicity, Singularity, Authorship, Authority, and Audience) pertain to specific beliefs about mathematical definitions and the act of defining. We also noticed the presence (and absence) of mathematical empathy in the reflections. In our efforts to analyze our data for evidence that our students exhibited mathematical empathy, it became clear to us that there were (at least) two types of empathetic work: Empathetic Awareness and Empathetic Comprehension. These two additional themes pertain to the ways in which our participants perceived others throughout the curricular experience. All seven themes are defined in Table 2.

Multiplicity is defined as the belief that definitions are not rigid and that many alternative definitions can be written for a given concept. Recalling the conversations in our whole-class discussions, Tanya writes in her reflection,

> I think that having those conversations about the quadrilaterals as a class was beneficial because then each of us got to explore the abstract ideas about different quadrilaterals and how they relate to each other. I think that when using definitions with children, we should be careful about [having] one single concrete definition so we don't limit children's learning. I think that we can have a definition but we should let the children explore other definitions.

This belief stands in contrast to Singularity, though it is possible to believe both that there are many possible definitions, and also that there is "one best" definition. Only one PSET in our study indicated this dichotomy of beliefs, favoring "definitions in the simplest terms" that leave out excess information.

Table 2
Framework for Examining the Beliefs of PSETs about Mathematical Definition and the Act of Defining

Theme	Description
Multiplicity	the belief that definitions are not rigid, and that many alternative definitions exist for a given concept
Singularity	the belief that definitions are rigid, and that there is one correct definition that exists for a given concept
Authorship	indicates ownership of a definition
Authority	indicates the power to decide which language, style, and properties are useful to include in a definition
Audience	the belief that a definition is influenced by who we intend to read and use it
Empathetic Awareness	indicates that the speaker believes that there is Multiplicity in mathematical perspectives; awareness can emerge as a belief that others see things differently than we do or that students will have different mathematical backgrounds, experiences, or understandings that are worthy of attention and understanding.
Empathetic Comprehension	indicates that the speaker can comprehend from someone else's mathematical perspective.

Singularity is defined as the belief that definitions are rigid, and that there is one correct definition that exists for a given concept. While instruction can focus on generating student ideas about what mathematical objects are and are not, a belief in Singularity indicates that a PSET places importance on standardization and revision toward one "universal" or "textbook" definition. As Matthew states, "there should be a universal definition for students to learn so everyone can know the same definition for standardized tests."

Authorship indicates ownership of a definition. This indicates a stance that definitions are personal articulations based on concept imagery that we hold as individuals within our community. To illustrate, Leila saw purpose and value in being able to author definitions based on our classroom activity and believes Authorship is akin to sense-making activity:

> I realized that our exploration of defining quadrilaterals in class had a unique purpose that would benefit our thinking and comprehension of

> mathematical concepts. It makes more sense to first create a definition of a mathematical concept through manipulation and discussion before being told the actual definition. This allows for children to actually contemplate definitions and decide what makes sense and what doesn't.

Authority indicates the power to decide which language, style, and properties are useful to include in a definition. If we understand meaning as negotiated, then Authority belongs to the knower, even as external sources are considered and evaluated critically. In the quote above, we also see evidence that Leila wants to give students Authority to decide what makes sense and what does not. This is echoed in Hagan's response when he said, "All of the definitions we decided were what made sense to us (and they were correct, which is a big part of it as well)." As he continued, he expressed "play" as a form of Authority and Authorship. "We got to play around with our own wording and what we felt it [sic] was important to know about each different quadrilateral, which makes it much more personal to us and easier to understand."

Audience honors the pedagogical context and is the belief that a definition is influenced by who we intend to read and use it. Exposing her beliefs about Audience, Aisha writes, "I think that while it's important to be specific enough to be able to distinguish two different types of quadrilaterals, sometimes if definitions are too specific then they are confusing, especially for children." In this reflection, Aisha expresses an appreciation for the pedagogical context in which a definition is used. Here, definitions are not just mathematical tools to be internalized, but teaching tools that help convey meaning to others.

Empathetic Awareness lives in a general space where the speaker understands that there are multiple mathematical perspectives. (Not to be confused with Multiplicity in concept definitions from above.) To illustrate from our data, John showed Empathetic Awareness when he wrote,

> When teaching complex concepts such as angles and quadrilaterals, I would want to provide a similar approach in which students and the teacher interact with each other to gain a better understanding of our peers' perceptions and views of these concepts.

As in this example, awareness can emerge as a belief that others see things differently than we do or that students will have different mathematical backgrounds, experiences, or understandings that are worthy of attention and understanding.

*Empathetic Comprehens*ion is observable when the speaker expresses understanding of another's perspective. Without a specific statement of understanding from another's perspective, awareness is not evidence of Comprehension. This is akin to active listening, or what Hufferd-Ackles et al. (2004) refer to as revoicing an idea shared by another. Henry expressed mathematical empathy for a student, Dave, from the article (Keiser, 2000), and went so far to say that our discussion was lacking because this perspective was not available.

> One idea that really interested me was the part of emphasis on the vertex. The article says, 'Dave still struggled to distinguish between points and angels [sic].' … I

> remember learning about this and being confused about similar concepts. Sometimes there is overlap and it is difficult to see the difference. Our conversation on quadrilaterals lacked this conversation because we all understand the difference but it is important to keep in mind that children are seeing this for the first time.

Using the framework to code all 71 reflections, we found evidence of all seven themes. A summary of these results and the overall frequency can be found in Table 3. We were able to find evidence of at least one theme in all but one reflection post, and the median number of themes coded given to a reflection post was 2. The most prevalent themes were Authorship (57.7%), Authority (43.7%) and Multiplicity (39.4%), while Empathetic Comprehension only appeared in 3 reflections (4.2%).

Results

As we coded, we found evidence of multiple themes in 76% of reflection posts. The most prevalent themes were Authorship and Authority. However, when examining the posts in these categories, the themes seemed tightly braided and difficult to parse because they appeared in posts that were coded with multiple themes. Given that Singularity was defined as the absence of Multiplicity, it makes sense that they would be almost mutually exclusive themes. If combined, Singularity and Multiplicity are found in 45 posts or 63.4% of the data. When we analyzed the body of posts that were coded for Singularity or Multiplicity, the complexity of these beliefs became evident.

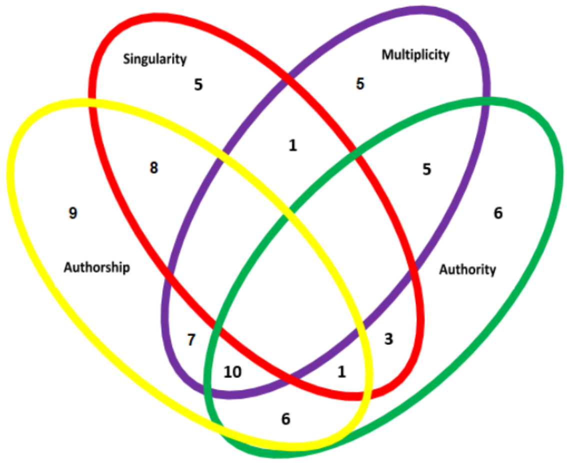
Table 3
Distribution of Coding Themes

Themes	Number of Posts Coded	Frequency (n=71)
Authorship	41	57.7%
Authority	31	43.7%
Multiplicity	28	39.4%
Audience	23	32.4%
Empathetic Awareness	23	32.4%
Singularity	18	25.3%
Empathetic Comprehension	3	4.2%

Some of the variations in the ways that Singularity and Multiplicity were expressed can be enhanced if we consider the intersection of these and other beliefs. In Figure 1, we have provided a Venn diagram visual representation representing the relationships between four main themes. Each reflection post is counted in the areas of the diagram according to the codes it received. To illustrate, there is just one post at the intersection of Singularity and Multiplicity, while there are ten posts that were coded for Multiplicity, Authority, and Authorship. No posts were coded for all four, which is why the center of the diagram is empty.

Figure 1

Intersections of beliefs about the definition and the act of defining in PSETs' reflection posts



We should note that Audience, though important in its role within the framework, is left out of this analysis as the signals we picked up about the relationship of Audience to the other beliefs were not as strong. It is possible that with a larger data set we could detect something more, but here we focus on strong relationships within the data we collected.

In this section, we will first explore Singularity and Multiplicity in more depth. Then, we will re-examine those reflection posts that exist at the intersections of these two themes with Authorship and Authority. We will conclude by examining the relationship of these beliefs with expressions of Mathematical Empathy.

Singularity

Singularity is defined as the belief that definitions are rigid, and that there is one correct definition that exists for a given concept. Of our 71 PSETs, 18 held this belief (25.3%). Framing her beliefs within the context of our class experience, Ke'yondrah wrote,

> Thinking about last week, we spent a whole class trying to define simple geometric shapes. However, in the article, they spent weeks trying to define the word angle. This made me think, at what point does this compromise student learning? If students spend this much time on one definition, is there material that has to be compromised at the end of the year? I understand the value of definition explorations, but at what point do we actually have to establish a definition?

Ke'yondrah portrays the work of defining as an inefficient instructional tool. She describes the work of her classmates and the children in the article as "trying to define" which varies greatly from what de Villiers (1998) refers to as constructive defining.

Matthew makes a clearer connection between defining and sense-making activity when he writes, "Teachers should combine the universal definition with lesson plans that leave creation of definitions up to students to enhance their knowledge and understanding of subject matter." Here, learning "the universal definition" seems to equate to an educational standard or lesson plan goal, while "defining" equates to the sense-making activities conducted to lead up to that goal.

The act of defining is an emotional pursuit and we found evidence of three strong pressures that seemed to influence beliefs. First, there is already a sense that instructional time is a resource best preserved. There were thinly veiled frustrations with defining activity, such as those shared by Ke'yondrah. Within her writing, she indicates an urgency that she felt that time spent developing a universal definition was wasted or unimportant. Five of 18 (28%) reflections explicitly mention limited classroom time as a pressure that PSETs feel when learning or imagining teaching. Second, testing is a force that was mentioned by about 4 out of 18 PSETs as a source of emotional and professional pressure that curtails their interest in the act of defining as an instructional activity. It is more important to them to attend explicitly to the definitions that will be tested. Third, when PSETs imagine themselves engaging students in defining activities, they are anxious about the knowledge and effort this teaching practice requires. As Sabah writes, "I think it is going to take a lot of self-control to not correct students right off the bat when they say something incorrect in discussion. I also think it will take a lot of effort on my part to effectively scaffold students so that they can figure out definitions on their own." Cindy echoes her worries about self-control, "I have to learn to fight my own urges to share the definition and encourage them to discover it for themselves."

Multiplicity

Multiplicity is defined as the belief that definitions are not rigid and that many alternative definitions exist for a given concept. Of our 71 PSETs, 28 held this belief. The belief in the Multiplicity of definitions is expressed in three different ways within our data: 1) differences in individual concept imagery, 2) refinement of concept images and definitions over time for an individual, and 3) contrasting locally constructed meaning with that presented by external authorities such as a textbook.

In her reflection, Shannon expressed a belief that each child has a unique personal concept definition, and that in a classroom context, there is value in negotiating with others about those definitions.

> After reading the article helped me understand how many different ways children think about the same thing. When the students have to come up with their own definition they have to decide what is important to them about the shape. They have to agree on what the key characteristics are that make that shape or angle what it is … When creating your own definition and comparing it with another you can find what works best for you.

Including Shannon, 12 PSETs expressed the importance of honoring personal concept images and allowing students to personalize these definitions.

Another way that a belief in Multiplicity was expressed was to call attention to the ways that definitions are refined by individuals over time. Henry and two other students expressed a belief that their own personal concept definitions are fluid and changing, being self-edited and clarified with each new experience. Henry states, “Each student should have their own definition which should be refined in order to move the definition from abstract to concrete in the student’s minds.”

The third way that Multiplicity was expressed in our data positions the definitions of a local community as separate from those from external authorities such as the textbook. In all, 8 PSETs expressed this. For example, Rafaela preferenced local definitions when she writes, “If students were given a definition that wouldn't be considered a textbook definition, I think it would be a lot easier for them to be able to understand.” Tanya and Nancy explicitly state that a single definition would limit student learning. Krupa seemed to agree with some frustration when she called out textbooks that portray a single definition as straightforward and rigid. She saw this as restricting instruction unnecessarily and described a freedom in being able to expand on the definitions in a local community.

> Something that really struck me was when Keiser discussed how definitions in the book are so straightforward and rigid while they need to be more loosely based so the students and teachers in each classroom can come up with what they think the definition truly is. I agree with this. Sometimes I feel that definitions in textbooks are too straight forward, and especially with definitions for shapes, there are so many different versions. I feel that if they (people who made definitions) agreed to make the definitions more loosely based, there wouldn't be as much controversy and teachers and students could expand on the definitions themselves.

In our study, we came across one response that indicated beliefs in both the Multiplicity and Singularity of definition. Sage writes, “Every student interprets things differently, therefore, different students may not thrive from the same definitions.” While this is indicative of Audience, it also indicates a belief in the Multiplicity of definition. However, she follows this statement with the following, indicating that these multiple definitions are just intellectually different versions of something more universal and singular, “We need to give definitions in the simplest terms in order for all of our students to truly understand and be able to apply them to their work. This includes using simple words and leaving out excess information.” Sage’s view of Multiplicity can be described as a series of singular definitions presented by an external authority who determines when students are ready. This is different from believing that an individual refines their concept definition over time because of the external Authority implied by her decision about which definition to expose.

Singularity and Authorship

Though it may seem incongruent, nine of the PSETs who maintained a belief in Singularity also believed in their role as an author of that definition. In that sense, though they were uncovering a known definition, they were taking an active role in that uncovering, and claiming ownership of that activity. There are two ways that PSETs maintained these disparate beliefs.

First, PSETs viewed Authorship as active learning. That appeared in a passive way as Katelyn writes, "I liked the activity because we were involved in forming the definitions, which kept us engaged and learning about the shapes." In other cases, PSETs like Gabriela stated it more directly while maintaining an external sense of a universal definition, "when you have to discover the information yourself rather than being told it that information sticks with you better because it's what you came up with and what you understand rather than just what you were told."

This sense of authoring the definition as active learning has a strong link with memorization for our PSETs. Authoring as a means to assist with memorization and understanding translated to their ideas about working with children. Sheila writes, "I believe that allowing children to explore possible definitions before just "handing" them one is a very powerful tool that can help expand their knowledge. Using definitions with children can be difficult and frustrating, however, allowing them to come up with their own version to define a term can make it easier for them to understand and memorize the term/definition." Of the nine students in this group, four mentioned better memorization as a motivation for the process of authoring definitions.

Second, authoring can be analogous to "coming up with" a definition from prior experience. Elan believed that asking children to author their own definitions would make the work of learning more difficult, citing a lack of prior experience with shapes and definition that was present for the adult learners. She noticed the children in the article struggling to describe angle in a formal way and suggested that this may be an unfair expectation. In this way, Authorship became less about ownership of the writing or meaning and more of an ability to pull a universal definition from personally-held memory and lived experience. Since the adult learners had previously encountered the material, they had more material from which to write the definitions.

Singularity and Authority

It is less likely that PSETs would maintain the beliefs of Singularity and Authority and of the 18 who maintained a belief in the Singularity of definitions, only four expressed simultaneous beliefs that they had the power and Authority to decide which language, style, and properties are useful to include in a definition. These beliefs were maintained by separating the process of defining from the sharing of a universal definition. Leila maintains the distinction between defining as a sense-making activity and learning a universal definition, but describes the process of defining as purposeful and important work where students have Authority.

> It makes more sense to first create a definition of a mathematical concept through manipulation and discussion before being told the actual definition. This allows for children to actually contemplate definitions and decide what makes sense and what doesn't. It also allows the teacher to understand where a child's thinking is coming from and whether the child grasps the concept.

Terry describes this phase of learning as "figuring out the "what's and what's not" of a certain term," claiming students' right to "reword and tweek [sic] for their personal understanding." In both of these cases, the PSET specified that this phase of learning only extended up to the point in which the universal definition was presented.

Multiplicity and Authorship

Recall that when combined with Singularity, Authorship took on two distinct meanings. Either PSETs equated it with active learning, or they viewed it as "coming up with" the correct universal definition. In both of these senses, Authorship conveyed less about ownership than it did the act of writing down a definition that was gradually funneled toward a predetermined ideal.

However, when combined with Multiplicity, Authorship does take on a sense of ownership. Hagan expresses a shift from one perspective to the other,

> In class when we were trying to define all of the different quadrilaterals, I felt really silly. I felt like it was an activity that didn't really need to be done and [the instructor] should have just given us the definitions to memorize like any other college level course. After reading this article I have a completely different opinion on the activity we did. All of the definitions we decided were what made sense to us (and they were correct which is a big part of it as well).

In this way, Authorship combined with Multiplicity becomes more about personalized definitions that grow out of negotiation and sense making and that are allowed to coexist in a non-hierarchical way.

Of the 28 PSETs whose reflections were coded for Multiplicity, 17 (61%) also made positive statements about Authorship. For some, Authorship was still equated with active learning. As Niles writes, "When students read a definition from a textbook, they tend to only go by that definition. An activity that involves the students to work and figure out a definition will really make them think. This also allows students to help others look at something in a new way."

The difference is in the lack of emphasis on standardization and the requirement that teachers "keep an open mind when it comes to students' various definitions" (Anya). While some still explicitly mentioned a universal or textbook definition that students and teachers could reference, what is common amongst these responses is the belief that student-authored definitions can coexist while differing in significant ways. In fact, the diversity is seen as adding value to a lesson. For example, Pat writes,

> Teachers can allow students to construct and manipulate their own mathematical vocabulary. They can do this as a class, by doing many activities that help the students collaborate and understand specific terms. … Students will make more meaningful connections by constructing their own definitions, than they would if a teacher just gave them a list of all the definitions.

Multiplicity and Authority

Of the 28 PSETs who believed in Multiplicity, 15 (54%) also believed in the Authority of learners. This is a much bigger proportion than those who believed in Singularity and Authority. When combined with Singularity, Authority was limited to that time of exploration and led up to the introduction of the universal definition. That focus on the process where teachers and students collaboratively define a term is present here, too. Hagan described this process as getting "to play around with our own wording and what we felt it [sic] was important to know about each different quadrilateral, which makes it much more personal to us and easier to understand." The difference here is that when one believes in Multiplicity, the constructed definition need not be compared to an external authoritative one. The Authority lives not only in the individually constructed definitions, but in those that are socially constructed in the classroom. As Riley advocates, "the teacher should bring the students together and work to create a classroom definition that pulls from each student's thoughts." Students and teachers with Authority get to decide what makes sense and what does not. As Hagan writes, "All of the definitions we decided were what made sense to us (and they were correct which is a big part of it as well)." Advocating for this collaborative process over providing a standardized definition for memorization, many expressed that this honored and valued critical thinking. Here, Authority was felt because the decisions made by individuals could exist simultaneously as "correct."

To conclude, we found that a belief in the Authority of students or their roles as Authors of mathematics took on different meanings depending on whether that belief was combined with a belief in Multiplicity or Singularity. It is important to note that we do not believe that these are rigid beliefs, nor do we believe they are hierarchical in nature. We believe that the proximity of classroom activity where these PSETs were positioned as authors and authorities was important and points to the potential impact of that positioning. The activities we shared and facilitated were certainly helpful in drawing out these beliefs and may give us the foundational experiences to generate new or stronger beliefs.

Mathematical Empathy

Looking now through the lens of mathematical empathy, there were only three reflection posts that showed evidence of Empathetic Comprehension, so patterns within the data were not apparent. However, 23 reflection posts showed evidence of Empathetic Awareness. Coincidentally, this is the same number of posts that were coded for Audience, however these two groups of posts show only slight overlap (n=7). Audience and Empathetic Awareness certainly exist as separate and distinct themes. Posts within the theme of Audience focus more on ways in which PSETs envisioned altering definitions for different populations, whether for different age ranges of students or, like Andre, children and adults. "It is easy for us as educators and older people to understand terms and we have to understand that kids or our students aren't necessarily going to see the definition in the same way."

This is very different from Empathy, even at the level of Awareness. Posts within the theme of Empathetic Awareness focused on the presence of multiple perspectives within one classroom, community, or even one individual. PSETs reflected on the benefits of listening to the ideas of others or setting aside your own ideas to do so. Niles wrote, "Hearing what someone else had to say allowed others to see it differently."

Empathetic Awareness is defined as the belief that there is a variety in mathematical perspectives or that students will have different mathematical backgrounds, experiences, or understandings that are worthy of attention and understanding. We used John's words to illustrate Empathic Awareness in our theoretical framework. However, there is more nuance that is apparent when more data are considered. There are two main ways that PSETs felt that the diverse perspectives were worthy of attention.

First, as with John (and later Niles) above, there was appreciation for the ways that exposure to different ideas deepened understanding. Jakkar concurred, "It was interesting to see how much your understanding of these shapes in general changes and develops through discussion instead of being given a definition and saying, 'this is what it is and nothing else.'" Others associated that deeper understanding with engagement and participation. They saw the value in having multiple students get opportunities to share their thinking throughout the lesson. Those expressing this viewpoint showed that they valued the impact of other viewpoints on their thinking. Hearing the ideas of others also had an impact on what individuals understood about the content. Maddy writes, "We all thought we knew exactly what certain things were until we started to hear other points of view about them."

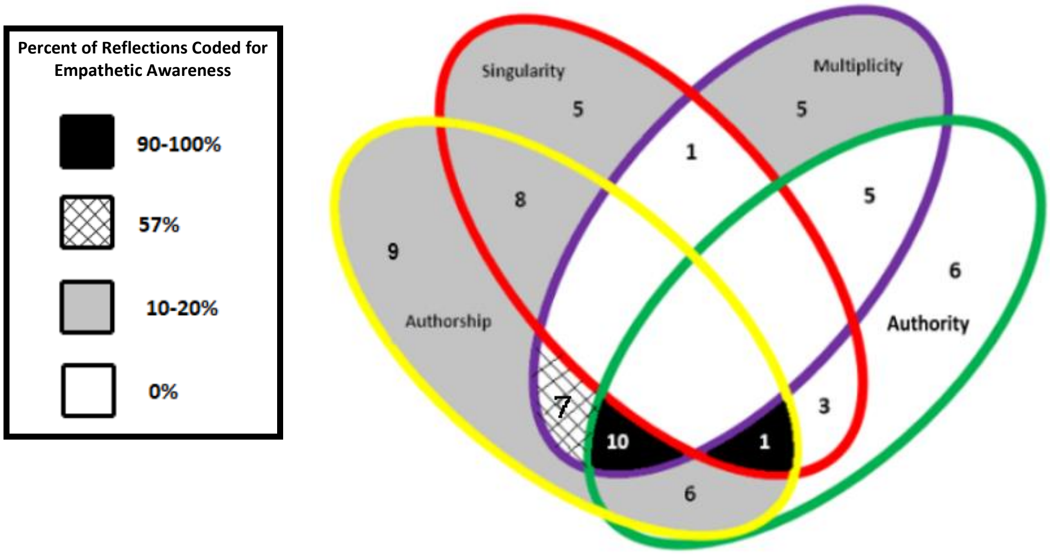
Second, PSETs who expressed Empathetic Awareness associated the exposure of individual concept imagery and definitions with formative assessment. This differed from the first viewpoint in that it focused on better understanding the mathematics of others, not just mathematics itself. As Maddy continues, "I think these conversations on definition are important to have, especially with a class of children because if as a teacher you just give a student a simple definition, you could definitely miss some parts of the student's thought process, perhaps missing vital information about how they think. Information that would make understanding why a student was struggling more apparent." It appears to us that those who are able to experience mathematical empathy might be better positioned to the type of teaching that achieves the vision of Jansen (2020) and others in the literature.

Intersections with Empathetic Awareness

We also wanted to better understand the relationship of Empathetic Awareness to the other beliefs. In order to do so, we examined where posts that were coded for Empathetic Awareness appeared in Figure 1. Figure 2 uses shading as a means to indicate the density of these reflections posts within the overall structure. For example, only 10-20% of the 5 posts in the section that is exclusively Singularity were coded for Empathetic Awareness, in contrast with 57% of the 7 posts at the intersection of Multiplicity and Authorship.

A closer look at Figure 2 indicates there are certain beliefs about the definition and the act of defining that are associated with Empathetic Awareness. Two areas of very high density stand out, those being the intersections where beliefs in Authorship and Authority also coincide with either Singularity or Multiplicity. Shaded in black, over 90% of the post reflections in each of these sections also showed evidence of Empathetic Awareness. We conclude that Empathetic Awareness is more likely to emerge when PSETs have other strong beliefs, specifically in the Authorship and Authority of learners.

Figure 2
Percent of PSETs' Reflections Coded for Empathetic Awareness



This confirms the work of Povey and Burton (1999) who described that "in mathematics classrooms in which the learner is the author/ity of knowledge, they have the opportunity to use their personal Authority both to produce and to critique meanings, to practise caring in a dialogic setting where the effectiveness of their own narrative(s) and also those of others is refined" (p. 237). The activity within our classroom matched the environment described there and did seem to impact PSETs' ability to practice care (Noddings, 1992). It is important to note that far more PSETs in our group may have practiced care or shown empathy and that our results only capture those who sought to make that act explicit within their reflection post. It is one thing to give PSETs author/ity, but quite another to have them recognize its value for others as well.

Discussion

In the beginning, we asked the question, what beliefs about mathematical definitions and the act of defining are exposed when reflecting on a classroom episode focusing on writing definitions for special quadrilaterals? We have found evidence of seven distinct beliefs in the data: Multiplicity, Singularity, Authorship, Authority, Audience, Empathetic Awareness and Empathetic Comprehension. We have also presented some analysis that points to further complexity within the ways Singularity, Multiplicity, and the ways Mathematical Empathy were expressed.

The framework that emerged from our work suggests that involving PSETs in the act of defining can encourage rich pedagogical insights concerning the role that defining or definition should play in mathematics curricula and in classroom instruction.

Our study implies that changes in curricula concerning the process of defining may be warranted. Often elementary textbooks introduce vocabulary early in each new section and then build upon those meanings with the assumption that, having been introduced, the terms used are understood. Perhaps certain terms (e.g., acute, equilateral, perpendicular) are more appropriate for early presentation while others (e.g., angle, parallelogram, polygon) can be defined in culminating activities after investigations of examples and non-examples have been explored.

One of the Common Core Standards (2010) for Mathematical Practice, Attend to Precision, suggests that proficient students should be able to communicate precisely with others using "clear definitions in discussion with others and in their own reasoning… In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions." This "use of definitions" is understood by many in the field of mathematics to mean that "we do not leave the meaning of a term to contextual interpretation; we declare our definition and expect there to be no variance in its interpretation in that particular work (Edwards & Ward, 2008, p. 224)." However, it is clear from the large majority of our student reflections that regardless of whether they were identified under the Multiplicity or Singularity categories, they valued the Authority given to them to be involved in the Authoring of definitions in the classroom.

By having informal defining experiences in their earlier grades, they will have already experienced the process of making a definition minimal and precise having already explored the many properties that result naturally from the final class-consensus definition. They may have been empathetic listeners to their classmates and be willing to adjust to their perspectives. They may also be more flexible in adjusting later to the paradigm used by mathematicians where a clear definition is stated and needs to be accepted as written. Therefore, changing the Common Core (2010) high school standards "explicit use of definitions" to mean that students understand more about the nature of the defining process and the role they play in the axiomatic system seems a more complete usage.

Evidence from this study further supports earlier literature that students benefit from being involved in the defining process, and that early exploration of concepts being defined (i.e., through the use of IGS) can lead to multiple ways to define the same concept. The process itself of drawing upon an individual's developing concept image and condensing their understandings to a clear precise concept definition is a mathematical skill as important as deductive reasoning or problem solving.

We acknowledge that we are in the beginning stages of exploring the act of defining and need more research that describes the characteristics of activities that are most effective in producing autonomy and agency in PSETs such that the act of defining becomes more comfortable, familiar and a natural part of classroom discourse.

Conclusion

Focusing on the process of defining seems to have a great deal of power to reveal existing beliefs and may play a role in establishing or shifting existing beliefs about definitions and the act of defining. Coming to a consensus about how to define a special quadrilateral exposed PSETs to more than just the properties of quadrilaterals, but also the process of defining. Experiencing, albeit in a vicarious way, the difficulties faced by sixth-grade students in defining angle created space for mathematical empathy, including both awareness and comprehension. As Henry stated above, the story of Dave's confusion resonated with our students and provided an opportunity to see from someone else's perspective. When combined with their own fraught experience negotiating the properties of kites and trapezoids, the article about angle (Keiser, 2000) enabled preservice elementary teachers to see far greater subjectivity in the discipline of mathematics and to consider, perhaps for the first time, that they, too, were both able and deserving of becoming authors of mathematical ideas.

There were some principles from which we were analyzing our data. First, it is important to us that we not reduce our PSETs' beliefs to the comments they made on this assignment. In our analysis, we have sought evidence of belief rather than the lack thereof. Second, it is important to us that we not view this study as an evaluation of a particular classroom episode. While we would wholeheartedly recommend experiences for PSETs that position them as authors and mathematical authorities, there are many ways to go about that work. In the pursuit of these principles, others may see limitations in the data we collected. Teaching the same lesson to a different group of PSETs or adopting the pedagogy and applying it to different content would likely impact the ways in which PSETs reflected on the activity and expressed their beliefs. Even now, asking the same group of PSETs to reflect on the same activity after time has passed would likely yield a completely different picture. However, we believe that our study represents a snapshot of something fluid and changing and something we would like to learn even more about.

Aside from learning the content inherent in various definitions of quadrilaterals, there are other aspects of this kind of activity (focusing on the process of defining) that can help PSETs envision the mathematics classroom in a new and different way. "The study of teaching and learning in the collaborative mathematics classroom can benefit from attention to the construction, organization, and distribution of intellectual authority among students, a focus that has the potential to be theoretically generative" (Langer-Osuna, 2017, p. 244). Our results suggest that involving students in the process of defining is just as valuable as realizing helpful strategies for problem solving, learning to pose conjectures based upon inductive reasoning, carefully navigating the steps of a proof—all of these ways of thinking and reasoning should be the underlying structure of mathematics instruction. As Chesler (2012) concluded, preservice mathematics teachers "may benefit from thoughtful modelling of and explicit attention to definition use by teacher educators" (Chesler, 2012, p. 38), resulting in "a deeper understanding of how knowledge about mathematical definitions interacts with or is subsumed by subject matter knowledge and pedagogical content knowledge" (Chesler, 2012, p. 39).

Ohtani (1996) argued that the traditional practice of simply telling definitions to students is a method of moral persuasion that focuses more on pedagogical control and uniformity. This circumvents a teacher's need for sustained interactions with children and their mathematics. We have seen that by involving students in the process of defining, the opposite seems to be the case.

Students present much more autonomy and agency over deciding upon what properties to include and exclude. Rather than creating conflict, there is more empathy and understanding of others' thoughts and perspectives. As we engage our PSETs in mathematical activity, we should pay attention to how they are positioned as learners of mathematics and help them do the same with their future students.

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