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Affine invariant signed-rank multivariate exponentially weighted moving average control chart for process location monitoring

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AN AFFINE INVARIANT SIGNED-RANK MULTIVARIATE EXPONENTIALLY WEIGHTED MOVING AVERAGE CONTROL CHART FOR PROCESS LOCATION MONITORING

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ABSTRACT

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Multivariate statistical process control (SPC) charts for detecting possible shifts in mean vectors assume that data observation vectors follow a multivariate normal distribution. This assumption is ideal and seldom met. Nonparametric SPC charts have increasingly become viable alternatives to parametric counterparts in detecting process shifts when the underlying process output distribution is unknown, specifically when the process measurement is multivariate. This study examined a new nonparametric signed-rank multivariate exponentially weighted moving average type (SRMEWMA) control chart for monitoring location parameters. The control chart was based on adapting a multivariate spatial signed-rank test. The test was affine-invariant and the weighted version of this test was used to formulate the charting statistic by incorporating the exponentially weighted moving average (EWMA) scheme. The test's in-control (IC) run length distribution was examined and the IC control limits were established for different multivariate distributions, both elliptically symmetrical and skewed. The average run length (ARL) performance of the scheme was computed using Monte Carlo simulation for select combinations of smoothing parameter, shift, and number of *p*-variate quality characteristics. The ARL performance was compared to the performance of the multivariate exponentially weighted moving average (MEWMA) and Hotelling T^2. The

control charts for observation vectors sampled the multivariate normal, multivariate *t*, and multivariate gamma distributions. The SRMEWMA control chart was applied to a real dataset example from aluminum smelter manufacturing that showed the SRMEWMA performed well. The newly investigated nonparametric multivariate SPC control chart for monitoring location parameters--the Signed-Rank Multivariate Exponentially Weighted Moving Average (SRMEWMA)--is a viable alternative control chart to the parametric MEWMA control chart and is sensitive to small shifts in the process location parameter. The signed-rank multivariate exponentially weighted moving average performance for data from elliptically symmetrical distributions is similar to that of the MEWMA parametric chart; however, SRMEWMA's performance is superior to the performance of the MEWMA and Hotelling's T^2 control charts for data from skewed distributions.

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DEDICATED

To Tsu Ai See and Miranda, Majeda, and Jibreel Zeinab

without whose unconditional love, patience, support, and understanding

this work would not have been possible.

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LIST OF ACRONYMS

ARL	Average run length
ARL_0	In-control average run length
ARL_1	Out-of-control average run length
COT	Cumulative sum of the *t* statistic
CUSUM	Cumulative sum
EWMA	Exponentially weighted moving average
FIR	Fast initial response
FMEWMA	Full-smoothing matrix-based exponentially weighted moving average
IML	Interactive matrix language
LCL	Lower control limit
MEWMA	Multivariate exponentially moving average
RANDGEN	Random query generator
SDRL	Standard deviation of run lengths
SPC	Statistical process control
SQC	Statistical quality control
SRMEWMA	Signed-rank multivariate exponentially weighted moving average

UCL	Upper control limit
WCUSUM	Weighted cumulative sum

CHAPTER I

INTRODUCTION

Statistical quality control (SQC) is a powerful set of problem solving tools that includes acceptance sampling, statistical process control, design of experiments, and capability analysis (Lowry, Woodall, Champ, & Rigdon, 1992). Statistical quality control plays a critical role in modern manufacturing and production environments and dominates every aspect of most processes in any discipline (Montgomery, 2009; Woodall & Montgomery, 1999). Statistical process control (SPC) utilizes the use of control charts that are useful in monitoring process variability and improving capability through reduction of variability. Since reducing process variability is the primary goal, control charts play a pivotal role in SPC and utilize statistical estimation, inference, and control schemes (Montgomery, 2009). These schemes evolved from monitoring a single quality characteristic to two or more related quality characteristics. Variability analysis is a statistical problem and control chart development has evolved with increasing modern day of data acquisition, distributional assumptions, and computational capabilities (Lowry et al., 1992). Therefore, the scope of SPC control charts is ever-changing to take advantage of the increased amount of available data and the use of nontraditional methods such as nonparametric methods, which are particularly useful in a data-rich environment (Woodall & Montgomery, 1999).

The purpose of this study was to develop a new affine invariant spatial signed-rank multivariate exponentially weighted moving average (SRMEWMA) control chart and compare the performance of the test to traditional parametric counterparts like the Hotelling's T^2 for different distributions. The author expanded on this purpose as well as defined and explained relevant terms used in this study in chapters II and III.

Statistical process control (SPC) was pioneered by Walter A. Shewhart (1931) in the early 1920s at Bell Telephone Laboratories and was later used by Edward Deming (1950) during World War II to improve quality in the manufacturing of war munitions and other products. Deming went on to introduce SPC to Japanese manufacturers in the 1950s after the war ended; he is widely credited with much success in quality improvements in the Japanese industry (Deming, 1950; Montgomery, 2009).

In product manufacturing, there are two sources of variation--chance causes and assignable causes. The first cannot be economically identified and corrected; whereas, the second can be identified and corrected (Chakraborti, van der Laan, & Bakir, 2001). Statistical process control is a statistical method used to monitor and control a process in order to improve process performance and reduce variability in key parameters (Montgomery, 2009). Shewhart (1931) introduced control charts as primary tools in SPC and pioneered statistical quality control through the design of experiments. The most common assumption since Shewhart introduced the control chart was that the underlying process behaved in statistically normal fashion or followed the theoretical normal distribution.

When a manufacturing process operates only under chance or random variation, it is said to be in a state of statistical control. Statistical process control and control charts

help to identify and eliminate assignable causes and insure that the process in a state of control. However, when there is a change in process, control charts are expected to quickly detect this change and signal out-of-control. The faster the chart signals, the more efficient it is (Chakraborti et al., 2001).

Univariate control charts were developed to monitor Phase I implementation of statistical process control where the process is likely to be out of control and experiencing assignable causes that result in large shifts in the monitored parameters (Montgomery, 2009). The Shewhart (1931) chart for monitoring the mean of a process consists of a centerline at the historical process level along with upper and lower control limits based on the mean +/- 3 sigma (standard error) limits where the standard error is estimated from the sample means. Process means are plotted over time; there is an out-of-control signal when any sample mean plots outside the 3-sigma control limits.

A major disadvantage of a Shewhart (1931) style control chart is that it uses the information from the last sample and ignores the information provided by the sequence of prior points (Prabhu & Runger, 1997). Shewhart charts are inefficient in detecting small shifts in the process mean. Two control charts, the exponentially weighted moving average (EWMA) and the cumulative sum (CUSUM), were developed to take advantage of all available observation sequences for a single variable of interest. Both charts are used when detecting a small shift is desired.

The cumulative sum or CUSUM control chart procedure was described by (Montgomery, 2009; Page, 1954) as the accumulation of the deviations from a process mean that is above the target with one statistic and the accumulation of the deviations that is below the target with another statistic. These two statistics are the upper and lower

CUSUMs. Another good alternative to the Shewhart control chart in detecting small shifts in the process mean is the EWMA control chart that was first proposed by Roberts (1959) at the Bell Telephone Laboratories. The EWMA's performance is equivalent to that of CUSUM but EWMA is easier to setup (Graham, Chakraborti, & Human, 2011). The EWMA scheme is a geometric weighted moving average of all past and current observations; according to Montgomery (2009), it is very robust to normality assumptions, which makes it an ideal control chart to use with individual observations.

Multivariate control charts are an extension of their univariate counterparts and are used to detect a shift in the process mean vector of several process variables. The chi-squared control chart was described by Hotelling (1947). The average run length (ARL) performance of a chi-squared control chart is easily analyzed from the central and non-central chi-squared distribution of the control statistic in in-control and out-of-control cases. The average run length has a geometric distribution with a mean equal to ARL = $1/p$, where p is the probability of a single observation being out of control limits (Prabhu & Runger, 1997; Runger & Prabhu, 1996). The Hotelling's chi-squared control chart is the multivariate extension of the classical Shewhart control chart. The Hotelling's chi-squared control chart also suffers from the same disadvantage because it uses the information from the most recent observation and ignores the information provided from the prior sequence of points. Unfortunately, the Hotelling's chi-square chart is not sensitive to small shifts of the mean vector. The MEWMA or multivariate exponentially weighted moving average is the multivariate extension of EWMA and is ideal for use when simultaneously monitoring two or more correlated quality

characteristics that are jointly described by a multivariate normal distribution (Montgomery, 2009).

In contemporary statistical process control, the use of control charts is a prevalent tool for monitoring manufacturing processes. The monitoring problem is closely related to the test of hypotheses for one-sample location problems (H_0: $\theta = \theta_0$ vs. H_1: $\theta \neq \theta_0$). An observation plotting within the control limits is equivalent to failing to reject the null hypothesis of statistical process control and an observation plotting outside the control limits is equivalent to rejecting the null hypothesis of statistical process control. These process-monitoring control charts are seldom based on single variables or characteristics; it is common to monitor several variables simultaneously (Stoumbos & Sullivan, 2002; Zou & Tsung, 2010). In practice, most processes involve simultaneously monitoring several related variables recorded with online computers or advanced data collection procedures (Montgomery, 2009). Designing individual control charts to monitor a process based on univariate variables when there are several related variables is misleading, inefficient, and leads to distorted control charts (Montgomery, 2009; Qiu & Hawkins, 2001). The probability of Type I error and the probability of a point correctly plotting in control are not equal to nominal levels.

Parametric multivariate control charts are designed based on assumptions of normality where *p* variables are jointly described by a multivariate normal distribution (Montgomery, 2009; Zou & Tsung, 2010). If a set of *p* variables is assumed to be independent and the probability of Type I error for joint control, $\alpha' = 1 - (1 - \alpha)^p$, as well as the probability that all variables means will simultaneously plot inside their control limits when the process is in control, $P\{\text{all p means plot in control}\} = (1 - \alpha)^p$

can be computed. However, if the variables are dependent, then the above probability functions do not hold and there is no easy way to measure the distortion in the joint control procedure (Montgomery, 2009). According to Montgomery (2009), the multivariate control charts work well when the number of variables is small (10 or less). As the number of variables grows, traditional multivariate control charts lose efficiency with regard to shift detection.

Nonparametric Perspective

In nonparametric analysis, very little or nothing can be said about the probability of obtaining future data beyond the largest sample observation or less than the smallest observation. For this reason, the actual measurements of a sample item mean less compared to its rank within the sample. In fact, nonparametric methods are typically based on ranks of the data and the properties of the population are deduced using order statistics (Kvam & Vidakovic, 2007).

Traditional statistical methods are based on parametric assumptions, i.e., the data can be assumed to be generated by some well-known family of distributions such as a normal, exponential, or Poisson distribution. Each of these distributions has one or more parameters, at least one of which is presumed unknown and must be inferred (Kvam & Vidakovic, 2007). For example, the famous classical multivariate inference method of Hotelling's T^2 is based on a sample mean vector and covariance matrix; therefore, it is optimal under multivariate normality assumptions, poor in efficiency for heavy-tailed distributions, and highly sensitive to extreme observations (Zou & Tsung, 2010). Nonparametric alternatives to Hotelling's T^2 based on sign and rank scores have promising efficiency and robustness properties for heavy-tailed and light-tailed

distributions (Nevalainen & Oja, 2005; Sirkia, Taskinen, Nevalainen, & Oja, 2007). The term nonparametric was first coined by Jacob Wolfowitz (1942), saying,

> We shall refer to this situation where a distribution is completely determined by the knowledge of its finite parameter set, as the parametric case, and denote the opposite case, where the functional forms of the distributions are unknown, as the nonparametric case. (p. 247)

The terms nonparametric and distribution-free are not synonymous. Popular usage, however, has equated them. A nonparametric test is one that makes no hypothesis about the value of a parameter in a statistical density function; whereas, a distribution-free test is one that makes no assumptions about the precise form of the sampled population (Bradley, 1968). According to Randles, Hettmansperger, and Casella (2004), "Nonparametric statistics can and should be broadly defined to include all methodology that does not use a model based on a single parametric family" (p. 561).

Analysts limited to basic statistical methods can be trapped into making parametric assumptions about the data that are not apparent in the analysis or the data. In the case where the analyst is not sure about the underlying distribution of the data, statistical techniques are needed that can be applied regardless of the true distribution of the data. These techniques are called nonparametric or distribution-free methods (Kvam & Vidakovic, 2007).

Purpose of the Study

The main purpose of this study was to explore the viability of a new nonparametric multivariate statistical control chart for process monitoring where the process monitoring p quality characteristics were not multivariate-normally distributed. Hence, the goal was to develop a multivariate exponentially moving average (MEWMA)-type control chart that used nonparametric signed-rank statistics as the unit of

measurement, weighted statistic, and charting statistic. Extending the research of Zou and Tsung (2010) using the affine sign and Zou, Zhou, Wang, and Tsung (2010) using the spatial rank, the control chart used an affine signed-rank test statistic (Hettmansperger, Mottonen, & Oja, 1997; Oja, 2010) to develop a new affine invariant spatial sign-rank MEWMA control chart or, hereafter named, the spatial signed-rank multivariate exponentially weighted moving average (SRMEWMA) control chart. The objective of the SRMEWMA control chart was to detect small shifts in the process location vector. Using simulation, the ARL performance of the SRMEWMA control chart was studied and compared to that of the Hotelling's T^2 and MEWMA.

As with any multivariate control chart, the goal of the SRMEWMA control chart was to quickly detect small shifts in the process location vector. The quick detection helped bring the process back into the in-control state earlier and avoided producing faulty products. The performance of the SRMEWMA control chart was evaluated based upon its average run length (ARL) using Monte Carlo simulation. In addition, the performance of the control chart or its ARL was compared to that of Hotelling's T^2 ARL and MEWMA's ARL.

Rationale for the Study

In practice, there is no assurance that the quality characteristics or variables are normally distributed and the multivariate control charts designed using the traditional methods relying on the normality assumption will provide misleading results and false alarms (Lowry et al., 1992). Multivariate nonparametric or robust control charts designed using spatial sign, rank, and signed-rank statistics seem to offer an attractive viable option to traditional methods (Zou & Tsung, 2010; Zou et al., 2010).

Asymptotic Relative Efficiency

Several forms of the spatial sign and sign-rank tests were developed based on different location estimates by Peters and Randles (1990), Hettmansperger et al. (1997), Oja (1999), Randles (2000), Hettmansperger and Randles (2002), and Oja and Randles (2004). The asymptotic relative efficiency (ARE) of both sign and signed-rank tests were extensively studied by Peters and Randles (1990), Mottonen, Oja, and Tienari (1997), Mottonen, Hettmansperger, Oja, and Tienari (1998), Mottonen, Oja, and Serfling (2004), Mahfoud and Randles (2005), and Nordhausen, Oja, and Tyler (2006).

Peters and Randles (1990) suggested a signed-rank test modifying Randles' (2000) sign test. Peters and Randles showed that the signed-rank test appeared to be robust and performed better than its competitors (Randles' sign and Hotelling's T^2 tests) for light-tailed distributions as well as Hotelling's T^2 test for the MVN distribution. However, for heavy-tailed distributions, Randles' sign test was more powerful, although the signed-rank test performed well relative to Hotelling's T^2. They went on to show that when $p = 2$ or 3, the power of the signed-rank test appeared to be uniformly high.

Mottonen et al. (1997) studied the efficiencies of the spatial sign and spatial signed-rank tests with respect to Hotelling's T^2 test for the multivariate *t*-distribution with selected values of degrees of freedom and selected dimensions and the multivariate normal distribution (df = ∞). They found that the signed-rank test dominated the asymptotic relative efficiencies (AREs) of the sign test for the multivariate normal case; however, for small values of degrees of freedom (heavy-tailed distributions) with high dimension, the sign test was better.

Mottonen et al. (1998) showed and calculated the AREs for a multivariate affine invariant signed-rank test under MVN and multivariate *t*-distributions (MV t). They showed that the signed-rank test had better ARE for multivariate normal and multivariate *t*-distribution with modest to large degrees of freedom (10+) when compared to the Hoteling's T^2 test.

Mahfoud and Randles (2005) introduced a signed-rank statistic and its null asymptotic distribution and demonstrated that it had strong efficiencies over a wide spectrum of distributions, ranging from very light-tailed to heavy-tailed ones. They showed that it performed as well or in many cases better than its competitors. Mahfoud and Randles also showed that their signed-rank test statistic, in which they modified Randles' (2000) affine sign test, had better ARE properties than Randles' sign test statistic, which was also used by Zou and Tsung (2010) to develop their multivariate sign-based EWMA control chart.

If the ARL performance of the spatial sign-rank MEWMA or SRMEWMA control chart could be efficient in monitoring and controlling a process location vector, a viable nonparametric multivariate control scheme alternative would provide a potential remedy for non-normally distributed data. An expanded rationale section follows in the review of literature in Chapter II.

Research Questions

This dissertation addressed the following questions:

Q1 How will the Spatial Signed-Rank MEWMA (SRMEWMA) control chart scheme be designed for the in-control average run length (ARL_0)?

Q2 What is the Spatial Signed-Rank MEWMA (SRMEWMA) control chart performance for different number, *p*, of monitored related quality characteristics?

Q3 What is the Spatial Signed-Rank MEWMA (SRMEWMA) control chart performance for different values of the smoothing parameter λ?

Q4 What is the Spatial Signed-Rank MEWMA (SRMEWMA) control chart performance for different sizes of shift in a process location vector?

Q5 What is the Spatial Signed-Rank MEWMA (SRMEWMA) control chart performance compared to the Hotelling's T^2 and MEWMA control chart scheme for elliptically symmetrical (multivariate normal and multivariate t) and skewed distributions (multivariate gamma)?

Delimitations of the Study

Two phases were used in multivariate statistical process control (SPC)--Phase I and Phase II. In Phase I, also called retrospective analysis, *m* samples of individual observations ($n = 1$) or sample means ($n > 1$) were used to estimate the location vector and covariance matrix in order to establish the control limits for Phase II--the monitoring process (Montgomery, 2009). This study focused on Phase II monitoring only and assumed that all of the historical observations used in establishing the in-control (IC) estimates of location and covariance matrix were independent and identically distributed (iid; Zou & Tsung, 2010). Therefore, this study did not address Phase I issues as they were beyond the scope of this study.

In addition, there were many multivariate distributions that could be simulated to study the performance of the SRMEWMA control chart; however, in the interest of brevity, this study relied on the recommendations of Stoumbos and Sullivan (2002) and Mottonen et al. (1997) for generating data from the MVN, *t*, and gamma distributions.

Definitions

Affine invariance. A property of a test that ensures the value of the test statistic remains unchanged following rotations of the observations about the origin.

Average run length (ARL). The average number of consecutive points that must be plotted before an out-of- control condition is signaled.

Center line (CL). An element of statistical process control corresponding to the average value of the quality characteristic that corresponds to the in-control process.

Distribution free test. A test statistic that does not depend on a specified probability density function or cumulative distribution function.

In-control ARL (ARL_0). The ARL of the control chart when process is in-control.

In-control process. A process that is operating with only the presence of chance (common) causes of variation.

i.i.d. Independent identically distributed observations.

Lower control limit (LCL). The smallest chosen value such that, if the process is in control, nearly all of the plotted points will fall above.

Phase I. A retrospective analysis phase in SPC where a set of samples or individual observations are used to estimate the parameters for Phase II.

Phase II. The phase of monitoring future production.

Out-of-control ARL (ARL_1). The ARL of the control chart when the process is out-of-control.

Out-of-control process. A process that is operating with the presence of assignable (special) causes.

Rational sub-group. A sample of data taken at some point in the process, e.g., a sample taken during a specific time period.

Robust statistic. **S**trictly speaking, a robust statistic is resistant to errors in the results produced by deviations from assumptions (e.g., of normality). This means that if the assumptions are only approximately met, the robust estimator will still have a reasonable efficiency, and reasonably small bias, as well as being asymptotically unbiased, meaning having a bias tending towards 0 as the sample size tends towards infinity.

Skewed distribution. Distribution of measurements that, when plotted, produce a nonsymmetrical curve. When the skewness of a group of measurements is zero, the distribution is symmetrical.

***t*-distribution.** A family of theoretical probability distributions used in hypothesis testing. As with normal distribution, t-distributions are unimodal, symmetrical, and bell-shaped. Their multivariate forms are also elliptically symmetrical. The *t*-distribution is especially important when the population variance is unknown. The larger the sample, the more closely the *t* approximates the normal distribution.

Target value. A pre-specified value of a quality characteristic.

Upper control limit (UCL). The largest chosen value that if the process is in-control, nearly all of plotted points will fall below.

CHAPTER II

REVIEW OF LITERATURE

Univariate Parametric Control Charts

Two sources of variation are associated with process change: chance causes and assignable causes. A process for which all variation is due to chance causes is operation under control (Montgomery, 2009). Chance causes cannot be economically identified and corrected. A process for which all variation is due to assignable causes is operating out-of-control. Assignable causes that are not part of the chance pattern can be identified and corrected by using control charts (Chakraborti et al., 2001; Montgomery, 2009). The most common quality control charting procedures include the Shewhart X-bar, the cumulative sum (CUSUM), and the exponentially weighted moving average (EWMA). These three procedures have in common their assumption that the underlying process distribution is normal or at least approximately normal. According to Shewhart (1931), the objective of the Shewhart X-bar method is trying to identify an assignable cause in order to improve product without changing the whole manufacturing process. This remains true with CUSUM and EWMA as well as any other proposed procedure.

The distribution of run length is traditionally used to characterize the performance of a chart. A popular measure of chart performance is the expected value of the run length distribution, called the average run length (ARL; Chakraborti et al., 2001; Montgomery, 2009). Average run length is the average number of points that must be

plotted before a point indicates an out-of-control condition (Montgomery, 2009). The ARL of an in-control process is equal to the reciprocal of the probability α of a signal at a given time period when the process is in-control or ARL = 1/α (Pappanastos & Adams, 1996). It is desirable that the ARL of a chart be large when the process is in-control and small when the process is out-of-control. The false alarm rate is the probability that a chart signals a process change when in fact there is no assignable change, i.e., the process is in-control. This is similar to the probability of a Type I error in the context of hypothesis testing. Two control charts are often compared on the basis of out-of-control ARL, such that their respective in-control ARLs are roughly the same. This parallels comparing two statistical tests on the basis of power against some alternative when they are roughly the same size (Chakraborti et al., 2001).

The Shewhart Control Chart

The earliest and simplest SPC control charts for monitoring location and dispersion are the Shewhart $\overline{\mathrm{X}}$ control chart, which signals whenever an observation plots outside the control limits of a sample mean, and the Shewhart R chart, which monitors process variability (Montgomery, 2009; Shewhart, 1931). In its simplest form, the Shewhart $\overline{\mathrm{X}}$ control charts uses multiples of the process standard deviation to establish control limits for the process mean and plots the sample means versus the sample number on a control chart. If the process standard deviation is unknown, then the standard deviation must be estimated from previous data. Another method for designing control limits is the range method. If $x_1, x_2, \dots, x_n$ is a sample of size *n*, then the range of the sample difference between the largest and the smallest observations is $= x_{max} - x_{min}$. Let $\mathrm{R}_1, \mathrm{R}_2, \dots, \mathrm{R}_\mathrm{m}$ be the ranges of the *m* samples. Then the average range is

$$\bar{R} = \frac{R_1 + R_2 + \cdots + R_m}{m}, \tag{1}$$

and the control limits for the Shewhart $\overline{\mathrm{X}}$ control charts are

$$UCL = \bar{\bar{X}} + A_2\bar{R}, \tag{2}$$

$$Center\ Line = \bar{\bar{X}}, \tag{3}$$

$$LCL = \bar{\bar{X}} - A_2\bar{R}. \tag{4}$$

The control limits for an R chart are

$$UCL = D_4\bar{R}, \tag{5}$$

$$Center\ Line = \bar{R}, \text{ and} \tag{6}$$

$$LCL = D_3\bar{R}. \tag{7}$$

The constants A_2, D_3, and D_4 are tabulated constants for various sample sizes (see Montgomery, 2009). When it is possible to specify the standard deviation and the mean of the process based on previous samples, then the control limits for the Shewhart $\overline{\mathrm{X}}$ control chart are

$$UCL = \mu + 3\frac{\sigma}{\sqrt{n}}, \tag{8}$$

$$Center\ Line = \mu, \text{ and} \tag{9}$$

$$LCL = \mu - 3\frac{\sigma}{\sqrt{n}}. \tag{10}$$

The control limits for the R chart are

$$UCL = d_2\sigma + 3d_3\sigma, \tag{11}$$

$$Center\ Line = d_2\sigma, \text{ and} \tag{12}$$

$$LCL = d_2\sigma - 3d_3\sigma \tag{13}$$

The constants d_2 and d_3 are tabulated constants for various sample sizes (see Montgomery, 2009). If σ is unknown, the sample standard deviation s is used. However,

the sample standard deviation s is not an unbiased estimator of σ. Montgomery (2009) suggests that s estimates $c_4\sigma$, where c_4 is a constant that depends on the sample size *n*. The average of *m* standard deviations is $\bar{s} = \frac{1}{m}\sum_{i=1}^{m} s_i$, and therefore, the control limits for $\bar{x}$ chart are

$$UCL = \bar{\bar{x}} + \frac{3\bar{s}}{c_4\sqrt{n}}, \tag{14}$$

$$Center\ Line = \bar{\bar{x}}, \text{and} \tag{15}$$

$$LCL = \bar{\bar{x}} - \frac{3\bar{s}}{c_4\sqrt{n}}. \tag{16}$$

Using $A_3 = \frac{\bar{s}}{c_4\sqrt{n}}$, then the $\bar{x}$ chart control chart control limits are

$$UCL = \bar{\bar{x}} + A_3\bar{s}, \tag{17}$$

$$Center\ Line = \bar{\bar{x}}, \text{ and} \tag{18}$$

$$LCL = \bar{\bar{x}} - A_3\bar{s}. \tag{19}$$

The constant A_3 is tabulated for various sample sizes (Montgomery, 2009).

To illustrate the control limits for $\bar{x}$ chart, we use an example for the inside diameter measurements (mm) for automobile engine piston rings from a data set borrowed from Montgomery (2009). Table 1 presents 25 observations from five samples of n = 5 each. The $\bar{x}$ control chart is shown in Figure 1. Since none of sample means plot outside the control limits, there is no indication that the process is out-of-control; therefore, those control limits could be adopted for Phase II monitoring of the process mean.

Table 1

Inside Diameter Measurement (mm) for an Automobile Engine Piston Rings

Sample Number	$\bar{x}_i$	s_i
1	74.010	0.0148
2	74.001	0.0075
3	74.008	0.0147
4	74.003	0.0091
5	74.003	0.0122
6	73.996	0.0087
7	74.000	0.0055
8	73.997	0.0123
9	74.004	0.0055
10	73.998	0.0063
11	73.994	0.0029
12	74.001	0.0042
13	73.998	0.0105
14	73.990	0.0153
15	74.006	0.0073
16	73.997	0.0078
17	74.001	0.0106
18	74.007	0.0070
19	73.998	0.0085
20	74.009	0.0080
21	74.000	0.0122
22	74.002	0.0074
23	74.002	0.0119
24	74.005	0.0087
25	73.998	0.0162

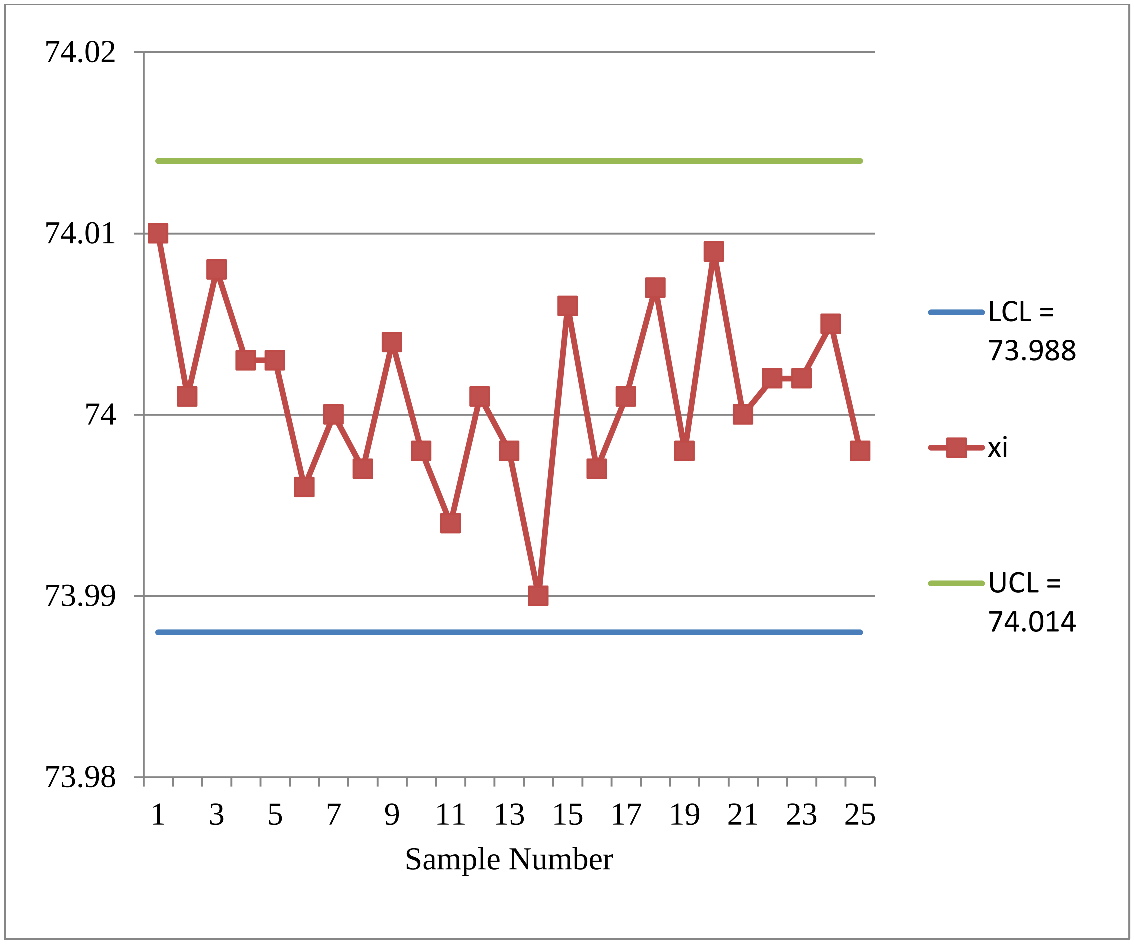

Figure 1. The $\bar{x}$ control chart for the example in Table 1.

The Shewhart $\bar{x}$ control chart for monitoring the mean of a process consists of a center line at the historical process level and upper and lower statistical control limits. Sample means are plotted over time. An out-of-control signal is detected when a sample mean falls outside the chart's control limits. The control limits are often set at the process mean with a width of 3σ, where σ is estimated using historical samples standard deviations. Woodall and Montgomery (1999) pointed out that other methods have been proposed to improve the sensitivity to small-sized and moderate-sized shifts in the process mean. Woodall and Montgomery suggested using run rules to signal for unusual

patterns on the chart. Although run rules improve the sensitivity of the chart, they increase the number of false alarms (Montgomery, 1999).

The Cumulative Sum Control Chart

The cumulative sum (CUSUM) and the exponentially weighted moving average (EWMA) control charts enjoy widespread popularity in practice because they are very effective in detecting small shifts quickly. Unlike the Shewhart chart, they use the information in the data from the beginning of the process and not the most recent time point only (Montgomery, 2009). Page (1954) introduced the CUSUM procedure as an alternative to Shewhart-style procedures that are based on a single point recorded on the control chart. Shewhart-style procedures fail to make use of all the information available from the process. While the Shewhart-type charts are probably most used because of their simplicity, CUSUM procedures are quite appropriate in view of the sequential nature of the process control problem (Chakraborti et al., 2001).

As an example of the advantages of the tabular CUSUM control charts, first consider the data in Table 2, column (a), from Montgomery (2009, pages 401-409). The first 20 observations are a random sample from a normal distribution with mean $\mu = 10$ and standard deviation $\sigma = 1$. These 20 observations have been plotted on a Shewhart control chart in Figure 2. The center line (CL) based on the first 20 observation labeled as μ_{20} and three sigma control limits on this chart are at

$$UCL = \mu_{20} + 3\sigma = 10 + 3(1) = 13, \tag{20}$$

$$CL = \mu_{20} = 10, \text{ and} \tag{21}$$

$$UCL = \mu_{20} - 3\sigma = 10 - 3(1) = 7. \tag{22}$$

Table 2

Data for Cumulative Sum Example

Sample, i	(a) x_i	(b) x_i - 10	(c) $C_i = (x_i - 10) + C_{i-1}$
1	9.45	-0.55	-0.55
2	7.99	-2.01	-2.56
3	9.29	-0.71	-3.27
4	11.66	1.66	-1.61
5	12.16	2.16	0.55
6	10.18	0.18	0.73
7	8.04	-1.96	-1.23
8	11.46	1.46	0.23
9	9.2	-0.8	-0.57
10	10.34	0.34	-0.23
11	9.03	-0.97	-1.2
12	11.47	1.47	0.27
13	10.51	0.51	0.78
14	9.4	-0.6	0.18
15	10.08	0.08	0.26
16	9.37	-0.63	-0.37
17	10.62	0.62	0.25
18	10.31	0.31	0.56
19	8.52	-1.48	-0.92
20	10.84	0.84	-0.08
21	10.9	0.9	0.82
22	9.33	-0.67	0.15
23	12.29	2.29	2.44
24	11.5	1.5	3.94
25	10.6	0.6	4.54
26	11.08	1.08	5.62
27	10.38	0.38	6
28	11.62	1.62	7.62
29	11.31	1.31	8.93
30	10.52	0.52	9.45

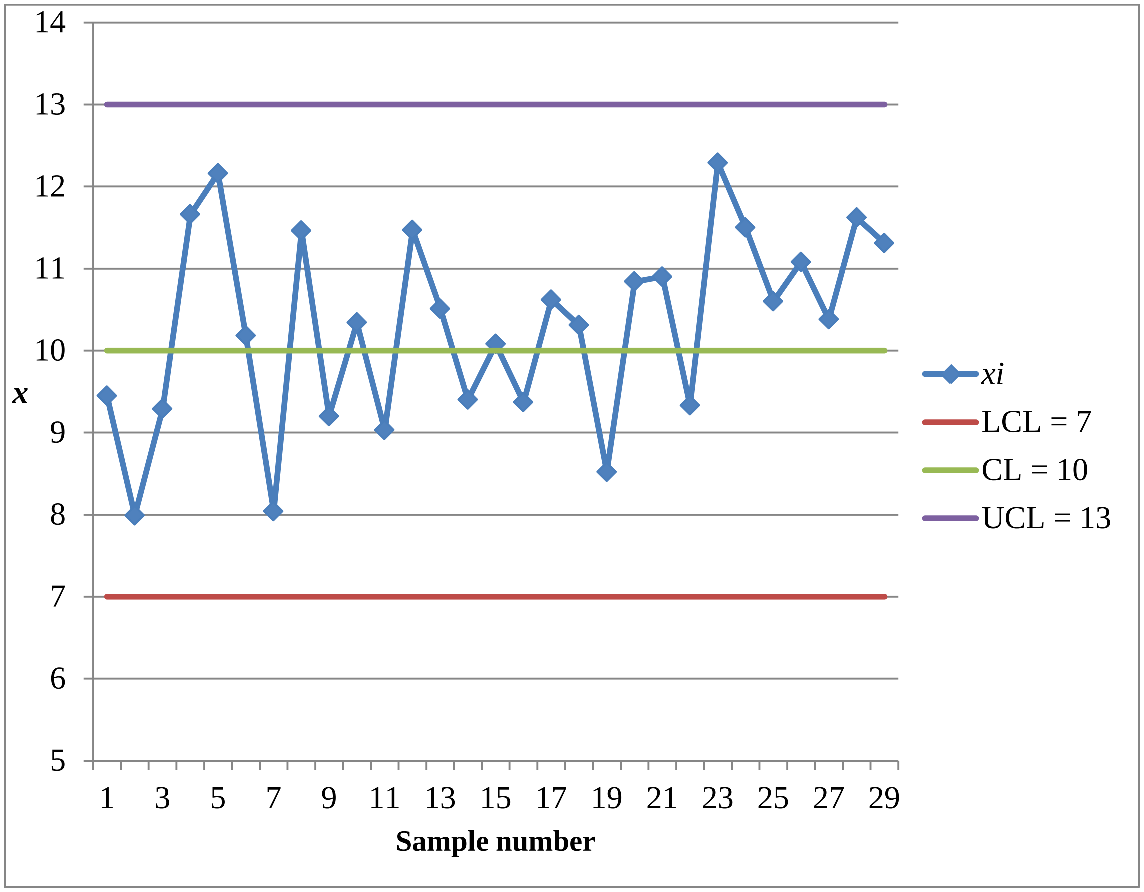

Figure 2. A Shewhart control chart for the data in Table 2.

Figure 2 shows that all 20 points plot are within the chart's control limits. The last 10 observations labeled as 21-30 are sampled from a normal distribution with mean $\mu = 11$ and standard deviation $\sigma = 1$. We can think of these observations as having been drawn from an out-of-control process with a mean shift of 1σ. These last 10 observations are also plotting within the chart's control limits. However, note that 9 out of 10 points plot above the center line, which is an indication of a process mean shift. The Shewhart control chart failed to detect this shift. The reason for this failure is that the Shewhart control chart is effective for large shifts in the range of 1.5σ to 2σ. The Shewhart control chart is ineffective for smaller shifts (Montgomery, 2009).

The CUSUM control chart incorporates all information in the sequence of sample observations by plotting the cumulative sums of the deviations of the sample observations from target value (Montgomery, 2009; Woodall, 1986; page 1954). Using the data in Table 3, column (a), if μ_0 is the target value, and $\overline{x_j}$ is the average of the jth sample (n ≥1) collected then we can define the cumulative sum as

$$C_i = \sum_{j=1}^{i} (\overline{x_j} - \mu_0). \tag{23}$$

The cumulative sum control chart is formed by plotting the C_i against the sample number i. Figure 3 shows that for the first 20 observations, there is a slow upward trend; however, after observation #20, the mean has shifted to μ = 11, and an upward trend has developed, which is evidence that the process mean has shifted (Crosier, 1986; Montgomery, 2009; Reynolds, 1975). Note the chart in Figure 3 is not a true control chart because it lacks statistical control limits. There are two ways to represent CUSUMs: the tabular CUSUM and the V-mask CUSUM (Bissell, 1969; Montgomery, 2009; Woodall, 1986). Using the current example of observations in Table 2, the tabular CUSUM is presented here. CUSUMs may be constructed both for individual observations and for averages of rational subgroups. The current example uses individual observations. The tabular CUSUM works by accumulating deviations from μ_0 above the target with one statistic C_i^+ and accumulating deviations from μ_0 below the target in another statistic C_i^- (Montgomery, 2009). The C_i^+ and C_i^- are called one-sided upper and lower CUSUMs. For the data in Table 2, they are computed as follows:

$$C_i^+ = \max[0, x_i - (\mu_0 + K) + C_{i-1}^+], \tag{24}$$

$$C_i^- = \max[0, (\mu_0 - K) - x_i + C_{i-1}^-], \text{ and} \tag{25}$$

$$C_0^+ =, C_0^- = 0. \tag{26}$$

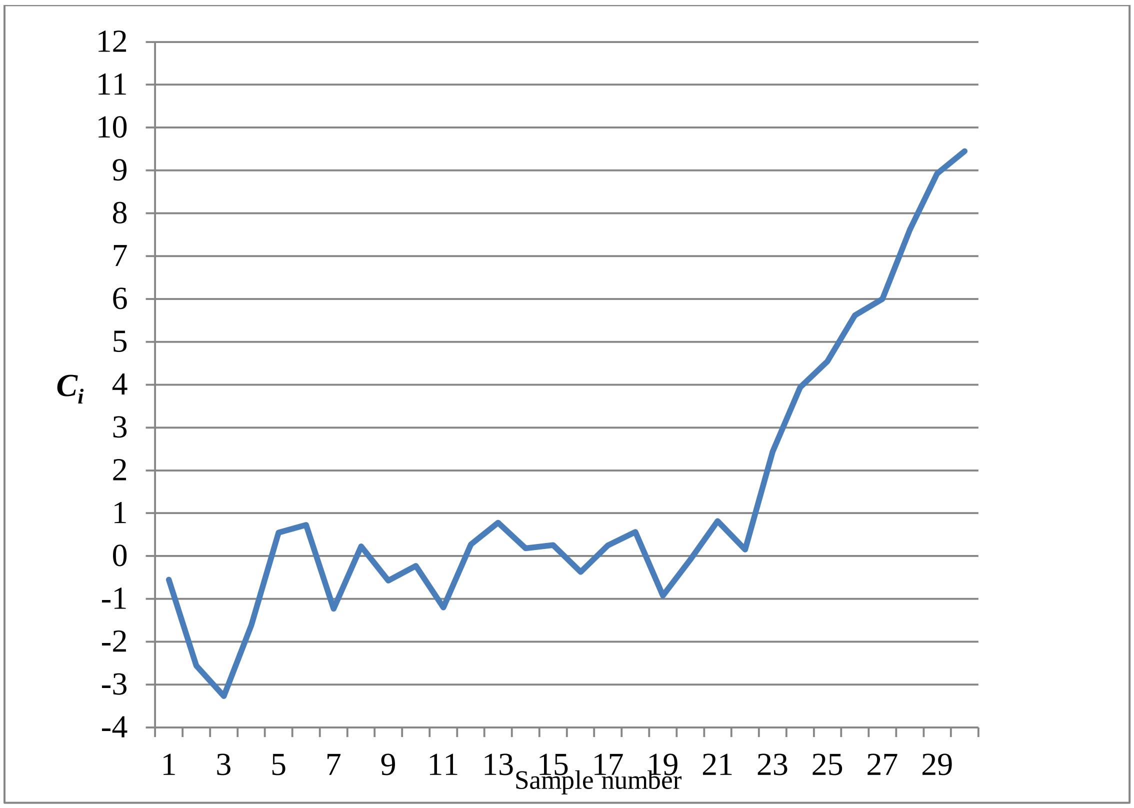

Figure 3. Plot of cumulative sum (CUSUM) from column c of Table 3.

K is usually a reference or allowance of the slack value, which is often chosen about halfway between μ_0 and the out-of-control value μ_1. Therefore, if the shift is expressed in standard deviations units as $\mu_1 = \mu_0 + \delta\sigma$, then K is one-half the shift or $K = \frac{\delta}{2}\sigma = \frac{|\mu_1 - \mu_0|}{2}$. So, C_i^+ and C_i^- accumulate deviations greater than K. Both C_i^+ and C_i^- are reset to 0 on becoming negative. If either of C_i^+ and C_i^- exceeds the decision

interval *H*, which has a reasonable value of five times the process standard deviation, the process is considered out-of-control (Montgomery, 2009). Using H = 5, Table 3 shows the calculations of both C_i^+ and C_i^- under the respective headings (a) and (b). The quantities N^+ and N^-, respectively, indicate the number of consecutive observations for which the CUSUMS C_i^+ and C_i^- have been greater than zero. The CUSUMs in Table 3 show that the upper CUSUM C_{29}^+ = 5.28 > H = 5, which is greater than the decision interval (*H = 5*); thus, we can conclude that the process is out-of-control at this point. Since N^+ =7 at period 29, this is an indication that the process shifted seven periods ago or at period 22. This can be seen in the CUSUM chart plotted in Figure 4.

Representing the tabular CUSUM graphically is both useful and convenient. This is done by plotting C_i^+ and C_i^- versus the sample number on a CUSUM status chart that resembles a Shewhart control chart (see Figures 3 or 4). Adding a Shewhart control chart to a CUSUM can improve the ARL properties of the combined control chart and can be designed to quickly detest large shifts in process mean. However, the combined scheme is not robust to outliers as a single outlier observation can cause an out-of-control signal (Lucas, 1982). A fast initial response (FIR) feature at the process startup was proposed by Lucas and Crosier (1982) in order to permit a faster response to an initial out-of-control signal. Lucas and Crosier studied the ARL properties of both a standard CUISUM and a FIR_CUSUM and found that if the process starts in-control, adding a FIR has negligible effect; however, if the process mean is not at the desired level, an out-of-control will be detected faster with an added FIR.

Table 3

The Tabular Cumulative Sum

		(a)			(b)		
Sample, i	x_i	x_i - 10.5	C_i^+	N^+	9.5 - x_i	C_i^-	N^-
1	9.45	-1.05	0	0	0.05	0.05	1
2	7.99	-2.51	0	0	1.51	1.56	2
3	9.29	-1.21	0	0	0.21	1.77	3
4	11.66	1.16	1.16	1	-2.16	0	0
5	12.16	1.66	2.82	2	-2.66	0	0
6	10.18	-0.32	2.5	3	-0.68	0	0
7	8.04	-2.46	0.04	4	1.46	1.46	1
8	11.46	0.96	1	5	-1.96	0	0
9	9.2	-1.3	0	0	0.3	0.3	1
10	10.34	-0.16	0	0	-0.84	0	0
11	9.03	-1.47	0	0	0.47	0.47	1
12	11.47	0.97	0.97	1	-1.97	0	0
13	10.51	0.01	0.98	2	-1.01	0	0
14	9.4	-1.1	0	0	0.1	0.1	1
15	10.08	-0.42	0	0	-0.58	0	0
16	9.37	-1.13	0	0	0.13	0.13	1
17	10.62	0.12	0.12	1	-1.12	0	0
18	10.31	-0.19	0	0	-0.81	0	0
19	8.52	-1.98	0	0	0.98	0.98	1
20	10.84	0.34	0.34	1	-1.34	0	0
21	10.9	0.4	0.74	1	-1.4	0	0
22	9.33	-1.17	0	0	0.17	0.17	1
23	12.29	1.79	1.79	1	-2.79	0	0
24	11.5	1	2.79	2	-2	0	0
25	10.6	0.1	2.89	3	-1.1	0	0
26	11.08	0.58	3.47	4	-1.58	0	0
27	10.38	-0.12	3.35	5	-0.88	0	0
28	11.62	1.12	4.47	6	-2.12	0	0
29	11.31	0.81	5.28	7	-1.81	0	0
30	10.52	0.02	5.3	8	-1.02	0	0

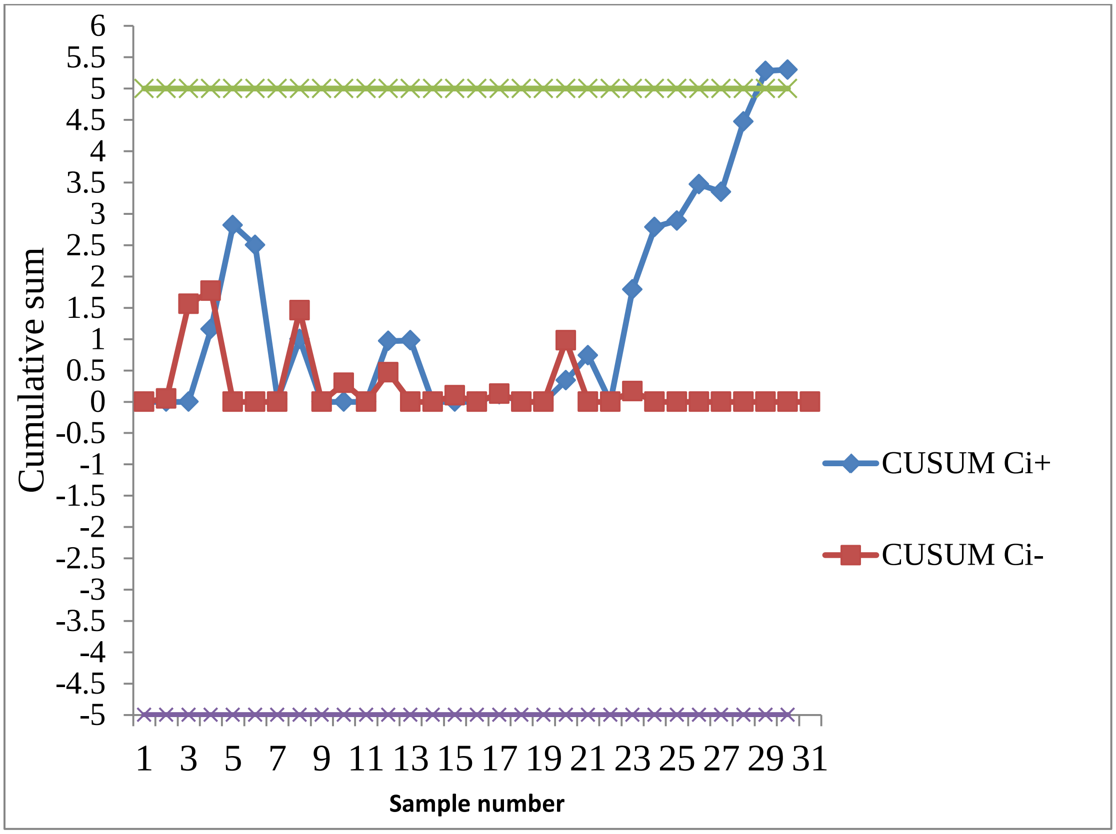

Figure 4. Cumulative sum status chart for data in Table 3.

The average run length properties of the standard CUSUM have been studied by Brook and Evans (1972), Reynolds (1975), Robinson and Ho (1978), Woodall (1983), Yashchin (1985), Crosier (1986), and many other authors. Reynolds used a Brownian motion approximation of CUSUM that does not require normality assumptions to derive an analytical form of ARL. This form can be used to determine optimal parameters to minimize ARL at a specified deviation from the mean. Montgomery (2009) used a method based on sequential analysis attributed to Siegmund (1985). Siegmund's approximation for one-sided CUSUM is

$$ARL = \frac{\exp(-2\Delta) + 2\Delta b - 1}{2\Delta^2}, \quad (27)$$

where $b = H + 1.116$ and $\Delta = \delta - K, \Delta \neq 0$. To calculate the ARL for a two-sided CUSUM, first obtain ARL^+ and ARL^- for the one-sided CUSUMs. Then use

$$\frac{1}{ARL} = \frac{1}{ARL^+} + \frac{1}{ARL^-}. \quad (28)$$

The last two decades provided an increased use of CUSUM as many authors studied and proposed various designs to tackle specific problems and situations. Some examples are outlined; however, the subject as a whole is very extensive and outside the scope of this study. Lucas (1985) described and implemented CUSUM schemes for counted data or CUSUM for attributes. CUSUM for attributes are used when the underlying process output is not continuous but rather is a count, e.g., the number of defects per unit. Lucas concluded that the CUSUM for attributes are simple to use, can be tailored to detect important shifts in count level, and use all the information in the data to quickly detect shifts. Shu, Jiang, and Tsui (2008) proposed a weighted cumulative sum (WCUSUM) procedure for monitoring a sequence with patterned mean shift and then used the estimates of a dynamic mean of the sequence for weighing the incremental in the conventional CUSUM chart. A WCUSUM is similar to the standard CUSUM chart and is less sensitive to large shifts. However, detection performance can be improved by using a combined WCUSUM-Shewhart control limits scheme. Mousavi and Reynolds (2009) investigated the problem of monitoring a proportion when there was a stream of autocorrelated binary observations with first order dependence. According to Mousavi and Reynolds, positive autocorrelation leads to false alarms that would be expected for independent observations. Mousavi and Reynolds showed that a Bernoulli

CUSUM and the Shewhart proportions control charts were not robust to autocorrelation and adjusting the control limits was not an efficient approach. Mousavi and Reynolds constructed a Markov binary cumulative sum control chart based on the log-likelihood ratio statistic and showed that the properties of this chart were calculable. The Markov binary cumulative sum control chart accounts for autocorrelations when present in the data and is most effective for detecting increases in proportions

The Exponentially Weighted Moving Average (EWMA) Control Chart

Exponentially weighted moving average EWMA control charts were developed by Roberts (1959) at Bell Telephone Laboratories where he presented an intuitive graphical technique for illustrating an EWMA design. Further design and ARL studies were presented by Crowder (1987a, 1987b, 1989). The EWMA method is useful for monitoring both the location and dispersion of a process as well as process for forecasting. Lucas and Saccucci (1990) have shown that the EWMA is as effective as the CUSUM in detecting periodic shifts in the process mean. Also, the EWMA is useful for forecasting gradual drift as highlighted by Hunter (1986). Hunter viewed the EWMA as an opportunity to begin to consider a real-time dynamic control of processes using discrete data and, if desired, to make the operator part of the feedback control loop. The EWMA design gives the most recent observation the greatest weight and every other observation receives a geometrically decreasing weight back to the first observation. EWMA charts take advantage of the sequentially accumulating nature of the data arising in a typical statistical process control environment and are known to be more efficient than the Shewhart control chart in detecting smaller shifts (Graham et al., 2011).

Since EWMA charts are known to be sensitive in detecting small changes in process mean and process variability, the EWMA method has gained a great deal of attention and has become widely used in many quality control applications (Chen, Cheng, & Xie, 2004). The EWMA is used mostly for monitoring process parameter shift, primarily the mean for using either the individual observations or the sample mean (Crowder, 1989; Lucas & Saccucci, 1990; Ng & Case, 1989). For an EWMA scheme using individual observation, see Montgomery (2009). Montgomery stated, "Since the EWMA can be viewed as a weighted average of all past and current observations, is very insensitive to the normality assumption. It is therefore an ideal control chart of individual observation" (p. 420). Crowder (1989) provided a simple procedure for designing an EWMA scheme for purposes of process monitoring and detection of shifts using sample means. For the sample means case $\bar{x}_i$, the EWMA control chart is based on the values

$$z_i = \lambda \bar{x}_i + (1 - \lambda)\bar{x}_{i-1}, \tag{29}$$

where $i = 1,2,3, \ldots$ and $z_0 = \bar{x}$, $0 < \lambda \leq 1$ and z_i is the EWMA charting statistic. In this scheme, λ is a smoothing constant and $\bar{x}_i$ are the sample means measured at time i. When $\lambda = 1$, the value of EWMA depends solely on the most recent observation as in the Shewhart X-bar chart case. If the observations x_i are independent random variables with variance σ^2, then the variance of z_i is

$$\sigma_{z_i}^2 = \sigma^2 \left(\frac{\lambda}{2 - \lambda}\right) \left[1 - (1 - \lambda)^{2i}\right]. \tag{30}$$

The EWMA control chart is constructed by plotting z_i versus the sample number *i*. The center line (*CL*) and control limits for the EWMA control chart are

$$UCL = \mu_0 + L\sigma\sqrt{\left(\frac{\lambda}{2-\lambda}\right)[1-(1-\lambda)^{2i}]}, \tag{31}$$

$$CL = \mu_{0,} \text{ and} \tag{32}$$

$$LCL = \mu_0 - L\sigma\sqrt{\left(\frac{\lambda}{2-\lambda}\right)[1-(1-\lambda)^{2i}]}. \tag{33}$$

In the above equations, the factor L is the width of the control limits in terms of standard deviations. As i get larger, the term $[1-(1-\lambda)^{2i}]$ approaches unity. This means that an EWMA has reached a steady-state and its center line and control limits are defined as follows:

$$UCL = \mu_0 + L\sigma\sqrt{\left(\frac{\lambda}{2-\lambda}\right)}, \tag{34}$$

$$CL = \mu_{0,} \text{ and} \tag{35}$$

$$LCL = \mu_0 - L\sigma\sqrt{\left(\frac{\lambda}{2-\lambda}\right)}. \tag{36}$$

The EWMA control chart signals out-of-control, detecting an off-target shift, if any sample mean plots outside the above control limits.

Zhang and Chen (2005) extended the exponentially weighted moving average (EWMA) technique by performing exponential smoothing twice; hence, they proposed a method called the double exponentially weighted moving average (DEWMA) technique for detecting process mean shifts. Zhang and Chen defined their DEWMA statistic by

$$\begin{cases} z_t = \lambda_1\bar{x}_t + \lambda_2 z_{t-1}, \quad t \geq 1 \\ z_0 = \mu_0 \\ w_t = \lambda_3\bar{z}_t + \lambda_4 w_{t-1}, \quad t \geq 1\text{'} \\ w_0 = \mu_0, \end{cases} \tag{37}$$

where $t = 1,2,3, \dots$ and $z_0 = w_0 = \bar{x},\ 0 < \lambda_i \leq 1,$ and z_t is the EWMA statistic and w_t is the DEWMA statistic. Based on $DEWMA\ (\lambda_1, \lambda_3),$ the DEWMA mean chart plots w_t against time *t* with starting target value μ_0 and control limits as follows:

$$UCL_t = \mu_0 + L\sqrt{var(w_t)}, \tag{38}$$

$$CL_t = \mu_0, \text{ and} \tag{39}$$

$$LCL_t = \mu_0 - L\sqrt{var(w_t)}\ . \tag{40}$$

Zhang and Chen used time-varying control limits to increase the sensitivity of the DEWMA chart in detecting mean shifts at the start of the process. Zhang and Chen used simulation studies to show that the DEWMA control charts outperform the EWMA control charts for small shifts in process mean, ranging from 0.1 to 0.5 of the process standard deviation. Both EWMA and DEWMA performed alike for process mean shifts greater than 0.5 of the process standard deviation. Shamma and Shamma (1992) also developed a DEWMA control chart that is based on a one λ and the fixed control limits. Shamma and Shamma studied the ARL properties of their proposed DEWMA chart using simulation. Shamma and Shamma concluded that their DEWMA chart had similar properties to the traditional EWMA control charts, which agrees with the conclusions of Zhang and Chen (2005).

There have been many different uses and studies based on the EWMA control chart method. For example, an EWMA control chart has been studied for monitoring process dispersion by Amin, Wolff, Besenfelder, and Baxley (1999), Shu and Jiang (2008), Chen et al. (2004), and Pascual (2010). Shu and Jiang developed an EWMA dispersion control chart by truncating the negative normalized observations to zero in the traditional EWMA statistic. Shu and Jiang found that by resetting the sample variance or

the logarithm of the sample variance before using it into the EWMA recursion, their proposed EWMA-type chart for monitoring increases in process variability outperformed the traditional EWMA chart by detecting small changes in dispersion.

Amin et al. (1999) proposed an EWMA chart based on the smallest and largest observation in each sample, which they called the MinMax EWMA control chart. The design of the MinMax EWMA used an EWMA for the minimum observation in the sample and an EWMA for the maximum observation in the sample and tracks observations that were farthest from the center line (target) of both the low-side and the upper-side. The advantages of the MinMax EWMA are as follows: (a) it allows for monitoring of the mean and standard deviation of the process, (b) it is a useful graphical tool, (c) it has good ARL properties for simultaneous changes in the process mean and standard deviation, (d) it allows the placement of specification limits on the chart, (e) it may be viewed as smoothed tolerance limits, and (f) it requires fewer measurements when rank ordering of observations is possible. Through a numerical example, Amin et al. showed the MinMax EWMA control chart had excellent ARL properties to detect changes in the mean and standard deviation simultaneously.

Monitoring both location and dispersion in one control chart was proposed by Chao and Cheng (1996) using the semicircle (SC) control chart. The SC control chart allows for the detection of the mean shift and variability change in one single chart. The SC chart's main advantage is its ease of attributing an out-of-control signal to shift in the mean or variability change. The SC chart is insensitive to small changes. Chen et al. (2004) combined the features of Chao and Cheng's semicircle chart with the EWMA method to develop a new control chart, EWMA-SC, which is very sensitive in detecting

small changes when a mean shift accompanies increased process variability. The EWMA-SC charting technique is efficient and easy to implement so one can quickly identify the sources and direction of an out-of-control signal on the plot (Chen et al., 2004).

Monitoring the shape parameter is not restricted to observations from the normal distribution. Other authors have used the EWMA technique to monitor shape parameters based on different statistical distributions. Pascual (2010) presented an EWMA control chart based on either a sample mean range or an unbiased estimator of the sample variance for monitoring the Weibull shape parameter. Pascual's proposed method allows for independent monitoring of both the shape and scale parameters and is recommended for monitoring small changes in the shape parameter. Pascual demonstrated through an example that ARLs are dependent on the shape parameter and are unbiased in the sense that they are expected to detect shifts sooner than the in-control ARL.

A standard assumption when using parametric control charts like the EWMA is that the observations coming from a process output are independent. Apley and Lee (2008) and Lu and Reynolds (1999) studied the ARL properties of EWMA charts for autocorrelated data. Lu and Reynolds considered the problem of detecting changes in a process in which observations modeled as an autoregressive moving average (AR1) plus a random error. Lu and Reynolds found that monitoring a process of the AR1 type is more difficult than monitoring a process in which the observations are independent normal random variables. The main reasons for the monitoring difficulty are (a) the autocorrelation appears to mask small changes in the process mean, and (b) the autocorrelated case requires more observations for parameter estimation than the

independent case as well as a more sophisticated model fitting. Lu and Reynolds concluded that for low to medium level autocorrelations, both the autocorrelated residual-based EWMA chart and the traditional EWMA chart have the same ARL properties. However, for high level autocorrelation and large shifts, the residual-based EWMA chart is better in detecting shifts. The results from Lu and Reynolds were corroborated by Apley and Lee (2008). Apley and Lee added that while the residual-based EWMA charts lacked robustness, they were more robust than independent EWMA when applied to autocorrelated process output observations.

Univariate Nonparametric Control Charts

A definition of a nonparametric or distribution-free control chart is given in terms of its in-control run length distribution. If the in-control run length distribution is the same for every continuous distribution, then the chart is called distribution-free (Chakraborti et al., 2001). In process control, chance causes are assumed to follow some parametric distribution; most often, it is the normal distribution (Chakraborti et al., 2001). The statistical properties of control charts are exact if the normality assumption is true; however, most underlying processes are not normal and, therefore, their statistical properties are not exact. This gives rise and justification to the idea of developing control charts for processes that do not depend on the normal distribution or any other parametric distribution (Bakir, 2004, 2006; Bakir & Reynolds, 1979; Lowry et al., 1992).

The recent development of a substantial number of distribution-free or nonparametric control charts where no-underlying probability distribution is assumed on the process output observations has been more available in the literature (Bakir, 2004). Many factors led to the development of nonparametric control charts. First, the

distributions of many process output observations are not known and, therefore, parametric or distribution-based control charts are not robust to these distribution-free process output observations. In addition, traditional or distribution-based control charts lack robustness when the data are skewed or when extreme or outlier data are present (Hackl & Ledolter, 1991). Finally, Hackl and Ledolter (1992) found that the average run length of a EWMA process is reduced when the data are heavily contaminated by outliers; hence, they argued that robust procedures are valuable.

Since the 1920s when Walter Shewhart (1931) developed the first control chart, statistical process control charts have been developed based on distribution-based procedures where the process output is assumed to follow a specified probability distribution such as the normal, binomial, or Poisson (Bakir, 2004). The review of literature provided a vast number of research studies proposing the use of univariate nonparametric or distribution-free control charts as an alternative to the traditional distribution-based control charts.

The use of nonparametric or distribution-free quality control charts for the univariate case was studied by many authors. Bakir and Reynolds (1979), Hackl and Ledolter (1991), Amin and Searcy (1991), Chakraborti et al., (2001), Amin, Reynolds, and Bakir (1995), Bakir (2004, 2006), Yang, Lin, and Cheng (2011) and Graham et al. (2011) proposed nonparametric control charts based on the sign, rank, or the signed-rank statistics. Once the sign, rank, or the signed-rank statistics are computed, they are used to construct control charts of the EWMA or CUSUM types. The conventional design of nonparametric charts replaces the parametric control statistic, e.g., the mean, with a

plausible statistic with an unknown distribution and uses the nonparametric statistics to study equivalents of the parametric charts (Chakraborti et al., 2001).

Chakraborti et al. (2001) presented an overview of nonparametric or distribution-free control charts for univariate data. They highlighted several advantages of using nonparametric control charts and pointed out some of the disadvantages of traditional or distribution-based control charts. The authors' goal was to present alternative control charts in the hope they would lead to wider acceptance of distribution-free charts and to understand the problems of practical statistical control without the confines of classical statistical estimation and hypothesis testing. The authors argued that nonparametric control charts had many advantages: (a) their simplicity, (b) lack of need to assume any particular parametric distribution for the underlying process, (c) same in-control length for all continuous distributions, (d) greater robustness to outliers, (e) efficiency in detecting changes when the distribution is not normal, and (f) the lack of need to estimate the variance to set up charts for the location parameter. What follows is a brief presentation of some highlights of univariate nonparametric control chart studies. For the most part, the use of the sign-test or signed-rank test was used to develop a nonparametric equivalent to the X-bar Shewhart, EWMA, or CUSUM charts. The performance and efficiency of the proposed charts were evaluated by comparing them to their parametric or distribution-based counterparts.

Bakir and Reynolds (1979), Amin and Searcy (1991), Bakir (2004, 2006) and Graham et al. (2011) proposed different nonparametric control chart procedures based on signed-ranks statistics. Bakir (2004, 2006) proposed a Shewhart style control chart based on the signed-rank statistic and then extended the procedure using signed-rank statistics

to develop cumulative sum (CUSUM) type control chart and an exponentially weighted moving average (EWMA) type control chart. Amin and Searcy (1991) and Graham et al. (2011) also used group signed-ranks to develop an exponentially weighted moving average (EWMA) type control chart.

Using a procedure based on the Wilcoxon signed-rank statistics where rankings are within a group, Bakir and Reynolds (1979) proposed a process control chart. The proposed statistic uses a nonparametric group signed-rank statistic (GSR) to compute a CUSUM type control chart. The GSRs are computed using the following procedure: Let $(X_{i1}, X_{12}, \ldots, X_{ig})$ for $i = 1,2, \ldots$, be groups of independent observations taken sequentially on the output on some process. If R_{ij} is the rank of $|X_{ij}|$ in the group $(|X_{i1}, |, |X_{i2}|, \ldots, |X_{ig}|)$ for $j = 1,2, \ldots, g$, then

$$U_{ij} = sign\,(X_{ij})R_{ij}, \quad j = 1,2, \ldots, g \qquad (41)$$

are the usual Wilcoxon signed-ranks of the observations within the ith group. Let $SR_i = \sum_j^g U_{ij}$ be the sum of the signed-ranks for the ith group. The values SR_i are a sequence of independent Wilcoxon signed-rank statistics, each based on g observations. The grouped-signed-rank (GSR) procedure uses the Wilcoxon statistics with a CUSUM stopping rule. Bakir and Reynolds showed that the average run length ARL for the GSR-CUSUM is slightly less efficient than the parametric procedure under the normality assumption since the null hypothesis is correct. However, the GSR-CUSUM is more efficient than the parametric procedures for non-normal distributions like the uniform, the double exponential, and the Cauchy distributions. A suitable subgroup size for this nonparametric procedure is suggested to be between *n* = 5 and 10, depending on the shift size and the desired in-control ARL. Furthermore, the GSR-CUSUM procedure requires

fewer assumptions about the distribution of the observations and the ARL of the GSR is the same for any continuous distribution that is symmetric about the control value.

The ARL properties of the nonparametric Wilcoxon group signed-rank statistic (GSR) developed by Bakir and Reynolds (1979) were investigated by Amin and Searcy (1991). Amin and Searcy developed a GRS-EWMA control chart using the GSR statistic and investigated the effect of autocorrelation on the average run length (ARL) properties of the GRS-EWMA. Amin and Searcy's simulation studies showed that the GSR-EWMA control chart was slightly less efficient than the traditional X-bar EWMA when the underlying distribution is normal and is more efficient at detecting smaller shifts when the process has a heavy-tailed distributions such as the double exponential.

Bakir (2004) proposed a nonparametric Shewhart style control chart based on signed-ranks for monitoring a process center. The exact false alarm rates and the in control average run lengths (ARL) for Bakir's proposed chart were calculated using the null hypothesis of the Wilcoxon's signed-rank statistic. The out-of-control ARLs were computed empirically by simulation for the light-tailed distributions (normal, uniform) and heavy-tailed distributions (double exponential and Cauchy) shift alternatives. The computing procedure for the signed-ranks is similar to the one used by Bakir and Reynolds (1979) in which they used group signed-ranks (GSR) applied to CUSUM type control chart. These same group signed-ranks statistics are used by Bakir (2004) in his proposed Shewhart style control chart. Bakir showed that control charts based on the univariate group signed-rank statistic are more efficient than the traditional Shewhart X-bar chart under heavy-tailed distributions but less efficient under light-tailed distributions.

Expanding on his initial research using GSR based control charts from 2004, Bakir (2006) proposed three different nonparametric or distribution-free control charts for monitoring a process mean when an in control target mean was specified: the Shewhart-type, the EWMA-type, and the CUSUM-type. Bakir found that the primary advantage of these signed-rank based control charts was having a constant in control ARL, regardless of the underlying distribution, as long as the distribution was continuous and symmetric. Furthermore, simulation studies showed that the signed-rank style control charts were robust against outliers while the traditional Shewhart, CUSUM, and EWMA control charts were not. Also, simulation studies showed that the signed-rank Shewhart style control chart was more efficient than its parametric counterpart for moderate to heavy-tailed distributions (Cauchy and double exponential). However, since the CUSUM and EWMA control charts were more efficient than the Shewhart control chart for detecting smaller shifts in the process mean, Bakir expected that the signed-rank EWMA and CUSUM control charts were more efficient than their parametric counterparts but further studies are required to confirm this expectation. Bakir’s results for the signed-rank EWMA match those of Amin and Searcy (1991) who found that the GRS-EWMA control chart performed well for non-normal and heavy-tailed data.

Graham et al. (2011) developed a new nonparametric EWMA control chart based on Wilcoxon’s signed-rank test (NPEWMA-SR) arguing that if normality is in doubt or cannot be justified for lack of information, then a control chart that combines the shift detection properties of EWMA with the robustness of nonparametric tests is desirable. They claimed that the Wilcoxon signed-rank test was efficient when compared to the standard *t*-test for testing hypotheses about the mean. Unlike the *t*-test, the SR test does

not require the assumption of normality and is very efficient. The asymptotic relative efficiency (ARE) for the SR test compared to the *t*-test is 0.955, 1, 1.097, and 1.5 for the normal, uniform, logistic, and Laplace distribution, respectively (Mottonen et al., 1997). Previous ARE values indicated that the SR test is more powerful for some heavy-tailed distributions like the uniform. Since the signed-rank test is more powerful than the sign test (Gibbons & Chakraborti, 2003), the SR test was used to construct the NPEWMA-SR control chart to monitor a process median of an asymptotic continuous distribution. Suppose that $X_{ij}, i = 1,2,3,..$ and $j = 1,2, \dots n$ denoted the jth observation in the ith subgroup of *n > 1*. Let us denote the rank R_{ij}^{+} of the absolute values of the differences $\left|X_{ij} - \theta_0\right|, j = 1,2, \dots n$ within the ith subgroup. Define

$$SR_i = \sum_{j=1}^{n} sign\left(X_{ij} - \theta_0\right) R_{ij}^{+}, \quad i = 1,2,3, \dots, n, \tag{42}$$

where sign (t) =1 if t > 0, 0 if t=0 and -1 if t < 0 and θ_0 is the known or specified value for the median, θ. R_{ij}^{+} is the difference between the sum of the ranks of the absolute differences corresponding to the positive and negative differences, respectively. The NPEWMA-SR control chart is constructed by accumulating statistics $R_1, R_2, R_3, \dots$ sequentially for each subgroup. The charting statistic is

$$Z_i = \lambda SR_i + (1 - \lambda) Z_{i-1} \; for \; i = 1,2,3, \dots, \tag{43}$$

where the starting values $Z_0 = 0$ and $0 < \lambda \leq 1$ is the smoothing parameter. T he control limits of the NPEWMA-SR for median are given by

$$LCL \,/\, UCL = \pm L \sqrt{\frac{n(n+1)(2n+1)}{6} \quad \frac{\lambda}{2-\lambda} \left(1 - (1-\lambda)^{2i}\right)}, \tag{44}$$

and *CL = 0*.

The steady-state $(as\ i\ \rightarrow \infty, \left(1-(1-\lambda)^{2i} \rightarrow 1\right)$ control limits and CL are given by

$$LCL\ /\ UCL = \pm L\sqrt{\frac{n(n+1)(2n+1)}{6} \quad \frac{\lambda}{2-\lambda}}\ , \tag{45}$$

and *CL = 0*. *L* is the width of the control limits, which is often expressed in multiples of the process standard deviation.

The NPEWMA-SR chart performs as well or better than its competitors when the median is known or specified (Gibbons & Chakraborti, 2003). On the basis of minimal assumptions, robustness of the in-control run-length and out-of-control distribution, the MPEWMA-SR chart is a viable alternative to parametric methods in SPC. It combines the advantages of the in-control robustness with the small shift detection capability of the EWMA-style charts. A disadvantage of the NPEWMA-SR is that its properties are unknown when the median is unknown or unspecified.

Other authors proposed and developed nonparametric control charts based on the sign-test statistic. Amine et al. (1995) used the sign-test statistic to develop nonparametric versions of CUSUM and EWMA control charts that were compared to their parametric counterparts. Yang et al. (2011) also used the sign-test statistics to develop a nonparametric EWMA sign control chart as well as an Arcsine EWMA control chart. Procedures using the sign-test statistic required that each observation be compared with a control value and the numbers of observations above and below the target mean, μ_0, be recorded for each sample. For a nonsymmetrical distribution, the sign-test is a test for a change in the median of observation, where μ_0 is the median (Amin et al. 1995). Let

$$SN_i = \sum_{j=1}^{n} sign\left(X_{ij} - \mu_0\right), \quad i = 1,2,3, \dots , \tag{46}$$

where sign (t) =1 if t > 0, 0 if t=0 and -1 if t < 0. SN_i is the difference between the number of observations above μ_0 and the number below μ_0. Amin et al. (1995) used SN_i as a charting statistic for detecting changes in the process location (e.g., mean, median) to develop Shewhart and CUSUM-style control charts.

Amin et al. (1995) found that if the distribution of observations is close to normal and the sample size is not too small, the distribution of $\bar{X}$ will be normal due to the central limit theory (CLT) and the resulting ARL will be approximately correct. However, if the distribution of observations is heavy-tailed (double exponential, Cauchy), then nonparametric control charts based on the sign-test seem to offer an advantage of fixed ARL when-in control and high efficiency in detecting shifts in μ. In addition, the variance does not need to be known or estimated to carry out the nonparametric sign-based procedure.

Yang et al. (2011) proposed a nonparametric EWMA control chart as follows. Assume that a quality characteristic, X, has a target value T. Let y = X – T and p = P(Y > 0) = the process proportion. When the process is in-control ($p = 0.5$) and when the process is out-of-control ($p \neq 0.05$). To monitor the deviation from the process target ant any given time, a random sample of size n, $X_1, X_2, \ldots, X_n$ is taken from

$$Y_i = X_j - T \; and \; I_i = \begin{cases} 1, & if \; Y_i > 0 \\ 0, & otherwise \end{cases} \text{ for } j = 1,2,\ldots,n. \qquad (47)$$

Let M be the total number of $Y_j > 0$, then $M = \sum_{j=1}^{n} I_j$ would follow a binomial distribution with parameters (n, 0.5) when the process is in control. Yang et al. found that for small n, the in-control ARL values were not always equal to the desired 370 ($ARL = \frac{1}{\alpha} = \frac{1}{0.0027} = 370$) where α is the 99.5% percentile of the observations. The reason was

that for small *n* values, the binomial distribution is asymmetric. The authors proposed a transformation using the arcsine function by letting $= \sin^{-1}\sqrt{\frac{M}{n}}$. Then *Y* is normally distributed with mean $\sin^{-1}\sqrt{p}$ and variance 1/ (4n). The resulting in-control *ARL* is 370.

Multivariate Parametric Control Charts

Multivariate control charts are used to monitor a process when more than one quality characteristic is being observed and to improve the detection of small shifts in SPC (Prabhu & Runger, 1997). Quality is seldom determined by a single quality characteristic but rather by several quality characteristics that are likely to be correlated. Multivariate control methods use the correlations between the variables to design more powerful control charts that are sensitive to assignable causes, which are poorly detected by individual variable control charts. Generalizations of the univariate control charts methods outlined above take this correlation into account when monitoring the mean vector or variance-covariance matrix (Woodall & Montgomery, 1999). As with the univariate case, we wish to design a control chart and assess its ARL performance in detecting a shift in the process mean vector, μ, or variance-covariance matrix Σ (Stoumbos & Sullivan, 2002).

Hotelling's χ^2 control chart. The first multivariate control chart was a Shewhart-type chart developed by Hotelling (1947). From the joint multivariate normal distribution, the squared standardized (generalized) distance from a vector $\boldsymbol{X}$ to the multivatiate mean $\boldsymbol{\mu}$ is

$$(\boldsymbol{X}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{X}-\boldsymbol{\mu}), \tag{48}$$

where Σ is the process variance-covariance matrix. For sub-grouped data, a modified form of the distance $(\overline{\boldsymbol{X}}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\overline{\boldsymbol{X}}-\boldsymbol{\mu})$ is used to construct a plotting statistic based on the χ^2 distribution,$\chi_0^2 = n(\overline{\boldsymbol{X}}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\overline{\boldsymbol{X}}-\boldsymbol{\mu})$ where $\overline{\boldsymbol{X}}$ is the vector of sample means and μ is the vector of in-control means (Montgomery, 2009; Zou & Tsung, 2010). This test statistic is plotted on the chi-square control chart for every sample. Since χ_0^2 statistic has a chi-square distribution, it is always positive; therefore, its control limits are *LCL=0* and $UCL = \chi^2_{\alpha,p}$. Hotelling's χ^2 signals that a statistically significant shift in the mean vector occurred or gave an out-of-control signal when

$$\chi_0{}^2 = n(\overline{\boldsymbol{X}}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\overline{\boldsymbol{X}}-\boldsymbol{\mu}) > h_1 \tag{49}$$

where $h_1 > 0$ is the specified control limit. Since this chart is based on only the most recent observation, it is insensitive to small to moderate shifts in the mean vector (Lowry et al., 1992).

Perhaps the best known parametric statistic used in multivariate statistical process control is Hotelling's T^2, which was developed by Hotelling (1947). For the one sample location problem, assume that $Y_1, \dots, Y_n$ are independent and identically distributed (iid) observation $\boldsymbol{Y}$, where $\boldsymbol{Y} = (Y_1, \dots Y_n)'$ has an absolutely continuous *p*-directional distribution with location parameter $\boldsymbol{\theta} = \left(\theta_1, \dots, \theta_p\right)'$. The Hotelling's T^2 chart is a direct extension of the Shewhart $\bar{x}$ chart and is used when $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are not known and must be estimated by $\overline{\boldsymbol{Y}}$ and $\boldsymbol{S}$ from preliminary samples taken when the process is assumed to be in-control. However, when μ and Σ are estimated from a large number of preliminary samples, it is customary to use *LCL=0* and $UCL = \chi^2_{\alpha,p}$. We wish to test

$$H_0\colon \boldsymbol{\theta} = 0\ vs.\ H_a\colon \boldsymbol{\theta} \neq 0. \tag{50}$$

The value 0 is used without loss of generality since if the null was $\boldsymbol{\theta} - \boldsymbol{\theta_0}$, the test would be performed on the random variables $Y_1 - \boldsymbol{\theta_0}, \ldots, Y_n - \boldsymbol{\theta_0}$. For SPC, this test is equivalent to testing whether an observation (sample) is in-control. The magnitude of the shift considered is

$$\boldsymbol{\delta} = (\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^{1/2} \tag{51}$$

In the context of multivariate normality, δ is called the non-centrality parameter (Stoumbos and Sullivan, 2002). The best known parametric test for this setting is the Hotelling's T^2 test statistic. The Hotelling's test is $T^2 = n\boldsymbol{\bar{Y}}'\boldsymbol{S}^{-1}\boldsymbol{\bar{Y}}$, where $\boldsymbol{\bar{Y}}$ and $\boldsymbol{S}$ are the sample mean vector and the unbiased estimate of the population covariance matrix, respectively.

There are two distinct phases of multivariate control charts. Phase I uses a set of observations to estimate the mean and covariance structure in order to obtain in-control limits for Phase II or the monitoring phase. Phase I control limits are given by

$$\boldsymbol{UCL} = \frac{\boldsymbol{p(m-1)(n-1)}}{\boldsymbol{mn-m-p+1}}\boldsymbol{F}_{\boldsymbol{\alpha,p,mn-m-p+1}}, \text{ and} \tag{52}$$

$$LCL = 0. \tag{53}$$

In Phase II, the monitoring control limits are

$$\boldsymbol{UCL} = \frac{\boldsymbol{p(m+1)(n-1)}}{\boldsymbol{mn-m-p+1}}\boldsymbol{F}_{\boldsymbol{\alpha,p,mn-m-p+1}}, \text{ and} \tag{54}$$

$$\boldsymbol{LCL} = \boldsymbol{0}\,. \tag{55}$$

The Hotelling's T^2 is directionally invariant with respect to nonsingular linear transformation of the observations. That is, if L is a *(p x p)* nonsingular matrix, then

$$T^2(\boldsymbol{Y}_1, \ldots., \boldsymbol{Y}_n) = T^2(L\boldsymbol{Y}_1, \ldots, L\boldsymbol{Y}_{\boldsymbol{n}}). \tag{56}$$

This invariance property ensures that the value of the test statistic remains unchanged following a rotation of the observations about the origin, reflections of the observations about a (*p - 1*) dimensional hyper plane through the origin, or changes in scale. Therefore, the performance of the Hotelling's T^2 and other similarly invariant procedures are independent of the structure of the population covariance matrix or the direction of the shift. This invariance property is referred to as affine-invariance (Peters & Randles, 1990) and is a desirable statistical property in any test statistic, parametric or nonparametric.

Another variation of the Hotelling's T^2 test was discussed by Lowry and Montgomery (1995) for individual observation or industrial settings where the subgroup size is $n = 1$. Let $\boldsymbol{x}$ be the sample vector of observation $n = 1$ and let $\overline{\boldsymbol{x}}$ and $\boldsymbol{S}$ be the sample mean vector and variance-covariance matrix for process with p observed quality characteristics in each sample. In the case of $n = 1$, the Hotelling's T^2 statistic is defined as

$$\boldsymbol{T^2} = (\boldsymbol{x} - \overline{\boldsymbol{x}})'\boldsymbol{S^{-1}}(\boldsymbol{x} - \overline{\boldsymbol{x}}). \tag{57}$$

The Phase II control limits for the above statistic are

$$\boldsymbol{UCL} = \frac{\boldsymbol{p(m+1)(m-1)}}{\boldsymbol{m^2-mp}}\boldsymbol{F_{\alpha,p,m-p}}\text{, and} \tag{58}$$

$$\boldsymbol{LCL} = \boldsymbol{0}\,. \tag{59}$$

However, when the number of preliminary samples is large, say > 100, Lowry and Montgomery (1995) and Montgomery (2009) suggested using

$$\boldsymbol{UCL} = \frac{\boldsymbol{p(m-1)}}{\boldsymbol{m-p}}\ \boldsymbol{F_{\alpha,p,m-p}}\,, \tag{60}$$

$$\boldsymbol{LCL} = \boldsymbol{0}, \tag{61}$$

or

$$\boldsymbol{UCL = \chi^2_{\alpha,p}}\,. \tag{62}$$

Lowry and Montgomery suggested that the chi-square limit should be used with caution only when the covariance matrix is known and with number of samples > 250. Tracey, Young, and Mason (1992) pointed out that when n = 1, Phase I limits should be based on the multivariate beta distribution where Phase I control limits are defined as

$$\boldsymbol{UCL = \frac{(m-1)^2}{m}\beta_{\alpha,p/2,\frac{m-p-1}{2}}}\text{, and} \tag{63}$$

$$\boldsymbol{LCL = 0}\,, \tag{64}$$

where $\boldsymbol{\beta_{\alpha,p/2,\frac{m-p-1}{2}}}$ is the upper α percentage point of a multivariate beta distribution with parameters $\frac{p}{2}\,and\,\frac{m-p-1}{2}$.

Multivariate cumulative sum (MCUSUM) control chart. Multivariate cumulative sum (MCUSUM) control charts are improvements on the multivariate Shewhart-type charts like the Hotelling's T^2 or χ^2 control charts; they use information from all samples and are more sensitive to small and moderate shifts in process mean vectors. As in the univariate CUSUM case, the MCUSUM chart is a Phase II procedure (Montgomery, 2009). Multivariate CUSUM (MCUSUM) charts have been studied by Woodall and Ncube (1985), Healy (1987), Crosier (1988), Pignatiello and Runger (1990), Hawkins (1991), Ngai and Zhang (2001), Runger and Testik (2004), Cheng (2007), and Golosnoy, Ragulin, and Schmid (2009).

Woodall and Ncube (1985) extended the univariate CUSUM charts to the multivariate case by monitoring multiple quality characteristics. To detect the shift in the mean vector of a p-variate normal distribution, construct multiple one-sided or two-sided CUSUM schemes simultaneously and evaluate the performance of the groups of

univariate CUSUM schemes. The ARL performance of the Woodall and Ncube depends on the direction of the mean vector shift. The process signals out-of-control once any one variable signals out-of-control. Woodall and Ncube showed that by using principal components analysis, the dependency of ARL on the direction of the shift can be reduced but not removed. Woodall and Ncube showed that in the bivariate case, the MCUSUM is preferable to the Hotelling's T^2 chart. They argued that since the performance of the MCUSUM depends less on correlations than that of the Hotelling's T^2 procedure, the use of MCUSUM to monitor correlations is less important provided that the variances are controlled. Healy (1987) also discussed the application of CUSUM to multivariate normal processes and showed that the MCUSUM procedure, which is related to sequential probability ratio tests, reduces to a univariate CUSUM. According to Healy, the specification of both a target value and a specific alternative for the mean vector of a multivariate normal distribution with known variance-covariance matrix yields a MCUSUM scheme.

Crosier (1988) and Pignatiello and Runger (1990) developed multiple MCUSUM schemes and compared their ARLs to each other and the Hotelling's T^2 chart. Crosier built on the work of Healy (1987) by using sequential probability ratio tests in developing his schemes. Crosier stated that there are two prevalent problems when deriving CUSUM schemes from theory of sequential tests. First, sequential theory requires two simple hypotheses to be tested instead of a composite one. Within quality control settings, it is the difference of requiring a simple hypothesis that the mean is at its desired level versus the composite hypothesis that the mean has shifted from the target value. Second, the logarithm of the sequential probability is often too complex to generate a

practical scheme. According to Healy, both problems have impacted development of the multivariate CUSUM. To address the first problem, Crosier proposed two multivariate CUSUM schemes as alternatives to the Hotelling T^2 chart. A first scheme forms a CUSUM by reducing the observations to a scalar Hotelling's *T* statistic and then forms a CUSUM of the *T* statistic. Crosier referred to this statistic as CUSUM of *T* or COT. Healy showed that a CUSUM of T^2 statistics is the appropriate sequential probability test for an inflation of the variance-covariance matrix $\mathbf{\Sigma}$. Interestingly, Hotelling (1947) suggested the plotting of T^2 instead of *T* to avoid the then-intensive effort to compute the square roots. The second multivariate CUSUM scheme developed by Crosier had smaller ARL and was based on the statistic

$$C_i = \{(\boldsymbol{S_{i-1}} + \boldsymbol{X_i})'(\boldsymbol{S_{i-1}} + \boldsymbol{X_i})\}^{1/2}, \tag{65}$$

where

$$\boldsymbol{S}_i = \begin{cases} 0, if\ C_i \le k \\ (\boldsymbol{S_{i-1}} + \boldsymbol{X_i})(1 - k/C_i), if\ C_i > k \end{cases}, \tag{66}$$

where $\mathbf{S}_0 = \mathbf{0}$, and $k > 0$. An out-of-control signal is generated when

$$Y_i = \left(\boldsymbol{S_i'}\boldsymbol{\Sigma^{-1}}\boldsymbol{S_i}\right)^{\frac{1}{2}} > H, \tag{67}$$

where *k* and *H* are the reference value and upper control limit for Y_i, respectively.

The ARL of both procedures depends on the mean vector and covariance structure of the data only through the non-centrality parameter, which allows these two procedures to be compared to the Hotelling T^2 chart. Crosier (1988) showed through simulations and Markov chain analyses that both MCUSUM schemes have smaller ARLs than the Hotelling's T^2 chart and the procedure developed by Woodall and Ncube (1985). Both procedures by Crosier allow for the use of the fast initial response (FIR) feature and

robustness enhancements. Crosier pointed out that the MCSUM procedure is preferred over COT because the CUSUM vector provides an indication of the direction of the shift.

Pignatiello and Runger (1990) also developed two different forms of the multivariate CUSUM schemes (which they called MC1 and MC2) for controlling the multivariate normal process. The MC1, which is the better of the two, is based on the following vectors of cumulative sums:

$$\boldsymbol{D_i} = \sum_{j=i-l_i+1}^{i} \boldsymbol{X}_j \tag{68}$$

and

$$\boldsymbol{MC_i} = max\left\{0, (\boldsymbol{D_i'}\Sigma^{-1}\boldsymbol{D}_i)^{1/2} - kl_i\right\}, \tag{69}$$

where $k > 0, l_i = l_{i-1} + 1\ if\ MC_{i-1} > 0$ and $l_i = 1$ otherwise. An out-of-control signal is generated if $\boldsymbol{MC}_i$ > H, where H is the upper control limit. The ARLs of both MC1 and MC2 were compared to those of multiple univariate CUSUM charts developed by Woodall and Ncube (1985) and multivariate Shewhart χ^2 charts. Pignatiello and Runger found that for shifts in the mean that are less than three standard deviations, both the MC1control and the multiple univariate CUSUM chart by Woodall and Ncube have better performance than the Shewhart χ^2 chart. For large shifts in the mean, the Shewhart χ^2 has smaller ARL than MC1.

Both of these multivariate CUSUM schemes have smaller ARL performance than the Hotelling T^2 or the chi-square control charts. However, the multivariate exponentially weighted moving average (MEWMA) has very similar ARL to both of these multivariate CUSUMs and is much easier to implement in practice; thus, it should be preferred

(Montgomery, 2009). The MEWMA control charts are discussed in the following section.

Multivariate exponentially weighted moving average (MEWMA) control chart. Computerized manufacturing processes make it possible to collect data on large amounts of correlated variables or characteristics for monitoring a manufactured part, product, or service process. Several types of multivariate quality control charts have been developed that perform well, providing these variables or characteristics are assumed to be normally distributed. Lowry et al. (1992) developed MEWMA or multivariate exponentially weighted moving average for monitoring the stability of a process. Suppose *(p x 1)* random vectors $\boldsymbol{X_1}, \boldsymbol{X_2}, \ \boldsymbol{X_3}, \ldots,$ each representing the p quality characteristics to be monitored, are observed over time. It will be assumed that the $\boldsymbol{X_i}, \ \boldsymbol{i = 1, 2, 3}, \ldots$ are independent multivariate normal vectors with mean vectors $\boldsymbol{\mu_i}$, respectively, and assume that each random vector has a known covariance matrix $\boldsymbol{\Sigma}$. It is further assumed, without loss of generality, that the in-control process mean vector is $\boldsymbol{\mu} = (\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, \ldots \boldsymbol{0})'$. Multivariate control charts procedures signal that a statistically significant shift in the mean (location) has occurred. Like the univariate EWMA control chart, the MEWMA control chart is based on the values

$$\boldsymbol{Z}_i = \boldsymbol{\Lambda}\, \boldsymbol{X}_i + (\boldsymbol{I} - \boldsymbol{\Lambda}\,)\boldsymbol{Z}_{i-1} \tag{70}$$

where $i = 1,2,3, \ldots$ and $\boldsymbol{Z_0} = \boldsymbol{0}$ and $\boldsymbol{\Lambda} = diag\left(\lambda_1, \lambda_2, .., \lambda_p\right), 0 < \lambda_j \leq 1, j = 1,2,3, \ldots, p$. If there is no prior reason to weigh past observations differently for the different *p* characteristics, then $\lambda_1 = \ \lambda_2 = \cdots = \lambda_p = \lambda$ and

$$\boldsymbol{Z}_i = \lambda \boldsymbol{X}_i + (1 - \lambda\,)\boldsymbol{Z}_{i-1}. \tag{71}$$

The MEWMA control chart signals out-of-control when

$$\boldsymbol{T}_i^2 = \boldsymbol{Z}_i'\boldsymbol{\Sigma}_{\boldsymbol{Z}}^{-\mathbf{1}}\boldsymbol{Z}_{\boldsymbol{i}} > h_2, \tag{72}$$

where $h_2 > 0$ is chosen to achieve a specified in control ARL.

$$\boldsymbol{\Sigma}_{\boldsymbol{Z}} = \frac{\boldsymbol{\lambda}}{\mathbf{2} - \boldsymbol{\lambda}}[\mathbf{1} - (\mathbf{1} - \boldsymbol{\lambda})^{\mathbf{2i}}]\boldsymbol{\Sigma}_{\boldsymbol{X}} \tag{73}$$

is the variance-covariance matrix of the recursive statistic Z_i and T^2_i is the charting statistic. Lowry et al. (1992) showed that the asymptotic variance-covariance matrix is

$$\boldsymbol{\Sigma}_{\boldsymbol{Z}} = \frac{\lambda}{2-\lambda}\boldsymbol{\Sigma}_{\boldsymbol{X}}. \tag{74}$$

The control limit h_2 is usually chosen to give an in-control ARL = 200, where the process is assumed to be in-control when the chart is started. This value is the zero-state ARL and the shift size is of the quantity

$$\boldsymbol{\delta} = (\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^{1/2}. \tag{75}$$

Again, in the context of multivariate normality, $\boldsymbol{\delta}$ is called the non-centrality parameter (Stoumbos & Sullivan, 2002). Larger values of **δ** correspond to larger shifts in the mean vector (Montgomery, 2009). When **δ = 0,** the process is in control and the chart can be constructed using standardized data. When λ = 1, MEWMA is equivalent to the T^2 or χ^2 control chart; the MEWMA is more sensitive to smaller shifts when λ ≠ 1. Since MEWMA is directionally invariant, we only need the **δ** values to examine the performance for any shift in the mean vector (Lowry et al., 1992; Montgomery, 2009). However, unlike Hotelling's T^2, MEWMA is not only based on the most recent observation and therefore is sensitive to small and moderate shifts in the mean vector (Lowry et al., 1992).

There have been various modifications to the MEWMA since it was first developed by Lowry et al. (1992). The nature of most research findings is theoretical;

however, general research demonstrates potential utility of many of the proposed methods and modifications to the original MEWMA scheme by Lowry et al. as well as providing encouragements for future researchers to further investigate the properties of the proposed alternatives thoroughly. For example, Hawkins, Choi, and Lee (2008**)** proposed using a full-smoothing matrix, Λ, instead of a diagonal one. Hawkins et al. stated that the traditional MEWMA with diagonal elements only is directionally invariant while the full-smoothing, matrix-based MEWMA (FMEWMA) is not. The FMEWMA is affected by the direction of the shift and the correlation structure, thereby complicating the control chart design. The FMEWMA scheme created additional computational requirements but provided tangible improvement in detecting a shift in the process mean vector. Hawkins et al. compared the ARL performance of FMEWMA to MEWMA and found FMEWMA to have shorter ARL performance.

Reynolds and Kim (2005) pointed out that the standard practice when using any control chart to monitor a process is to take samples of fixed size at regular intervals. Reynolds and Kim investigated MEWMA based on sequential sampling where observations at a sampling point were taken in groups of one or more observations and the number of groups taken was a random variable that depended on the data. The MEWMA chart was based on sequential sampling, which used the standard MEWMA statistic at point *k* for variable *i* as

$$\mathrm{E}_{ki} = (1-\lambda)\boldsymbol{E}_{k-1,i} + \lambda\overline{\boldsymbol{X}}_{ki} \ , \tag{76}$$

where $E_0 = 0$ and λ is a weighting smoothing parameter such that $0 < \lambda < 1$. At the sampling point k, the EWMA vector was formed by the E_{ki} statistics as

$$\boldsymbol{e}_k = \left(\boldsymbol{E}_{k1}, \boldsymbol{E}_{k2}, \ldots., \boldsymbol{E}_{kp}\right)', \tag{77}$$

and the control statistic

$$\boldsymbol{Y}_k = c_k^{-1}(\boldsymbol{e_k} - \boldsymbol{\mu_0})'\boldsymbol{\Sigma_0^{-1}}(\boldsymbol{e_k} - \boldsymbol{\mu_0}), \tag{78}$$

where the constant $c_k = \frac{\lambda[1-(1-\lambda)^{2k}]}{(2-\lambda)n}$, $k = 1,2, \ldots,$ and an out-of-control signal is given when $Y_k > h$, where h is the control limit.

Reynolds and Kim (2005) found that the performance of MEWMA based on sequential sampling is much more efficient in detecting process mean vector shifts than standard MEWMA. Reynolds and Kim showed that for small to moderate shifts, both the MEWMA based on sequential sampling and the standard MEWMA were more efficient than the Hotelling's Shewhart chart. For large shifts, they found that the performance of the control chart based on MEWMA sequential sampling was close to the performance of the Hotelling's Shewhart chart.

Reynolds and Cho (2006) investigated the performance of MEWMA control charts for simultaneous monitoring of the mean vector and variance-covariance matrix compared to the of standard multivariate Shewhart chart and to combinations of univariate EWMA charts applied to each of the variables. Reynolds and Cho found that using combinations of MEWMA-type charts based on the mean and on the sum of squared regression adjusted deviation from the target performed best. They concluded that the chart based on squared deviations from target would detect large shifts both in process mean vector as well as variance-covariance matrix. Huwang, Yeh, and Wu (2007) developed two new control charts for monitoring process variability for individual observations. Huwang et al. argued that any changes in the mean vector or the process variability were more likely to occur within rational subgroups than between subgroups. Huwang et al. developed two control charts to monitor process variability: the

multivariate exponentially weighted mean squared deviation (MEWMS) and multivariate exponentially weighted moving variance (MEWMV) charts. When monitoring process variability, it is assumed that the process mean is constant; otherwise, the shift is subject to confounding of both mean vector and variance-covariance matrices. Huwang et al. showed through simulation studies that if the process means vector remained in control, the MEWMS chart outperformed MEWMV and both standard MEWMA and MCUSUM. The authors pointed out that MEWMS and MEWMV charts could be applied to cases when the number of observation in the rational subgroups $n > 1$.

Lowry et al. (1992) used simulation to estimate the ARLs of the MEWMA charts; whereas, others used approaches based on Markov chains or integral equations (Reynolds & Kim, 2005). Markov chains can be applied when the multivariate control statistic can be modeled and the run length performance depends on the off-target mean through the non-centrality parameter. Runger and Prabhu (1996) used Markov chain approximation to determine the run length performance of the MEWMA chart. They used symmetry and orthogonal invariance to provide a Markov chain analysis of a multivariate control procedure. They demonstrated the Markov chain analysis for a bivariate case is extendible to multivariate processes by changing the chi-square degrees of freedom to obtain transitional probabilities. They obtained MEWMA chart ARL estimates similar to those obtained by Lowry et al. using simulation studies. Prabhu and Runger (1997) also used Markov chain analyses to provide recommended values for weighting or smoothing parameters for the zero-state, steady-state, and worst-state cases. ARL performance results obtained from simulation studies by Lowry et al. have been limited. Prabhu and Runger's objective was to select the parameters for the design of a MEAMA by using the

Markov chain method to provide design recommendations for a MEWMA control chart. Molnau et al. (2001) also provided a program using Markov chain analysis that calculates ARL for MEWMA control charts. The program returns the ARL given values of the shift in the mean vector. The Markov chain program is dependent on the number of states used in the approximation with a greater number of states providing more accurate approximation of ARL but at the expense of increased computing resources and time. But the ARL performance of MEWMA obtained using the Markov chain program compares favorably with those obtained by Lowry et al. using simulation.

A very important research area in SPC is the robustness of statistical control charts to non-normality. Stoumbos and Sullivan (2002) investigated the effects of non-normality on the ARL performance of the MEWMA control chart and its special case--the Hotelling's χ^2 in which the smoothing parameter $\lambda = 1$. Stoumbos and Sullivan showed that the Hotelling's χ^2 chart is highly sensitive to non-normality assumptions but also showed that for individual observation and by extension for subgroups of size greater than one, the MEWMA can be designed to be robust for elliptical symmetrical distributions like the multivariate *t* distribution, the highly skewed such as the multivariate gamma, and extremely heavy-tailed distributions. Stoumbos and Sullivan demonstrated through simulation studies that for the bivariate *t* distribution with three or more degrees of freedom, when the smoothing parameter $\lambda \leq 0.046$, the in-control ARL is at least as large as those obtained when the underlying process output is bivariate normal. Also, Stoumbos and Sullivan demonstrated through simulation studies that the bivariate gamma distribution Gam_2 with different values of the shape parameter α and scale parameter β=1, values of $\lambda \leq 0.046$ give an in-control ARL values that are close

to design values. When the shape parameter α increases and the gamma distribution approaches normality, the in-control ARL is close 200 for any value of λ.

Testik, Runger and Borror (2003) also examined the robustness of the MEWMA control charts to non-normal data, specifically the symmetric multivariate *t* and the skewed multivariate gamma distributions. Testik et al. demonstrated through simulation studies MEWMA control charts' in-control ARL outperformed its competitor control charts for both the multivariate *t* and the multivariate gamma distributions with different dimensions and $\lambda \leq 0.05$. Overall, the authors showed that the MEWMA chart $\lambda \leq 0.05$ was insensitive to the underlying distribution, which agreed with Stoumbos and Sullivan's (2002) results. Testik et al. also demonstrated through simulation studies that the MEWMA control chart performance, when $\lambda \leq 0.05$, was less sensitive and more robust to non-normality. Testik et al. recommended using a MEWMA with $\lambda = 0.5$ as a control chart for individual multivariate process outputs across many applications.

Zou and Tsung (2008**)** developed a directional multivariate exponentially weighted moving average (directional MEWMA) by integrating the MEWMA with the generalized likelihood ratio test (GLRT), which incorporates directional information based on a multistage state-space model and effectively monitors the process mean vector shift. Zou and Tsung's scheme provides an SPC solution that incorporates both inter-stage and intra-stage correlations as well as resolving the confounding issue of monitoring cumulative effects from stage to stage such as in automotive body assembly processes. The authors showed through simulation studies that the proposed directional MEWMA procedure outperformed all existing SPC procedures in a multistage process.

Multivariate parametric control charts have been studied extensively and have been shown to provide a potential solution to many monitoring situations and settings. However, although these methods are robust and efficient with modifications to chart design, they all are based on the assumption that the underlying process output data are sampled from a known parametric distribution such as the multivariate normal distribution. This is rarely true and more robust methods are needed when the underlying process output is distribution-free or the distribution is not known.

Multivariate Nonparametric Control Charts

Although multivariate SPC problems are important in their own right, the field of multivariate nonparametric statistical process control techniques is not sufficiently developed (Qiu & Hawkins, 2001). A partial review of this area is presented next. There are two main components to multivariate nonparametric control charts. The first are the various nonparametric test statistics used in place of parametric statistics when the underlying distribution is unknown. The second are the control chart schemes employed using these nonparametric statistics. Among the most prevalent multivariate nonparametric statistics involve spatial statistics. The statistical process control chart schemes utilizing these spatial statistics are extensions of the parametric MEWMA and MCUSUM.

Multivariate signs, ranks, and signed-ranks statistics are designed based on different mathematical concepts. Marginal multivariate signs and ranks are based on the Manhattan-distance or norm $|.|$,$|\boldsymbol{x}| = |\boldsymbol{x_1}| + |\boldsymbol{x_2}| + \cdots + |\boldsymbol{x_k}|$; whereas, spatial signs and ranks are based on the Euclidean distance, $||\boldsymbol{x}|| = \left(\boldsymbol{x_1^2} + \cdots + \boldsymbol{x_k^2}\right)^{\boldsymbol{1/2}}$ and the affine signs and ranks are based on average determinants of subsets of variables

(Hettmansperger & Randles, 2002; Oja, 1999; Paindaveine, 2008). Using marginal signs and ranks yield a vector of marginal medians, using spatial signs and ranks yield spatial medians, and when using affine signs and ranks, the Oja median is produced(Oja, 1983)--a location estimate satisfying the affine invariance and consistency conditions. The spatial median is not scale equivariant and the marginal median does not satisfy rotation equivariance (Visuri, Koivunen, & Oja, 2000). Depending on which signs and ranks concepts are used, the covariance matrices have different statistical properties (consistency, limiting distribution efficiencies, and influence functions, etc.).

Several nonparametric spatial multivariate statistics have been incorporated in the design of multivariate control charts. Randles (1989, 2000) developed a multivariate sign test based on the transformation-retransformation approach (Chakraborti, Chaudhuri, & Oja, 1998) together with the directional transformation developed by Tyler (1987). Tyler's transformation is to find a *p x p* positive-definite matrix that has a trace $(\boldsymbol{V}_x) = p$, which satisfies that for any $\boldsymbol{A}_x'\boldsymbol{A}_X = \boldsymbol{V}_x^{-1}$. Randles (2000) proposed $\boldsymbol{A}_X$ that is convenient to calculate and produces a sign test with desirable characteristics. Randles' $\boldsymbol{A}_X$ is nonsingular and is affine-invariant such that

$$\boldsymbol{A}(X_1, \ldots\ldots, X_n) = \boldsymbol{A}(\delta_1 X_1, \ldots\ldots, \delta_n X_n), \tag{79}$$

for all $\delta_i = \pm 1$ and $x_i = 1,2, \ldots\ldots, n$. This property simplifies the conditional distribution of the test statistic satisfying

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\boldsymbol{A}_x\boldsymbol{X}_i}{\|\boldsymbol{A}_x\boldsymbol{X}_i\|}\right)\left(\frac{\boldsymbol{A}_x\boldsymbol{X}_i}{\|\boldsymbol{A}_x\boldsymbol{X}_i\|}\right)' = \frac{1}{p}\boldsymbol{I}, \tag{80}$$

for which

$$\frac{1}{n}\sum_{i=1}^{n}(\boldsymbol{V}_i)(\boldsymbol{V}_i)' = \frac{1}{p}\boldsymbol{I}, \tag{81}$$

where $\boldsymbol{V}_i$ denotes

$$\frac{\boldsymbol{A}_x\boldsymbol{X}_i}{\|\boldsymbol{A}_x\boldsymbol{X}_i\|}, \tag{82}$$

and

$$\overline{\boldsymbol{V}} = \frac{1}{n}\sum_{i}^{n}\boldsymbol{V}_i. \tag{83}$$

After obtaining $\boldsymbol{V}_i$, Randles proposed to use the test statistic

$$S_n = n\overline{\boldsymbol{V}}'\left(n^{-1}\sum_{i=1}^{n}\boldsymbol{V}_{\boldsymbol{i}}\boldsymbol{V}_{\boldsymbol{i}}'\right)^{-1}\overline{\boldsymbol{V}}. \tag{84}$$

Randles showed that if $\boldsymbol{A}_X$ is replaced by $\boldsymbol{A}_d$, which is calculated based on interdirections, then

$$S_n = np\overline{\boldsymbol{V}}\,\overline{\boldsymbol{V}}' \tag{85}$$

The test statistic S_n was shown by Randles to be affine-invariant and distribution-free for the class of distributions with elliptical directions. In addition, Randles proved that $S_n \xrightarrow{d} \chi_p^2$. Tyler (1987) demonstrated that $\boldsymbol{A}_x$ is unique and the estimator $\widehat{\boldsymbol{A}}_x$ is consistent with asymmetrical normal and provided an algorithm to compute $\widehat{\boldsymbol{A}}_x$. Randles stated that his test makes minimal assumptions, is directional, and showed that the test not only has a small sample distribution-free property over broad class of distributions but performs well in comparison with Hotelling's T^2.

Hettmansperger et al. (1997) developed an affine-invariance (equivariant) signed-rank test (estimates) based on the generalized median of Oja and Oja signs and signed-ranks. Oja (1983) extended the affine-invariant multivariate sign test of Hettmansperger, Nyblom, and Oja (1994). The authors used the Oja criterion function to develop an affine-invariant multivariate vector sign, then used the vector sign to develop a vector rank, and finally defined the vector signed-rank and an affine-equivariant estimate of the location. Their test needs a symmetry assumption but no assumptions about the covariance structure are required. Their statistic proves more efficient than Hotelling's T^2 when the underlying distribution of the variables is the multivariate *t*-distribution with small degrees of freedom. For higher degrees of freedom, Hettmansperger et al. (1997) showed that the performance of their signed-rank statistic improves and compares favorably with that of Hotelling's T^2.

Hallin and Paindaveine (2002) proposed several multivariate location tests based on interdirections and pseudo-Mahalanobis ranks under elliptical symmetry. Hallin and Paindaveine developed an alternative to their multivariate location tests in which the interdirections were replaced by "Tyler angles" or the angles between observations standardized via Tyler's estimator of scatter (Tyler, 1987). The tests developed using Tyler's angles are computationally preferable in terms of CPU to those developed using interdirections. However, the authors showed via simulation studies that the two-versions are asymptotically equivalent. Hallin and Paindaveine's tests, which are a generalization of the univariate signed-rank tests, are affine-invariant under elliptical symmetry. Oja (2010) points out that the sign and signed-ranks of Hettmansperger et al. (1997), which are based on the Oja signs and signed-ranks, are asymptotically equivalent

to spatial signs and signed-ranks tests in the spherical case and are affine-invariant. However, unlike the tests developed by Hallin and Paindaveine, their performance under the elliptical case may be poor.

Peters and Randles (1990) developed an affine-invariant signed-rank test and signed sum test for the one-sample multivariate location problem, respectively. Both tests are modifications of Randles' (1989) multivariate sign test and were developed based on the principle of interdirections introduced by Randles. Randles introduced a sign test based on interdirections that used the direction of the observations from 0 rather than the distances from 0. Consider a pair of observations X_j and X_k in a sample of size *n*. Let C_{jk} denote the number of hyperplanes formed by the origin and *p -1* other points (excluding X_j and X_k) such that X_j and X_k are on the opposite sides of the formed hyperplane. Therefore, given a sample of size n, C_{jk} is an integer between 0 and $\binom{n-2}{p-1}$ inclusive. A value $C_{jk} = 0$ implies that the points X_j and X_k are adjacent. The C_{jk} counts are referred to as interdirections, which measure the angular distance between X_j and X_k relative to the origin and other data points. To describe this test, consider the test for general *p* that rejects H_0 statistic for large values of the statistic

$$V_n = \left(\frac{p}{n}\right) \sum_{j=1}^{n} \sum_{k=1}^{n} \cos(\hat{p}_{jk}), \tag{86}$$

where

$$\hat{p}_{jk} = \begin{cases} \dfrac{(c_{jk} + d_n)}{\binom{n}{p-1}} \ if \ j \neq k \\ 0, if \ j = k, \end{cases} \tag{87}$$

and

$$\mathrm{d_n} = \frac{1}{2}\left\{\binom{\mathrm{n}}{\mathrm{p}-1} - \binom{\mathrm{n}-2}{\mathrm{p}-1}\right\}, \tag{88}$$

and c_{jk} denotes the number of hyper-planes formed by the origin and p-1 observations. The proportion $\hat{p}_{jk}$ is the observed fraction of times that $X_j$and X_k fall on the opposite sides of the formed hyperplanes. Randles showed that the sign test based on interdirections in invariant under non-singular linear transformations and that under H_0, V_n has small-sample distribution-free properties over the broad class of elliptical distributions. Randles also showed that under H_0, V_n has an asymptotic chi-square distribution with p degrees of freedom.

Peters and Randles (1990) developed a signed-rank statistic as a special case based on Randles' (1989) V_n statistic. Using the original estimated Mahalanobis distances of the original Y vector observations, $\widehat{D}_i = Y_i'\hat{\Sigma}^{-1}Y_i$, where $\hat{\Sigma} = \frac{1}{n}Y_iY_i'$ is a constant estimator of the variance and letting $R_i = rank\left(\widehat{D}_i\right)$, Peters and Randles' signed-rank statistic is defined as

$$W_n = \frac{3p}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n}\cos(\hat{p}_{jk})\frac{R_j}{n}\frac{R_k}{n}. \tag{89}$$

Peters and Randles showed through Monte Carlo simulations that their signed-rank test static was robust and it performed better than its competitors when the distribution was light-tailed and as well as the Hotelling's T^2 under multivariate normality. For heavy-tailed distributions, the signed-rank statistic performed better than Hotelling's T^2 but not as well as Randles' statistic.

Hossjer and Croux (1995) and Hallin and Paindaveine (2002) developed a class of optimal procedures based on Randles' (1989) interdirections and pseudo-Mahalanobis ranks. Hossjer and Croux developed a method to generalize signed-rank statistics to higher dimensions. They suggested that when the underlying distribution of the data was elliptically symmetric, transforming the observations using an equivariant estimate of the population covariance matrix and then calculating the test statistic using the transformed observation, the corresponding location estimator is affine-invariant if the signed-rank statistic is applied to standardized data. For any sample $\boldsymbol{X} = (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$, let $a_n(1), \ldots, a_n(n)$ be a sequence of non-negative scores, define the test statistic

$$T_n(\boldsymbol{X}) = \sum_{i=1}^{n} a_n R(||\boldsymbol{X}_i||) U(\boldsymbol{X}_i), \qquad (90)$$

where $R(||\boldsymbol{X}_i||)$ and $U(\boldsymbol{X}_i) = \boldsymbol{x}/||\boldsymbol{x}||$ are the rank and sign of the vector $\boldsymbol{X}$. Hossjer and Croux provided formulas for calculating the asymptotic relative efficiency (ARE) of these generalized tests with respect to Hotelling's T^2. Hossjer and Croux showed through simulation studies that the performance of the proposed signed-rank test was as good as that of the Hotelling's T^2 and more robust to contamination for spherical and elliptically symmetrical distributions.

Mahfoud and Randles (2005) argued that the test statistics by Hossjer and Croux (1995) were too complicated to compute; while their performance was excellent for the distribution for which they were optimal, their performance might not be good over a broad spectrum of distributions. Mahfoud and Randles proposed a class of affine-invariant multivariate signed-rank test based on the affine-invariant sign test that was originally developed by Randles (2000). Using Randles' original statistic

$$S_n = n\overline{\boldsymbol{V}}'\left(n^{-1}\sum_{i=1}^{n}\boldsymbol{V}_i\boldsymbol{V}_i'\right)^{-1}\overline{\boldsymbol{V}}\ , \tag{91}$$

Mahfoud and Randles proposed and developed a signed-rank statistic

$$W_{np} = n\overline{\boldsymbol{V}_s}'\left(n^{-1}\sum_{i=1}^{n}\boldsymbol{V}_{si}\boldsymbol{V}_{si}'\right)^{-1}\overline{\boldsymbol{V}_s}\,. \tag{92}$$

The statistic W_{np} uses the ranks of the distances of the transformed observations from the origin. Mahfoud and Randles defined the ranks Q_i to be the rank of $\|A_dY_i\|$ among $\|A_dY_1\|, \ldots \|A_dY_n\|$ and $V_{si} = \phi\left(\frac{Q_i}{n+1}\right)V_i$, where $\phi\,(.)$ is a nonnegative, non-decreasing, uniformly bounded continuous function that may depend on the dimension p. Mahfoud and Randles showed that W_{np} is affine-invariant as long as $n > p(p-1)$. Mahfoud and Randles demonstrated that W_{np} has strong efficiencies over a wide spectrum of distributions, ranging from very light-tailed distributions to very heavy-tailed ones.

Efficiency of Multivariate Spatial Sign and Rank Tests

To use the sign and rank test statistics in statistical process control and the design of control charts as an alternative to parametric ones when the distributional assumption is violated, the efficiency of the nonparametric spatial sign and rank statistics must be comparable to their parametric counterparts. Since the asymptotic relative efficiency plays a very important role in the development of nonparametric tests, it is critical to consider multivariate efficiencies when developing multivariate location models. Mottonen et al. (1997) derived the asymptotic relative efficiency formulae for the multivariate spatial sign and signed-rank methods. The efficiency of affine invariant multivariate rank tests under similar conditions was also developed by Mottonen et al.

(1998). Mottonen et al. (2004) also calculated relative efficiencies for the spatial sign and signed-rank methods with respect to Hotelling's T^2 under the multivariate *t*-distribution with selected values of degrees of freedom ν and with selected dimensions *k*. Mottonen et al. (1997) found that in the multivariate normal case (ν = ∞), the efficiencies of the spatial signed-rank test dominated the efficiencies of the spatial sign test; however, for small values of degrees of freedom (heavy-tailed distributions) with high dimensions, the sign test was better. They also found that both tests had good efficiencies over broad class of multivariate *t*-distributions and efficiencies were better for higher dimensions. Mottonen et al. (1997) found that the efficiencies of the spatial sign test agreed with efficiencies of the sign test based on the Oja median (Brown & Hettmansperger, 1987a, 1987b; Oja, 1983; Hettmansperger et al., 1994). Oja and Randles (2004) also calculated the asymptotic relative efficiency (ARE) for the multivariate sign test and the signed-rank test relative to Hotelling's T^2 under the multivariate *t*-distribution with selected values of degrees of freedom ν and with selected dimensions *k.*

Oja and Randles (2004) also investigated the ARE properties of some affine-invariant sign and signed-rank tests. They found that as the dimension increased and as the distribution got heavier tailed or when the degrees of freedom got smaller, the performance of the sign and signed-rank tests improved relative to Hotelling's T^2, indicating that the multivariate nonparametric test were clearly better in heavy-tailed cases. For example, for the heavy-tailed multivariate *t*-distribution with degrees of freedom ν= 3, the AREs of the affine -invariant signed-rank test relative to Hotelling's T^2 were 1.9, 1.95, 2.02, and 2.09 for dimensions 1, 2, 4, and 10, respectively (Oja & Randles, 2004). These AREs demonstrated that the affine-invariant spatial signed-rank

test is a viable alternative to the Hotelling's T^2, indicating when the underlying distribution was not multivariate normal.

Liu (1995) introduced three robust nonparametric multivariate control charts (RMVCC): the *r*, *Q*, and *S* charts. Liu's robust multivariate control charts are based on the notion of data depth and do not require any underlying distributional assumptions. The principal idea in constructing these RMVCC is the reduction of each vector of observations $\boldsymbol{X}' = (X_1, \dots, X_p)$ to a univariate rank based on the notion of data depths. These ranks are then used to construct the multivariate control charts. Liu suggested that for any point $\boldsymbol{X} \in \mathbb{R}^p$, the simpilical depth of $\boldsymbol{X}$ with respect to a distribution G is given by

$$SD_G(\boldsymbol{X}) = P_G\,\{\boldsymbol{X} \in s[X_1, \dots, X_{k+1}]\}, \tag{93}$$

where $s[X_1, \dots, X_{k+1}]$ is a simplex whose vertices $X_1, \dots, X_{k+1}$ are k+1 random observations from G. The simpilical depth SD_G is a measure of how deep or central the point X is with respect to the distribution G. However, most often, G is unknown and SD_G will be estimated empirically from a sample of points $X_1, \dots, X_m$. The empirical simpilical depth with respect to the sample $X_1, \dots, X_m$ is given by

$$SD_{Gm}(\boldsymbol{X}) = \binom{m}{k+1}^{-1} \sum_{All\ subsets\ of\ X_1,\dots,X_m\ of\ size\ (k+1)} I\,\{\boldsymbol{X} \in s[X_{i1}, \dots, X_{ik+1}]\}, \tag{94}$$

where *Gm* is the empirical distribution of $X_1, \dots, X_m$, I is the indicator function; that is , *I(A)* =1 if A occurs and *I(A)* = 0 otherwise. In the RMVCC case, the sample $X_1, \dots, X_m$ is considered to be the base period sample and the point X is considered to be an observation from the control period. The Phase I sample is assumed to come from a distribution *G* while the Phase II sample is assumed to come from a distribution *F*. Both

distributions *G* and *F* distributions are assumed to be unknown and if the process is in-control, then *G* = *F*. Otherwise, *G* ≠ *F*.

The robust multivariate control charts proposed by Liu (1995) are summarized as follows: For an assumed in-control process in Phase I, take a sample $X_1, \ldots, X_m$, and then for each observation X in Phase II, consider the test statistic:

$$r_{Gm}(\boldsymbol{X}) = \frac{1}{m}\sum_{j=q}^{m} I\left(D_{Gm}(X_j) \le D_{Gm}(\boldsymbol{X})\right), \tag{95}$$

Where I is the indicator function such that *I (A)* =1 if the data depth X_j is less than or equal to the data depth of **X** and *I(A)* = 0 otherwise. The quantity $r_{Gm}(\boldsymbol{X})$ measures the outlying of the point X with respect to the sample $X_1, \ldots, X_m$. Smaller values of $r_{Gm}(\boldsymbol{X})$ are desired. Liu's robust nonparametric multivariate control charts (the *r*, *Q*, and *S* charts) are based on the quantity $r_{Gm}(\boldsymbol{X})$.

The *r* control chart is constructed by taking a Phase I sample of *m* observations $X_1, \ldots, X_m$. For each observation X_t^* in Phase II, compute the charting statistic $r_{Gm}(\boldsymbol{X}^*)$ versus time (*t =1, 2, …*). The control limits are defined by setting the center line *CL = 0.5* and the lower control limit *LCL = α*. These control limits are based on the asymptotic distribution of $r_{Gm}(\boldsymbol{X}^*) \sim U[0,1]$. The asymptotic distribution of $r_{Gm}(\boldsymbol{X}^*)$ suggests that *LCL = α*. The process is out-of-control whenever the value $r_{Gm}(\boldsymbol{X}^*)$ is below *LCL = α*.

Similarly, the *Q* control chart is constructed by taking a Phase I sample of *m* observations $(X_1, \ldots, X_m)$. Consider taking several samples of size *n* $(X_1^*, \ldots, X_n^*)$ from Phase II. For each observationX_t^* in Phase II, compute the charting statistic $r_{Gm}(\boldsymbol{X}^*)$.

The Q chart is constructed based on plotting the average of $r_{Gm}(\boldsymbol{X}^*)$ versus time for the points taken from Phase II, such that

$$Q^t(G_m, F_n) = \frac{1}{n}\sum_{i=1}^{n} r_{Gm}(X_i^*), \tag{96}$$

where G_m and F_n are the empirical distributions of $(X_1, \ldots, X_m)$ and $(X_1^*, \ldots, X_n^*)$, respectively. Liu and Singh (1993) have shown by simulation studies that the statistic $Q^t(G_m, F_n)$ is asymptotically distributed as $N(\frac{1}{2}, \frac{1}{12}\left(\frac{1}{m}+\frac{1}{n}\right))$. The control limits of Q are set as

$$CL = 0.5, \text{and}$$
$$LCL = 0.5 - Z_\alpha\sqrt{\frac{1}{12}[\frac{1}{m}+\frac{1}{n}]}. \tag{97}$$

The process is out-of-control whenever the value $Q^t(G_m, F_n)$ falls below LCL.

The S control chart is analogous to a univariate CUSUM chart for process location and is constructed by taking a Phase I sample of *m* observations $(X_1, \ldots, X_m)$. Then for each observation X_t^* in Phase II, compute the statistic $r_{Gm}(\boldsymbol{X}^*)$. The S control chart is based on the charting statistic

$$S^t(G_m) = \sum_{i=1}^{t}\left[r_{Gm}(X_i^*) - \frac{1}{2}\right], \tag{98}$$

which also can be written as

$$S^t(G_m) = n\left[Q^t(G_m, F_n) - \frac{1}{2}\right]. \tag{99}$$

Liu (1995) suggested that the statistic $S^t(G_m)$ is asymptotically distributed as $N\left(0, \frac{n^2[\frac{1}{m}+\frac{1}{n}}{12}\right)$. Therefore, the S control chart limits are defined as

$$CL = 0.5, \text{and}$$

$$LCL = -Z_\alpha \sqrt{\frac{n^2[\frac{1}{m} + \frac{1}{n}}{12}}. \quad (100)$$

The process is out-of-control whenever the value $S^t(G_m)$ falls below LCL.

Several multivariate statistical process control methods are based on parametric and nonparametric assumptions. The parametric methods were designed with the assumption of multivariate normality to describe the sample. The nonparametric methods do not assume a known joint distribution of the *p* quality characteristics, although it requires the assumption that the sample units are iid and exchangeable. The parametric methods assume that the joint probability distribution of the *p* quality characteristics is the *p*-variate normal distribution from a sample of size *n* (Montgomery, 2009; Stoumbos & Sullivan, 2002). There is considerable dichotomy of opinions regarding the statistical performance of MEWMA when normality assumptions are violated. Stoumbos and Sullivan (2002) argue that statistical performance of an appropriately designed MEWMA control chart is robust under non-normality and comes close to being distribution free. However, Stoumbos and Sullivan continue to argue that the MEWMA chart would be preferable to the nonparametric MEWMA control charts that are less powerful, are computationally more intensive than their multi-normal counterparts, and do not apply to heavily skewed multivariate distributions like the multivariate gamma distribution. On the other hand, others argue for the need of a distribution-free MEMA control chart (Woodall & Montgomery, 1999).

According to Oja (2010), consider a model

$$\boldsymbol{y}_i = \boldsymbol{\mu} + \boldsymbol{\Omega\varepsilon_i}, i = 1, \ldots, n. \quad (101)$$

We assume that the errors are independent and identically distributed random vectors from spherically symmetrical and continuous distribution. We say that the distribution of the error vector $\boldsymbol{\varepsilon}$ is spherically symmetrical around the origin if the density function $f(\varepsilon)$ depends on $\boldsymbol{\varepsilon}$ only through the modulus $|\boldsymbol{\varepsilon}|$. The modulus $r_i = |\boldsymbol{\varepsilon}|$ and direction $\boldsymbol{u}_i = |\varepsilon_i|^{-1}\varepsilon_i$ are independent and the direction of the vector $\boldsymbol{u}_i$ is uniformly distributed on the *p*-dimensional unit sphere where

$$\boldsymbol{E}(\boldsymbol{u_i}) = \mathbf{0} \tag{102}$$

and

$$\boldsymbol{COV}(\boldsymbol{u}_i) = E(\boldsymbol{u}_i\boldsymbol{u}_i') = \frac{1}{p}I_p\,. \tag{103}$$

$\boldsymbol{\Omega}$ is a full rank p x p transformation matrix and the regular covariance matrix of the multivariate normal distribution is $\boldsymbol{\Sigma} = \boldsymbol{\Omega\Omega'}$. Under these assumptions, the random sample $\boldsymbol{Y} = (\boldsymbol{y}_1, \dots, \boldsymbol{y}_n)'$ comes from *p*-variate elliptical distribution with probability density function

$$f_y(\boldsymbol{y}) = |\boldsymbol{\Sigma}|^{-\frac{1}{2}} f\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{y} - \mu)\right), \tag{104}$$

where μ is the symmetry center and $\boldsymbol{\Sigma} > 0$ is the scatter matrix. The matrix $\boldsymbol{\Sigma}^{-\frac{1}{2}}$ is chosen to be symmetric. The location parameter μ is the mean vector, the scatter matrix $\boldsymbol{\Sigma}$ is proportional to the regular covariance matrix, and the correlation matrix $= [diag(\boldsymbol{\Sigma})]^{-\frac{1}{2}}\boldsymbol{\Sigma}\,[diag(\boldsymbol{\Sigma})]^{-\frac{1}{2}}$. Then we have

$$\boldsymbol{y}_i \sim E_p(\mu, \boldsymbol{\Sigma}, \rho). \tag{105}$$

Methods for accomplishing the monitoring task are usually based on the following quadratic formulation of the test statistics:

$$(\boldsymbol{y_i} - \widehat{\boldsymbol{\mu}}_\mathbf{0}))' \, \widehat{\boldsymbol{\Sigma}^{-1}}_\mathbf{0}(\boldsymbol{y_i} - \widehat{\boldsymbol{\mu}}_\mathbf{0}) \,, \tag{106}$$

where $\widehat{\boldsymbol{\mu}}_\mathbf{0}$ and $\widehat{\Sigma}_0$ are the mean vector and covariance matrix estimated from the in-control sample.

Several authors showed that a nonparametric control chart can be designed while being computationally efficient and as statistically powerful or better than the multivariate normal counterparts. Qui and Hawkins (2001) proposed and developed a nonparametric MCUSUM procedure for detecting shifts in the location vector of a multivariate measurement statistical process based both on the order information among measurements components and on the order information between the measurement components and their in-control means. Qui and Hawkins' nonparametric MCUSUM is distribution-free in the sense that its properties depend only on the ordering of the measurements components and the ordering between the measurement components and their in-control means. The distribution-free property makes the nonparametric MCUSUM appropriate to use when the potential shifts in mean vectors of the process can occur in all possible directions and the underlying distribution is not multivariate normal.

Zou and Tsung (2010) used a nonparametric multivariate exponentially weighted moving average (MSEWMA) control chart for monitoring location parameters. Zou and Tsung modified Randles' (2000) nonparametric directional multivariate sign test statistic by estimating an affine equivariant multivariate median or AEM-median $\boldsymbol{\theta_0}$ (Hettmansperger & Randles, 2002) to develop a multivariate sign exponentially weighted moving average (MSEWMA) control chart. After standardizing and transforming the $\boldsymbol{X_i}$ and calculating a modified statisic

$$V_i = \frac{\boldsymbol{A}_x(\boldsymbol{X_i} - \boldsymbol{\theta_0})}{\|\boldsymbol{A}_x(\boldsymbol{X_i} - \boldsymbol{\theta_0})\|}, \tag{107}$$

they define a EWMA sequence

$$\boldsymbol{W}_i = (I - \lambda)\boldsymbol{W}_{i-1} + \lambda \boldsymbol{V}_i. \tag{108}$$

They proposed a control chart that triggers a signal if

$$Q_i = \frac{2-\lambda}{\lambda} p \boldsymbol{W}_i' \boldsymbol{W}_i > L, \tag{109}$$

where L > 0 is the control limit chosen to achieve a specific in-control average run length (IC ARL).

MSEWMA adapts the multivariate sign test using an affine equivariant multivariate median (AEM-Median) developed by Hettmansperger and Randles (2002) to create a new test statistic. The resulting statistic is used as a charting statistic to develop a new nonparametric counterpart of the MEWMA. MSEWMA is easy to implement because only the affine-equivariant multivariate median and the transformation-retransformation matrix need to be estimated from the reference (Phase I) data set using an algorithm developed by Tyler (1987) in the same manner as estimating the mean and covariance matrix in a parametric MEWMA setting. Zou and Tsung (2010) showed that MSEWMA is robust in attaining the in-control (IC) ARL and is efficient in detecting small to moderate location shifts for heavy-tailed or skewed distributions. MSEWMA is computationally fast, easy to implement, and it outperformed the MEWMA control chart in detecting small to moderate shifts of data from the multivariate *t*-distribution and multivariate gamma distribution..

Zou et al. (2011) argued that the MSEWMA control chart developed by Zou and Tsung (2010) could result in a significant uncertainty in parameter estimation when Phase

I sample size is small. Zou et al. developed a spatial rank-based MEWMA or empirical rank-based EWMA (EREWMA) control chart using spatial ranks where the weighted version of the rank test was used to compute the charting statistic by incorporating a MEWMA scheme. In the univariate case, signs and ranks are based on the ordering of the data; however, in the multivariate case, there is no natural ordering of data points (Oja, 1983; Randles, 2000). Oja (2010) developed the concept of spatial signs and ranks. The spatial sign function was defined by Oja as

$$U_X(\boldsymbol{x}) = \begin{cases} \|\boldsymbol{x}\|^{-1}x, & x \neq 0 \\ 0, & x = 0' \end{cases} \tag{110}$$

where $\|\boldsymbol{x}\| = (\boldsymbol{x}'\boldsymbol{x})^{\frac{1}{2}}$ is the Euclidean length of the vector $\boldsymbol{x}$. The theoretical spatial rank and signed-rank are

$$R_x(\boldsymbol{x}) = \frac{1}{n}\sum_{1}^{n}\{U_x(\boldsymbol{x} - \boldsymbol{x_i})\}, \tag{111}$$

and

$$Q(\boldsymbol{x}) = \frac{1}{2}[R_X(\boldsymbol{x}) + R_{-X}(\boldsymbol{x})], \tag{112}$$

where $R_{-X}(\boldsymbol{x}) = -R_X(\boldsymbol{x})$.

Zou et al. (2011) used an affine invariant version of a test statistic based on the ranks $R_x(\boldsymbol{x})$ to define a MEWMA-type control chart where the MEWMA sequence is

$$\boldsymbol{y_t} = (1-\lambda)\boldsymbol{y}_{t-1} + \lambda R_x(\boldsymbol{M}\boldsymbol{x_i}), \text{ and} \tag{113}$$

where $\boldsymbol{M} = \boldsymbol{\Omega^{-1}}$ such that a scatter matix $\boldsymbol{\Sigma} = \boldsymbol{\Omega\Omega}' > 0$ is used to make the test affine invariant. vThe charting statistic is

$$Q = \frac{(2-\lambda)}{\lambda}\boldsymbol{y}_t'\{covR_x(M\boldsymbol{x_i}\}^{-1}\boldsymbol{y}_{t\,.} \tag{114}$$

Zou et al. called this the theoretical rank-based MEWMA or TRMEWMA. The authors showed that their scheme is more appropriate in self-starting situations. The theoretical rank-based nonparametric MEWMA has distribution-free properties in the sense that its IC ARL is very close to the nominal ARL for parametric MEWMA. The empirical rank-based MEWMA charts are easy to compute, computationally efficient, robust to non-normality, and very efficient in detecting multivariate process shifts, especially when data come from a heavy-tailed or skewed distribution. The authors proceed to show that this rank-based MEWMA scheme is more robust in its in-control (IC) performance and generally more sensitive to small and moderate shifts in location parameters.

CHAPTER III

METHODOLOGY

The Spatial Nonparametric Signed-Ranks

I utilized a nonparametric test statistic based on the work of Hettmansperger et al. (1997). Hettmansperger et al. (1997) developed a multivariate affine-invariant (equivariant) signed-rank test (estimates) by extending the work of Brown and Hettmansperger (1987a, 1987b). Hettmansperger and McKean (2011) stated that this is a delicate problem since there is no natural way to order or rank vectors. Let $\boldsymbol{X_1}, \ldots \boldsymbol{X_n}$ be a random sample from a continuous *k*-varite distribution. Let

$$P = \{p = (i_1, \ldots, i_k): 1 \leq i_1 < \cdots < i_k \leq n\} \tag{115}$$

be the set of $N_P = \binom{n}{k}$ different *k*-tuples of an index set $\{1, \ldots, n\}$. Index $p \in P$ then refers to a *k*-subset of the original observations. The volume of the simplex determined by $p \in P$ along with $\boldsymbol{X}$ is

$$V_p(\boldsymbol{x}) = V(\boldsymbol{x}_{i1}, \ldots, \boldsymbol{x}_{ik}, \boldsymbol{x}) = \frac{1}{k!}\left|d_{0p} + \boldsymbol{X}^T \boldsymbol{d}_p\right|, \tag{116}$$

where $d_{0p} = (-1)^k det(X_{i1}, \ldots, X_{ik})$ and $\boldsymbol{d}_p$ is the *k*-dimensional vectors of cofactors of $\boldsymbol{X}$ in $det\begin{pmatrix} 1 & \cdot\cdot & 1 & 1 \\ \boldsymbol{x}_{i1} & \cdot\cdot & \boldsymbol{x}_{ik} & \boldsymbol{x} \end{pmatrix}$. The Oja (1983) objective function is

$$D_n(\boldsymbol{\theta}) = \frac{1}{N_p} = \frac{1}{\binom{n}{k}} \sum_{p \in P} V_p(\boldsymbol{\theta}), \tag{117}$$

and the Oja median $\hat{\boldsymbol{\theta}}$ minimizes $D_n(\boldsymbol{\theta})$. The Oja median is affine equivariant and has efficiency at the multivariate normal distribution that increases with dimension k (Oja & Niinimaa, 1985). Following Hettmansperger et al. (1994), Hettmansperger et al. (1997) took the gradient of $k!\,V_p(\boldsymbol{X})$ to be the sign vector of X relative to the hyper-plane $p = (i_1, \dots, i_k) \in P$. Let $\boldsymbol{Q}_p(\boldsymbol{X}) = k!\,V_p(\boldsymbol{X})$; then

$$\boldsymbol{Q}_p(\boldsymbol{x}) = \boldsymbol{S_p}(\boldsymbol{x})\boldsymbol{d}_{P\,,} \tag{118}$$

where

$$\boldsymbol{S_p}(\boldsymbol{x}) = sgn\left(d_{0p} + \boldsymbol{x}^T \boldsymbol{d}_p\right). \tag{119}$$

We say that $\boldsymbol{x}$ is above the hyper-plane p if $\boldsymbol{S_p}(\boldsymbol{x}) > 0$ and below if $\boldsymbol{S_p}(\boldsymbol{x}) < 0$. The centered vector rank function, analogous to the univariate case, is defined to be the mean of the signs with respect to all possible p's (variables),

$$\boldsymbol{R}_n(x) = N_P^{-1} \sum_{p \in P} \boldsymbol{Q}_p(\boldsymbol{x}). \tag{120}$$

Now, let A be the set of 2^k possible vectors $(\pm 1, \dots, \pm 1)$ and define

$$\boldsymbol{Q}_p^+(\boldsymbol{x}) = 2^{-k} \sum_{a \in A} \boldsymbol{S_{pa}}(\boldsymbol{x})\boldsymbol{d}_{pa}, \tag{121}$$

where $\boldsymbol{S_{pa}}(\boldsymbol{x})\boldsymbol{d}_{pa}$ is the gradient of

$$\frac{1}{k!} abs\{\det\begin{pmatrix} 1 & \cdot\cdot & 1 & 1 \\ a_1 \boldsymbol{x}_{i1} & \cdot\cdot & a_k \boldsymbol{x}_{ik} & \boldsymbol{x} \end{pmatrix}\} \text{ and } p = (i_1, \dots, i_k). \tag{122}$$

Now, define the ***signed-rank*** function as

$$\boldsymbol{R}_n^+(\boldsymbol{x}) = N_P^{-1} \sum_{p \in P} \boldsymbol{Q}_p^+(\boldsymbol{x}). \tag{123}$$

Hettmansperger et al. (1997) showed that both $\boldsymbol{R}_n^+(\boldsymbol{x})$ and $\boldsymbol{Q}_p^+(\boldsymbol{x})$ are odd: $\boldsymbol{R}_n^+(\boldsymbol{x}) = -\,\boldsymbol{R}_n^+(\boldsymbol{x})$ and $\boldsymbol{Q}_p^+(-\boldsymbol{x}) = -\boldsymbol{Q}_p^+(\boldsymbol{x})$.

**A Computational Example of the Signed-
Ranks Observation Vectors**

We now illustrate the computation of the signed-ranks on a data set. The data consist of thickness of cork borings for 28 trees from four different directions: North, East, South, and West (Hettmansperger et al., 1997; Rao, 1948). We first reduce the data to tri-variate (contrast) observations N-E, E-S, and S-W. Table 4 provides the original quad-variate data along with the tri-variate data. Table 5 gives the signed-rank vectors of the tri-variate data. Appendix B displays the SAS code and interactive matrix language (IML) routines for the computation of the signed-rank vectors. The signed-ranks vectors in Table 5 are duplicate results validated by Hettmansperger et al. (1997) in their example using the same data.

Table 4

Cork Boring Data: North (N), East (E), South (S), West (W), and the Difference N-E, E-S, and S-W

Tree	N	E	S	W	N-E	E-S	S-W
1	72	66	76	77	6	-10	-1
2	60	53	66	63	7	-13	3
3	56	57	64	58	-1	-7	6
4	41	29	36	38	12	-7	-2
5	32	32	35	36	0	-3	-1
6	30	35	34	26	-5	1	8
7	39	39	31	27	0	8	4
8	42	43	31	25	-1	12	6
9	37	40	31	25	-3	9	6
10	33	29	27	36	4	2	-9
11	32	30	34	28	2	-4	6
12	63	45	74	63	18	-29	11
13	54	46	60	52	8	-14	8
14	47	51	52	43	-4	-1	9
15	91	79	100	75	12	-21	25
16	56	68	47	50	-12	21	-3
17	79	65	70	61	14	-5	9
18	81	80	68	58	1	12	10
19	78	55	67	60	23	-12	7
20	46	38	37	38	8	1	-1
21	39	35	34	37	4	1	-3
22	32	30	30	32	2	0	-2
23	60	50	67	54	10	-17	13
24	35	37	48	39	-2	-11	9
25	39	36	39	31	3	-3	8
26	50	34	37	40	16	-3	-3
27	43	37	39	50	6	-2	-11
28	48	54	57	43	-6	-3	14

Table 5

Signed-Rank Vectors for the N-E, E-S, and S-W Original Observation in Table 4

Tree	SR(N-E)	SR(E-S)	SR(S-W)
1	5.2	-66.9	-47.2
2	-4.5	-70.8	-17.0
3	-68.6	-70.9	24.6
4	96.9	5.8	-28.7
5	-42.8	-54.4	-30.6
6	-48.3	15.4	73.2
7	74.1	94.5	60.5
8	69.7	107.4	71.5
9	28.0	82.9	81.7
10	62.9	25.3	-65.2
11	8.6	-3.0	54.7
12	22.7	-64.5	2.6
13	10.0	-48.2	24.9
14	-44.4	-0.8	76.7
15	15.6	-8.5	88.2
16	-2.2	84.8	36.4
17	113.3	71.0	63.9
18	83.0	105.9	89.5
19	122.3	44.4	26.3
20	116.9	77.0	12.7
21	78.7	41.8	-13.3
22	37.3	12.4	-16.0
23	19.4	-32.7	54.1
24	-93.2	-92.2	25.0
25	42.4	35.4	81.1
26	129.7	58.2	-12.8
27	37.7	-30.5	-95.6
28	-68.1	-19.3	86.0

A Detailed Signed-Rank Numerical Example

The computations of the invariant signed-ranks are highly intensive and a stochastic algorithm is used to calculate the signed-rank estimates by sampling observation hyperplanes (Oja, 1983). The following example is presented in summary form; the full example can be found in Appendix B.

Let $\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}$ be a random sample from a continuous *3*-variate distribution. $\boldsymbol{X}$ is defined in (124) as a (*3 x 3*) matrix of $n = 3$ observations on 3 variables. Hence, $k = 3$ and $n = 3$, and

$$\boldsymbol{X} = \begin{bmatrix} 6 & -10 & 12 \\ -7 & 13 & -11 \\ 5 & 7 & 15 \end{bmatrix} \tag{124}$$

$$P = \{p = (i_1, i_2, i_3) : i_1 < i_2 < i_3 \leq n\} \tag{125}$$

be the set of $N_p = \binom{n}{k}$ different k-tuples of index set {1,2,3}. In this example, there is only one set of $N = \binom{3}{3} = \frac{3!}{3!} = 1$. Therefore, the set $P = \{p = (1,2,3)\}$ and the index $p \in P$ refer to a k-subset of the original observations.

Recall the multivariate sign Equation 121 defined below as 126.

$$\boldsymbol{Q}_p^{+}(\boldsymbol{x}) = 2^{-k} \sum_{a \in A} \boldsymbol{S_{pa}}(\boldsymbol{x}) \boldsymbol{d}_{pa}. \tag{126}$$

Since k=3,

$$\boldsymbol{Q}_p^{+}(\boldsymbol{X}) = \frac{1}{8} \sum_{a \in A} \boldsymbol{S_{pa}}(\boldsymbol{x}) \boldsymbol{d}_{pa}., \tag{127}$$

Also, the signed-rank function from Equation 123 for $N_P^{-1} = 1$is

$$\boldsymbol{R}_n^{+}(\boldsymbol{x}) = \sum_{p \in P} \boldsymbol{Q}_p^{+}(\boldsymbol{x}) \,. \tag{128}$$

Substituting (126) into (127), we get the empirical signed-rank function

$$\boldsymbol{R}_n^+(\boldsymbol{x}) = \frac{1}{8}\sum_{\boldsymbol{a}\in \boldsymbol{A}} \boldsymbol{S}_{\boldsymbol{pa}}(\boldsymbol{x})\boldsymbol{d}_{\boldsymbol{pa}}\,. \tag{129}$$

The multivariate sign is an average of all possible vector set *A*, such that *A* is the set of 2^k possible vectors$(\pm 1\,,\quad \pm 1,\quad \pm 1)$. Since k = 3, we have a set *A* with 8 possible vectors:

$$\begin{aligned} A = \{a_0 &= [-1 \quad -1 \quad -1], \\ a_1 &= [1 \quad -1 \quad -1], \\ a_2 &= [-1 \quad 1 \quad -1], \\ a_3 &= [1 \quad 1 \quad -1], \\ a_4 &= [-1 \quad -1 \quad 1], \\ a_5 &= [1 \quad -1 \quad 1], \\ a_6 &= [-1 \quad 1 \quad 1], \\ a_7 &= [1 \quad 1 \quad 1]\}. \end{aligned} \tag{130}$$

Let

$$\boldsymbol{X}` = \begin{bmatrix} 6 & -7 & 5 \\ -10 & 13 & 7 \\ 12 & -11 & 15 \end{bmatrix}. \tag{131}$$

1. The signed-rank vector $\boldsymbol{R}_n^+(\boldsymbol{x})$ for $\boldsymbol{x}`_{\boldsymbol{1}}$.

We start by calculating the sign vector $\boldsymbol{Q}_1^+(\boldsymbol{x})$ and signed-rank vector $\boldsymbol{R}_n^+(\boldsymbol{x})$ for $\boldsymbol{x}`_{\boldsymbol{1}} = \begin{bmatrix} 6 \\ -10 \\ 12 \end{bmatrix}$. We will calculate the sign vector $\boldsymbol{Q}_1^+(\boldsymbol{x})$ and signed-rank vector $\boldsymbol{R}_n^+(\boldsymbol{x})$ for the other vector components, $\boldsymbol{x}`_{\boldsymbol{2}} = \begin{bmatrix} -7 \\ 13 \\ -11 \end{bmatrix}$, and $\boldsymbol{x}`_{\boldsymbol{3}} = \begin{bmatrix} 5 \\ 7 \\ 15 \end{bmatrix}$ in the same manner.

Calculate all vectors $a_t p\, :\, t = 0,1,2,3,4,5,6,7$ and check if $i = 1\ \in \boldsymbol{a_i p}$. If $i = 1\ \in \boldsymbol{a_i p}$, then $\boldsymbol{S}_{\boldsymbol{pa}}(\boldsymbol{x})\boldsymbol{d}_{pa} = 0$. So by using element–wise multiplication, #, we have

$$\boldsymbol{a_0 p} = [-1 \quad -1 \quad -1]\#[1 \quad 2 \quad 3] = [-1 \quad -2 \quad -3],$$

$$\boldsymbol{a_1 p} = [1 \quad -1 \quad -1]\#[1 \quad 2 \quad 3] = [\mathbf{1} \quad -2 \quad -3],$$

$$\boldsymbol{a_2 p} = [-1 \quad 1 \quad -1]\#[1 \quad 2 \quad 3] = [-1 \quad 2 \quad -3],$$

$$\boldsymbol{a_3 p} = [1 \quad 1 \quad -1]\#[1 \quad 2 \quad 3] = [\mathbf{1} \quad 2 \quad -3],$$

$$\mathbf{a_4}\boldsymbol{p} = [-1 \quad -1 \quad 1]\#[1 \quad 2 \quad 3] = [-1 \quad -2 \quad 3],$$

$$\boldsymbol{a_5 p} = [1 \quad -1 \quad 1]\#[1 \quad 2 \quad 3] = [\mathbf{1} \quad -2 \quad 3],$$

$$\boldsymbol{a_6 p} = [-1 \quad 1 \quad 1]\#[1 \quad 2 \quad 3] = [-1 \quad 2 \quad 3],$$

$$\boldsymbol{a_7 p} = [1 \quad 1 \quad 1]\#[1 \quad 2 \quad 3] = [\mathbf{1} \quad 2 \quad 3],$$

Note that since $i = 1 \ \in \boldsymbol{a_i p}$ for $i = 1, 3, 5, \& \ 7$. Therefore, $\boldsymbol{Q}_p^{+}(\boldsymbol{x}) = \frac{1}{8}\sum_{a \in A} \boldsymbol{S_{pa}}(\boldsymbol{x})\boldsymbol{d}_{pa}$ is determined by vectors $\boldsymbol{a_0}, \boldsymbol{a_2}, \boldsymbol{a_4}, \& \ \boldsymbol{a_6}$only. We now calculate the component $\boldsymbol{S_{pa}}(\boldsymbol{x})\boldsymbol{d}_{pa}$ for $\boldsymbol{a_0}, \boldsymbol{a_2}, \boldsymbol{a_4}, \& \ \boldsymbol{a_6}$only.

Define

$$\boldsymbol{Y_i} = \boldsymbol{a_i}\#\boldsymbol{X}`,$$

$$\boldsymbol{W_1} = (\boldsymbol{Y_i} = [\ \boldsymbol{y_2} \quad \boldsymbol{y_3}]) - [\ \boldsymbol{x_i} \quad \boldsymbol{x_i}] : \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{or}\ \mathbf{3},$$

$$\boldsymbol{W_2} = \boldsymbol{Y_0} - [\boldsymbol{x_i} \quad \boldsymbol{x_i} \quad \boldsymbol{x_i}] : \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{or}\ \mathbf{3},$$

(132)

$$\boldsymbol{d_{pa}} = \begin{bmatrix} (-1)^1 \left|\boldsymbol{W_1} = \begin{bmatrix} \boldsymbol{w_2} \\ \boldsymbol{w_3} \end{bmatrix}\right| \\ (-1)^2 \left|\boldsymbol{W_1} = \begin{bmatrix} \boldsymbol{w_1} \\ \boldsymbol{w_3} \end{bmatrix}\right| \\ (-1)^3 \left|\boldsymbol{W_1} = \begin{bmatrix} \boldsymbol{w_1} \\ \boldsymbol{w_2} \end{bmatrix}\right| \end{bmatrix}, or$$

$$d_{pa} = \begin{bmatrix} -1 \left| W_1 = \begin{bmatrix} w_2 \\ w_3 \end{bmatrix} \right| \\ 1 \left| W_1 = \begin{bmatrix} w_1 \\ w_3 \end{bmatrix} \right| \\ -1 \left| W_1 = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right| \end{bmatrix}, \text{ and}$$

$$S_{pa}d_{pa} = sign(|W_2|d_{pa}).$$

We now calculate W_1, W_2, d_{p0}, and $S_{p0}d_{p0}$ as follows:

$$Y_0 = [-1 \quad -1 \quad -1]\# \begin{bmatrix} 6 & -7 & 5 \\ -10 & 13 & 7 \\ 12 & -11 & 15 \end{bmatrix} = \begin{bmatrix} -6 & 7 & -5 \\ 10 & -13 & -7 \\ -12 & 11 & -15 \end{bmatrix},$$

we have

$$W_1 = (Y_0 = [\, y_2 \quad y_3]) - [\, x_i \quad x_i],$$

$$W_1 = \begin{bmatrix} 7 & -5 \\ -13 & -7 \\ 11 & -15 \end{bmatrix} - \begin{bmatrix} -6 & -6 \\ 10 & 10 \\ -12 & -12 \end{bmatrix} = \begin{bmatrix} 13 & 1 \\ -23 & -17 \\ 23 & -3 \end{bmatrix},$$

$$W_2 = Y_0 - [x_1 \quad x_1 \quad x_1],$$

$$W_2 = \begin{bmatrix} -6 & 7 & -5 \\ 10 & -13 & -7 \\ -12 & 11 & -15 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ -10 & -10 & -10 \\ 12 & 12 & 12 \end{bmatrix} = \begin{bmatrix} -12 & 1 & -11 \\ 20 & -3 & 3 \\ -24 & -1 & -27 \end{bmatrix},$$

$$d_{p0} = \begin{bmatrix} -1 \begin{vmatrix} -23 & -17 \\ 23 & -3 \end{vmatrix} \\ 1 \begin{vmatrix} 13 & 1 \\ 23 & -3 \end{vmatrix} \\ -1 \begin{vmatrix} 13 & 1 \\ -23 & -17 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -1(460) \\ 1(-62) \\ -1(-198) \end{bmatrix} = \begin{bmatrix} -460 \\ -62 \\ 198 \end{bmatrix},$$

and the sign vector is

$$\boldsymbol{S_{p0}d_{p0}} = sign\left(|\boldsymbol{W_2}|\boldsymbol{d_{p0}}\right) = sign\left(\left|\begin{bmatrix} -12 & 1 & -11 \\ 20 & -3 & 3 \\ -24 & -1 & 27 \end{bmatrix}\right| * \begin{bmatrix} -460 \\ -62 \\ 198 \end{bmatrix}\right) = \begin{bmatrix} -460 \\ -62 \\ 198 \end{bmatrix}.$$

Using the same formulas in Equation 132, we calculate

$\boldsymbol{W_1}, \boldsymbol{W_2}, \boldsymbol{d_{p2}},$ and $\boldsymbol{S_{p2}d_{p2}}$ for $\mathbf{Y_2}, \boldsymbol{Y}_4,$ and $\boldsymbol{Y_6}$ as

$$\boldsymbol{Y_2} = [-1 \quad 1 \quad -1] \# \begin{bmatrix} 6 & -7 & 5 \\ -10 & 13 & 7 \\ 12 & -11 & 15 \end{bmatrix} = \begin{bmatrix} -6 & -7 & -5 \\ 10 & 13 & -7 \\ -12 & -11 & -15 \end{bmatrix},$$

$$\boldsymbol{W_1} = (\boldsymbol{Y_2} = [\,\boldsymbol{y_2} \quad \boldsymbol{y_3}]) - [\,\boldsymbol{x_i} \quad \boldsymbol{x_i}]$$

$$\boldsymbol{W_1} = \begin{bmatrix} -7 & -5 \\ 13 & -7 \\ -11 & -15 \end{bmatrix} - \begin{bmatrix} -6 & -6 \\ 10 & 10 \\ -12 & -12 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3 & -17 \\ 1 & -3 \end{bmatrix},$$

$$\boldsymbol{W_2} = \boldsymbol{Y_2} - [\boldsymbol{x_1} \quad \boldsymbol{x_1} \quad \boldsymbol{x_1}],$$

$$\boldsymbol{W_2} = \begin{bmatrix} -6 & -7 & -5 \\ 10 & 13 & -7 \\ -12 & -11 & -15 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ -10 & -10 & -10 \\ 12 & 12 & 12 \end{bmatrix} = \begin{bmatrix} -12 & -13 & -11 \\ 20 & 23 & 3 \\ -24 & -23 & -27 \end{bmatrix},$$

$$\boldsymbol{d_{p2}} = \begin{bmatrix} -1\begin{vmatrix} 3 & -17 \\ 1 & -3 \end{vmatrix} \\ 1\begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} \\ -1\begin{vmatrix} -1 & 1 \\ 3 & -17 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -1(8) \\ 1(2) \\ -1(14) \end{bmatrix} = \begin{bmatrix} -8 \\ 2 \\ -14 \end{bmatrix},$$

and the sign vector is

$$\boldsymbol{S_{p2}d_{p2}} = sign(|\boldsymbol{W_2}|\boldsymbol{d_{p2}}) = sign\left(\left|\begin{bmatrix} -12 & -13 & -11 \\ 20 & 23 & 3 \\ -24 & -23 & -27 \end{bmatrix}\right| * \begin{bmatrix} -8 \\ 2 \\ -14 \end{bmatrix}\right) = \begin{bmatrix} 8 \\ -2 \\ 14 \end{bmatrix}.$$

$$\boldsymbol{Y_4} = [-1 \quad -1 \quad 1]\# \begin{bmatrix} 6 & -7 & 5 \\ -10 & 13 & 7 \\ 12 & -11 & 15 \end{bmatrix} = \begin{bmatrix} -6 & 7 & 5 \\ 10 & -13 & 7 \\ -12 & 11 & 15 \end{bmatrix},$$

$$\boldsymbol{W_1} = (\boldsymbol{Y_4} = [\, \boldsymbol{y_2} \quad \boldsymbol{y_3}]) - [\, \boldsymbol{x_i} \quad \boldsymbol{x_i}],$$

$$\boldsymbol{W_1} = \begin{bmatrix} 7 & -5 \\ -13 & -7 \\ 11 & -15 \end{bmatrix} - \begin{bmatrix} -6 & -6 \\ 10 & 10 \\ -12 & -12 \end{bmatrix} = \begin{bmatrix} 13 & 11 \\ -23 & -3 \\ 23 & 27 \end{bmatrix},$$

$$\boldsymbol{W_2} = \boldsymbol{Y_4} - [\boldsymbol{x_1} \quad \boldsymbol{x_1} \quad \boldsymbol{x_1}],$$

$$\boldsymbol{W_2} = \begin{bmatrix} -6 & 7 & 5 \\ 10 & -13 & 7 \\ -12 & 11 & 15 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ -10 & -10 & -10 \\ 12 & 12 & 12 \end{bmatrix} = \begin{bmatrix} -12 & 1 & -1 \\ 20 & -3 & 17 \\ -24 & -1 & 3 \end{bmatrix},$$

$$\boldsymbol{d_{p4}} = \begin{bmatrix} -1\begin{vmatrix} -23 & -3 \\ 23 & 27 \end{vmatrix} \\ 1\begin{vmatrix} 13 & 11 \\ 23 & 27 \end{vmatrix} \\ -1\begin{vmatrix} 13 & 11 \\ -23 & -3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -1(-552) \\ 1(98) \\ -1(214) \end{bmatrix} = \begin{bmatrix} 552 \\ 98 \\ -214 \end{bmatrix},$$

and the sign vector is

$$\boldsymbol{S_{p4}d_{p4}} = sign(|\boldsymbol{W_2}|\boldsymbol{d_{p4}}) = sign\left(\left|\begin{bmatrix} -12 & 1 & -1 \\ 20 & -3 & 17 \\ -24 & -1 & 3 \end{bmatrix}\right| * \begin{bmatrix} 552 \\ 98 \\ -214 \end{bmatrix}\right) = \begin{bmatrix} -552 \\ -98 \\ 214 \end{bmatrix}$$

$$\boldsymbol{Y_6} = [-1 \quad 1 \quad 1]\# \begin{bmatrix} 6 & -7 & 5 \\ -10 & 13 & 7 \\ 12 & -11 & 15 \end{bmatrix} = \begin{bmatrix} -6 & -7 & 5 \\ 10 & 13 & 7 \\ -12 & -11 & 15 \end{bmatrix},$$

$$\boldsymbol{W_1} = (\boldsymbol{Y_6} = [\,\boldsymbol{y_2} \quad \boldsymbol{y_3}]) - [\,\boldsymbol{x_i} \quad \boldsymbol{x_i}],$$

$$\boldsymbol{W_1} = \begin{bmatrix} -7 & -5 \\ 13 & -7 \\ -11 & -15 \end{bmatrix} - \begin{bmatrix} -6 & -6 \\ 10 & 10 \\ -12 & -12 \end{bmatrix} = \begin{bmatrix} -1 & 11 \\ 3 & -3 \\ 1 & 27 \end{bmatrix},$$

$$\boldsymbol{W_2} = \boldsymbol{Y_6} - [\boldsymbol{x_1} \quad \boldsymbol{x_1} \quad \boldsymbol{x_1}],$$

$$\boldsymbol{W_2} = \begin{bmatrix} -6 & -7 & 5 \\ 10 & 13 & 7 \\ -12 & -11 & 15 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ -10 & -10 & -10 \\ 12 & 12 & 12 \end{bmatrix} = \begin{bmatrix} -12 & -13 & -1 \\ 20 & 23 & 17 \\ -24 & -23 & 3 \end{bmatrix},$$

$$\boldsymbol{d_{p6}} = \begin{bmatrix} -1\begin{vmatrix} 3 & -3 \\ 1 & 27 \end{vmatrix} \\ 1\begin{vmatrix} -1 & 11 \\ 1 & 27 \end{vmatrix} \\ -1\begin{vmatrix} -1 & 11 \\ 3 & -3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -1(84) \\ 1(-38) \\ -1(-30) \end{bmatrix} = \begin{bmatrix} -84 \\ -38 \\ 30 \end{bmatrix},$$

and the sign vector is

$$\boldsymbol{S_{p6}d_{p6}} = sign\big(|\boldsymbol{W_2}|\boldsymbol{d_{p6}}\big) = sign\left(\left|\begin{bmatrix} -12 & -13 & -1 \\ 20 & 23 & 17 \\ -24 & -23 & 3 \end{bmatrix}\right| * \begin{bmatrix} -84 \\ -38 \\ 30 \end{bmatrix}\right) = \begin{bmatrix} -84 \\ -38 \\ 30 \end{bmatrix}.$$

Applying Equations 128 through 132, the signed-rank vector for the original $\boldsymbol{X`_1} = \begin{bmatrix} 6 \\ -10 \\ 12 \end{bmatrix}$ is

$$\boldsymbol{R}^{+}(\boldsymbol{x`_1}) = \sum_{p \in P} \boldsymbol{Q}_1^{+}(\boldsymbol{x}) = \frac{1}{8}\sum_{a \in A} S_{pa}(\boldsymbol{x})\boldsymbol{d}_{pa}$$

$$\boldsymbol{R}^{+}(\boldsymbol{x`_1}) = \frac{1}{8}\big(\boldsymbol{S_{p0}d_{p0}} + \boldsymbol{S_{p2}d_{p2}} + \boldsymbol{S_{p4}d_{p4}} + \boldsymbol{S_{p6}d_{p6}}\big)$$

$$\boldsymbol{R}^{+}(\boldsymbol{x}`_{\mathbf{1}}) = \tfrac{1}{8}\left(\begin{bmatrix}-460\\-62\\198\end{bmatrix} + \begin{bmatrix}8\\-2\\14\end{bmatrix} + \begin{bmatrix}-552\\-98\\214\end{bmatrix} + \begin{bmatrix}-84\\-38\\30\end{bmatrix}\right) = \begin{bmatrix}-136\\-25\\57\end{bmatrix}.$$

The same algorithm was applied to the other two vectors in the matrix. The complete step-by-step computations are available in Appendix B. The following is a summary of the results for vectors $\boldsymbol{x}`_{\mathbf{2}} = \begin{bmatrix}-7\\13\\-11\end{bmatrix}$ and $\boldsymbol{x}`_{\mathbf{3}} = \begin{bmatrix}5\\7\\15\end{bmatrix}$.

2. The signed-rank vector $\boldsymbol{R}_n^{+}(\boldsymbol{x})$ for $\boldsymbol{x}`_{\mathbf{2}}$

In the same manner, we calculate the sign vector $\boldsymbol{Q}_2^{+}(\boldsymbol{x})$ and signed-rank vector $\boldsymbol{R}_n^{+}(\boldsymbol{x})$ for $\boldsymbol{x}`_{\mathbf{2}} = \begin{bmatrix}-7\\13\\-11\end{bmatrix}$. Again, applying Equations 128 through 132, the signed-rank vector for the original $\boldsymbol{X}`_{\mathbf{2}} = \begin{bmatrix}-7\\13\\-11\end{bmatrix}$, we get

$$\boldsymbol{R}^{+}(\boldsymbol{x}`_{\mathbf{2}}) = \sum_{p \in P} \boldsymbol{Q}_2^{+}(\boldsymbol{x}) = \frac{1}{8}\sum_{a \in A} \boldsymbol{S}_{\boldsymbol{pa}}(\boldsymbol{x})\boldsymbol{d}_{pa}$$

$$\boldsymbol{R}^{+}(\boldsymbol{X}`_{\mathbf{2}}) = \frac{1}{8}\left(\boldsymbol{S}_{\boldsymbol{p0}}\boldsymbol{d}_{\boldsymbol{p0}} + \boldsymbol{S}_{\boldsymbol{p1}}\boldsymbol{d}_{\boldsymbol{p1}} + \boldsymbol{S}_{\boldsymbol{p4}}\boldsymbol{d}_{\boldsymbol{p4}} + \boldsymbol{S}_{\boldsymbol{p5}}\boldsymbol{d}_{\boldsymbol{P5}}\right)$$

$$\boldsymbol{R}^{+}(\boldsymbol{X}`_{\mathbf{2}}) = \frac{1}{8}\left(\begin{bmatrix}-460\\-62\\198\end{bmatrix} + \begin{bmatrix}84\\38\\-30\end{bmatrix} + \begin{bmatrix}-552\\-98\\214\end{bmatrix} + \begin{bmatrix}-8\\2\\-14\end{bmatrix}\right) = \begin{bmatrix}-117\\-15\\46\end{bmatrix}$$

3. The signed-rank vector $\boldsymbol{R}_n^{+}(\boldsymbol{x})$ for $\boldsymbol{x}`_{\mathbf{3}}$

And finally, in the same manner, we calculate the sign vector $\boldsymbol{Q}_3^{+}(\boldsymbol{x})$ and signed-rank vector $\boldsymbol{R}_n^{+}(\boldsymbol{x})$ for $\boldsymbol{x}`_{\mathbf{3}} = \begin{bmatrix}5\\7\\15\end{bmatrix}$. Once again, applying Equations 128 through 132, the signed-rank vector for the original $\boldsymbol{x}`_{\mathbf{3}} = \begin{bmatrix}5\\7\\15\end{bmatrix}$, we get

$$\boldsymbol{R}^{+}(\boldsymbol{x}`_{\boldsymbol{3}}) = \sum_{p \in P} \boldsymbol{Q}_3^{+}(\boldsymbol{x}) = \frac{1}{8}\sum_{a \in A} \boldsymbol{S}_{\boldsymbol{pa}}(\boldsymbol{x})\boldsymbol{d}_{pa}$$

$$\boldsymbol{R}^{+}(\boldsymbol{x}`_{\boldsymbol{3}}) = \frac{1}{8}\left(\boldsymbol{S}_{\boldsymbol{p0}}\boldsymbol{d}_{\boldsymbol{p0}} + \boldsymbol{S}_{\boldsymbol{p1}}\boldsymbol{d}_{\boldsymbol{p1}} + \boldsymbol{S}_{\boldsymbol{p2}}\boldsymbol{d}_{\boldsymbol{p2}} + \boldsymbol{S}_{\boldsymbol{p2}}\boldsymbol{d}_{\boldsymbol{p3}}\right)$$

$$\boldsymbol{R}^{+}(\boldsymbol{x}`_{\boldsymbol{3}}) = \frac{1}{8}\left(\begin{bmatrix} -460 \\ -62 \\ 198 \end{bmatrix} + \begin{bmatrix} 84 \\ 38 \\ -30 \end{bmatrix} + \begin{bmatrix} 8 \\ -2 \\ 14 \end{bmatrix} + \begin{bmatrix} 552 \\ 98 \\ -214 \end{bmatrix}\right) = \begin{bmatrix} 23 \\ 9 \\ -4 \end{bmatrix}.$$

We now have $R_n^{+} = [\boldsymbol{R}^{+}(\boldsymbol{x}`_{\boldsymbol{1}}) \quad \boldsymbol{R}^{+}(\boldsymbol{x}`_{\boldsymbol{2}}) \quad \boldsymbol{R}^{+}(\boldsymbol{x}`_{\boldsymbol{3}})] = \begin{bmatrix} -136 & -117 & 23 \\ -25 & -15 & 9 \\ 57 & 46 & -4 \end{bmatrix}$.

The calculated signed-rank vectors $\boldsymbol{R}^{+}(\boldsymbol{x}`_1), \boldsymbol{R}^{+}(\boldsymbol{x}`_2)$, and $\boldsymbol{R}^{+}(\boldsymbol{x}`_{\boldsymbol{3}})$ were generated for the transposed $\boldsymbol{X}$ or $\boldsymbol{X}` = \begin{bmatrix} 6 & -7 & 5 \\ -10 & 13 & 7 \\ 12 & -11 & 15 \end{bmatrix}$ and the signed-rank vectors or matrix is then transposed to give the final signed-ranks matrix

$$\boldsymbol{R}_{\boldsymbol{n}}^{+} = (\boldsymbol{R}_{\boldsymbol{n}}^{+})` = \begin{bmatrix} -136 & -25 & 57 \\ -117 & -15 & 46 \\ 23 & 9 & -4 \end{bmatrix}.$$

The above result is identical to the signed-rank vectors or matrix obtained by applying the SAS code and interactive matrix language (IML) routines for the purpose of calculating the Oja invariant signed-rank vectors in Appendix A.

The Signed-Rank Based Test Statistic

Again, let $\boldsymbol{X_1}, \dots \boldsymbol{X_n}$ be a random sample from a continuous and symmetric *k*-varites distribution with density $f(x-\theta)$ so that $f(x)$ is symmetric about the origin and θ is the unknown center of symmetry. Without loss of generality, we assume then the null hypothesis is $H_0: \boldsymbol{\theta} = \boldsymbol{0}$. Now we consider a one-sample randomization or sign change test. From above, since $\boldsymbol{R}_n^+(\boldsymbol{x}) = -\boldsymbol{R}_n^+(\boldsymbol{x})$ and under H_0, $\boldsymbol{x}$ and $-\boldsymbol{x}$ are equally probable. Let H be a fixed half-space such that $\boldsymbol{x} \neq 0$ belongs to H, then $-\boldsymbol{x}$ does not. We now write

$$\boldsymbol{x}_i = s_i \boldsymbol{y}_i, \tag{133}$$

where $\boldsymbol{y}_i \in H$. Hence $s_i = \pm 1$ as $\boldsymbol{x}_i \in H \; or \; \boldsymbol{x}_i \in H^c$. If $H_0: \boldsymbol{\theta} = \boldsymbol{0}$ is true, then $s_1, \dots, s_n$ and $\boldsymbol{y}_1, \dots, \boldsymbol{y}_n$ are mutually independent. Write

$$\boldsymbol{r}_{ni}^+ = \boldsymbol{R}_n^+(\boldsymbol{y}_i). \tag{134}$$

The vector-valued, multivariate one-sample signed-rank test statistic is the sum of signed ranks of the observations

$$\boldsymbol{T}_n = \sum_{i=1}^{n} \boldsymbol{R}_n^+(\boldsymbol{x}_i) = \sum_{i=1}^{n} s_i \boldsymbol{r}_{ni}^+. \tag{135}$$

Under $H_0: \boldsymbol{\theta} = \boldsymbol{0}$ and given $(\boldsymbol{y}_1, \dots, \boldsymbol{y_n})$, the "signs" $s_1, \dots, s_n$ are iid Bernoulli random variables with $P(s_i = 1) = P(s_i = -1) = \frac{1}{2}$. Hence, conditionally, $E(\boldsymbol{T}_n) = \boldsymbol{0}$ and

$$\boldsymbol{B} = cov\left(n^{-\frac{1}{2}} \boldsymbol{T}_n\right) = \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{r}_{ni}^+ \boldsymbol{r}_{ni}^{+\,T} = \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{R}_n^+(\boldsymbol{x}_i) \; \boldsymbol{R}_n^{+\,T}(\boldsymbol{x}_i). \tag{136}$$

The conditional, approximately size α, randomization test is then carried out by rejecting $H_0: \boldsymbol{\theta} = \boldsymbol{0}$ when $n^{-1} \boldsymbol{T}_n^T \boldsymbol{B}_N^{-1} \boldsymbol{T}_n \geq \chi_\alpha^2(k)$, where $\chi_\alpha^2(k)$ is the $(1-\alpha)$-percentile from a

chi-squared distribution with *k* degrees of freedom. Hettmansperger et al. (1997) showed that

$$\boldsymbol{B}_n \xrightarrow{p} \boldsymbol{B} = E[\boldsymbol{R}_n^+(\boldsymbol{X}_i)\boldsymbol{R}_n^{+T}(\boldsymbol{X}_i)]. \tag{137}$$

Now the research questions outlined in Chapter 1 are revisited and address the research methods corresponding to each question specifically. Once again, this dissertation addressed the following questions:

Q1 How will the Spatial Signed-Rank MEWMA (SRMEWMA) control chart scheme be designed for the in-control average run length (ARL_0)?

Q2 What is the Spatial Signed-Rank MEWMA (SRMEWMA) control chart performance for different number, *p*, of monitored related quality characteristics?

Q3 What is the Spatial Signed-Rank MEWMA (SRMEWMA) control chart performance for different values of the smoothing parameter λ?

Q4 What is the Spatial Signed-Rank MEWMA (SRMEWMA) control chart performance for different sizes of shift in a process location vector?

Q5 What is the Spatial Signed-Rank MEWMA (SRMEWMA) control chart performance compared to the Hotelling's T^2 and MEWMA control chart scheme for elliptically symmetrical (multivariate normal and multivariate t) and skewed distributions (multivariate gamma)?

A new multivariate nonparametric MEWMA control chart combined with the signed-rank test is derived next.

A Spatial Signed-Rank Based Multivariate Exponentially Weighted Moving Average Control Chart

This section provides the necessary background to answer the first research question:

Q1 How will the Spatial Signed-Rank MEWMA (SRMEWMA) control chart scheme be designed for the in-control average run length (ARL_0)?

The monitoring problem is closely related to the nonparametric statistical tests of hypothesis of the one-sample location problem. Therefore, to facilitate the derivation of the proposed charting statistic, we start by assuming the underlying in-control (IC) distribution $F(\boldsymbol{X}-\boldsymbol{\mu_0})$ is completely known, where *F* represents a continuous *p*-dimensional distribution located at the vector $\boldsymbol{\mu_0}$. Given a random observed vector $\boldsymbol{x} \sim F(\boldsymbol{X}-\boldsymbol{\mu_0})$, we want to test the null hypothesis, H_0, that $\boldsymbol{\mu}=\boldsymbol{\mu_0}$ against H_1 that $\boldsymbol{\mu} \neq \boldsymbol{\mu_0}$. By definition, it is easy to see that under H_0, $E[\boldsymbol{R}_n^+(\boldsymbol{X}_i)]=0$. Thus, it is straightforward to consider the test statistic

$$\boldsymbol{Q}^R=\boldsymbol{R}_n^{+T}(\boldsymbol{X}_i)\left\{Cov\left(\boldsymbol{R}_n^+(\boldsymbol{X}_i)\right)\right\}^{-1}\boldsymbol{R}_n^+(\boldsymbol{X}_i), \tag{138}$$

as a reasonable candidate for testing. When H_0 is true, the test statistic should be small and thus not lead to rejection of the null hypothesis. Now, define a MEWMA sequence

$$\boldsymbol{w}_t=(1-\lambda)\boldsymbol{w}_{t-1}+\lambda\boldsymbol{R}_n^+(\boldsymbol{x}_i), \tag{139}$$

where $\boldsymbol{w}_0=0$ and λ is a smoothing parameter.

The charting statistic is given by

$$\boldsymbol{Q}_t^R=\boldsymbol{w}_t^T\{Cov(\boldsymbol{w}_t)\}^{-1}\boldsymbol{w}_t\text{ , }or \tag{140}$$

$$\boldsymbol{Q}_t^R=\frac{2-\lambda}{\lambda}\boldsymbol{w}_t^T\left\{Cov\left(\boldsymbol{R}_n^+(\boldsymbol{X}_i)\right)\right\}^{-1}\boldsymbol{w}_t, \tag{141}$$

such that

$$\boldsymbol{\Sigma}_{\boldsymbol{w}_t}=cov(\boldsymbol{w}_t)=\frac{\lambda}{2-\lambda}Cov\left(\boldsymbol{R}_n^+(\boldsymbol{X}_i)\right). \tag{142}$$

Since $\Sigma_{\boldsymbol{R}_n^+(\boldsymbol{X}_i)}=Cov\left(\boldsymbol{R}_n^+(\boldsymbol{X}_i)\right)$, equation 142 can be written as

$$\boldsymbol{\Sigma}_{\boldsymbol{w}_t}=\frac{\lambda}{2-\lambda}\boldsymbol{\Sigma}_{\boldsymbol{R}_n^+(\boldsymbol{X}_i)}. \tag{143}$$

The covariance of $\boldsymbol{w}_t$ is derived below.

Recall the MEWMA sequence using the signed-ranks

$$\boldsymbol{w}_t = (1-\lambda)\boldsymbol{w}_{t-1} + \lambda \boldsymbol{R}_n^+(\boldsymbol{X}_i). \tag{144}$$

Let us suppose that we are using a full smoothing parameters matrix Λ such that

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_{11} & \cdots & \lambda_{1t} \\ \vdots & \ddots & \vdots \\ \lambda_{n1} & \cdots & \lambda_{nt} \end{bmatrix}, \tag{145}$$

then Equation 144 can be represented as

$$\boldsymbol{w}_t = (1-\boldsymbol{\Lambda})\boldsymbol{w}_{t-1} + \boldsymbol{\Lambda} \boldsymbol{R}_n^+(\boldsymbol{X}_i). \tag{146}$$

By repeated substitution in Equation 146, it can be shown that

$$\boldsymbol{w}_t = \sum_{j=1}^{t} \boldsymbol{\Lambda}(1-\boldsymbol{\Lambda})^{t-j}\ \boldsymbol{w}_{t,} \tag{147}$$

thus

$$\boldsymbol{\Sigma}_{w_t} = \sum_{j=1}^{t} var\{\boldsymbol{\Lambda}(1-\boldsymbol{\Lambda})^{t-j}\ \boldsymbol{w}_t\} \tag{148}$$

$$= \sum_{j=1}^{t} [\boldsymbol{\Lambda}(1-\boldsymbol{\Lambda})^{t-j}\, Cov\left(\boldsymbol{R}_n^+(\boldsymbol{X}_i)\right) \boldsymbol{\Lambda}(1-\boldsymbol{\Lambda})^{t-j}\, \boldsymbol{w}_t] \tag{149}$$

because $\boldsymbol{\Lambda}$ and $(1-\boldsymbol{\Lambda})$ are diagonal matrices, the (*k*,*l*)th element of $\boldsymbol{\Sigma}_{w_t}$ is

$$\lambda_k\lambda_l[1 - \frac{(1-\lambda_k)^i(1-\lambda_l)^i}{[\lambda_k + \lambda_l - \lambda_k\lambda_l]\sigma_{k,l}}, \tag{150}$$

where $\sigma_{k,l}$ is the (*k*,*l*)th element of $Cov\left(\boldsymbol{R}_n^+(\boldsymbol{X}_i)\right)$. If $\lambda_1 = \lambda_2 = \cdots = \lambda_t = \lambda$, then the expression in (A.7) simplifies to

$$\frac{\lambda\left(1-(1-\lambda)^{2i}\right)}{2-\lambda}\sigma_{k,l}. \tag{151}$$

Such that

$$\boldsymbol{\Sigma}_{\boldsymbol{w_t}} = cov(\boldsymbol{w_t}) = \frac{\lambda\left(1-(1-\lambda)^{2i}\right)}{2-\lambda} Cov\left(\boldsymbol{R}_n^+(\boldsymbol{X}_i)\right); \tag{152}$$

however, as $i \to \infty$, the asymptotic covariance matrix is

$$\boldsymbol{\Sigma}_{\boldsymbol{w_t}} = cov(\boldsymbol{w_t}) = \frac{\lambda}{2-\lambda} Cov\left(\boldsymbol{R}_n^+(\boldsymbol{X}_i)\right) \text{ or} \tag{153}$$

$$\boldsymbol{\Sigma}_{\boldsymbol{w_t}} = \frac{\lambda}{2-\lambda} \boldsymbol{\Sigma}_{\boldsymbol{R}_n^+}. \tag{154}$$

Equation 141 can be written as

$$\boldsymbol{Q}_t^R = \boldsymbol{w}_t^T (\boldsymbol{\Sigma}_{\boldsymbol{w}_t})^{-1} \boldsymbol{w}_t > L. \tag{155}$$

where $L > 0$ is the control limit chosen to achieve a specific IC ARL or ARL_0. We refer to this method as the Spatial Signed-Rank MEWMA or SRMEWMA.

The SRMEWMA chart is affine invariant. For any $p\ x\ p$ nonsingular matrix $\boldsymbol{M}$, the charting statistic $\boldsymbol{Q}_t^R$, based on $\boldsymbol{X}_t$ and $\boldsymbol{X}_t^* = \boldsymbol{M}\boldsymbol{X}_t$ are the same. This property is intuitively appealing and it also ensures that the performance of SRMEWMA is the same for any initial covariance matrix and location. The value of the charting statistics remains the same for any of the following conditions: (a) the data points are rotated, (b) the data points are reflected around a p-1 dimensional hyperplane, or (c) the scales of measurement are altered (Hettmansperger et al., 1997; Zou & Tsung, 2010).

Proof: we know from Equation 139 that

$$\boldsymbol{w}_t = (1-\lambda)\boldsymbol{w}_{t-1} + \lambda \boldsymbol{R}_n^+(\boldsymbol{X}_i). \tag{156}$$

Now, define

$$\boldsymbol{w}_t^* = (1-\lambda)\boldsymbol{M}\boldsymbol{w}_{t-1} + \lambda\boldsymbol{M}\boldsymbol{R}_n^+(\boldsymbol{X}_i). \tag{157}$$

Then

$$\boldsymbol{w}_t^* = \boldsymbol{M}[(1-\lambda)\boldsymbol{w}_{t-1} + \lambda\boldsymbol{R}_n^+(\boldsymbol{X}_i)], \quad (158$$

$$\boldsymbol{w}_t^* = \boldsymbol{M}\boldsymbol{w}_t, \tag{159}$$

and the covariance matrix of $\boldsymbol{w}_t^*$ is

$$\boldsymbol{\Sigma}_{w_t^*} = \boldsymbol{\Sigma}_{\boldsymbol{M}w_t} = \boldsymbol{M}\boldsymbol{\Sigma}_{w_t}\boldsymbol{M}'. \tag{160}$$

now, define a new charting statistic

$$\boldsymbol{Q}_t^{R*} = \boldsymbol{w}_t^{T*}\left(\boldsymbol{\Sigma}_{\boldsymbol{w}_t^*}\right)^{-1}\boldsymbol{w}_t^* \tag{161}$$

$$= (\boldsymbol{M}\boldsymbol{w}_t)'\left(\boldsymbol{M}\boldsymbol{\Sigma}_{\boldsymbol{w}_t}\boldsymbol{M}'\right)^{-1}(\boldsymbol{M}\boldsymbol{w}_t) \tag{162}$$

$$= \boldsymbol{w}_t^T(\boldsymbol{M}\boldsymbol{M}'^{-1})\left(\boldsymbol{\Sigma}_{\boldsymbol{w}_t}\right)^{-1}(\boldsymbol{M}^{-1}\boldsymbol{M})\boldsymbol{w}_t \tag{163}$$

$$= \boldsymbol{w}_t^T\left(\boldsymbol{\Sigma}_{\boldsymbol{w}_t}\right)^{-1}\boldsymbol{w}_t \tag{164}$$

$$= \boldsymbol{Q}_t^R. \tag{165}$$

This completes the proof.

Spatial Signed-Rank Multivariate Exponentially Weighted Average Run Length Performance

Choice of multivariate distributions for simulation. Following the robustness analyses by Stoumbos and Sullivan (2002), Stoumbos and Reynolds (2000), Borror, Montgomery, and Runger (1999), Zou and Tsung (2010), and Zou et al. (2010), data were generated from the following distributions: (a) multivariate normal (elliptically symmetrical) ; (b) multivariate t-distribution with $\boldsymbol{\nu}$ degrees of freedom (elliptically symmetrical), denoted as $\boldsymbol{t_{p,\nu}}$; and (c) multivariate gamma (skewed) with shape parameter $\boldsymbol{\alpha}$ and scale parameter $\boldsymbol{\beta} = \mathbf{1}$, without any loss of generality, denoted

as $\boldsymbol{Gam_2}\,(\boldsymbol{\alpha}, \boldsymbol{\beta} = \mathbf{1})$. Further statistical details on the general multivariate *t*-distribution and gamma distribution can be found in the Appendix to Stoumbos and Sullivan's study (2002). As discussed by Stoumbos and Sullivan, since the multivariate normal and *t* distributions were elliptically symmetrical, the MEWMA's OC performance depended on a shift in the process mean vector only through a non-centrality parameter. This was still true for the SRMEWMA chart because of its affine invariance. However, with the other distributions, such as multivariate gamma, the performance was not invariant to the covariance matrix of the "implicit" multivariate normal observation. The number and variety of covariance matrices and shift directions were too large to allow a comprehensive, all-encompassing comparison. The goal of this study was to show the effectiveness, robustness, and sensitivity of the SRMEWMA chart; thus, only certain representative models were chosen for illustration. Specifically for the three distribution cases, the covariance matrix $\boldsymbol{\Sigma_0} = (\boldsymbol{\sigma_{ij}})$ was estimated from a large reference sample of 10,000 sample vectors for each distribution. In the interest of brevity, a shift of size $\boldsymbol{\delta}$ in only the first component is used, say, $\boldsymbol{x_i} + \boldsymbol{\delta e_1}$ with $\boldsymbol{e_1} = (\mathbf{1}, \mathbf{0}, \dots, \mathbf{0})'$, unless stated otherwise.

Two literal bodies of work support the choice of distributions for simulation. The first is based on the robustness to non-normality of MEWMA control charts for heavy-tailed and skewed distributions and the second is based on the efficiencies of multivariate signs and ranks tests relative to Hotelling's T^2.

Mottonen et al. (1997) examined the asymptotic relative efficiencies (ARE) of the spatial sign test and the spatial signed-rank test with respect to the Hotelling's T^2 for the *t*-distribution with selected values of degrees of freedom ν and with selected dimensions

p. Both tests seemed to have good ARE over a broad class of *t*-distributions with the signed–rank test exhibiting better AREs; the higher the dimension the higher the ARE. Peters and Randles (1990) found that the signed-rank statistic appeared to be robust and performed better than its competitors when the data came from light-tailed distributions--better than Hotelling's T^2 for the heavy-tailed distribution and Hotelling's T^2 for the multivariate normal distribution.

The Design of the Signed-Rank Multivariate Exponentially Weighted Average Control Scheme

This section provides the necessary background to answer the second, third, fourth, and fifth research questions:

Q2 What is the Spatial Signed-Rank MEWMA (SRMEWMA) control chart performance for different number, *p*, of monitored related quality characteristics?

Q3 What is the Spatial Signed-Rank MEWMA (SRMEWMA) control chart performance for different values of the smoothing parameter λ?

Q4 What is the Spatial Signed-Rank MEWMA (SRMEWMA) control chart performance for different sizes of shift in a process location vector?

Q5 What is the Spatial Signed-Rank MEWMA (SRMEWMA) control chart performance compared to the Hotelling's T^2 and MEWMA control chart scheme for elliptically symmetrical (multivariate normal and multivariate *t*) and skewed distributions (multivariate gamma)?

Using SAS 9.2, Monte Carlo simulation techniques were used to generate simulated *p*-variate normal, *t*, and gamma observations vectors as described above and to compute the signs, ranks, and signed-ranks. The following SAS functions were used to generate the random *p*-variate observation vectors:

1. The Random Query Generator (RANDGEN) function call was used to generate multivariate normal *p*-variate observation vectors,
2. The RANDGEN function call was used to generate multivariate student's *t*-distribution *p*-variate observation vectors, and
3. The RANDGEN function call was used to generate multivariate *t*-distribution *p*-variate observation vectors.

Practical Guidelines on Choosing the Reference Sample Size

The choice of sample size plays an important role in any simulation study and affects parameter estimation. If the sample size is too small, there will be considerable uncertainty in parameter estimates, which in turn will distort the in-control ARL (Zou et al., 2010). Since collecting large reference samples is both costly and not time feasible, Zou et al. (2010) suggested using a reference sample size of at least $m_0 \geq 2p$. To achieve satisfactory performance, Zou et al. suggested using 50 observations or more. However, the signed-ranks algorithm is computationally intensive when $n > 5$; therefore, I used a universal sample size of 5.

Average run length$_1$ and upper control limit properties for signed-rank multivariate exponentially weighted moving average. For a desired $ARL_0 \approx \mathbf{200}$, ARL_1 and L values (see Equations 141 and 144) were calculated; where $L > 0$ was the control limit chosen to achieve the desired ARL_0 of the SRMEWMA ($\boldsymbol{Q_t^R}$) control scheme for all combinations of following parameters' values:

1. The number of quality variables to be simultaneously monitored, p = 2, 3, 4, and 5 for the multivariate normal and t distributions and p = 2 only for the multivariate gamma distribution.

2. The smoothing parameter, λ ∈ [0.01, 0.02, 0.03, 0.05, 0.10, and 0.50]. A smaller λ led to a quicker detection (Lucas & Saccucci, 1990; Prabhu & Runger, 1997).

3. Shifts in the process mean vector, δ ∈ [0, 0.25, .05, 1.0, 1.5, and 2.5].

The simulation process first determined the *L* values and based upon the simulated *L* values, the associated ARL_1 values were calculated using simulation in Phase II. As an example, Table 6 illustrates the ARL_1 denoted by "x" and *L* values, denoted by "*h*" for the SRMEWMA ($\boldsymbol{Q}_t^R$) control scheme for $ARL_0 \approx 200$, *p* = 2, and all values of λ and δ mentioned above in 2 and 3. All shifts in process mean vector were introduced in the first variable or component, such that the shift in mean vector was from $\boldsymbol{\mu_0} = (0, \dots, 0)$ to $\boldsymbol{\mu} = (\delta, 0, \dots, 0)$.

Table 6

Average Running Length₁ Values of the Signed-Rank Multivariate Exponentially Weighted Moving Average ($\boldsymbol{Q}_t^R$) Control Scheme for Average Running Length₀ ≈ 200 and p = 2

	λ			
	0.01	0.02	0.05	0.5
			L	
δ	*h*	*h*	*h*	*h*
0.00	x	x	x	x
0.25	x	x	x	x
1.50	x	x	x	x
2.50	x	x	x	x

In addition, the performance of the SRMEWMA ($\boldsymbol{Q}_t^R$) control scheme was investigated for $ARL_0 \approx 500$ for the same simulation condition associated with $ARL_0 \approx 200$. Both ARL_1 and L values were displayed in the same manner as Table 6 but for a desired in control $ARL_0 \approx 500$.

Average running length comparisons to Hotelling's T^2 and multivariate exponentially weighted moving average. The performance of the SRMEWMA ($\boldsymbol{Q}_t^R$) control scheme was compared to the ARL values of both the Hotelling's T^2 and MEWMA control schemes for the same sampling distributions and simulation parameters listed above in "Simulation parameters" 1-3 listed previously. The following SAS functions were used to generate the random *p*-variate observation vectors:

1. The RANDGEN function call was used to generate multivariate normal *p*-variate observation vectors,
2. The RANDGEN function call was used to generate multivariate student's *t*-distribution *p*-variate observation vectors, and
3. The RANDGEN function call was used to generate multivariate *t*-distribution *p*-variate observation vectors.

The ARL and upper control values L, h_1 (see Equation 49) and h_2 (see Equation 72) for the Hotelling's χ^2, and MEWMA control schemes, respectively, will be generated according to Table 7. The UCL (h_1) for the Hotelling's χ^2 for which $ARL_0 \approx 200$ are obtained from the χ_p^2 tables only for *p*-variate observation vectors from the multivariate normal distribution. All other UCL for all control schemes will be obtained by using Monte Carlo simulation. The performance of the SRMEWMA ($\boldsymbol{Q}_t^R$) control scheme was compared to the ARL values of both the Hotelling's χ^2 and MEWMA control schemes

only for observations from the multivariate normal distribution. Additionally, the performance of the SRMEWMA ($\boldsymbol{Q}_t^R$) control scheme was compared to the ARL values of both the MEWMA control scheme for observation samples from both the multivariate *t*-distribution and the multivariate gamma distributions.

Table 7

Average Run Length and Upper Control Limit (L, h_1, & h_2) Determination Methods for Control Schemes and Selected Sampling Distribution

Distribution	Control Scheme		
	SRMEWMA	MEWMA	Hotelling's χ^2
Multivariate Normal $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}_p)$	Simulation	Simulation	$\chi_p^2\ Tables$
Multivariate t $t_{p,\nu}$	Simulation	Simulation	Simulation
Multivariate gamma $Gam_{2,(\alpha,\beta=1)}$	Simulation	Simulation	Simulation

Table 8 (for *p*-variate observations from the multivariate normal distribution), Table 9 (for *p*-variate observations from the multivariate student's *t*-distribution), and Table 10 (for *p*-variate observations from the multivariate gamma distribution) are examples of the *ARL* comparison for the SRMEWMA ($\boldsymbol{Q}_t^R$) control scheme, the Hotelling's T^2, and MEWMA control schemes for *ARL* Comparisons for $p = 2$, $\lambda = 0.10$ and $ARL_0 \approx 200$.

Table 8

Average Run Length Comparisons for p = 2, λ = 0.10 and In-Control Average Run Length ≈ 200 for p-Variate Observation Vectors from the Multivariate Normal Distribution

	SRMEMA	Hotelling's T^2*	MEWMA
δ	$L = h$	$h_1 = 10.59$*	$h_2 = h$
0.00	**x**	x	**x**
0.25	**x**	**x**	**x**
0.50	**x**	**x**	**x**
1.00	**x**	**x**	**x**
1.50	**x**	x	**x**
2.50	**x**	x	**x**

*Values obtained from Lowry et al. (1992).

Table 9

Average Run Length Comparisons for p = 2, λ = 0.10 and In-Control Average Run Length ≈ 200 for p-Variate Observation Vectors from the Multivariate Student's t-Distribution

	SRMEMA	Hotelling's T^2*	MEWMA
δ	$L = h$	$h_1 = 10.59$*	$h_2 = h$
0.00	**x**	x	**x**
0.25	**x**	**x**	**x**
0.50	**x**	**x**	**x**
1.00	**x**	**x**	**x**
1.50	**x**	**x**	**x**
2.50	**x**	**x**	**x**

Table 10

Average Run Length Comparisons for p = 2, λ = 0.10 and In-Control Average Run Length ≈ 200 for p-Variate Observation Vectors from the Multivariate Gamma Distribution

δ	SRMEMA $L = h$	Hotelling's T^2* $h_1 = 10.59$	MEWMA $h_2 = h$
0.00	**x**	x	**x**
0.25	**x**	**x**	**x**
0.50	**x**	**x**	**x**
1.00	**x**	**x**	**x**
1.50	**x**	x	**x**
2.50	**x**	x	**x**

Simulation Process

The simulation process was conducted in two phases as follows:

Step 1 for Phase I: determine the UCLs for the SRMEWMA (Q_t^R) control scheme, the Hotelling's χ^2 (multivariate normal only), and MEWMA control schemes for the desired ARL_0 values.

1. The following SAS function was used to generate the random *p*-variate observation vectors: The RANDGEN function call was used to generate multivariate normal observation vectors, the multivariate student's *t*-distribution *p*-variate observation vectors, and the multivariate gamma distribution).

2. Using the observation vectors from 1 above, the spatial signs, ranks and signed-ranks were computed using SAS IML routines and macros based on the work of Hettmansperger et al. (1997).

3. The $\boldsymbol{Q}_t^R$ control statistic from Equation 155, the MEWMA control statistic $\boldsymbol{T}_i^2$ from Equation 72, and the Hotelling's T^2 in Equation 49 (only for the multivariate normal distribution) were computed. The computed signed-ranks were used to compute the $\boldsymbol{Q}_t^R$ control statistic, while the original *p*-variate was used to calculate both the MEWMA control statistic $\boldsymbol{T}_i^2$ from equation and the Hotelling's T^2 .

4. The computed $\boldsymbol{Q}_t^R$, $\boldsymbol{T}_i^2$, and χ^2 were compared to the corresponding L, h_1, and h_2 values, respectively. Once $\boldsymbol{Q}_t^R > L$ (similarliy $\boldsymbol{T}_i^2 > h_2$ and $\chi^2 > h_1$) (i.e., the process signals out-of-control), the run length for the i^{th} simulation RL_0 (the number of samples simulated before first out-of-control signals occur when process is operating in-control state) was recorded.

5. The above process was repeated 10,000 times for each combination of conditions (e.g., see Table 6).

6. At the end of the 10,000 simulation, the computed average run length ARL_0 is obtained as

$$ARL_0 = \frac{1}{10{,}000}\sum\nolimits_{1}^{10{,}000} RL_0 \, . \tag{166}$$

7. Once the ARL_0 was approximately equal to 200, the process stopped and then the corresponding values of L or h_1 were recorded.

Step 2: Determining the ARL_1 values for Phase II.

1. The following SAS function was used to generate the random *p*-variate observation vectors: The RANDGEN function call was used to generate multivariate normal *p*-variate observation vectors, the multivariate student's *t*-distribution *p*-variate observation vectors, and the multivariate gamma distribution. The shift $\boldsymbol{\delta}$ value was added to the first component of the standardized signed-ranks of the simulated observation vectors.

2. The $\boldsymbol{Q}_t^R$ control statistic, the MEWMA control statistic $\boldsymbol{T}_i^2$ from Equation 72, and the Hotelling's χ^2 in Equation 49 (only for the multivariate normal distribution) were computed. The computed signed-ranks were used to compute the $\boldsymbol{Q}_t^R$ control statistic; whereas, the original *p*-variate was used to calculate both the MEWMA control statistic $\boldsymbol{T}_i^2$ from equation and the Hotelling's T^2 .

3. The computed $\boldsymbol{Q}_t^R$, $\boldsymbol{T}_i^2$, and χ^2 were compared to the corresponding L, h_1, and h_2 values, respectively. Once $\boldsymbol{Q}_t^R > L$ (similarliy $\boldsymbol{T}_i^2 > h_2$ and $\chi^2 > h_1$) (i.e., the process signals out-of-control), the run length for the i^th^ simulation RL_1 (the number of samples simulated before first out-of-control signals occur when process is operating in-control state) was recorded.

4. The above process was repeated 10,000 times for each combination of conditions.

5. At the end of the 10,000 simulations, the computed average run length ARL_1 was obtained as

$$ARL_1 = \frac{1}{10{,}000}\sum_{1}^{10{,}000} RL_1 \, . \tag{167}$$

The calculated ARL_1 values for all three control statistics, the $\boldsymbol{Q}_t^R$ control statistic from Equation 155, the MEWMA control statistic $\boldsymbol{T}_i^2$ from Equation 72, and the Hotelling's χ^2 in Equation 49 were compared to one another for the different values of the simulation parameters, the number of variables p, the shift parameter δ, and the smoothing parameter λ. The control statistic and control chart with the lowest ARL_1 was considered the favorable control chart.

CHAPTER IV

RESULTS

The goal of this study was twofold. First, a new nonparametric multivariate SPC control chart for monitoring location parameters--the Signed-Rank Multivariate Exponentially Weighted Moving Average (SRMEWMA)--was developed as outlined in Chapter III. Second, the average run length (ARL_1) performance of SRMEWMA was compared with those of other known parametric control charts, specifically the Multivariate Exponentially Weighted Moving Average (MEWMA) and Hotelling's T^2 control charts.

To facilitate ARL_1 comparisons, data were generated using the Monte Carlo simulation technique using the interactive matrix language (IML) of the Statistical Analysis System (SAS®) Windows 7 version 9.3 TSM10 running on an Intel core i7-3930K CPU @ 3.2GHZ/64GB RAM-based system. Interactive matrix language-based simulation algorithms were used to generate process observations in the form of vector-means samples from the multivariate normal, *t*, and gamma distributions as outlined in Table 7. For phase I, in-control (IC) data were simulated from the abovementioned multivariate distribution in order to compute the upper control limits (UCL) for the SRMEWMA, MEWMA, and Hotelling's T^2 statistics and control charts. The upper control limits (UCLs) that achieved an in-control (IC) $ARL_0 \cong 200, 500, \& 1{,}000$ (equivalent $\alpha = 0.005, 0.002, \& 0.001$ respectively) were generated. Then the vector

signed-ranks were computed using IML macros. All UCL values were computed using simulated data samples with the exception of the UCL values for the Hotelling's T^2 control chart for data from the multivariate normal distribution. Those UCL values were used from the χ_p^2 quintile tables.

Second, in phase II, simulated data samples of vector means were generated using the same set of algorithms in phase I; then a shift of magnitude δ as a multiple of the standard deviations was introduced in the first component of simulation-generated data samples for MEWMA and the signed-ranks for SRMEWMA in order to compute the average run length (ARL_1) for the SRMEWMA, MEWMA, and Hotelling's T^2 control charts. Finally, the ARL_1 values from phase II simulation results were compared for the three control charts above.

This dissertation addressed the following research questions:

Q1 How will the Spatial Signed-Rank MEWMA (SRMEWMA) control chart scheme be designed for the in-control average run length (ARL_0)?

Q2 What is the Spatial Signed-Rank MEWMA (SRMEWMA) control chart performance for different number, p, of monitored related quality characteristics?

Q3 What is the Spatial Signed-Rank MEWMA (SRMEWMA) control chart performance for different values of the smoothing parameter λ?

Q4 What is the Spatial Signed-Rank MEWMA (SRMEWMA) control chart performance for different sizes of shift in a process location vector?

Q5 What is the Spatial Signed-Rank MEWMA (SRMEWMA) control chart performance compared to the Hotelling's T^2 and MEWMA control chart scheme for elliptically symmetrical (multivariate normal and multivariate t) and skewed distributions (multivariate gamma)?

Research question 1 was answered in Chapter III. Questions 2, 3, and 4 were answered simultaneously using phase I simulation. Question 5 was answered using phase

II simulation results. Finally, a real-data manufacturing example was used to illustrate the application of SRMEWMA.

Phase I Simulation Results

Using IML and the random query generator (RANDGEN) function in SAS®, 10,000 samples-per-run were generated from the multivariate normal, *t*, and gamma distributions for each run of the following combinations of study parameters:

1. The number of variables, p = 2, 3, 4, and 5 for the multivariate normal and *t* distributions and p = 2 only for the multivariate gamma distribution, and
2. The smoothing parameter, $\lambda \in$ [0.01, 0.02, 0.03, 0.05, 0.10, 0.2, and 0.50].

To estimate the vector mean and variance-covariance matrix in in-control phase I, 10,000 samples were generated using IML function RANGEN from the multivariate normal, *t*, and gamma distributions for p = 2, 3, 4, and 5. Then the UCL values that achieved an ARL_0 = 200, 500, and 1,000 for the MEWMA ($\boldsymbol{T_i^2}$), SRMEWMA ($\boldsymbol{Q_t^R}$), and Hotelling's T^2 control chart statistics were computed.

Signed-Rank Multivariate Exponentially Weighted Moving Average Upper Control Limit and Multivariate Exponentially Weighted Moving Average Upper Control Limit from the Multivariate Normal Distribution

Tables 11-13 and Tables 14-16 show the computed IC UCL values for the MEWMA and SRMEWMA, respectively, for data generated from the multivariate normal distribution for p = 2, 3, 4, and 5 and $\lambda \in$ [0.01, 0.02, 0.03, 0.05, 0.10, 0.2, and 0.50] for ARL_0 = 200, 500, & 1,000, respectively. In addition to the UCL and nominal ARL_0 values, the standard deviation of run lengths (SDRL) is shown for all p and λ

combinations. Additionally, Figures 5-7 show the UCL values for both MEWMA and SRMEWMA for p = 2 and ARL_0 = 200, 500, & 1,000, respectively. Additional tables and figures summarizing the UCL results for the SRMEWMA, MEWMA, and Hotelling's T^2 control charts for p = 3, 4, and 5 are available in Appendix C (Figures 21-29). The results revealed the following:

1. The UCLs values, L and h_1, for the SRMEWMA and MEWMA control charts, respectively, increased as p increased for any λ value.
2. The UCLs values, L and h_1, for the SRMEWMA and MEWMA control charts, respectively, increased as λ increased for any p value.
3. For p = 2, the UCL values for the SRMEWMA control chart were slightly larger than those of the MEWMA control charts as λ increased.
4. For any given p, the UCL values for SRMEWMA got larger than those of MEWMA as λ got larger.
5. The UCLs for SRMEWMA got increasingly larger than those of MEWMA as λ got larger for p = 3, 4, and 5.
6. The SRMEWMA and MEWMA UCL values, L and h_1, respectively, increased as ARL_0 increased for any p and λ value.

Table 11

The Upper Control Limits of the Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 200 from the Multivariate Normal Distribution

	p											
	2			3			4			5		
λ	h_1	ARL_0≅	SDRL	h_1	ARL_0≅	SDRL	h_1	ARL_0≅	SDRL	h_1	ARL_0≅	SDRL
0.01	5.30	193	288	7.20	196.	295	8.8	201	301	10.4	200	289
0.02	6.20	197	245	8.20	200	244	9.9	199	246	11.6	200	241
0.03	6.80	194	223	8.80	193	220	10.7	203	232	12.3	195	222
0.05	7.70	201	214	9.70	198	214	11.6	193	209	13.4	200	218
0.10	8.80	201	205	11.0	209	214	12.9	200	203	14.7	199	201
0.20	9.70	202	203	12.0	204	207	13.9	199	199	15.8	199	197
0.50	10.40	198	200	12.6	195	194	14.7	201	201	16.6	197	195

Table 12

The Upper Control Limits of the Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 500 from the Multivariate Normal Distribution

	p											
	2			3			4			5		
λ	h_1	$ARL_0 \cong$	SDRL	h_1	$ARL_0 \cong$	SDRL	h_1	$ARL_0 \cong$	SDRL	h_1	$ARL_0 \cong$	SDRL
0.01	7.0	505	625	9.0	504	635	10.8	500	623	12.5	499	625
0.02	8.1	494	551	10.2	488	54	12.1	492	546	14.0	504	560
0.03	8.8	501	537	11.0	495	521	13.0	503	530	14.8	494	520
0.05	9.7	502	501	11.9	483	496	14.0	506	523	15.9	499	514
0.10	10.8	498	505	13.0	473	475	15.2	505	513	17.1	488	489
0.20	11.6	482	488	14.0	498	497	16.2	501	500	18.1	485	485
0.50	12.3	504	504	14.6	490	485	16.8	496	502	18.8	495	493

Table 13

The Upper Control Limits of the Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 1,000 from the Multivariate Normal Distribution

	p											
	2			3			4			5		
λ	h_1	ARL_0≅	SDRL	h_1	ARL_0≅	SDRL	h_1	ARL_0≅	SDRL	h_1	ARL_0≅	SDRL
0.01	8.3	967	1,100	10.6	1,027	1,184	12.5	1,010	1,150	14.3	1,001	1,128
0.02	9.6	976	1,030	11.8	942	1,002	13.9	962	1,011	15.8	996	1,049
0.03	10.3	971	999	12.6	955	972	14.8	1,006	1,029	16.7	996	1,008
0.05	11.2	968	996	13.6	995	1,004	15.8	1,004	1,032	17.7	980	1,015
0.10	12.3	981	993	14.8	1,014	1,022	16.9	973	978	18.9	974	959
0.20	13.1	987	973	15.6	994	992	17.8	1,006	1,003	19.8	974	965
0.50	13.7	1,011	1,012	16.2	1,014	1,019	18.3	985	975	20.4	985	983

Table 14

The Upper Control Limits of the Signed-Rank Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 200 from the Multivariate Normal Distribution

	p											
	2			3			4			5		
λ	*L*	ARL_0≅	SDRL	*L*	ARL_0≅	SDRL	*L*	ARL_0≅	SDRL	*L*	ARL_0≅	SDRL
0.01	5.30	196	287	7.30	204	307	9.00	202	305	10.60	199	301
0.02	6.30	206	253	8.30	197	257	10.30	201	263	12.60	202	273
0.03	6.90	193	233	9.10	200	239	11.45	202	256	14.10	200	248
0.05	7.70	198	213	10.20	201	221	13.05	200	230	16.90	201	231
0.10	8.85	197	200	12.00	202	214	16.25	200	218	22.80	199	216
0.20	9.90	196	197	14.45	200	206	21.45	200	210	33.50	201	208
0.50	10.90	201	203	19.10	200	197	33.50	200	200	58.30	200	200

Table 15

The Upper Control Limits of the Signed-Rank Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 500 from the Multivariate Normal Distribution

	p											
	2			3			4			5		
λ	*L*	$ARL_0 \cong$	SDRL	*L*	$ARL_0 \cong$	SDRL	*L*	$ARL_0 \cong$	SDRL	*L*	$ARL_0 \cong$	SDRL
0.01	6.90	496	599	9.10	487	634	11.15	494	659	13.30	495	669
0.02	8.10	492	556	10.50	497	568	12.90	494	585	15.90	500	615
0.03	8.85	499	539	11.40	502	558	14.20	498	563	18.05	501	575
0.05	9.75	494	517	12.65	500	533	16.30	498	541	21.90	503	556
0.10	10.95	502	507	14.70	497	520	20.50	499	522	30.60	501	527
0.20	11.95	491	484	17.60	497	505	27.65	500	515	46.90	501	507
0.50	12.75	482	479	23.80	504	500	44.50	500	505	84.00	493	497

Table 16

The Upper Control Limits of the Signed-Rank Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 1,000 from the Multivariate Normal Distribution

	p											
	2			3			4			5		
λ	*L*	ARL_0≅	SDRL	*L*	ARL_0≅	SDRL	*L*	ARL_0≅	SDRL	*L*	ARL_0≅	SDRL
0.01	8.40	1,007	1,151	10.70	996	1,180	13.00	988	1,187	15.55	1,005	1,269
0.02	9.60	985	1,047	12.20	993	1,019	15.00	988	1,121	18.60	1,010	1,178
0.03	10.40	990	1,024	13.20	996	1,056	16.50	997	1,096	21.30	998	1,103
0.05	11.35	1,003	1,033	14.20	1,004	1,035	18.85	1,000	1,078	26.10	1,006	1,092
0.10	12.50	995	1,007	16.90	993	1,010	23.95	1,001	1,053	37.50	998	1,046
0.20	13.50	992	982	20.25	992	998	32.80	992	1,020	58.40	983	997
0.50	14.20	967	960	27.40	994	1,019	53.60	989	989	107.9	983	973

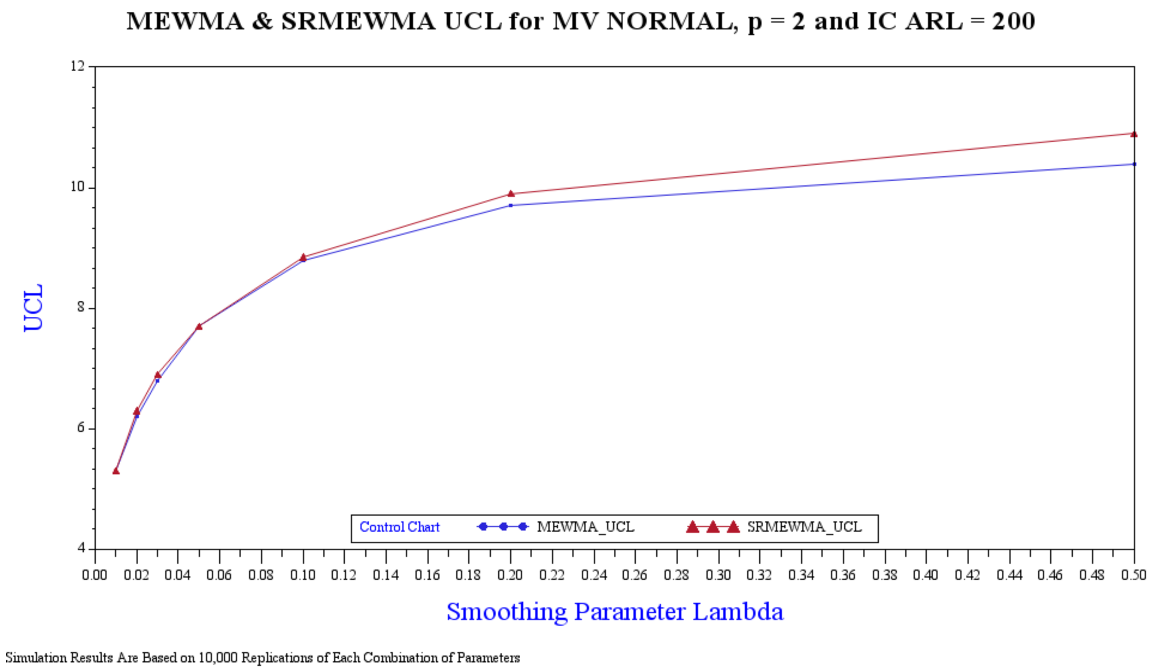

Figure 5. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate normal, p = 2 and in-control average run length = 200.

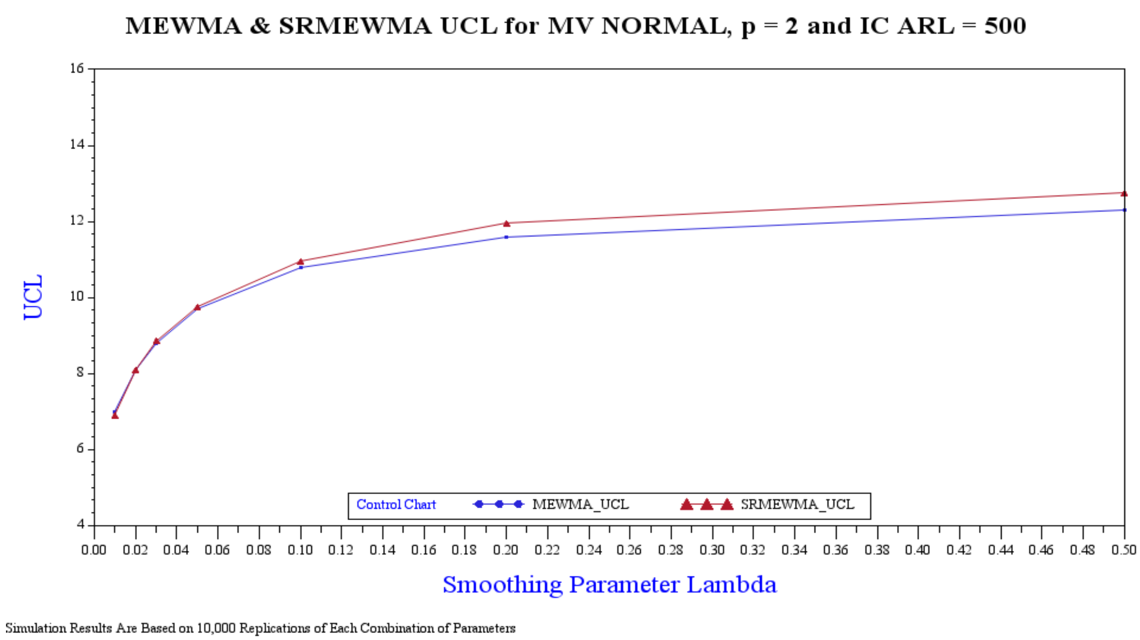

Figure 6. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate normal, p = 2 and in-control average run length = 500.

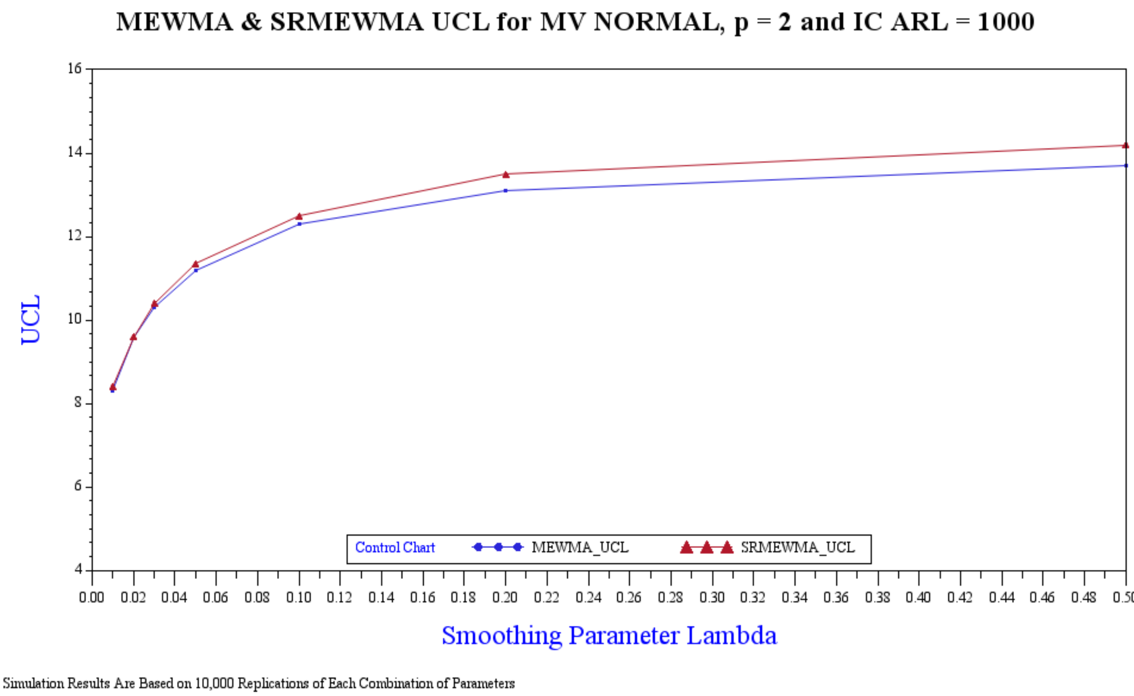

Figure 7. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate normal, p = 2 and in-control average run length = 1,000.

Signed-Rank Multivariate Exponentially Weighted Moving Average Upper Control Limit and Multivariate Exponentially Weighted Moving Average Upper Control Limit from the Multivariate *t* Distribution

Tables 17-19 and Tables 20–22 show the computed IC UCL values for the MEWMA and SRMEWMA, respectively, for data generated from the multivariate t_p (df = 5) distribution for p = 2, 3, 4, and 5 and $\lambda \in$ [0.01, 0.02, 0.03, 0.05, 0.10, 0.2, and 0.50] for ARL_0 = 200, 500, and 1,000, respectively. In addition to the UCL and nominal ARL_0 values, the standard deviation of run lengths (SDRL) is shown for all p and λ combinations. Additionally, Figures 8-10 show the UCL values for both MEWMA and SRMEWMA for p = 2 and ARL_0 = 200, 500, and 1,000, respectively. Additional tables

and figures summarizing the UCL results for the SRMEWMA, MEWMA, and Hotelling's T^2 control charts for p = 3, 4, and 5 are available in Appendix C (Figures 33-41). The results for the UCL values generated from multivariate t distribution were similar to those generated from the multivariate normal distribution. The results show the following:

1. The UCL values, L and h_1, for the SRMEWMA and MEWMA control charts, respectively, increased as p increased for any λ value.
2. The UCL values, L and h_1, for the SRMEWMA and MEWMA control charts, respectively, increased as λ increased for any p value.
3. For $p = 2$, the UCL values for the SRMEWMA control chart were slightly larger than those of the MEWMA control charts as λ increased.
4. For any given p, the UCL values for SRMEWMA got larger than those of MEWMA as λ got larger.
5. The UCLs for SRMEWMA got increasingly larger than those of MEWMA as λ got larger for p = 3, 4, and 5.
6. The SRMEWMA and MEWMA UCL values, L and h_1, respectively, increased as ARL_0 increased for any p and λ value.

Table 17

The Upper Control Limits of the Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 200 from the Multivariate $t_p(5)$-Distribution

	p											
	2			3			4			5		
λ	h_1	$ARL_0\cong$	SDRL	h_1	$ARL_0\cong$	SDRL	h_1	$ARL_0\cong$	SDRL	h_1	$ARL_0\cong$	SDRL
0.01	5.3	200	290	7.1	199	294	8.8	192	280	10.4	192	280
0.02	6.2	198	246	8.2	200	247	10.0	201	247	11.6	196	248
0.03	6.8	195	224	8.9	201	234	10.8	201	233	12.5	200	229
0.05	7.7	197	212	9.9	202	219	11.8	200	214	13.6	194	214
0.10	9.0	202	208	11.3	199	206	13.2	196	204	15.2	201	209
0.20	10.3	198	198	12.7	200	198	14.8	198	200	16.8	200	200
0.50	12.3	200	200	15.0	198	199	17.5	199	197	19.7	197	194

Table 18

The Upper Control Limits of the Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 500 from the Multivariate $t_p(5)$-Distribution

	p											
	2			3			4			5		
λ	h_1	$ARL_0 \cong$	SDRL	h_1	$ARL_0 \cong$	SDRL	h_1	$ARL_0 \cong$	SDRL	h_1	$ARL_0 \cong$	SDRL
0.01	6.9	493	605	9.0	507	634	10.8	488	617	12.6	496	619
0.02	8.1	488	544	10.3	500	553	12.2	491	545	14.1	501	562
0.03	8.9	492	528	11.1	492	434	13.2	498	547	15.0	496	534
0.05	9.8	474	486	12.2	491	510	14.2	482	504	16.2	494	519
0.10	11.2	489	496	13.7	493	500	15.9	504	521	17.8	490	498
0.20	12.7	486	491	15.4	492	490	17.7	501	501	19.8	503	507
0.50	15.7	500	501	18.8	507	510	21.5	493	494	23.9	486	489

Table 19

The Upper Control Limits of the Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 1,000 from the Multivariate $t_p(5)$-Distribution

	p											
	2			3			4			5		
λ	h_1	$ARL_0 \cong$	SDRL	h_1	$ARL_0 \cong$	SDRL	h_1	$ARL_0 \cong$	SDRL	h_1	$ARL_0 \cong$	SDRL
0.01	8.3	956	1,091	10.6	998	1,161	12.5	983	1,145	14.4	1,001	1,152
0.02	9.7	975	1,047	12.0	985	1,076	14.0	971	1,029	16.0	1,007	1,068
0.03	10.5	1,006	1,054	12.9	1,007	1,060	15.0	1,005	1,078	16.9	983	1,013
0.05	11.5	984	1,010	14.0	1002	1,038	16.2	1,002	1,022	18.1	1,017	1,058
0.10	13.0	984	982	15.6	1,006	1,026	17.9	1,002	1,019	19.9	992	995
0.20	14.8	995	1,022	17.6	1,003	1,007	20.0	995	1,013	22.2	984	986
0.50	18.8	997	993	22.1	964	966	25.3	985	1,002	27.9	996	1,090

Table 20

The Upper Control Limits of the Signed-Rank Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 200 from the Multivariate $t_p(5)$- Distribution

	p											
	2			3			4			5		
λ	*L*	$ARL_0 \cong$	SDRL	*L*	$ARL_0 \cong$	SDRL	*L*	$ARL_0 \cong$	SDRL	*L*	$ARL_0 \cong$	SDRL
0.01	5.35	203	300	7.10	196	283	8.75	199	296	10.30	200	297
0.02	6.30	201	248	8.35	200	253	10.35	198	252	12.69	201	262
0.03	6.90	197	226	9.25	200	241	11.70	199	240	14.70	201	243
0.05	7.85	200	221	10.70	200	227	14.00	201	229	18.60	202	231
0.10	9.20	201	208	13.30	200	215	19.20	201	211	27.55	199	209
0.20	10.70	199	205	17.65	200	206	28.30	199	206	43.80	200	208
0.50	13.40	200	204	27.40	200	203	49.80	200	199	81.90	198	201

Table 21

The Upper Control Limits of the Signed-Rank Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 500 from the Multivariate $t_p(5)$- Distribution

	p											
	2			3			4			5		
λ	*L*	ARL_0 ≅	SDRL	*L*	ARL_0 ≅	SDRL	*L*	ARL_0 ≅	SDRL	*L*	ARL_0 ≅	SDRL
0.01	6.95	495	609	9.20	504	650	11.30	501	667	13.70	498	663
0.02	8.20	500	558	10.75	498	579	13.62	502	604	17.60	497	589
0.03	8.95	497	539	11.95	503	564	15.60	506	578	20.90	497	563
0.05	10.00	501	527	13.80	501	546	19.20	503	541	27.60	502	548
0.10	11.55	493	500	17.50	500	516	27.60	501	521	43.40	498	519
0.20	13.50	496	504	24.15	495	496	43.00	501	505	72.90	499	507
0.50	17.50	501	501	40.10	496	493	79.00	497	493	142.30	500	500

Table 22

The Upper Control Limits of the Signed-Rank Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 1,000 from the Multivariate $t_p(5)$- Distribution

	p											
	2			3			4			5		
λ	*L*	ARL_0≅	SDRL	*L*	ARL_0≅	SDRL	*L*	ARL_0≅	SDRL	*L*	ARL_0≅	SDRL
0.01	8.40	991	1,132	10.90	997	1,179	13.50	998	1,246	16.90	996	1,238
0.02	9.80	1,010	1,087	12.75	997	1,093	16.45	992	1,132	22.20	1,000	1,152
0.03	10.60	998	1,042	14.15	1,006	1,082	19.05	992	1,099	27.20	995	1,091
.05	11.75	993	1,020	16.40	998	1,076	23.95	999	1,041	37.30	996	1,059
0.10	13.50	1,007	1,021	21.20	982	1,016	35.50	992	1,007	61.20	1,000	1,010
0.20	15.80	997	1,015	30.20	996	1,002	58.00	988	984	106.9	1,000	1,009
0.50	21.15	981	972	52.20	999	985	111.1	997	987	212.10	1,006	1,006

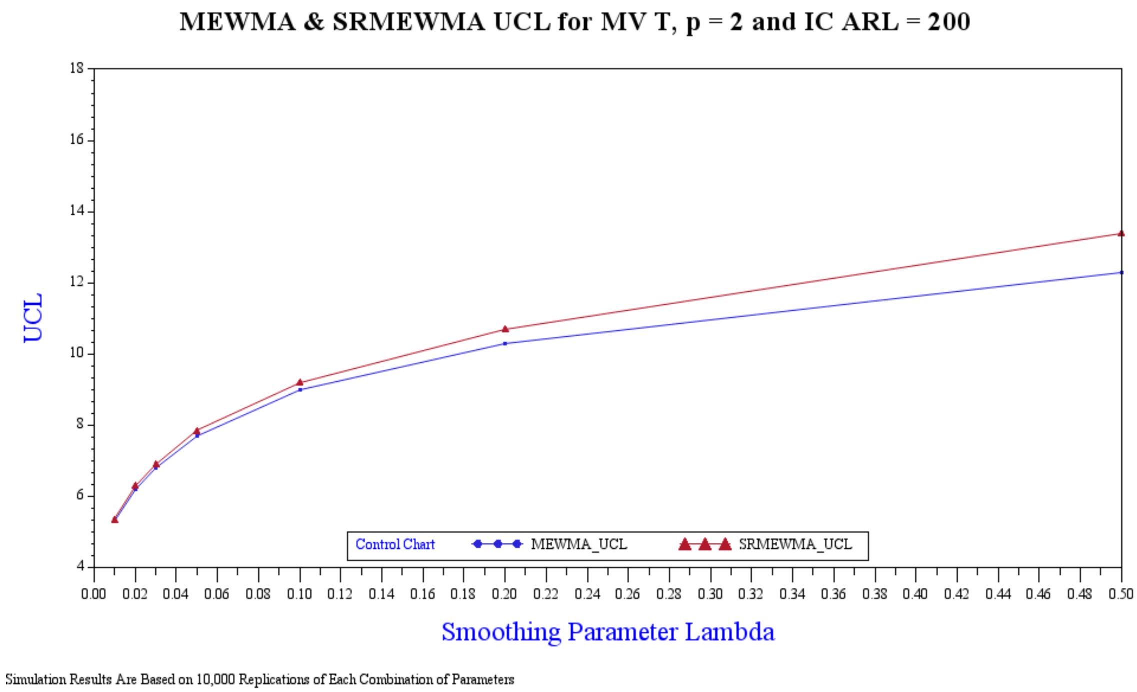

Figure 8. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate normal t_p (*df* = 5), *p* = 2 and in-control average run length = 200.

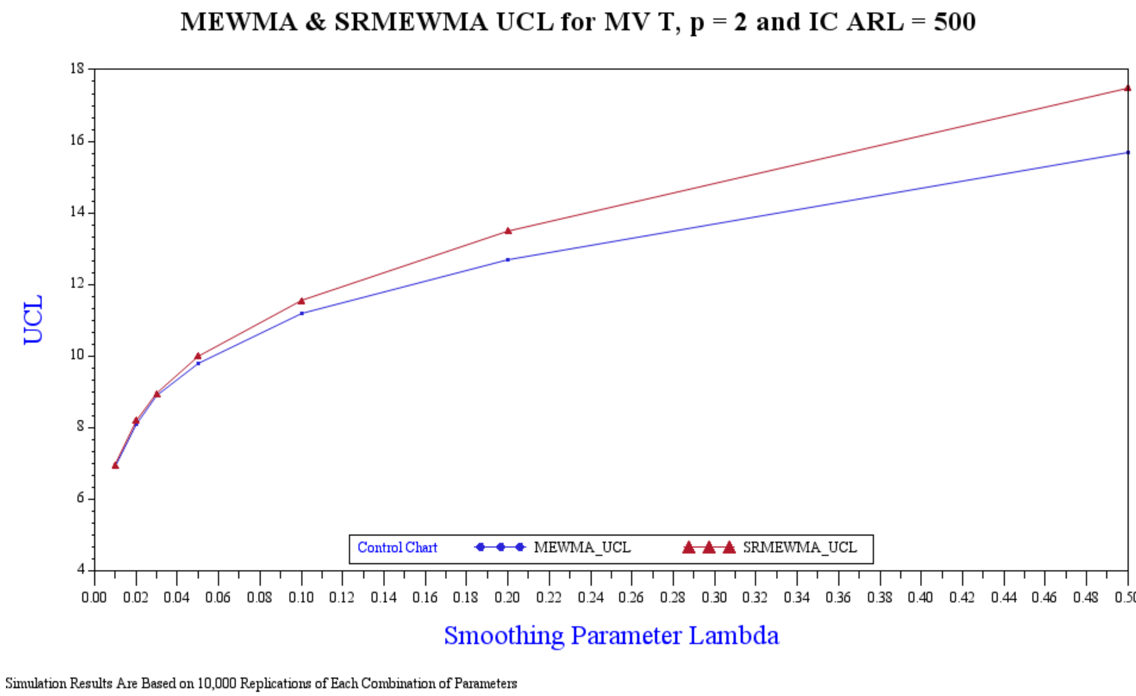

Figure 9. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate normal t_p (df = 5), p = 2 and in-control average run length = 500.

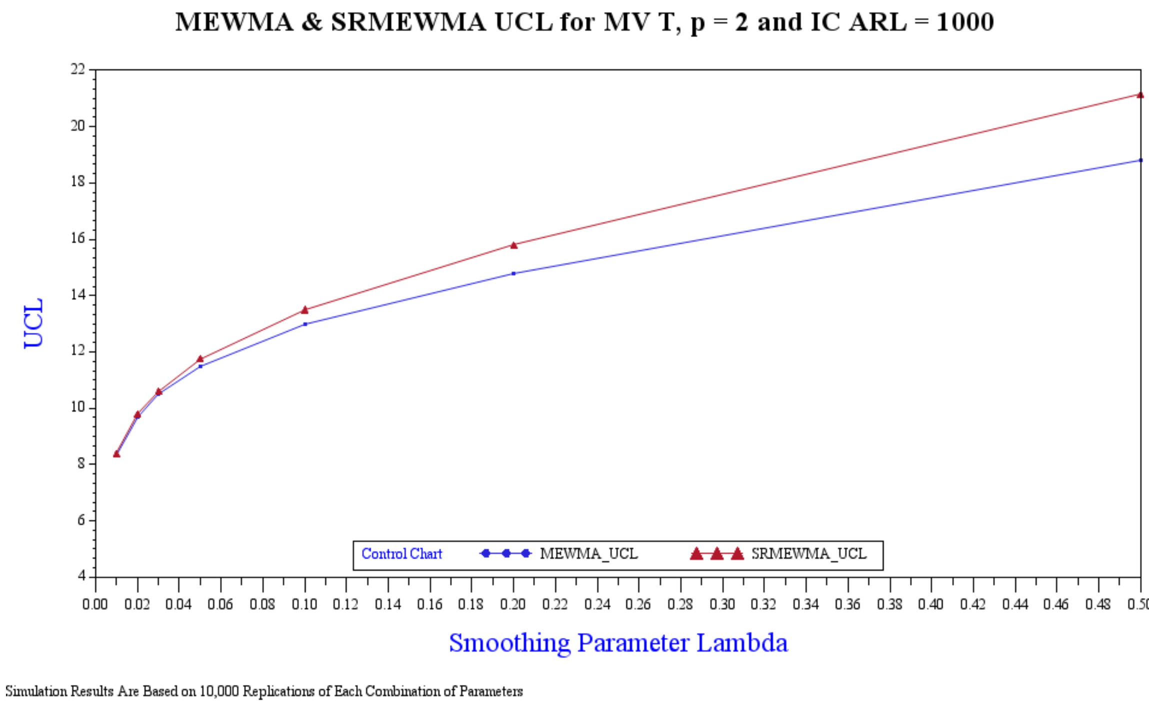

Figure 10. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate normal t_p (df = 5), p = 2 and in-control average run length = 1,000.

Signed-Rank Multivariate Exponentially Weighted Moving Average Upper Control Limit (*L*) and Multivariate Exponentially Weighted Moving Average Upper Control Limit (*h_1*) from the Multivariate Gamma$_2$ (α = 3, β = 1) Distribution

Tables 23 and 24 show the computed IC UCL values for the MEWMA and SRMEWMA, respectively, for data generated from the multivariate $gamma_2$ $(\alpha=3,\beta=1)$ distribution for p = 2 and $\lambda \in$ [0.01, 0.02, 0.03, 0.05, 0.10, 0.2, and 0.50] for ARL_0 = 200, 500, and 1,000, respectively. In addition to the UCL and nominal ARL_0 values, the standard deviation of run lengths (SDRL) is shown for all p and λ combinations. Additionally, Figures 11-13 show the UCL values for both

MEWMA and SRMEWMA for p = 2 and ARL_0 = 200, 500, and 1,000, respectively. Only p = 2 was considered because the computation of the centered signed-ranks, which was necessary to compute the SRMEWMA charting statistic, was very intensive due to the number of vector combinations that were evaluated from the simulated variables to calculate the vector signed-ranks. For example, when p = 3 and n = 5, there are 10 vector combinations to be analyzed. However, when n = 20, the number of vector combinations to be analyzed is 1,140, a multiple of 114. For a detailed explanation of this limitation, see Chapter V, Table 56. The results showed the following:

1. The UCL values, L and h_1, for the SRMEWMA and MEWMA control charts, respectively, increased as λ increased.
2. The UCL values for SRMEWMA got smaller than those of MEWMA as λ got larger.
3. The SRMEWMA and MEWMA UCL values, L and $h_{1,}$ respectively, increased as ARL_0 increased for any p and λ value.
4. The UCL values, h_1, for the MEWMA were larger than the UCL values, L, for the SRMEWMA for any given λ and ARL_0 values.

Table 23

The Upper Control Limits of the Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length $\approx 200,\ 500,\ and\ 1{,}000$ *from the Gamma₂ (α = 3, β = 1) Distribution with* $\rho_{12} = 0.5$

	IC ARL=200			IC ARL=500			IC ARL=1,000		
λ	h_1	$ARL_0 \cong$	SDRL	h_1	$ARL_0 \cong$	SDRL	h_1	$ARL_0 \cong$	SDRL
0.01	4,124.1	200.05	215.23	4,209.5	499.98	485.80	4,268.4	1,002	959.10
0.02	2,116.6	200.1	201.53	2,175.5	500.45	480.18	2,211.7	1,002	993.84
0.03	1,442.9	200.5	195.80	1.486.7	500.1	480.08	1,515.7	997.5	993.66
0.05	896.6	200.0	192.00	928.5	500.3	490.50	948.8	1,000	995.40
0.10	476.8	200.2	193.90	498.3	501.0	499.80	512.0	1,007	1,003.3
0.20	257.8	200.2	196.60	272.6	498.0	503.80	282.8	1,001	993.05
0.50	116.1	200.7	202.30	125.8	503.7	502.16	132.7	999	982.17

Table 24

The Upper Control Limits of the Signed-Rank Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length $\approx 200,\ 500,$ $and\ 1{,}000$ *from the Gamma₂ (α=3,β=1) Distribution with* $\rho_{12} = 0.5$

	IC ARL=200			IC ARL=500			IC ARL=1,000		
λ	L	$ARL_0 \cong$	SDRL	L	$ARL_0 \cong$	SDRL	L	$ARL_0 \cong$	SDRL
0.01	*1,366.3*	201	236.48	*1419.9*	499.4	513.19	*1,460.0*	994.0	988.60
0.02	*723.1*	200	209.37	*761.7*	494.7	494.26	*798.1*	999.4	980.70
0.03	*505.6*	200	200.50	*537.4*	498.5	495.26	*559.3*	999.4	966.63
0.05	*328.5*	200	196.52	*352.7*	500	500.16	*369.2*	1,001	1,002.63
0.10	*190.3*	200	199.56	*207.7*	500	497.38	*219.4*	997.0	990.75
0.20	*116.2*	200	196.41	*129.3*	500	509.81	*138.7*	997.0	994.06
0.50	*65.9*	201	200.60	*76.7*	502	502.64	*84.5*	1,000	997.33

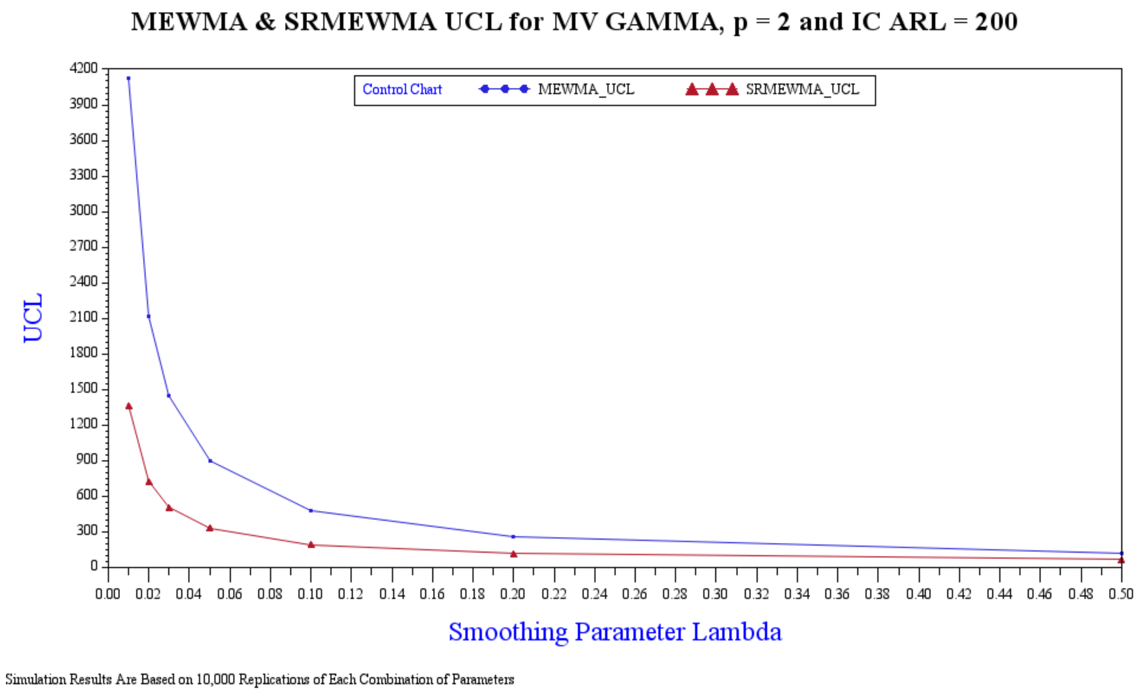

Figure 11. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate gamma$_2$ ($\alpha = 3, \beta = 1$), p = 2 and in-control average run length = 200.

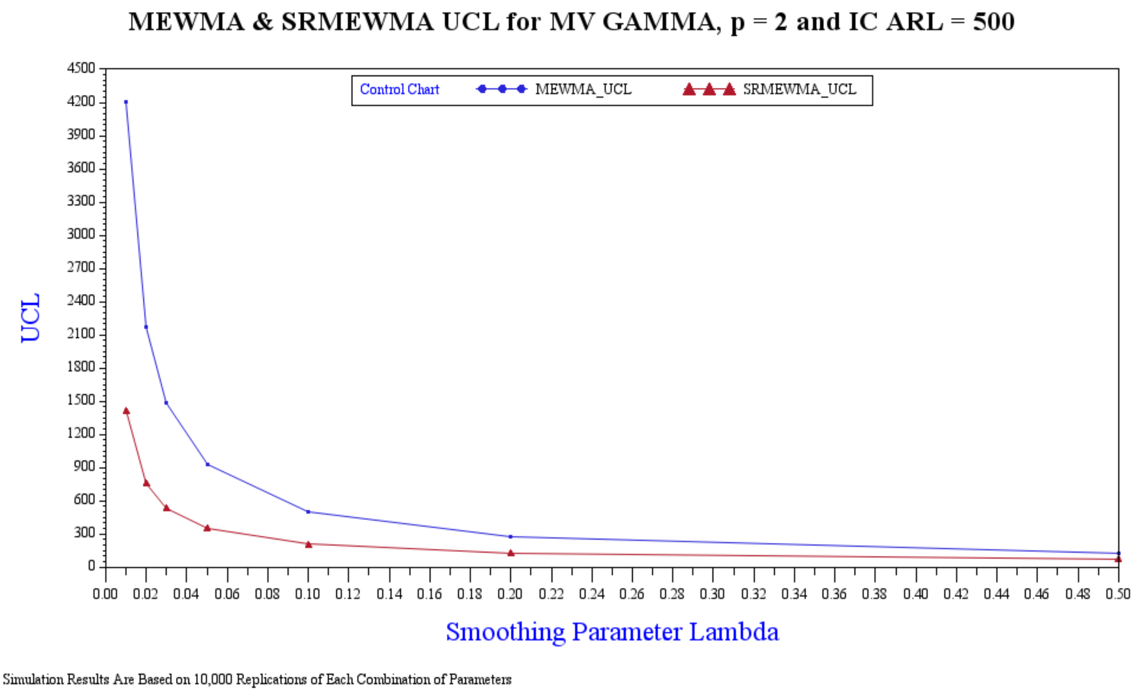

Figure 12. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate gamma$_2$ ($\alpha = 3, \beta = 1$), p = 2 and in-control average run length = 500.

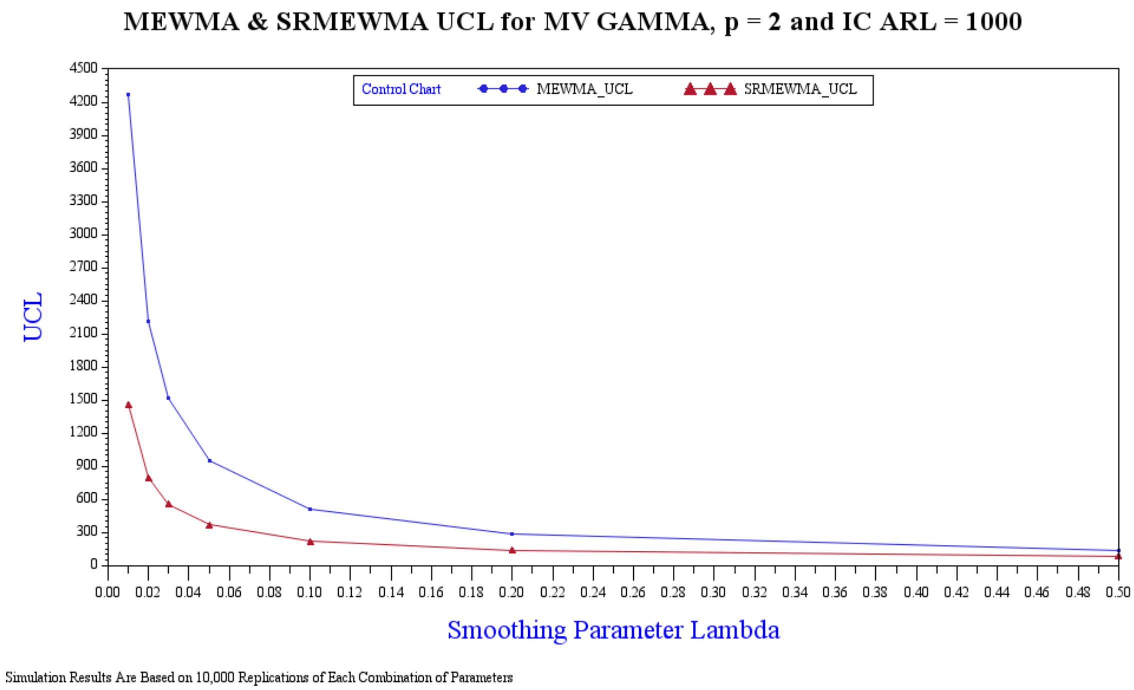

Figure 13. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate gamma_2 ($\alpha = 3, \beta = 1$), $p = 2$ and in-control average run length = 1,000.

Hotelling's T^2 (h_2) Upper Control Limit

In addition to the MEWMA and SRMEWMA control charts, the UCL values for the Hotelling's T^2 control chart were generated according to Table 6. Tables 25-27 show the computed IC Hotelling's T^2 UCL (h_2) values for data generated from the multivariate normal, t, and $Gamma_2$ (α=3,β=1) distribution for p = 2, 3, 4, and 5 for ARL_0 = 200, 500, and 1,000, respectively. Recall that for the multivariate normal distribution, the UCL (h_2) in Table 25 for the Hotelling's T^2 control chart was obtained from χ_p^2 quintiles using the SAS CINV function. The results show the following:

1. The UCLs values, h_2, for the Hotelling's T^2 control chart increased as p increased for the MV normal and t distributions.

2. For a fixed p, the UCLs values, h_2, for the Hotelling's T^2 control chart increased as ARL_0 increased for the MV normal and t distributions.

Table 25

The Upper Control Limits (h_2) of the Hotelling's χ^2 That Achieved an In-Control Average Run Length $\approx 200,\ 500,\ and\ 1{,}000$ under p-variates from the Multivariate Normal Distribution

	ARL_0		
p	200	500	1,000
2	10.59	12.42	13.88
3	12.83	14.79	16.26
4	14.86	16.92	18.46
5	16.74	18.90	20.51

Table 26

The Upper Control Limits (h_2) of the Hotelling's χ^2 That Achieved an In-Control Average Run Length $\approx 200,\ 500,\ and\ 1{,}000$ under p-variates from the Multivariate $t_p(5)$-Distribution

	IC ARL=200			IC ARL=500			IC ARL=1,000		
p	h_2	$ARL_0\cong$	SDRL	h_2	$ARL_0\cong$	SDRL	h_2	$ARL_0\cong$	SDRL
2	13.50	200	196.61	17.60	488	483.73	21.50	992	986.65
3	16.60	200	196.61	21.10	494	493.82	25.60	998	978.42
4	19.20	201	202.63	24.30	500	501.81	29.20	988	984.37
5	21.60	201	198.70	27.00	494	493.68	32.40	1,000	990.61

Table 27

The Upper Control Limits (h_2) of the Hotelling's χ^2 That Achieved an In-Control Average Run Length ≈ 200, 500, and 1,000 under p-variates from the Multivariate Gamma$_2$(α = 3, β = 1) Distribution

	IC ARL=200			IC ARL=500			IC ARL=1,000		
p	h_2	ARL_0≅	SDRL	h_2	ARL_0≅	SDRL	h_2	ARL_0≅	SDRL
2	25.30	201	197.98	34.82	503	505.74	42.7	993	1,000.55
3	15.50	200	202.82	18.95	501	507.26	21.6	995	983.80
4	17.93	200	199.38	21.5	500	501.74	24.28	1,000	1,000.13
5	20.00	200	197.66	23.6	500	491.81	26.45	999	1,011.04

Phase II Average Run Length Simulation Results

The upper control limits (UCLs) that were generated in phase I were used in the phase II ARL_1 simulation. Using IML and the RANDGEN function in SAS, 10,000 samples per run were generated from the multivariate normal, *t*, and gamma distributions for each run of the following combinations of study parameters:

1. The number of variables, p = 2, 3, 4, and 5 for the multivariate normal and *t* distributions and p = 2 only for the multivariate gamma distribution;
2. The smoothing parameter, $\lambda \in$ [0.01, 0.02, 0.03, 0.05, 0.10, 0.2, and 0.50]; and
3. Shift parameter, $\delta \in$ [0.0, 0.25, 0.50, 1.00, 1.50, and 2.50].

The full phase II ARL_1 simulation results are available in Appendix D (Tables 74-96) for all three control charts and three multivariate distributions based upon the abovementioned conditions in steps 1 through 3; thus, they provide the answers to

research questions 3 and 4. In this section, the ARL_1 performance results of the SRMEWMA and MEWMA control charts from the multivariate normal distribution for ARL_0 = 200, p = 2, 3, 4, and 5, and $\delta \in$ [0.0, 0.25, 0.50, 1.00, 1.50, and 2.50] are summarized using Tables 28 (MEWMA) and 29 (SRMEWMA) for data from the multivariate normal distribution for ARL_0 = 200. The results showed the following:

1. For any given values of p and shift parameter δ, the ARL_1 and SDRL values increased as the smoothing parameter λ increased.
2. For any given values of p and smoothing parameter λ, the ARL_1 and SDRL values decreased as the shift parameter δ increased.
3. For any given value of the shift parameter δ and smoothing parameter λ, the ARL_1 values increased as p increased.
4. For any given values of δ and $\lambda > 0.02$, the SDRL values increased as p increased.
5. The ARL_1 values increased as ARL_0 increased.

Table 28

The Upper Control Limits and Average Run Length Values of the Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 200 from the Multivariate Normal Distribution

			λ						
p			0.01	0.02	0.03	0.05	0.1	0.2	0.5
2	δ	h_1	5.30	6.20	6.80	7.7	8.8	9.7	10.40
	0.00		193	197	194	201	201	202	198
			(288)	(245)	(223)	(214)	(205)	(203)	(200)
	0.25		35.91	45.42	49.77	59.51	74.91	94.88	133.82
			(42.77)	(76.75)	(48.67)	(57.43)	(71.68)	(92.43)	(131.22)
	0.50		13.06	16.12	18.19	20.77	25.17	33.84	62.70
			(13.57)	(14.56)	(15.51)	(16.87)	(20.79)	(30.71)	(61.51)
	1.00		4.39	5.33	5.92	6.73	7.78	8.94	15.37
			(3.75)	(4.02)	(4.30)	(4.61)	(5.13)	(6.31)	(13.55)
	1.50		2.44	2.92	3.16	3.58	4.01	4.46	6.06
			(1.67)	(1.92)	(2.02)	(2.17)	(2.34)	(2.56)	(4.44)
	2.50		1.32	1.48	1.57	1.71	1.87	2.04	2.25
			(0.60)	(0.72)	(0.76)	(0.83)	(0.92)	(0.99)	(1.20-)
3	δ	h_1	7.20	8.20	8.80	9.70	11.00	12.00	12.60
	0.00		196	200	193	198	209	204	195
			(295)	(244)	(220)	(214)	(214)	(207)	(195)
	0.25		42.86	51.46	56.38	65.63	83.18	110.95	142.83
			(48.77)	(52.06)	(54.32)	(62.80)	(80.16)	(109.87)	(143.54)
	0.50		15.78	19.23	21.01	23.17	29.07	40.67	73.98
			(15.30)	(16.49)	(17.36)	(18.70)	(24.04)	(36.61)	(72.18)
	1.00		5.51	6.48	6.97	7.68	8.73	10.49	18.70
			(4.30)	(4.67)	(4.83)	(5.06)	(5.70)	(7.56)	(16.91)
	1.50		3.06	3.47	3.69	4.04	4.49	5.06	7.02
			(2.03)	(2.16)	(2.30)	(2.40)	(2.59)	(2.95)	(5.31)
	2.50		1.60	1.79	1.79	1.92	2.07	2.23	2.45
			(0.78)	(0.83)	(0.88)	(0.94)	(0.99)	(1.06)	(1.317)
4	δ	h_1	8.80	9.90	10.70	11.60	12.90	13.90	14.70
	0.00		201	199	203	193	200	199	201
			(301)	(246)	(232)	(209)	(203)	(199)	(201)
	0.25		44.09	55.30	61.93	71.22	89.81	116.74	156.31
			(52.03)	(55.12)	(60.20)	(67.50)	(86.98)	(115.13)	(15.30)
	0.50		16.06	20.26	22.11	24.94	31.44	43.76	84.06
			(16.39)	(17.69)	(18.33)	(20.02)	(25.71)	(40.22)	(83.19)
	1.00		5.42	6.69	7.48	8.28	9.57	11.34	21.64
			(4.48)	(5.00)	(5.24)	(5.51)	(6.19)	(8.07)	(19.52)
	1.50		2.99	3.58	3.95	4.27	4.86	5.33	8.06
			(2.10)	(2.32)	(2.44)	(2.56)	(2.77)	(3.11)	(6.31)
	2.50		1.53	1.75	1.87	2.01	2.18	2.35	2.70
			(0.76)	(0.88)	(0.93)	(0.98)	(1.05)	(1.12)	(1.47)
5	δ	h_1	10.40	11.60	12.30	13.40	14.70	15.8	16.6
	0.00		200	200	195	200	199	199	197
			(289)	(241)	(222)	(218)	(201)	(197)	(185)
	0.25		49.80	59.12	66.05	77.00	95.78	123.33	160.37
			(55.27)	(58.57)	(64.05)	(74.76)	(93.40)	(122.15)	(160.06)
	0.50		18.98	21.94	24.03	27.23	34.32	49.28	92.99
			(17.64)	(18.42)	(19.63)	(21.39)	(29.07)	(45.03)	(92.07)
	1.00		6.65	7.66	8.11	8.91	10.05	12.40	25.05
			(4.99)	(5.35)	(5.52)	(5.83)	(6.43)	(8.74)	(23.05)
	1.50		3.69	4.03	4.24	4.69	5.12	5.77	9.08
			(2.37)	(2.49)	(2.57)	(2.69)	(2.83)	(3.33)	(7.31)
	2.50		1.83	1.94	2.04	2.18	12.34	2.50	2.86
			(0.91)	(0.95)	(1.00)	(1.05)	(1.12)	(1.18)	(1.56)

Note. Standard deviation of run length is in parentheses.

Table 29

The Upper Control Limits of the Signed-Rank Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 200 from the Multivariate Normal Distribution

			λ						
p			0.01	0.02	0.03	0.05	0.1	0.2	0.5
2	δ	*L*	5.30	6.30	6.90	7.70	8.85	9.90	10.90
	0.00		196	206	193	198	197	196	201
			(287)	(253)	(233)	(213)	(200)	(197)	(203)
	0.25		38.65	46.86	51.45	59.11	74.92	97.13	141.65
			(43.62)	(47.19)	(49.27)	(57.49)	(73.52)	(93.66)	(142.57)
	0.50		13.98	16.93	18.72	20.52	25.62	35.10	68.02
			(13.59)	(14.81)	(15.57)	(16.60)	(21.27)	(32.32)	(66.90)
	1.00		4.96	5.71	6.21	6.88	7.81	9.36	16.82
			(3.84)	(4.17)	(4.47)	(4.71)	(5.14)	(6.62)	(15.41)
	1.50		2.73	3.07	3.32	3.61	4.07	4.60	6.57
			(1.76)	(1.94)	(2.06)	(2.16)	(2.36)	(2.68)	(4.99)
	2.50		1.43	1.56	1.63	1.74	1.90	2.08	2.34
			(0.66)	(0.74)	(0.79)	(0.84)	(0.90)	(0.98)	(1.21)
3		*L*	7.30	8.30	9.10	10.20	12.00	14.45	19.10
	0.00		204	197	200	201	202	200	200
			(307)	(257)	(239)	(221)	(214)	(206)	(197)
	0.25		40.40	50.11	56.84	69.70	96.00	136.97	180.71
			(49.26)	(53.77)	(59.74)	(69.97)	(96.21)	(138.34)	(181.89)
	0.50		14.19	18.15	20.79	24.29	33.60	61.03	132.92
			(15.12)	(16.79)	(17.95)	(20.31)	(29.27)	(59.21)	(133.35)
	1.00		4.66	5.93	6.79	7.84	9.74	14.53	51.80
			(4.02)	(4.53)	(4.85)	(5.22)	(6.37)	(11.05)	(49.62)
	1.50		2.47	3.21	3.60	4.12	4.92	6.32	19.03
			(1.71)	(2.08)	(2.22)	(2.41)	(2.71)	(3.65)	(17.17)
	2.50		1.26	1.55	1.73	1.92	2.21	2.63	4.38
			(0.57)	(0.73)	(0.82)	(0.88)	(1.00)	(1.15)	(2.62)
4		*L*	9.00	10.30	11.45	13.05	16.25	21.45	33.50
	0.00		202	201	202	200	200	200	200
			(305)	(263)	(256)	(230)	(218)	(210)	(200)
	0.25		46.90	56.33	67.32	84.7	123.19	171.09	196.13
			(54.48)	(61.76)	(71.02)	(87.25)	(131.13)	(174.35)	(195.50)
	0.50		17.09	21.13	24.61	30.43	50.77	102.06	174.16
			(16.60)	(18.66)	(21.07)	(25.94)	(47.12)	(102.95)	(176.10)
	1.00		6.08	7.33	8.20	9.62	13.10	28.11	114.80
			(4.55)	(5.13)	(5.49)	(6.23)	(8.60)	(24.39)	(114.95)
	1.50		3.37	3.89	4.33	4.97	6.37	10.13	63.05
			(2.04)	(2.26)	(2.46)	(2.72)	(3.36)	(6.30)	(61.71)
	2.50		1.67	1.88	2.05	2.28	2.75	3.65	14.35
			(0.73)	(0.81)	(0.87)	(0.97)	(1.16)	(1.51)	(11.92)
5		*L*	10.60	12.60	14.10	16.90	22.80	33.50	58.30
	0.00		199	202	200	201	199	201	200
			(301)	(273)	(248)	(231)	(216)	(208)	(200)
	0.25		51.67	67.46	81.28	111.61	156.56	187.51	194.70
			(59.17)	(73.35)	(88.24)	(120.36)	(169.48)	(194.42)	(198.98)
	0.50		19.86	25.44	30.00	41.03	82.16	147.13	191.50
			(18.21)	(22.09)	(25.34)	(36.27)	(80.09)	(147.88)	(189.93)
	1.00		7.29	8.70	9.86	12.33	20.78	67.24	161.04
			(5.03)	(5.86)	(6.39)	(7.66)	(14.85)	(65.52)	(161.80)
	1.50		4.08	4.73	5.18	6.28 (3.21)	9.02	23.43	124.96
			(2.22)	(2.52)	(2.78)		(4.59)	(18.56)	(124.57)
	2.50		1.99	2.25	2.43	2.81	3.66	5.79	55.24
			(0.76)	(0.98)	(0.96)	(1.11)	(1.41)	(2.30)	(52.50)

Note. Standard deviation of run length is in parentheses.

It should be noted that these findings also applied to the ARL_1 performance results of the SRMEWMA and MEWMA control charts from the multivariate *t* distribution as summarized in Tables 80-85 in appendix D for ARL_0 = 200, 500, and 1,000, respectively.

Tables 30 and 31 summarize the ARL_1 & SDRL performance for both the MEWMA and SRMEWMA control charts for p = 2 and ARL_0 = 200, 500, and 1,000 from the multivariate gamma distribution. The results were similar across all values ARL_0. For example, for ARL_0 = 200, the top parts of Table 30 (MEWMA) and 31 (SRMEWMA) are reproduced below and show the following results:

1. For any given values of p and shift parameter δ, the ARL_1 and SDRL values increased as the smoothing parameter λ increased.
2. For any given values of p and smoothing parameter λ, the ARL_1 and SDRL values decreased as the shift parameter δ increased.
3. For any given value of the shift parameter δ and smoothing parameter λ, the ARL_1 values increased as p increased.

Table 30

The Upper Control Limits and Average Run Length Values of the Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 200 from the Multivariate Gamma$_2$ (α = 3, β = 1) Distribution

			λ						
ARL_0	δ		0.01	0.02	0.03	0.05	0.1	0.2	0.5
200		h_1	4,124.1	2,116.6	1,442.9	896.6	476.8	257.8	116.1
	0.00		200	200	200	200	200	200	200
			(215.23)	(201.53)	(195.80)	(192.00)	(193.90)	(196.60)	(202.30)
	0.25		58.24	62.89	65.75	73.06	80.47	95.80	130.19
			(47.74)	(50.77)	(53.43)	(63.22)	(73.11)	(90.70)	(129.27)
	0.50		29.78	31.05	32.08	32.95	37.47	47.43	78.21
			(18.94)	(19.65)	(21.15)	(23.37)	(31.23)	(43.78)	(76.99)
	1.00		14.50	14.73	14.52	14.22	13.79	15.53	29.06
			(6.43)	(6.53)	(6.66)	(7.13)	(8.10)	(11.27)	(27.52)
	1.50		6.70	9.54	9.35	8.68	7.97	7.90	12.34
			(3.47)	(3.41)	(3.35)	(3.30)	(3.46)	(4.36)	(10.50)
	2.50		5.88	5.71	5.50	5.03	4.36	3.81	3.88
			(1.60)	(1.57)	(1.49)	(1.41)	(1.33)	(1.34)	(2.16)

Note. Standard deviation of run length is in parentheses.

4. For any given values of shift δ and smoothing parameter λ, the SDRL values increased as *p* increased.

5. The ARL_1 and SDRL values increased as ARL_0 increased for any fixed values of shift δ and smoothing parameter λ.

Table 31

The Upper Control Limits and Average Run Length Values of the Signed-Rank Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 200 for Data from the Multivariate GAMMA$_2$ (α=3, β=1) Distribution

			λ						
ARL_0	δ		0.01	0.02	0.03	0.05	0.1	0.2	0.5
200		*L*	1,366.3	723.1	505.6	328.5	190.3	116.2	65.9
	0.00		201	200	200	200	200	200	201
			(236.48)	(209.37)	(200.50)	(196.52)	(199.56)	(196.41)	(200.60)
	0.25		10.46	11.40	11.53	11.72	12.38	15.24	30.93
			(4.60)	(5.10)	(5.28)	(5.81)	(7.25)	(11.33)	(29.17)
	0.50		5.24	5.61	5.56	5.40	5.18	5.18	8.04
			(1.60)	(1.68)	(1.71)	(1.77)	(1.93)	(2.42)	(6.23)
	1.00		2.97	2.92	2.87	2.74	2.51	2.24	2.14
			(0.59)	(0.61)	(0.61)	(0.62)	(0.61)	(0.62)	(0.85)
	1.50		2.00	2.05	2.02	1.96	1.83	1.60	1.26
			(0.27)	(0.32)	(0.29)	(0.28)	(0.38)	(0.49)	(0.44)
	2.50		1.21	1.34	1.26	1.01	1.00	1.00	1.00
			(0.41)	(0.47)	(0.44)	(0.31)	(0.528)	(0)	(0)

Note. Standard deviation of run length is in parentheses.

For comparison purposes, the ARL_1 performance results of the Hotelling's T^2 control chart were generated by simulation from the multivariate normal, *t*, and gamma distributions. Using IML and the RANDGEN function in SAS, 10,000 samples per run were generated for each run of the following combinations of study parameters:

1. The number of variables, p = 2, 3, 4, and 5 for the multivariate normal and *t* distributions and p = 2 only for the multivariate gamma distribution;
2. Shift parameter, δ ∈ [0.0, 0.25, 0.50, 1.00, 1.50, and 2.50]; and
3. ARL_0 = 200, 500, and 1,000.

The ARL performance results of the Hotelling's T^2 control chart were consistent with those of the MEWMA and SRMEWMA control chart results. The full ARL_1 and SDRL performance simulation results of the Hotelling's T^2 control chart are available in Appendix D (see Tables 88-96). As an example, Table 32 below summarizes the Hotelling's T^2 control chart's ARL_1 and SDRL performance for p = 2, 3, 4, and 5; δ ∈

[0.0, 0.25, 0.50, 1.00, 1.50, and 2.50]; and ARL_0 = 200. Table 32 shows that the values of ARL_1 and SDRL increased as *p* increased for fixed shift parameter δ and decreased as the shift parameter δ increased for fixed *p*.

Table 32

Average Run Length Values of the Hotelling T^2 That Achieved an In-Control Average Run Length ≈ 200 under p-variates Multivariate Normal Distribution

	p							
	2		3		4		5	
δ	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL
0.00	200.00		200.00		200.00		200.00	
0.25	162.11	160.46	175.80	174.36	182.42	180.29	186.44	186.26
0.5	112.45	113.48	130.19	129.89	140.16	140.78	144.74	147.05
1.0	41.50	41.28	53.42	53.31	60.59	60.09	68.62	69.67
1.5	15.67	15.24	20.77	19.84	25.18	24.87	28.94	28.21
2.5	3.61	3.08	4.46	3.95	5.27	4.72	6.03	5.58

Furthermore, the ARL_1 and SDRL performances for ARL_0 = 500 and 1000 (see Tables 89-90 in Appendix D) were consistent with those for ARL_0 = 200. Additionally, the ARL_1 and SDRL performance results for data simulated from the multivariate *t* and gamma distribution were consistent with the ARL_1 and SDRL behavior from the multivariate normal distribution.

Average Run Length Performance Comparisons of Signed- Rank Multivariate Exponentially Weighted Moving Average, Multivariate Exponentially Weighted Moving Average, and Hotelling's T^2

The combined simulation results from phase I where the UCL was computed and phase II where the ARL_1 values were generated are presented here to answer the fifth and final research question:

Q5 What is the Spatial Signed-Rank MEWMA (SRMEWMA) control chart performance compared to the MEWMA control chart and Hotelling's T^2 control chart schemes for elliptically symmetrical (multivariate normal and *t*) and skewed distribution (multivariate gamma)?

In this section, the average run length (ARL_1) performance results are compared for SRMEWMA, MEWMA, and Hotelling's T^2 control charts for data from the multivariate normal, *t*, and gamma distributions. First, the ARL_1 comparisons for the three control charts from the multivariate normal distribution are presented. Second, the ARL_1 comparisons for the three control charts from the multivariate t_p (df = 5) distribution are presented. And finally, the ARL_1 comparisons for the three control charts from the multivariate gamma ($\alpha = 3$, $\beta = 1$) distribution are presented for $p = 2$ only. For brevity, only results for p = 2, 3, 4, and 5, $ARL_0 = 200$ for the multivariate normal and *t* distributions, and p =2 for the multivariate gamma distribution are discussed below. Please refer to Appendix E for comprehensive ARL_1 comparisons (Tables 97-150) of all study parameters combinations for IC ARL_0 = 500 and 1,000.

Average Run Length Comparisons for the Multivariate Normal Distribution

First, the ARL_1 simulation results from the multivariate normal distribution are presented. Tables 33-40 show the ARL_1 comparisons for SRMEWMA, MEWMA and Hotelling's T^2 for the following conditions:

1. The number of variables, p = 2, 3, 4, and 5;
2. The smoothing parameter, $\lambda \in$ [0.02, 0.03, 0.05, 0.10, 0.2, and 0.50]; and
3. Shift parameter, $\delta \in$ [0.0, 0.25, 0.50, 1.00, 1.50, and 2.50].

Tables 33–40 show the ARL_1 comparisons for in-control (IC) ARL_0 = 200 only. Simulation results are shown in Tables 33 and 34, 35 and 36, 37 and 38, and 39 and 40 for p = 2, 3, 4, and 5, respectively. Please note that for shift parameter δ = 0.0, the results represented the in-control (IC) ARL_0 values from phase I simulation and were included for comparison purposes. Furthermore, comparisons for ARL_0 = 500 and ARL_0 = 1,000 are available in Appendix E.

The simulation results for SRMEWMA, MEWMA, and Hotelling's T^2 with λ = 0.02, 0.03, and 0.05 and Hotelling's T^2 for p = 2 are presented in Table 33 and results with λ = 0.1, 0.2, and 0.5 and Hotelling's T^2 for p = 2 are presented in Table 34. In addition to the ARL_1 values, the corresponding standard deviations of the run lengths (SDRL) are also included in these two tables. Tables 33-40 show that the MEWMA control chart had better efficiency in detecting mean shifts as expected since the parametric hypothesis was the correct one in this case (Zou & Tsung, 2010). However, the SRMEWMA control chart offered reasonably comparable ARL_1 performance and the difference between MEWMA and SRMEWMA was not significant for p = 2, 3, and 4

and all δ shift values; but in $p = 5$ and $\lambda \geq 0.1$, MEWMA was superior to SRMEWMA for all δ shift values as shown in Table 40. Additionally, both SRMEWMA and MEWMA were superior to Hotelling's T^2 for all p, λ, and δ values. Hackl and Ledolter (1991) and Zhou et al. (2010) pointed out that MEWMA becomes more significant for large δ values. Signed-Rank Multivariate Exponentially Weighted Moving Average is based on signs and ranks and even for large shifts, and the observations may not grow large; hence, SRMEWMA is not as significant as MEWMA for large shift δ values.

Table 33

Average Run Length Comparisons for p = 2, λ = .02, .03, and .05, and In-Control Average Run Length ≈ 200 *from the Multivariate Normal Distribution*

			λ						
p			0.02		0.03		0.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	6.30	6.20	6.90	6.80	7.70	7.7	10.59
	0.00		206 (253)	197 (245)	193 (233)	194 (223)	198 (213)	201 (214)	200
	0.25		46.86 (47.19)	45.42 (76.75)	51.45 (49.27)	49.77 (48.67)	59.11 (57.49)	59.51 (57.43)	162.11 (160.46)
	0.50		16.93 (14.81)	16.12 (14.56)	18.72 (15.57)	18.19 (15.51)	20.52 (16.60)	20.77 (16.87)	112.45 (113.48
	1.00		5.71 (4.17)	5.33 (4.02)	6.21 (4.47)	5.92 (4.30)	6.88 (4.71)	6.73 (4.61)	41.50 (41.28
	1.50		3.07 (1.94)	2.92 (1.92)	3.32 (2.06)	3.16 (2.02)	3.61 (2.16)	3.58 (2.17)	15.67 (15.42)
	2.50		1.56 (0.74)	1.48 (0.72)	1.63 (0.79)	1.57 (0.76)	1.74 (0.84)	1.71 (0.83)	3.61 (3.08)

Note. Standard deviation of run length is in parentheses.

Table 34

Average Run Length Comparisons for p = 2, λ = 0.1, 0.3, and 0.5, and In-Control Average Run Length ≈ 200 from the Multivariate Normal Distribution

p			λ						
			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	8.85	8.8	9.90	9.7	10.90	10.40	10.59
	0.00		197 (200)	201 (205)	196 (197)	202 (203)	201 (203)	198 (200)	200
	0.25		74.92 (73.52)	74.91 (71.68)	97.13 (93.66)	94.88 (92.43)	141.65 (142.57)	133.82 (131.22)	162.11 (160.46)
	0.50		25.62 (21.27)	25.17 (20.79)	35.10 (32.32)	33.84 (30.71)	68.02 (66.90)	62.70 (61.51)	112.45 (113.48
	1.00		7.81 (5.14)	7.78 (5.13)	9.36 (6.62)	8.94 (6.31)	16.82 (15.41)	15.37 (13.55)	41.50 (41.28
	1.50		4.07 (2.36)	4.01 (2.34)	4.60 (2.68)	4.46 (2.56)	6.57 (4.99)	6.06 (4.44)	15.67 (15.42)
	2.50		1.90 (0.90)	1.87 (0.92)	2.08 (0.98)	2.04 (0.99)	2.34 (1.21)	2.25 (1.20)	3.61 (3.08)

Note. Standard deviation of run length is in parentheses.

Table 35

Average Run Length Comparisons for p =3, λ = .02, .03, and .05, and In-Control Average Run Length ≈ 200 from the Multivariate Normal Distribution

p			λ						
			0 .02		0.03		0.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
3	δ	*UCL*	8.30	8.20	9.10	8.80	10.20	9.70	12.83
	0.00		197 (257)	200 (244)	200 (239)	193 (220)	201 (221)	198 (214)	200
	0.25		50.11 (53.77)	51.46 (52.06)	56.84 (59.74)	56.38 (54.32)	69.70 (69.97)	65.63 (62.80)	175.80 (174.36)
	0.50		18.15 (16.79)	19.23 (16.49)	20.79 (17.95)	21.01 (17.36)	24.29 (20.31)	23.17 (18.70)	130.19 (129.89)
	1.00		5.93 (4.53)	6.48 (4.67)	6.79 (4.85)	6.97 (4.83)	7.84 (5.22)	7.68 (5.06)	53.42 (53.31)
	1.50		3.21 (2.08)	3.47 (2.16)	3.60 (2.22)	3.69 (2.30)	4.12 (2.41)	4.04 (2.40)	20.77 (19.84)
	2.50		1.55 (0.73)	1.79 (0.83)	1.73 (0.82)	1.79 (0.88)	1.92 (0.88)	1.92 (0.94)	4.46 (3.95)

Note. Standard deviation of run length is in parentheses.

Table 36

Average Run Length Comparisons for p =3, λ = 0.1, 0.3, and 0.5, and In-Control Average Run Length ≈ 200 from the Multivariate Normal Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
3	δ	*UCL*	12.00	11.00	14.45	12.00	19.10	12.60	12.83
	0.00		202 (214)	209 (214)	200 (206)	204 (207)	200 (197)	195 (195)	200
	0.25		96.00 (96.21)	83.18 (80.16)	136.97 (138.34)	110.95 (109.87)	180.71 (181.89)	142.83 (143.54)	175.80 (174.36)
	0.50		33.60 (29.27)	29.07 (24.04)	61.03 (59.21)	40.67 (36.61)	132.92 (133.35)	73.98 (72.18)	130.19 (129.89)
	1.00		9.74 (6.37)	8.73 (5.70)	14.53 (11.05)	10.49 (7.56)	51.80 (49.62)	18.70 (16.91)	53.42 (53.31)
	1.50		4.92 (2.71)	4.49 (2.59)	6.32 (3.65)	5.06 (2.95)	19.03 (17.17)	7.02 (5.31)	20.77 (19.84)
	2.50		2.21 (1.00)	2.07 (0.99)	2.63 (1.15)	2.23 (1.06)	4.38 (2.62)	2.45 (1.317)	4.46 (3.95)

Note. Standard deviation of run length is in parentheses.

Table 37

Average Run Length Comparisons for p = 4, λ = .02, .03, and .05, and In-Control Average Run Length ≈ 200 from the Multivariate Normal Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
4	δ	*UCL*	10.30	9.90	11.45	10.70	13.05	11.60	14.86
	0.00		201 (263)	199 (246)	202 (256)	203 (232)	200 (230)	193 (209)	200
	0.25		56.33 (61.76)	55.30 (55.12)	67.32 (71.02)	61.93 (60.20)	84.7 (87.25)	71.22 (67.50)	182.42 (180.29)
	0.50		21.13 (18.66)	20.26 (17.69)	24.61 (21.07)	22.11 (18.33)	30.43 (25.94)	24.94 (20.02)	140.16 (140.78)
	1.00		7.33 (5.13)	6.69 (5.00)	8.20 (5.49)	7.48 (5.24)	9.62 (6.23)	8.28 (5.51)	60.59 (60.09)
	1.50		3.89 (2.26)	3.58 (2.32)	4.33 (2.46)	3.95 (2.44)	4.97 (2.72)	4.27 (2.56)	25.18 (24.87)
	2.50		1.88 (0.81)	1.75 (0.88)	2.05 (0.87)	1.87 (0.93)	2.28 (0.97)	2.01 (0.98)	5.27 (4.72)

Note. Standard deviation of run length is in parentheses.

Table 38

Average Run Length Comparisons for p = 4, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 200 from the Multivariate Normal Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
4	δ	*UCL*	16.25	12.90	21.45	13.90	33.50	14.70	14.86
	0.00		200 (218)	200 (203)	200 (210)	199 (199)	200 (200)	201 (201)	200
	0.25		123.19 (131.13)	89.81 (86.98)	171.09 (174.35)	116.74 (115.13)	196.13 (195.50)	156.31 (15.30)	182.42 (180.29)
	0.50		50.77 (47.12)	31.44 (25.71)	102.06 (102.95)	43.76 (40.22)	174.16 (176.10)	84.06 (83.19)	140.16 (140.78)
	1.00		13.10 (8.60)	9.57 (6.19)	28.11 (24.39)	11.34 (8.07)	114.80 (114.95)	21.64 (19.52)	60.59 (60.09)
	1.50		6.37 (3.36)	4.86 (2.77)	10.13 (6.30)	5.33 (3.11)	63.05 (61.71)	8.06 (6.31)	25.18 (24.87)
	2.50		2.75 (1.16)	2.18 (1.05)	3.65 (1.51)	2.35 (1.12)	14.35 (11.92)	2.70 (1.47)	5.27 (4.72)

Note. Standard deviation of run length is in parentheses.

Table 39

Average Run Length Comparisons for p = 5, λ=.02, .03, and .05, and In-Control Average Run Length ≈ 200 from the Multivariate Normal Distribution

			λ						
p			0.02		0.03		0.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
5	δ	*UCL*	12.60	11.60	14.10	12.30	16.90	13.40	16.74
	0.00		202 (273)	200 (241)	200 (248)	195 (222)	201 (231)	200 (218)	200
	0.25		67.46 (73.35)	59.12 (58.57)	81.28 (88.24)	66.05 (64.05)	111.61 (120.36)	77.00 (74.76)	186.44 (186.26)
	0.50		25.44 (22.09)	21.94 (18.42)	30.00 (25.34)	24.03 (19.63)	41.03 (36.27)	27.23 (21.39)	144.74 (147.05)
	1.00		8.70 (5.86)	7.66 (5.35)	9.86 (6.39)	8.11 (5.52)	12.33 (7.66)	8.91 (5.83)	68.62 (69.67)
	1.50		4.73 (2.52)	4.03 (2.49)	5.18 (2.78)	4.24 (2.57)	6.28 (3.21)	4.69 (2.69)	28.94 (28.21)
	2.50		2.25 (0.98)	1.94 (0.95)	2.43 (0.96)	2.04 (1.00)	2.81 (1.11)	2.18 (1.05)	6.03 (5.58)

Note. Standard deviation of run length is in parentheses.

Table 40

Average Run Length Comparisons for p = 5, λ=0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 200 from the Multivariate Normal Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
5	δ	*UCL*	22.80	14.70	33.50	15.8	58.30	16.6	16.74
	0.00		199	199	201	199	200	197	200
			(216)	(201)	(208)	(197)	(200)	(185)	
	0.25		156.56	95.78	187.51	123.33	194.70	160.37	186.44
			(169.48)	(93.40)	(194.42)	(122.15)	(198.98)	(160.06)	(186.26)
	0.50		82.16	34.32	147.13	49.28	191.50	92.99	144.74
			(80.09)	(29.07)	(147.88)	(45.03)	(189.93)	(92.07)	(147.05)
	1.00		20.78	10.05	67.24	12.40	161.04	25.05	68.62
			(14.85)	(6.43)	(65.52)	(8.74)	(161.80)	(23.05)	(69.67)
	1.50		9.02	5.12	23.43	5.77	124.96	9.08	28.94
			(4.59)	(2.83)	(18.56)	(3.33)	(124.57)	(7.31)	(28.21)
	2.50		3.66	12.34	5.79	2.50	55.24	2.86	6.03
			(1.41)	(1.12)	(2.30)	(1.18)	(52.50)	(1.56)	(5.58)

Note. Standard deviation of run length is in parentheses.

Average Run Length Comparisons
for the Multivariate t_p *(df = 5)*

Next, the ARL_1 simulation results from the multivariate t_p *(df = 5)* distribution are presented. Tables 41-48 show the ARL_1 comparisons for SRMEWMA, MEWMA, and Hotelling's T^2 for the following conditions:

1. The number of variables, p = 2, 3, 4, and 5;
2. The smoothing parameter, λ ∈ [0.02, 0.03, 0.05, 0.10, 0.2, and 0.50]; and
3. Shift parameter, δ ∈ [0.25, 0.50, 1.00, 1.50, and 2.50].

Tables 41-48 show the ARL_1 comparisons for in-control ARL_0 = 200 only. Simulation results are shown in Tables 41 and 42, 43 and 44, 45 and 46, and 47 and 48 for p = 2, 3, 4, and 5, respectively. Comparisons with ARL_0 = 500 and ARL_0 = 1,000 are available in Appendix E.

The simulation results for SRMEWMA, MEWMA with λ = 0.02, 0.03, and 0.05 and Hotelling's $T^2$for p = 2 are presented in Table 41 and results with λ = 0.1, 0.2, and 0.5 and Hotelling's $T^2$for p = 2 are presented in Table 42. In addition to the ARL_1s, the corresponding standard deviations of the run lengths (SDRL) are also included in these two tables. Tables 41–48 show that the MEWMA control chart had better efficiency. However, the SRMEWMA control chart offers reasonably comparable ARL_1 performance and the difference between MEWMA and SRMEWMA was not significant for p = 2, 3, and 4, for $\lambda \leq 0.1$, and all δ shift values; but for p = 5 and $\lambda \geq 0.1$, MEWMA was superior to SRMEWMA for all δ shift values. Additionally, MEWMA was superior to Hotelling's T^2 for all p, λ, and δ values. However, SRMEWMA was only significantly superior to Hotelling's T^2 for p = 2 and 3 for all δ shift values but Hotelling's T^2 was superior to SRMEWMA for p =4 and 5 and λ = 0.5.

Table 41

Average Run Length Comparisons for p = 2, λ = .02, .03, and .05, and In-Control Average Run Length ≈ 200 from the Multivariate t_p(5) Distribution

			λ						
p			0.02		0.03		0.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	6.30	6.20	6.90	6.80	7.85	7.70	13.5
	0.00		201 (248)	198 (246)	197 (226)	195 (244)	200 (221)	197 (212)	200 (196.61)
	0.25		47.36 (47.58)	45.63 (46.42)	51.34 (49.45)	50.12 (49.19)	60.82 (59.76)	59.11 (55.92)	173.44 (170.68)
	0.50		17.05 (14.92)	16.67 (14.55)	18.72 (15.79)	18.54 (15.41)	21.49 (17.22)	20.80 (16.79)	142.94 (142.15)
	1.00		5.80 (4.14)	5.62 (4.11)	6.28 (4.32)	6.13 (4.33)	7.03 (4.74)	6.93 (4.63)	76.65 (75.09)
	1.50		3.17 (1.94)	2.99 (1.90)	3.33 (2.00)	3.30 (2.04)	3.72 (2.17)	3.62 (2.15)	34.06 (33.18)
	2.50		1.56 (0.74)	1.50 (0.70)	1.63 (0.76)	1.60 (0.77)	1.78 (0.84)	1.73 (0.83)	6.78 (6.23)

Note. Standard deviation of run length is in parentheses.

Table 42

Average Run Length Comparisons for p = 2, λ = 0.1, 0.3, and 0.5, and In-Control Average Run Length ≈ 200 from the Multivariate $t_p(5)$ Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	9.20	9.00	10.70	10.30	13.40	12.30	13.5
	0.00		201 (208)	202 (208)	199 (205)	198 (198)	200 (204)	200 (200)	200 (196.61)
	0.25		80.56 (77.97)	77.35 (74.18)	112.22 (112.97)	105.85 (104.85)	164.98 (164.07)	160.7 (159.49)	173.44 (170.68)
	0.50		27.12 (22.43)	26.06 (21.37)	41.68 (38.00)	38.20 (35.52)	101.41 (101.40)	93.25 (90.85)	142.94 (142.15)
	1.00		8.35 (5.36)	8.04 (5.27)	10.63 (7.57)	9.80 (6.90)	29.88 (27.75)	24.24 (22.49)	76.65 (75.09)
	1.50		4.24 (2.40)	4.11 (2.33)	4.98 (2.80)	4.76 (2.73)	10.16 (8.33)	8.38 (6.53)	34.06 (33.18)
	2.50		1.99 (0.92)	1.93 (0.92)	2.19 (1.00)	2.12 (0.99)	2.91 (1.53)	2.64 (1.38)	6.78 (6.23)

Note. Standard deviation of run length is in parentheses.

Table 43

Average Run Length Comparisons for p = 3, λ = .02, .03, and .05, and In-Control Average Run Length ≈ 200 from the Multivariate $t_p(5)$ Distribution

			λ						
p			0.02		0.03		0.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
3	δ	*UCL*	8.35	8.20	9.25	8.90	10.70	9.90	16.6
	0.00		200 (253)	200 (247)	200 (241)	201 (234)	200 (227)	202 (219)	200 (196.61)
	0.25		53.22 (55.90)	51.54 (51.58)	61.27 (61.59)	56.16 (55.02)	79.69 (81.16)	67.63 (65.77)	183.13 (180.89)
	0.50		19.71 (17.08)	18.82 (16.23)	22.38 (18.44)	20.69 (17.30)	27.18 (22.02)	23.58 (19.25)	159.21 (159.41)
	1.00		6.75 (4.63)	6.31 (4.63)	7.42 (4.95)	6.90 (4.80)	8.72 (5.52)	7.70 (5.03)	100.52 (101.41)
	1.50		3.59 (2.06)	3.32 (2.13)	3.94 (2.22)	3.60 (2.20)	4.51 (2.40)	4.07 (2.40)	48.77 (47.86)
	2.50		1.75 (0.76)	1.65 (0.80)	1.87 (0.81)	1.78 (0.87)	2.08 (0.89)	1.89 (0.91)	10.04 (9.52)

Note. Standard deviation of run length is in parentheses.

Table 44

Average Run Length Comparisons for p = 3, λ = 0.1, 0.3, and 0.5, and In-Control Average Run Length ≈ 200 from the Multivariate $t_p(5)$ Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
3	δ	*UCL*	13.30	11.30	17.65	12.70	2.40	15.00	16.6
	0.00		200 (215)	199 (206)	200 (206)	200 (198)	200 (203)	198 (199)	200 196.61)
	0.25		115.66 (118.31)	88.96 (86.71)	163.46 (167.63)	118.18 (116.92)	188.95 (190.11)	166.60 (163.88)	183.13 (180.89)
	0.50		43.25 (38.55)	30.25 (25.09)	91.76 (90.89)	45.71 (42.39)	168.17 (166.17)	106.47 (104.71)	159.21 (159.41)
	1.00		11.35 (7.08)	9.11 (5.84)	22.39 (18.19)	11.37 (8.15)	105.31 (103.6)	30.70 (29.04)	100.52 (101.41)
	1.50		5.60 (2.91)	4.61 (2.63)	8.48 (4.92)	5.42 (3.10)	53.01 (51.02)	10.46 (8.64)	48.77 (47.86)
	2.50		2.45 (1.03)	2.12 (1.01)	3.18 (1.28)	3.34 (1.08)	9.74 (7.29)	3.02 (1.63)	10.04 (9.52)

Note. Standard deviation of run length is in parentheses.

Table 45

Average Run Length Comparisons for p = 4, λ = .02, .03, and .05, and In-Control Average Run Length ≈ 200 from the Multivariate $t_p(5)$ Distribution

			λ						
p			0.02		0.03		0.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
4	δ	*UCL*	10.35	10.00	11.70	10.80	14.00	11.80	19.2
	0.00		198 (252)	201 (247)	199 (240)	201 (233)	201 (229)	200 (214)	201 (202.62)
	0.25		60.28 (64.29)	57.57 (56.97)	75.31 (79.62)	64.36 (62.77)	105.20 (113.01)	72.97 (71.38)	185.73 (184.26)
	0.50		22.00 (18.81)	21.37 (18.29)	26.73 (21.96)	23.24 (19.00)	36.33 (30.93)	26.13 (20.85)	166.28 (163.48)
	1.00		7.29 (4.88)	7.31 (5.10)	8.59 (5.54)	7.80 (5.33)	10.71 (6.47)	8.56 (5.54)	101.41 (110.60)
	1.50		3.79 (2.08)	3.93 (2.36)	4.38 (2.32)	4.17 (2.47)	5.36 (2.76)	4.53 (2.61)	59.15 (58.04)
	2.50		1.76 (0.71)	1.94 (0.94)	1.99 (0.79)	1.99 (0.95)	2.41 (0.95)	2.10 (0.99)	12.70 (12.42)

Note. Standard deviation of run length is in parentheses.

Table 46

Average Run Length Comparisons for p = 4, λ = 0.1, 0.3, and 0.5, and In-Control Average Run Length ≈ 200 from the Multivariate $t_p(5)$ Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
4	δ	*UCL*	19.20	13.20	28.30	14.80	49.80	17.50	19.2
	0.00		201 (211)	196 (204)	199 (206)	198 (200)	200 (199)	199 (197)	201 (202.62)
	0.25		155.38 (164.26)	92.65 (92.28)	191.56 (192.02)	125.48 (127.11)	196.88 (197.67)	176.02 (171.98)	185.73 (184.26)
	0.50		75.55 (74.50)	32.21 (28.33)	146.52 (148.49)	51.51 (47.89)	188.51 (186.32)	120.17 (121.42)	166.28 (163.48)
	1.00		17.85 (11.50)	9.87 (6.36)	58.89 (56.08)	12.76 (9.23)	160.96 (162.93)	38.23 (36.96)	101.41 (110.60)
	1.50		7.91 (3.86)	5.02 (2.78)	18.75 (13.60)	5.88 (3.37)	122.07 (122.30)	12.68 (10.68)	59.15 (58.04)
	2.50		3.20 (1.21)	2.28 (1.06)	4.94 (1.85)	2.52 (1.16)	49.29 (48.18)	3.41 (1.90)	12.70 (12.42)

Note. Standard deviation of run length is in parentheses.

Table 47

Average Run Length Comparisons for p = 5, λ = .02, .03, and .05, and In-Control Average Run Length ≈ 200 from the Multivariate $t_p(5)$ Distribution

			λ						
p			0.02		0.03		0.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
5	δ	*UCL*	12.69	11.60	14.70	12.50	18.60	13.60	21.60
	0.00		201 (262)	196 (248)	201 (243)	200 (229)	202 (231)	194 (214)	201 (198.70)
	0.25		76.16 (78.89)	58.18 (60.50)	96.06 (104.47)	65.21 (64.69)	134.85 (150.44)	77.10 (75.39)	189.25 (187.77)
	0.50		28.08 (22.45)	21.83 (18.83)	34.70 (28.02)	24.04 (19.98)	54.74 (48.94)	27.63 (22.63)	176.03 (174.21)
	1.00		9.27 (5.66)	7.31 (5.38)	11.01 (6.36)	8.09 (5.56)	14.86 (8.46)	8.92 (5.84)	120.05 (120.40)
	1.50		4.81 (2.37)	3.91 (2.46)	5.65 (2.71)	4.30 (2.62)	7.17 (3.32)	4.75 (2.76)	67.38 (67.27)
	2.50		2.19 (0.80)	1.89 (0.93)	2.53 (0.91)	2.02 (0.99)	3.10 (1.10)	2.19 (1.04)	15.88 (15.19)

Note. Standard deviation of run length is in parentheses.

Table 48

Average Run Length Comparisons for p = 5, λ = 0.1, 0.3, and 0.5, and In-Control Average Run Length ≈ 200 from the Multivariate t_p(5) Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
5	δ	*UCL*	27.55	15.20	43.80	16.80	81.90	19.70	21.60
	0.00		199 (209)	201 (209)	200 (208)	200 (200)	198 (201)	197 (194)	201 (198.70)
	0.25		178.49 (185.49)	102.13 (102.23)	190.74 (196.67)	130.92 (12.85)	198.43 (197.72)	176.93 (175.00)	189.25 (187.77)
	0.50		120.79 (124.97)	36.92 (31.70)	176.05 (178.26)	57.13 (54.52)	196.04 (194.71)	125.52 (123.27)	176.03 (174.21)
	1.00		32.98 (25.04)	10.56 (6.74)	111.64 (110.42)	13.97 (10.09)	183.36 (184.40)	44.35 (43.08)	120.05 (120.40)
	1.50		11.98 (5.61)	5.33 (2.92)	51.88 (47.46)	6.27 (3.64)	162.28 (160.67)	14.62 (12.65)	67.38 (67.27)
	2.50		4.46 (1.53)	2.40 (1.13)	8.53 (3.37)	2.65 (1.25)	107.64 (109.46)	3.72 (2.14)	15.88 (15.19)

Note. Standard deviation of run length is in parentheses.

Average Run Length Comparisons
for the Multivariate *Gamma*$_p$
(α = 3, β = 1) Distribution

Finally, the ARL_1 simulation results from the multivariate *gamma*$_p$ (α = 3, β= 1) distribution are presented. Tables 49-54 show the ARL_1 comparisons for SRMEWMA, MEWMA, and Hotelling's T^2 for the following conditions:

1. The number of variables, p = 2 only;
2. The smoothing parameter, λ ∈ [0.02, 0.03, 0.05, 0.10, 0.2, and 0.50]; and
3. Shift parameter, δ ∈ [0.25, 0.50, 1.00, 1.50, and 2.50].

Tables 49 and 50, 51 and 52, and 53 and 54 show the ARL_1 comparisons for in-control ARL_0 = 200, 500, and 1,000, respectively.

The simulation results for SRMEWMA; MEWMA with λ = 0.02, 0.03, and 0.05; and Hotelling's $T^2$for ARL_0 = 200 are presented in Table 49. Results with λ = 0.1, 0.2,

and 0.5, and Hotelling's $T^2$for ARL_0 = 200 are presented in Table 50. In addition to the ARL_1s, the corresponding standard deviations of the run lengths (SDRL) are also included in these two tables. Tables 49-54 show that the SRMEWMA control chart had superior efficiency in detecting mean shifts by a large margin. This showed that the SRMEWMA control chart was more sensitive to process shift from normality for skewed distributions. Furthermore, the SRMEWMA control chart was superior to the Hotelling's T^2 control chart for all ARL_0, λ, and δ values. The MEWMA control chart was superior to Hotelling's T^2 for all shift values $\delta \leq 2.5$.

Table 49

Average Run Length Comparisons for p = 2, λ = .02, .03, and .05, and In-Control Average Run Length ≈ 200 with Multivariate Gamma Distribution

			λ						
p			0.02		0.03		0.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	723.1	2,116.6	723.1	1,442.9	723.1	896.6	25.3
	0.00		200 (209.37)	200 (201.53)	200 (209.37)	200 (195.80)	200 (209.37)	200 (192.00)	201 (198.98)
	0.25		11.40 (5.10)	62.89 (50.77)	11.40 (5.10)	65.75 (53.43)	11.40 (5.10)	73.06 (63.22)	146.77 (148.19)
	0.50		5.61 (1.68)	31.05 (19.65)	5.61 (1.68)	32.08 (21.15)	5.61 (1.68)	32.95 (23.37)	92.89 (93.04)
	1.00		2.92 (0.61)	14.73 (6.53)	2.92 (0.61)	14.52 (6.66)	2.92 (0.61)	14.22 (7.13)	30.32 (29.92)
	1.50		2.05 (0.32)	9.54 (3.41)	2.05 (0.32)	9.35 (3.35)	2.05 (0.32)	8.68 (3.30)	8.73 (8.33)
	2.50		1.34 (0.47)	5.71 (1.57)	1.34 (0.47)	5.50 (1.49)	1.34 (0.47)	5.03 (1.41)	1.05 (0.22)

Note. Standard deviation of run length is in parentheses.

Table 50

Average Run Length Comparisons for p = 2, λ = 0.1, 0.3, and 0.5, and In-Control Average Run Length ≈ 200 with Multivariate Gamma Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	190.3	2,116.6	116.2	1,442.9	65.9	896.6	25.3
	0.00		200 (199.56)	200 (201.53)	200 (196.41)	200 (195.80)	201 (200.60)	200 (192.00)	201 (198.98)
	0.25		12.38 (7.25)	62.89 (50.77)	15.24 (11.33)	65.75 (53.43)	30.93 (29.17)	73.06 (63.22)	146.77 (148.19)
	0.50		5.18 (1.93)	31.05 (19.65)	5.18 (2.42)	32.08 (21.15)	8.04 (6.23)	32.95 (23.37)	92.89 (93.04)
	1.00		2.51 (0.61)	14.73 (6.53)	2.24 (0.62)	14.52 (6.66)	2.14 (0.85)	14.22 (7.13)	30.32 (29.92)
	1.50		1.83 (0.38)	9.54 (3.41)	1.60 (0.49)	9.35 (3.35)	1.26 (0.44)	8.68 (3.30)	8.73 (8.33)
	2.50		1.00 (0.528)	5.71 (1.57)	1.00 (0)	5.50 (1.49)	1.00 (0)	5.03 (1.41)	1.05 (0.22)

Note. Standard deviation of run length is in parentheses.

Table 51

Average Run Length Comparisons for p = 2, λ = .02, .03, and .05, and In-Control Average Run Length ≈ 500 with Multivariate Gamma Distribution

			λ						
p			0.02		0.03		0.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	761.7	2,175.5	537.4	1,486.7	352.7	928.5	34.82
	0.00		494 (494.26)	500 (480.18)	498 (495.26)	500 (480.08)	500 (500.16)	500 (490.50)	503 (505.74)
	0.25		16.79 (6.56)	115.43 (88.44)	16.44 (6.90)	122.14 (102.09)	16.15 (7.60)	135.68 (120.26)	363.96 (363.06)
	0.50		7.95 (2.05)	49.42 (28.52)	7.53 (2.04)	48.56 (30.40)	7.05 (2.13)	51.75 (37.01)	234.19 (236.49)
	1.00		4.03 (0.72)	21.62 (8.29)	3.79 (0.71)	20.25 (8.46)	3.44 (0.70)	18.91 (9.03)	76.57 (77.11)
	1.50		2.82 (0.43)	13.69 (4.28)	2.65 (0.48)	12.49 (4.03)	2.37 (0.49)	11.28 (3.97)	21.76 (21.00)
	2.50		1.97 (0.14)	7.98 (1.84)	1.94 (0.22)	7.15 (1.73)	1.77 (0.42)	6.30 (1.60)	1.65 (1.04)

Note. Standard deviation of run length is in parentheses.

Table 52

Average Run Length Comparisons for p = 2, λ = 0.1, 0.3, and 0.5, and In-Control Average Run Length ≈ 500 with Multivariate Gamma Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	207.7	498.3	129.3	272.6	76.7	125.8	34.82
	0.00		500 (497.38)	501 (499.80)	500 (509.81)	498 (503.80)	502 (502.64)	503 (502.16)	503 (505.74)
	0.25		17.48 (10.46)	170.89 (161.73)	23.76 (19.03)	217.21 (213.21)	66.31 (64.08)	316.65 (318.75)	363.96 (363.06)
	0.50		6.52 (2.3)	63.85 (53.76)	6.69 (3.25)	93.26 (86.92)	13.71 (11.65)	177.47 (173.15)	234.19 (236.49)
	1.00		3.00 (0.68)	19.18 (11.23)	2.65 (0.71)	24.13 (18.92)	2.73 (1.17)	58.86 (56.08)	76.57 (77.11)
	1.50		2.04 (0.33)	10.24 (4.42)	1.85 (0.37)	10.55 (6.14)	1.53 (0.51)	21.04 (18.51)	21.76 (21.00)
	2.50		1.25 (0.43)	5.23 (1.50)	1.00 (.045)	4.53 (1.57)	1.00 (0.01)	5.16 (3.21)	1.65 (1.04)

Note. Standard deviation of run length is in parentheses.

Table 53

Average Run Length Comparisons for p = 2, λ = .02, .03, and .05, and In-Control Average Run Length ≈ 1,000 with Multivariate Gamma Distribution

			λ						
p			0.02		0.03		0.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	798.1	2,211.7	559.3	1,515.7	369.2	948.8	42.70
	0.00		1,000 (980.70)	1,002 (993.84)	1,000 (966.63)	998 (993.66)	1,001 (1,002.63)	1,000 (995.40)	993 (1,000.55)
	0.25		22.22 (7.90)	171.30 (135.49)	20.09 (7.98)	185.61 (157.16)	19.64 (9.27)	216.14 (197.23)	731.07 (737.23)
	0.50		10.23 (2.41)	63.78 (36.21)	8.96 (2.32)	64.41 (40.64)	8.15 (2.33)	70.43 (52.68)	453.92 (455.09)
	1.00		5.06 (0.83)	25.95 (9.43)	4.42 (0.78)	24.17 (9.80)	3.90 (0.75)	22.71 (10.55)	148.01 (148.15)
	1.50		3.48 (0.53)	16.15 (4.55)	3.03 (0.43)	14.70 (4.52)	2.71 (0.47)	13.00 (4.43)	43.76 (43.17)
	2.50		2.08 (0.28)	9.36 (1.99)	1.99 (0.07)	8.33 (1.88)	1.95 (0.22)	7.08 (1.72)	3.2 (2.65)

Note. Standard deviation of run length is in parentheses.

Table 54

Average Run Length Comparisons for p = 2, λ = 0.1, 0.3, and 0.5, and In-Control Average Run Length ≈ 1,000 with Multivariate Gamma Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	219.4	512.0	138.7	282.8	84.5	132.7	42.70
	0.00		997 (990.75)	1,007 (1,003.3)	997 (994.06)	1,001 (993.05)	1,000 (997.33)	999 (982.17)	993 (1,000.55)
	0.25		21.79 (13.54)	288.32 (278.74)	34.21 (28.56)	411.96 (404.02)	115.95 (115.52)	601.31 (603.94)	731.07 (737.23)
	0.50		7.50 (2.67)	93.85 (83.42)	7.96 (3.89)	156.50 (149.59)	20.60 (18.14)	331.37 (327.96)	453.92 (455.09)
	1.00		3.33 (0.74)	23.47 (14.41)	2.93 (0.76)	33.68 (27.83)	3.27 (1.51)	98.39 (95.80)	148.01 (148.15)
	1.50		2.25 (0.45)	11.75 (4.96)	1.97 (0.35)	12.90 (7.76)	1.70 (0.51)	32.58 (30.41)	43.76 (43.17)
	2.50		1.58 (0.49)	5.18 (1.59)	1.05 (0.23)	5.11 (1.78)	1.00 (0.00)	6.53 (4.28)	3.2 (2.65)

Note. Standard deviation of run length is in parentheses.

A Real Data Manufacturing Industry Example

The performance of the SRMEWMA control chart methodology along with the parametric MEWMA control chart methodology for SPC location monitoring was demonstrated using a data set from an aluminum electrolyte capacitor manufacturing data example by Qiu and Hawkins (2001). The same data set was also used by Zou and Tsung (2010) to illustrate their nonparametric multivariate sign EWMA (MSEWMA) control chart methodology.

The goal of an aluminum electrolyte capacitor (AEC) process is to transform the raw materials into AECs. The three most important characteristics in the process are the capacitance, dissipation, and leakage. The three variables were measured electronically at given voltage, frequency, and temperature.

The data set contained 200 data vectors (see Table 151 in Appendix E). Initially, using all 200 data vectors, the vector singed-ranks were computed and labeled as SR1, SR2, and SR3 and were used in computing the SRMEWMA control chart plotting statistic ($\boldsymbol{Q_t^R}$). The raw data vectors (capacitance, dissipation, and leakage) were used to compute the MEWMA control chart statistic ($\boldsymbol{T_i^2}$). The first 170 vectors (both raw data and signed-ranks) were used as a reference sample to estimate the process mean and variance-covariance matrix. The reference data set of 170 perhaps was smaller than optimal but it was sufficient to illustrate the SRMEWMA scheme in an industry setting.

The normal Q-Q plots of the raw data vectors based on the 170 phase I vectors are shown in figures 14, 15, and 16, respectively. The Q-Q plots showed that the three variables (capacitance, dissipation, and leakage) were not normal. The Shapiro-Wilk, Kolmogorov-Smirnov, Cramer-von Mises, and Anderson-Darling goodness-of-fit tests for normality concluded that all three raw variables were *not* normally distributed (the *p*-values were smaller than 0.0001, 0.001, 0.005, and 0.005, respectively). The results of the four goodness-of-fit tests of normality along with the normal Q-Q plots (Figures 14-16) showed that the multivariate normality assumption was not valid. Therefore, the nonparametric SRMEWMA control chart would be more powerful than the MEWMA control chart, which was based on normal parametric assumptions.

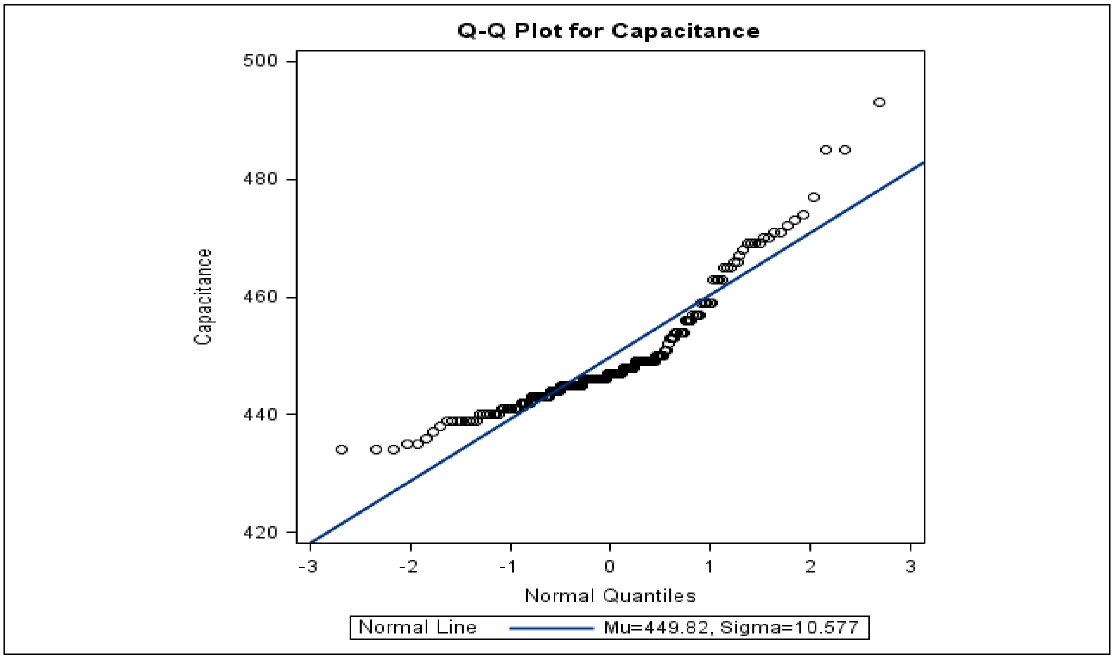

Figure 14. The normal Q-Q plot for capacitance.

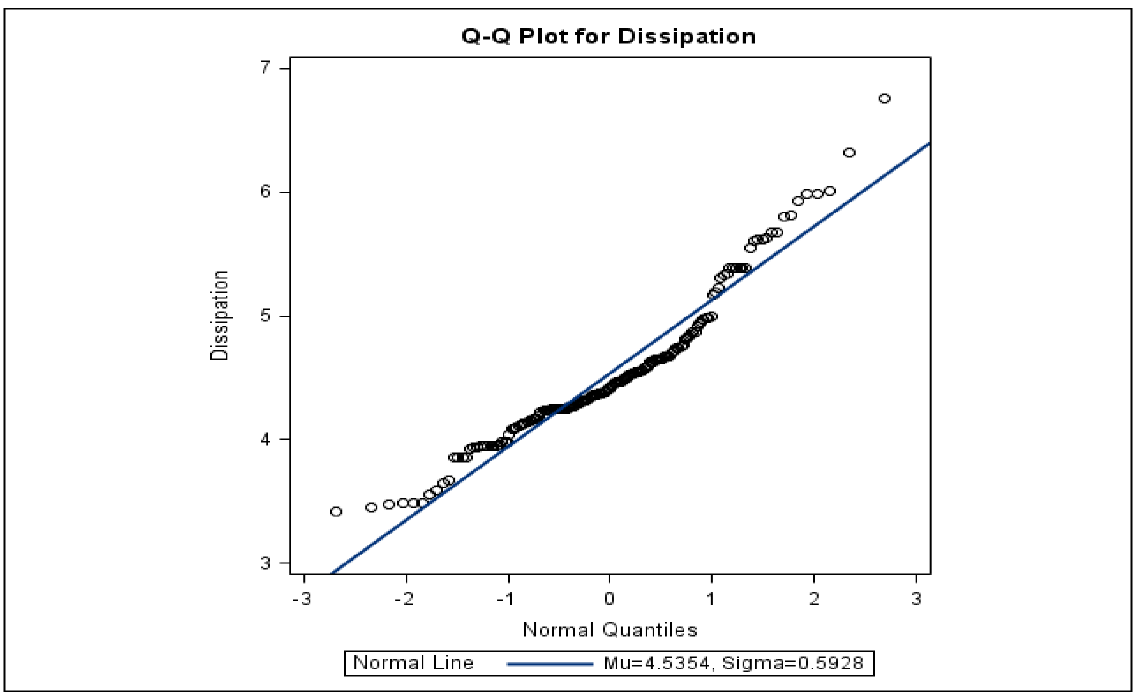

Figure 15. The normal Q-Q plot for dissipation.

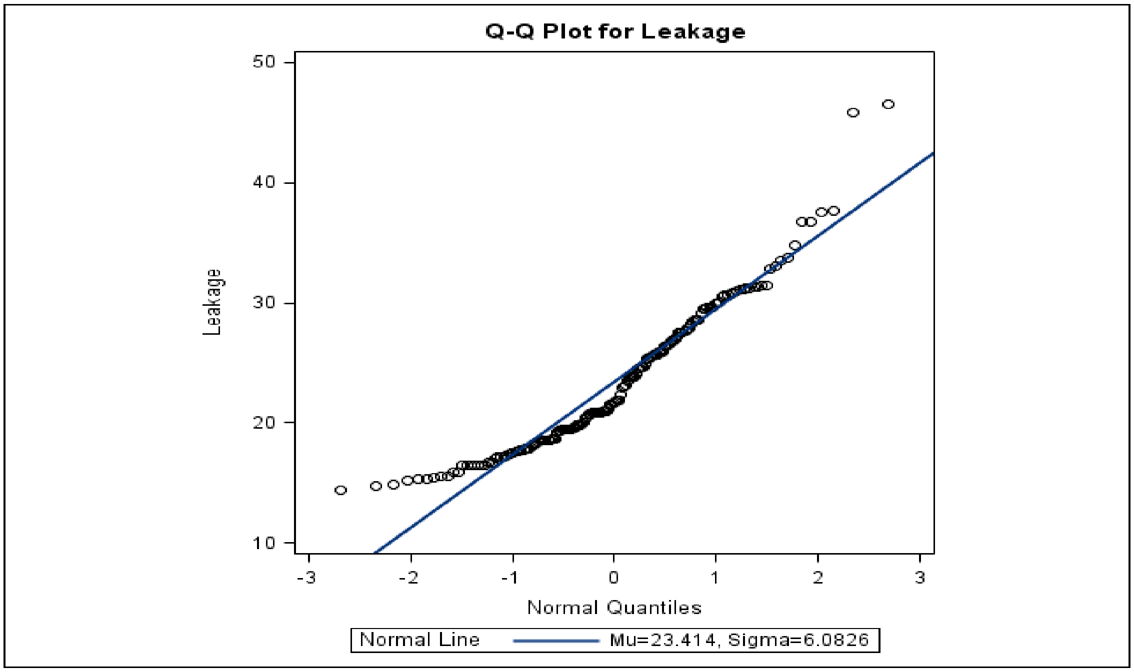

Figure 16. The normal Q-Q plot for leakage.

The first 170 data vectors were used in to compute the vector signed-ranks corresponding to the original measurements and the IC parameter estimates of the mean vectors and variance-covariance matrices for the raw data and signed-ranks respectively. The IC ARL_0 was fixed at 200 and the smoothing parameter λ was chosen to be 0.03 in order to make MEWMA robust to non-normality (Zou & Tsung, 2010). The control limits for MEWMA and SRMEWMA were 8.80 and 9.10, respectively. A shift δ = 0.25 multiples of the standard deviation was added to the first variable (capacitance) of the remaining 30 vectors for phase II analysis. Table 55 shows the phase II analysis sample of 30, the original raw data vectors (labeled as observations 171 – 200), the computed vector signed-ranks, the MEWMA control chart statistic ($\boldsymbol{T_i^2}$), and the SRMEWMA control chart statistic ($\boldsymbol{Q_t^R}$). Figure 17 shows the plotted SRMEWMA and MEWMA

control chart statistics along with their corresponding upper control limits of 8.80 and 9.10. It can be seen from the table or plot that the SRMEWMA signaled an out-of-control at the 177th observation, it stayed in control until the 193rd observation, and finally it remained above the control limit. In contrast, the MEWMA did not signal an out-of-control until the 191st observation where it remained above the control limit until the 196th observation before it shifted below the control limit.

Table 55

The Phase II Signed-Rank Multivariate Exponentially Weighted Moving Average and Multivariate Exponentially Weighted Moving Average Control Chart Plotting Statistics

Obs.	Capacitance	Dissipation	Leakage	MEWMA (T_i^2)	SR1	SR2	SR3	SRMEWMA (Q_t^R)
171	456.26	4.36	15.30	2.52	22.29	-62.17	-365.66	3.21
172	449.26	4.35	18.90	2.56	12.16	-98.12	-138.18	5.63
173	459.26	4.19	27.50	1.71	17.93	-2530.83	220.67	5.93
174	449.26	4.05	25.60	1.46	20.74	-2641.82	180.24	7.04
175	446.26	5.63	21.90	0.63	-28.59	3543.35	-71.58	6.11
176	445.26	4.35	18.30	0.86	8.84	385.20	-176.45	8.19
177	443.26	4.69	26.50	0.32	-18.90	1462.22	156.94	**9.63****
178	437.26	4.45	15.70	1.16	1.75	1750.51	-328.27	8.59
179	441.26	4.56	35.00	0.69	-18.83	161.74	378.52	5.06
180	447.26	3.47	19.50	1.56	41.72	-3781.94	-26.12	3.79
181	440.26	4.08	20.50	2.71	20.90	-1749.13	1.63	4.83
182	439.26	5.73	24.30	2.16	-37.40	3711.42	52.62	3.58
183	436.26	3.92	17.40	3.94	32.47	-2156.98	-161.19	3.25
184	440.26	4.52	15.40	5.09	0.52	1970.24	-346.87	3.40
185	446.26	5.62	19.90	4.66	-23.65	3586.85	-183.35	4.17
186	439.26	4.47	16.70	6.09	0.13	1727.67	-285.34	5.32
187	445.26	4.32	19.50	6.59	9.82	-123.14	-91.38	6.96
188	439.26	4.27	20.10	7.79	7.45	-142.60	-45.70	8.18
189	442.26	4.11	21.30	8.63	19.49	-1770.31	35.67	8.74
190	442.26	4.98	29.00	8.03	-32.23	2361.72	227.79	8.14
191	438.26	3.83	14.70	**10.43****	37.73	-2093.52	-291.13	6.76
192	453.26	4.27	17.30	**10.11**	25.66	-923.62	-253.07	8.55
193	448.26	4.93	17.50	**10.04**	-9.16	2677.23	-280.87	**10.59**
194	447.26	4.39	19.80	**10.25**	4.98	369.20	-87.44	**13.30**
195	447.26	4.15	30.30	**9.15**	10.43	-2334.56	308.85	**10.75**
196	447.26	4.52	17.50	**9.55**	1.36	1532.02	-256.96	**13.10**
197	465.26	4.37	24.70	7.17	16.80	-1728.24	102.43	**16.35**
198	447.26	4.47	17.20	7.81	4.35	1303.59	-272.98	**18.59**
199	443.26	4.73	31.30	6.71	-26.10	1367.00	299.62	**16.44**
200	456.26	4.37	16.70	6.59	20.69	-163.78	-304.57	**18.69**

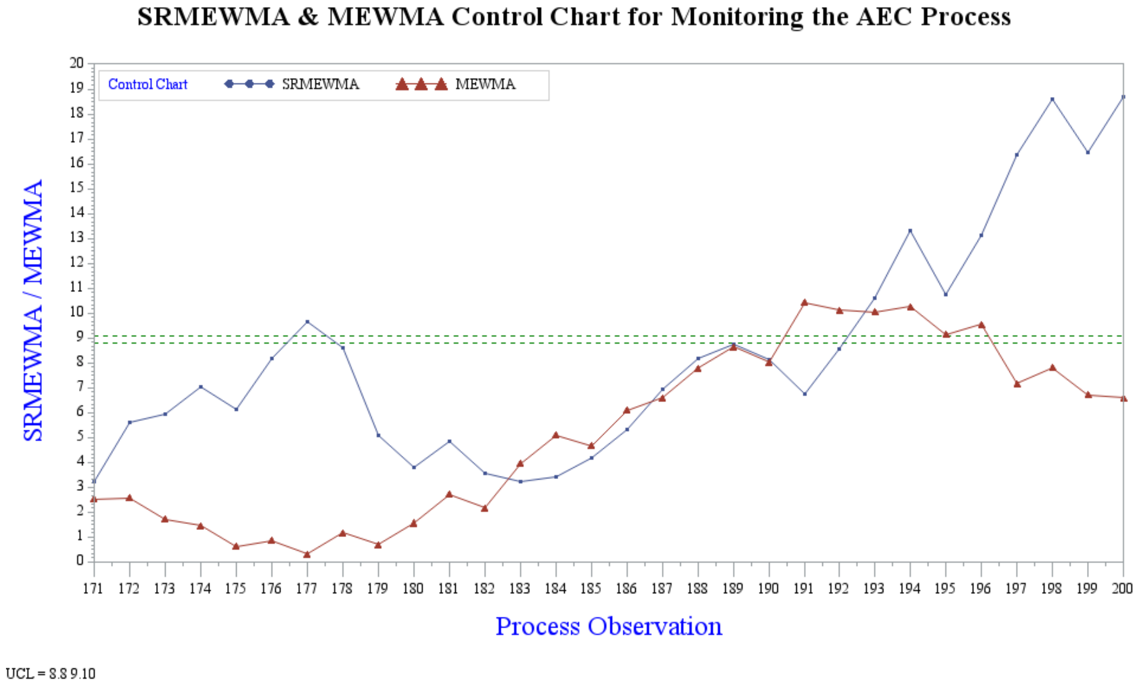

Figure 17. The signed-rank multivariate exponentially weighted moving average and multivariate exponentially weighted moving average control charts for monitoring the aluminum electrolyte capacitor process.

CHAPTER V

CONCLUSIONS AND DISCUSSION

The purpose of this study was to develop a new affine invariant spatial signed-rank multivariate exponentially weighted moving average control chart (the SRMEWMA control chart) and to compare its performance to traditional parametric counterparts like the multivariate exponentially moving average (MEWMA) and Hotelling's T^2 for different distributions, mainly the multivariate normal, *t*, and gamma using the concept of average run length (ARL_1). The control chart integrated a signed-rank test (Hettmansperger et al., 1997) and exponentially weighted moving average (EWMA) process monitoring. Finally, a real data example from the manufacturing industry showed that SRMEWMA performance was robust and effective.

To achieve the first goal as presented in Chapter III, the theoretical development of the new SRMEWMA control charts was shown based on the work of Hettmansperger et al. (1997), Mottonen et al. (1998), and Oja (2010). Central to the process of developing SRMEWMA was the concept of centered signed-rank vectors (Oja, 1983, 1999, 2010), which was illustrated using a numerical example that computed vector signed-ranks from original observations using SAS® IML macros originally developed by Mottonen et al. (1997). Additionally, like MEWMA and Hotelling's T^2, the new SRMEWMA control chart was shown to have the intuitively appealing property of affine invariance for distributions with elliptical directions, insuring that the performance of

SRMEWMA was the same for any initial covariance matrix. Also, the affine invariance property insured that for elliptically symmetrical distributions, the performance of MEWMA, SRMEWMA, and Hotelling's T^2 depended on a shift in process mean vector only through the non-centrality parameter $\delta = [\mu'\Sigma^{-1}\mu]^{1/2}$ (Lowry et al., 1992; Stoumbos & Sullivan, 2002).

In comparison with MEWMA and Hotelling's T^2, SRMEWMA's ARL_1 performance was robust to non-normality and sensitive to small shifts in the process mean vector. It performed better than Hotelling's T^2 and MEWMA for vector observations from the multivariate normal and *t* distributions (elliptically symmetrical) and better than MEWMA for observations from multivariate gamma (skewed) distributions.

To achieve the second goal of ARL_1 performance comparisons of SRMEWMA, MEWMA, and Hotelling's T^2, a Monte Carlo simulation study was designed to compute the UCLs of the three competing control charts for variations to the IC ARL_0 and compare the ARL_1 performance of the three charts for observation vectors from the multivariate normal, *t*, and gamma distributions for the number of variables, p = 2, 3, 4, and 5 for the multivariate normal and *t* distributions, $p = 2$ only for the multivariate gamma distribution and the smoothing parameter, $\lambda \in$ [0.01, 0.02, 0.03, 0.05, 0.10, 0.2, and 0.50], and IC ARL_0 = 200, 500, and 1,000. The UCLs for Hotelling's T^2 from the multivariate normal distribution were obtained using the CINV function in SAS®, which were equivalent to the same values obtained from the χ^2 statistical tables.

Based on phase II simulated ARL_1 and SRDL values presented in Chapter IV, SRMEWMA was shown to be robust and equally as powerful as MEWMA; it

outperformed Hotelling's T^2 for location process monitoring when the underlying process observations-vectors came from a multivariate normal distribution or more precisely, the marginal distributions were normal. In addition, SRMEWMA was also demonstrated to be robust and as powerful as MEWMA for location process monitoring when the underlying process observations vectors came from the multivariate *t* distribution. Both the multivariate normal and *t* distributions were elliptically symmetrical distributions. However, SRMEWMA was shown to be superior to both MEWMA and Hotelling's T^2 when the underlying process observations vectors came from a member of the family multivariate *gamma* distributions--a skewed distribution.

There was one major limitation to computing SRMEWMA. As with most higher-dimension methods, SRMEWMA suffered from what is known as the "curse of dimensionality," a term coined by Richard Bellman (1961). As the number of monitored quality variables p increased, the number of estimable parameters increased exponentially. Hence, larger numbers of observations n were needed in order to estimate those parameters. This "curse of dimensionality" becomes a significant obstacle in high dimension data analysis, computation, and estimation. The computation of the centered signed-ranks, which were necessary to compute the SRMEWMA charting statistic, was very intensive due to the number of vector combinations that were evaluated from the simulated variables to calculate the vector signed-ranks. For example, when $p = 3$ and $n = 5$, there are 10 vector combinations to be analyzed. However, when $n = 20$, the number of vector combinations to be analyzed is 1,140, a multiple of 114. The multiples increased geometrically as p and n increased. This limitation made simulation very intensive and almost prohibitive with current technology to practically use any sample

size $n > 5$ and $p > 5$ when computing the vector-signed-ranks. Table 56 illustrates the exponential nature of the number of vector combinations that must be evaluated from the simulated variables to calculate the vector signed-ranks. Therefore, due to the extensive computation requirements of the signed-ranks for SRMEWMA, p = 2, 3, 4, and 5 and n = 5 only were considered.

Table 56

Number of Vector Combinations That Must Be Evaluated from the Simulated Variables to Calculate the Vector Signed-Ranks

	Number of Simulated Variables					
p	***1***	***2***	***3***	***4***	***5***	***6***
n						
5	5	10	10	5	1	
10	10	45	120	210	252	210
15	15	105	455	1,365	3,003	5,005
20	20	190	1,140	4,845	15,504	38,760
25	25	300	2,300	12,650	53,130	177,100
30	30	435	4,060	27,405	142,506	593,775
35	35	595	6,545	52,360	324,632	1,623,160
40	40	780	9,880	91,390	658,008	3,838,380
45	45	990	14,190	148,995	1,221,759	8,145,060
50	50	1,225	19,600	230,300	2,118,760	15,890,700
55	55	1,485	26,235	341,055	3,478,761	28,989,675
60	60	1,770	34,220	487,635	5,461,512	50,063,860
65	65	2,080	43,680	677,040	8,259,888	82,598,880
70	70	2,415	54,740	916,895	12,103,014	131,115,985
75	75	2,775	67,525	1,215,450	17,259,390	201,359,550
80	80	3,160	82,160	1,581,580	24,040,016	300,500,200
85	85	3,570	98,770	2,024,785	32,801,517	437,353,560
90	90	4,005	117,480	2,555,190	43,949,268	622,614,630
95	95	4,465	138,415	3,183,545	57,940,519	869,107,785
100	100	4,950	161,700	3,921,225	75,287,520	1,192,052,400

Data were generated using the Monte Carlo Simulation technique using the interactive matrix language (IML) of the Statistical Analysis System (SAS) Windows 7 version 9.3 TSM10 running on an Intel core i7-3930K CPU @ 3.2GHZ/64GB RAM-based system. The system specifications are mentioned here to highlight the number of parameter limitations and intensive simulation requirements needed to generate data from various distributions in order to compute the vector-centered spatial signed-ranks and the SRMEWMA control chart charting statistic. Due to the large number of parameter level combinations that needed to be simulated, most simulation runs ran for more than 30 days and a full study became unattainable particularly with higher values of p using current technology. With certain improved processing power in the future, this study should be more thorough and insightful.

Recommendations for Future Research

A number of relevant issues and topics that were not addressed in this study could and should be addressed in future research:

1. Study the phase I UCL distribution and compare the ARL_1 performance of SRMEWMA and MEWMA control charts for different IC ARL_0 other than 200, 500, and 1,000.
2. Compare the ARL_1 performance of SRMEWMA and MEWEMA for the case where the smoothing matrix Λ_{pxp} was not diagonal and/or the individual smoothing parameter components $\lambda_{ij} \neq \lambda_{ij}$.
3. Study the ARL_1 performance for other SPC possible and likely continuous and discrete distributions.

4. Study the ARL_1 performance of the SRMEWMA control chart to monitor and detect shifts in other location shifts, e.g., median, percentile, and process variability. This is possible with reasonable modifications to the methodology utilized in this study.
5. Investigate the performance of SRMEWMA for higher order dimensionality.
6. Investigate the optimal smoothing parameter $\lambda > 0.2$ in more detail for nonparametric control charts like SRMEWMA.
7. Investigate the use of variable selection techniques to reduce dimensionality and increase computational efficiency.
8. Compare the performance of SRMEWMA to other nonparametric sign- and rank-based control charts.

Final Thoughts

As was demonstrated in Chapter II, a survey of multivariate nonparametric control charts in the field of nonparametric multivariate process control revealed few commercially available and utilized control charts in practice. This was due in part to many reasons; among them was the difficulty of their computation, the curse of dimensionality, and infancy of the multivariate nonparametric statistics field in terms of software and hardware dependence.

The newly investigated nonparametric multivariate SPC control chart for monitoring location parameters--the Signed-Rank Multivariate Exponentially Weighted Moving Average (SRMEWMA)--is a viable alternative control chart to the parametric MEWMA control chart and is sensitive to small shifts in the process location parameter.

Recommendations of its use are mixed based solely on this study. Among its advantages are its affine-invariant properties, its parallel performance to MEWMA for data from elliptically symmetrical distributions, and its superiority to MEWMA and Hotelling's T^2 control charts for data from skewed distributions. Among its disadvantages are complexity of computations for higher dimensions and lack of commercially available software. Most developed software methods for computing multivariate signs, ranks, and signed-ranks are in their infancy and are designed for academic research and low dimensional vector observations. Additionally, SRMEWMA is not as efficient as parametric charts for detecting large shifts. As the number of simultaneously monitored quality characteristics have dramatically increased in manufacturing, software and capable hardware must advance to take advantage of newly presented nonparametric control charts like SRMEWMA.

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APPENDIX A

SAS IML SIMULATION CODES

```
/*--------------------------------------------------------------------*/
/*--------------------------------------------------------------------*/
/* >>>>>>>>>>>>>>>>>>  MEWMA Phase I UCL Determination <<<<<<<<<<<<< */
/*--------------------------------------------------------------------*/
/*-- A doctoral dissertation supplemental SAS code by Jamil Zeinab --*/
/*--------------------------------------------------------------------*/
/*--------------------------------------------------------------------*/
/* 1. The IML MEWMA code generates 10,000 iterations of              */
/*    of up to 15,000 p-variate samples from a                       */
/*    sampling distribution and computes MEWMA                       */
/* 2. It creates a MEWMA chating statistic T                         */
/* 3. It calculates the UCLs for MEWMA  for which the                */
/*    in-control average run length is ARL_0 = 200, 500, & 1000.     */
/* 4. Several versions of this code will exist based on the sampling */
/*    criteria below (distribution, p-vartiates).                    */
/*--------------------------------------------------------------------*/
/*                >>>> Sampling criteria <<<<<                       */
/* Distribution: Multivariate Normal (Normal Marginals)              */
/* Number of qualirty variables: p=2, n=5                            */
/* Estimated Covariance matrix & mean vector, from 100,000 samples  */
/*--------------------------------------------------------------------*/

/*--------------------------------------------------------------------*/
/*  >>>>>>>>>>>>>>>>>>>    Notes for usage <<<<<<<<<<<<<<<<<<<              */
/* 1. Change the sampling distribution as needed, I used Normal, T,  */
/*    & Gamma.                                                       */
/* 2. Change the sampling size as desired.                           */
/* 3. Change the smoothing parameter "lambda" as desired.            */
/*--------------------------------------------------------------------*/;
dm 'output'  clear ;
dm 'log'  clear ;
options   mlogic mprint FULLSTIMER compress=yes nonumber THREADS CPUCOUNT=ACTUAL ;
libname out 'C:\SIMULATION_N5\MEWMA\MEWMA_UCLI_P2_N5';


/** -- to avoid the problem of filling SAS Log
   Window in display save the SAS log and listing to files;*/
proc printto  new
 log  =  'C:\SIMULATION_N5\MEWMA\MEWMA_UCLI_P2_N5\MEWMAP1N_P3N5_Xn.txt'  ;
  print = "C:\SIMULATION_N5\MEWMA\MEWMA_UCLI_P2_N5\Temp_LIST.txt";

RUN; title 'MEWMA UCL PHASE I Simulation - Normal for (Est. Cov.e matrix and mean vector,
from 100,000 samples) , p=2, n=5';

/*------------------------------------------------------------------------*/;
/** Clear old SAS data sets with teh same name                           */;
/*------------------------------------------------------------------------*/;


proc datasets library=out;
 delete   MEWMA_PhaseIUCL_Normal_NP2_N5XSn ;
Run;


proc iml;

/*------------------------------------------------------------------------*/;
/** START Estimate the covariance matrix and mean from 100,000 samples   */;
/*------------------------------------------------------------------------*/;

       NumSim=100000 ;
       n=5;
       p=2;
       RESULT_MEAN = j(NumSim,p,.);
       RESULT_COV = j(p,p,0);
       call randseed(1);                               /* set seed for RANDGEN */
       XX =  j(n,p,.);                                 /* initialize X matrix with n
rows and p columns (n x p)    */
       do iii=1 to NumSim;
```

```
                call RANDGEN(XX,'NORMAL');            /* fill X values from normal
distribution default parameters */
        X_MEAN=XX[:,];                                /* compute the column means of X
matrix */
                RESULT_MEAN[iii,]=X_MEAN;             /* save as the ith row of means
matrix */
                COV_X=cov (XX);                               /* compute the covariance
matrix of X matrix */
                RESULT_COV = RESULT_COV + COV_X;
        end;
        X_BAR_BAR_EST=RESULT_MEAN[:,];                /* calculate the samples means */
        COV_EST = RESULT_COV / (NumSim*n) ;   /* Calculate the average covariance matrix
to estimate sample covariance matrix */
        print   COV_EST;                                      /* This is the estimated
covariance matrix of X samples */
        print   X_BAR_BAR_EST;                                /* This is the estimated mean
vextor matrix of X samples, use as initial Z value */

/*-----------------------------------------------------------------------------*/;
/**  END Estimatation of the covariance matrix and mean from 100,000 samples   */
/*-----------------------------------------------------------------------------*/;

         varNames = {"lambda" "K" "h" "T" "RL"};

         create out.MEWMA_PhaseIUCL_Normal_NP2_N5XSn var varNames;
         SIMS=10000 ;
         Z =  j(1,p,.);               /* initialize Z matrix with 1 row and p columns (1 x
2 ) */
     X =  j(n,p,.);           /* initialize X matrix with n rows and p columns (5 x 2) */
     RL=.       ;
         RESULT=j(SIMS,5,.);   /* Result*/
     call randseed(12345);    /* set seed for RANDGEN */

/*----------------------------------------------------------------------*/;
/**  Start MEWMA simulation to find UCL for ARL0=200,500, &1000         */
/*----------------------------------------------------------------------*/

 do l= 4 to 4;                /* (1) Start iteration counter "do" loop */
        if l=1  then lambda=0.01;
        if l=2  then lambda=0.02;
        if l=3  then lambda=0.03;
        if l=4  then lambda=0.05;
        if l=5  then lambda=0.1 ;
        if l=6  then lambda=0.2 ;
        if l=7  then lambda=0.5 ;

    do h= 6.8  to 12.3 by  0.1;  /*FOR LAMBDA=.05*/ /* (2) Test UCL limits from 7 to 40
for each lambda */
        *do h= 5  to 15 by  0.1;                      /* (2) Test UCL limits from 7 to 40
for each lambda */
                do k = 1 to SIMS   ;                              /* (3) DO 10,000
ITERATIONS */
                        do m= 1 to 15000   until (flag=0) ; /* (4) generate up to 15000
sample or until T > h */
                    if m=1 then flag=1;
                call RANDGEN(X,'NORMAL');     /* fill x1 values from normal distribution
default parameters */
                X_MEAN=X[:,];                                     /* compute the column
means of X matrix */
                    if m=1 then Z=X_BAR_BAR_EST;              /* usew instead of Z={0 0} */
                    Zm=(lambda*X_MEAN) + (1-lambda)*Z;        /* compute the Z vectors
recrusive MEWMA structure */
                    Z=Zm;                                                     /* save
Z lag vectors for next vector in series */
                    COV_Z=(lambda/(2-lambda)) *
                        (1-(1-lambda)**(2*m))*COV_EST ;              /*compute the
covariance matrix of Z matrix */
                        T =  Z * SOLVE(COV_Z,Z`);                 /* compute the MEWMA
charting statistic T**2 */
```

```
                              if (T > h & flag=1) then  do; /* start T > h condition test
loop   */
                              RL=m     ;                                   /* set RL =
number of subgroups or samples */
                                     R = lambda||K||h||T||RL ;
                                     RESULT[k,]=R;
                      flag=0;
                      end;                                 /* >> close T > h condition
test loop << */
               end;                                               /* (4) close/end  (m)
subgroup "do" loop */
       end;                                                       /* (3) close/end 10K
iteration loop */
       append from RESULT;                                 /*      Append 10000 iteration
for eack lambda and UCL into SAS data set */
  end;                                                            /* (2) close/end UCL
loop */
end;                                                              /* (1) close/end
lambda loop */


title  'MEWMA UCL PHASE I Simulation';
title2 'Normal for (Est. Cov. matrix & mean vector, from 100k samples), p=2, n=5';
proc means data=out.MEWMA_PhaseIUCL_Normal_NP2_N5XSn  ;
class lambda H;
var RL;
output out=averages mean=ARL std=SDRL;
run;

data averages ;
 keep Lambda UCL ARL STDERROR_ARL _FREQ_;
 set averages ;
 where _type_=3;
 STDERROR_ARL=sdrl/100;
 UCL=h;
run;

proc print data=averages    ;
run;


/*-------------------------------------------------------------------*/
/*-------------------------------------------------------------------*/
/* >>>>>>>>>>>>>>  MEWMA Phase II ARL Determination <<<<<<<<<<<<<<<< */
/*-------------------------------------------------------------------*/
/*-- A doctoral dissertation supplemental SAS code by Jamil Zeinab --*/
/*-------------------------------------------------------------------*/
/*-------------------------------------------------------------------*/
/* 1. The IML MEWMA code generates 10,000 iterations of              */
/*    of up to 15,000 p-variate samples from a                       */
/*    sampling distribution and computes MEWMA                       */
/* 2. It creates a MEWMA chating statistic T                         */
/* 3. It calculates the out of control ARL for each combonation of   */
/*    lambda, shift, and UCL obtaioned from Phase I for ARL0=200     */
/* 4. Several versions of this code will exist based on the sampling */
/*    criteria below (distribution, p-vartiates).                    */
/*-------------------------------------------------------------------*/
/*                >>>> Sampling criteria <<<<<                       */
/* Distribution: Multivariate Normal (Normal Marginals)              */
/* Number of qualirty variables: p=2, n=5                            */
/* Estimated Covariance matrix & mean vector, from 100,000 samples   */
/*-------------------------------------------------------------------*/

/*-------------------------------------------------------------------*/
/*  >>>>>>>>>>>>>>>>>>>    Notes for usage <<<<<<<<<<<<<<<<<<<              */
/* 1. Change the sampling distribution as needed, I used Normal, T,  */
/*    & Gamma.                                                       */
/* 2. Change the sampling size as desired.                           */
/* 3. Change the smoothing parameter "lambda" as desired.            */
/*-------------------------------------------------------------------*/
```

```
options   mlogic mprint SYMBOLGEN FULLSTIMER
compress=yes nonumber THREADS CPUCOUNT=ACTUAL ;

/** to avoid the problem of filling SAS Log Window   */
/** in display save the SAS log and listing to files */
proc printto  new
 log  =  'C:\SIMULATION_N5\MEWMA\MEWMA_PHASE_II_UCL_P2_N5\MEWMAP2N_P2N5_Xn.txt'  ;
 print = "C:\SIMULATION_N5\MEWMA\MEWMA_PHASE_II_UCL_P2_N5\Temp_LIST.txt";
RUN;

/*-----------------------------------------------------------------------*/;
/** Clear old SAS data sets with the same name                          */;
/*-----------------------------------------------------------------------*/;


libname out 'F:\ASRM2\SRM799_SIMULATION_WORK\PHASE_II\MEWMA\MEWMA_PHASE_II_ARL_P2_N5';

%macro mewmaphase2(arl0,ucl1,ucl2,ucl3,ucl4,ucl5,ucl6,ucl7,variates);

proc iml;

/*-----------------------------------------------------------------------*/;
/** START Estimate the covariance matrix and mean from 100,000 samples   */;
/*-----------------------------------------------------------------------*/;

       NumSim=10000 ;
       n=5;
       p=%eval(&variates);
       var_x1=0;
       RESULT_MEAN = j(NumSim,p,.);
       RESULT_COV = j(p,p,0);
       call randseed(1);                           /* set seed for RANDGEN */
       XX =  j(n,p,.);                             /* initialize X matrix with n rows
and p columns (n x p)    */
       do iii=1 to NumSim;
              call RANDGEN(XX,'NORMAL');    /* fill X values from normal distribution
default parameters */
       X_MEAN=XX[:,];                       /* compute the column means of X matrix */
              RESULT_MEAN[iii,]=X_MEAN;     /* save as the ith row of means matrix */
              COV_X=cov (XX);                      /* compute the covariance matrix of X
matrix */
              RESULT_COV = RESULT_COV + COV_X;
       end;
       X_BAR_BAR_EST=RESULT_MEAN[:,];                      /* calculate the samples
means */
       COV_EST = RESULT_COV / (NumSim*n ) ; /* Calc. the avg. covariance matrix to est.
sample cov matrix */

       var_x1=COV_EST[1,1];
       print   result_cov COV_EST;                 /* This is the estimated covariance
matrix of X samples */
       print   X_BAR_BAR_EST var_x1;               /* This is the estimated mean vextor
matrix of X samples, use as initial Z value */

/*-----------------------------------------------------------------------------*/;
/**  END Estimatation of the covariance matrix and mean from 100,000 samples   */
/*-----------------------------------------------------------------------------*/;

        varNames = {"lambda" "Delta" "H" "T" "RL"};
        /* varNames = {"lambda" "K" "h" "X1_BAR" "X2_BAR" "Z1" "Z2" "T" "RL"}; */
        create out.MEWMA_%eval(&arl0)_PII_ARL_N_P%eval(&variates)  var varNames;
        SIMS=10000 ;
        Z =  j(1,p,.);       /* initialize Z matrix with 1 row and p columns (1 x 2)
*/
     X =  j(n,p,.);    /* initialize X matrix with n rows and p columns (5 x 2)    */
        X1=  j(n,1,.);
        X2=  j(n,1,.);
     RL=.      ;
        RESULT=j(SIMS,5,.);                                /* Result*/
```

```
     call randseed(12345);                                /* set seed for RANDGEN */

/*-----------------------------------------------------------------------*/;
/**  Start MEWMA To Generate Phase II ARL values for ARL0=200,500, &1000 */
/*-----------------------------------------------------------------------*/


 do l= 1 to 7;

       if l=1  then lambda=0.01;      /* (1) Start iteration counter "do" loop */
       if l=2  then lambda=0.02;
       if l=3  then lambda=0.03;
       if l=4  then lambda=0.05;
       if l=5  then lambda=0.1 ;
       if l=6  then lambda=0.2 ;
       if l=7  then lambda=0.5 ;
       if l=1  then H=&ucl1 ; /*5.3*/;
       if l=2  then H=&ucl2 ; /*6.2*/;
       if l=3  then H=&ucl3 ; /*6.8*/;
       if l=4  then H=&ucl4 ; /*7.7*/;
       if l=5  then H=&ucl5 ; /*8.8*/;
       if l=6  then H=&ucl6 ; /*9.7*/;
       if l=7  then H=&ucl7 ; /*10.4*/;

       do ss= 1 to 5;                      /* (2) Test shift in first variable for each
lambda */
               if ss=1 then Delta=0.25* sqrt(var_x1);
               if ss=2 then Delta=0.5 * sqrt(var_x1);
               if ss=3 then Delta=1.0 * sqrt(var_x1);
               if ss=4 then Delta=1.5 * sqrt(var_x1);
               if ss=5 then Delta=2.5 * sqrt(var_x1);


               do k = 1 to SIMS   ;                             /* (3) DO 10,000
ITERATIONS */
                       do m= 1 to 15000   until (flag=0) ;   /* (4) generate up to 15000
sample or until T > h */
                   if m=1 then flag=1;
               call RANDGEN(X1,'NORMAL');   /* fill x1 values from normal distribution
default parameters */
               call RANDGEN(X2,'NORMAL');    /* fill x1 values from normal distribution
default parameters */
                       X1=X1+Delta;
                       X[,1]=X1;
                       X[,2]=X2;
                       X_MEAN=X[:,];                        /* compute the column means
of X matrix */
                   if m=1 then Z=X_BAR_BAR_EST;             /* usew instead of Z={0 0} */
                   Zm=(lambda*X_MEAN) + (1-lambda)*Z;       /* compute the Z vectors
recrusive MEWMA structure */
                   Z=Zm;                                            /* save Z lag vectors
for next vector in series */
                   COV_Z=(lambda/(2-lambda)) *
                       (1-(1-lambda)**(2*m))*COV_EST;       /*compute the covariance
matrix of Z matrix */
                       T =  Z * SOLVE(COV_Z,Z`);            /* compute the MEWMA charting
statistic T**2 */

                               if (T > h & flag=1) then  do; /* start T > h condition test
loop   */
                               RL=m     ;                               /* set RL =
number of subgroups or samples */
                                       R = lambda||Delta||H||T||RL;
                                       RESULT[k,]=R;
                       flag=0;
                       end;                          /* >> close T > h condition test loop
<< */
               end;                                         /* (4) close/end  (m)
subgroup "do" loop */
```

```
       end;                                              /* (3) close/end 10K
iteration loop */
       append from RESULT;                       /*      Append 10000 iteration for
eack lambda and UCL into SAS data set */
  end;                                                   /* (2) close/end UCL loop */
end;                                                     /* (1) close/end lambda loop
*/

quit;

title "MEWMA UCL PHASE II Simulation - Normal for: (P=&variates) and (IC ARL0=&arl0)";

proc means data=out.MEWMA_%eval(&arl0)_PII_ARL_N_P%eval(&variates)  ;
class lambda h Delta;

var RL;
output out=out.MEWMA_%eval(&arl0)_PII_ARL_N_P%eval(&variates)_AVG mean=ARL std=SDRL;
run;
proc print data=out.MEWMA_%eval(&arl0)_PII_ARL_N_P%eval(&variates)_AVG;
run;


data out.MEWMA_%eval(&arl0)_PII_ARL_N_P%eval(&variates)_AVG2 ;
 keep Lambda UCL Delta ARL sdrl _FREQ_;
 set out.MEWMA_%eval(&arl0)_PII_ARL_N_P%eval(&variates)_AVG ;
   UCL=h;
   where _type_=7;
run;

proc print data=out.MEWMA_%eval(&arl0)_PII_ARL_N_P%eval(&variates)_AVG2    ;
run;
%mend;

%mewmaphase2(200,5.3,6.2,6.8,7.7,8.8,9.7,10.4,2);
%mewmaphase2(500,7,8.1,8.8,9.7,10.8,11.6,12.3,2);
%mewmaphase2(1000,8.3,9.6,10.3,11.2,12.3,13.1,13.7,2);


/*-----------------------------------------------------------------*/
/*-----------------------------------------------------------------*/
/* >>>>>>>>>>>>>>>>  SRMEWMA Phase I UCL Determination <<<<<<<<<<<<< */
/*-----------------------------------------------------------------*/
/*-- A doctoral dissertation supplemental SAS code by Jamil Zeinab --*/
/*-----------------------------------------------------------------*/
/*-----------------------------------------------------------------*/
/* 1. The IML SRMEWMA code generates 10,000 iterations of          */
/*    of up to 15,000 p-variate samples from a                     */
/*    sampling distribution and computes MEWMA                     */
/* 2. It creates a SRMEWMA chating statistic Qt                    */
/* 3. It calculates the UCLs for MEWMA  for which the              */
/*    in-control average run length is ARL_0 = 200, 500, & 1000.  */
/* 4. Several versions of this code will exist based on the sampling */
/*    criteria below (distribution, p-vartiates).                  */
/*-----------------------------------------------------------------*/
/*                 >>>> Sampling criteria <<<<<                    */
/* Distribution: Multivariate Normal (Normal Marginals)            */
/* Number of qualirty variables: p=5, n=5                          */
/* Estimated Covariance matrix & mean vector, from 100,000 samples   */
/*-----------------------------------------------------------------*/

/*-----------------------------------------------------------------*/
/*  >>>>>>>>>>>>>>>>>>>    Notes for usage <<<<<<<<<<<<<<<<<<<              */
/* 1. Change the sampling distribution as needed, I used Normal, T,  */
/*    & Gamma.                                                     */
/* 2. Change the sampling size as desired.                         */
/* 3. Change teh smoothing parameter "lambda" as desired.          */
/*-----------------------------------------------------------------*/

dm 'output'  clear ;
dm 'log'  clear ;
```

```
options   mlogic mprint FULLSTIMER compress=yes nonumber THREADS CPUCOUNT=ACTUAL ;
libname out 'C:\SIMULATION_N5\SRMEWMA\SRMEWMA_PAHSEI_UCL_P5_N5';

/** -- to avoid the problem of filling SAS Log
Window in display save the SAS log and listing to files;*/
proc printto  new
log  =  'C:\SIMULATION_N5\SRMEWMA\SRMEWMA_PAHSEI_UCL_P5_N5\SRMEWMAP1N_P2N5_Xn.txt'  ;
print = "C:\SIMULATION_N5\SRMEWMA\SRMEWMA_PAHSEI_UCL_P5_N5\Temp_LIST.txt";

RUN;

title 'SRMEWMA UCL PHASE I Simulation - Normal for (Estimated Covariance matrix and mean
vector, from 100,000 samples) , p=5, n=5';

/*-----------------------------------------------------------------------*/;
/** Clear old SAS data sets with tehe same name                          */;
/*-----------------------------------------------------------------------*/;



/*----------------------------------------------------------------*/
/* >>>>>>>>>>> Oja Centered Signed-Sank Vector  <<<<<<<<<<<<<<<<<< */
/*                                                                */
/* infile:   Specifies a name for the input SAS data set. The ith */
/*           observation is the ith observation vector. The jth   */
/*           variable is the jth element of the observation       */
/*           vector.                                              */
/* outfile:  Specifies a name for the output SAS data set.        */
/*           This data set will contain the Oja signed-rank       */
/*           vectors (Hettmansperger, Mottonen and Oja (1996).    */
/*           Affine invariant multivariate one-sample signed-rank */
/*           tests. Mathematics, University of Oulu).             */
/*----------------------------------------------------------------*/
/*----------------------------------------------------------------*/
/*----------------------------------------------------------------*/
/* Courtesy of:                                                   */
/*          Jyrki Mottonen, PhD                                   */
/*          Department of Mathematical Sciences,                  */
/*          University of Oulu,                                   */
/*          FIN-90570 Oulu,                                       */
/*          Finland                                               */
/*----------------------------------------------------------------*/
/*----------------------------------------------------------------*/
/** >>>>>>>>>>>>>>>>> Module "Rn" <<<<<<<<<<<<<<<<<<<<<<<<<<<<<<**/
/*----------------------------------------------------------------*/

proc iml;

start Rn(n,k,i,X); /* Module "Rn" gives the signed-rank */
p=1:k;   s=j(k,1,0);   cnt=0;
do until(lc=0);
  cnt=cnt+1;
  s=s+Qp(k,i,p,X);
  run nextp(n,k,p,lc);   /* take next p */
end;
return(s/(cnt*2**k)); /* cnt = N_p = "n choose k" */
/**/print s;
finish Rn;
/*----------------------------------------------------------------*/
/** >>>>>>>>>>>>>>>>>>>>>>> Module "nextp" <<<<<<<<<<<<<<<<<<<<<<**/
/*----------------------------------------------------------------*/
start nextp(N,k,p,lc);                      /* Module "nextp" gives the next k-subset of
an N-set  */
                                                               /* lc=0 if the previous k-
subset was the last k-subset */
lc=k;
if(p[k]<N)then do;
   p[k]=p[k]+1;
```

```
end;                                              /* returns the p-set combination
{1,2,3}.  */
else do;                                  /*for example. Order is not important */
  flag=0;
  do until((lc=0)|(flag=1));
    lc=lc-1;
    if(lc>0)then do;
      if(p[lc]<(N-k+lc))then flag=1;
    end;
  end;
  if(lc>0) then do;
    p[lc]=p[lc]+1;
    do i=(lc+1) to k;
      p[i]=p[i-1]+1;
    end;
  end;
end;
finish nextp;
/*----------------------------------------------------------------*/
/** >>>>>>>>>>>>>>>>>>>> Module "Qp" <<<<<<<<<<<<<<<<<<<<<<<<<<<<<<**/
/*----------------------------------------------------------------*/
start Qp(k,i,p,X);                     /* Module "Qp" gives the vector (2.8) times 2**k */
s=j(k,1,0);
temp=2##(0:(k-1));                     /* temp = (1 2 4 ... 2**(k-1)) */
do j=0 to 2**k-1;
  a=2*(band(temp,j)>0)-1;      /* a[m]=+1 or -1, m=1,...k */
  if(all(a#p-i))then do;       /* if i is in a#p, Spdp=0  */
    Y=repeat(a,k,1)#X[,p];
    s=s+Spdp(k,Y,X[,i]);
  end;
end;
return(s);
finish Qp;
/*----------------------------------------------------------------*/
/** >>>>>>>>>>>>>>>>>>>>>>>> Module "Spdp" <<<<<<<<<<<<<<<<<<<<<<<**/
/*----------------------------------------------------------------*/
/* Module "Spdp" computes the vector S_pa()d_pa in (2.8) Mottenen paper */

start Spdp(k,Y,xi);
W1=Y[,2:k]-repeat(Y[,1],1,k-1); /* (xi2-xi1,...,xik-xi1) */
W2=Y-repeat(xi,1,k);            /* (xi1-xi,...,xik-xi)   */
dp=j(k,1,0);
do j=1 to k;
  dp[j]=((-1)**j)*det(W1[remove(1:k,j),]);
end;
return(sign(det(W2))*dp);              /*sign function return 1,-1, or 0 */
finish Spdp;
/*----------------------------------------------------------------*/
/** >>>>>>>>>>>>>>>>>>>>>>>> Module "score" <<<<<<<<<<<<<<<<<<<<<<**/
/*----------------------------------------------------------------*/
start score;                              /*print"Oja signed-rank vectors";*/
                                                        /*print"Signed-Rank Vectors";
*/
*use X;
*read all into X;
X=X`;
N=ncol(X);                                        /* n = # of cols = # of observations
*/
k=nrow(X);                                        /* k = # of rows = dimension
*/
srank=j(k,N,0);
do j=1 to N;
  srank[,j]=Rn(n,k,j,X);
end;
Srank=Srank`;
/*create OSR from srank [colname={SR1,SR2}];
append from srank;*/
finish score;
/*----------------------------------------------------------------*/
/** >>>>>>>>>>>>>>>>>>>>>>>> Module "score" <<<<<<<<<<<<<<<<<<<<<<**/
```

```
/*-------------------------------------------------------------------*/
start score2;                                    /*print"Oja signed-rank vectors";*/
                                                             /*print"Signed-Rank Vectors";
*/
*use X;
*read all into X;
G=X`;
n=ncol(G);                                              /* n = # of cols = # of observations
*/
k=nrow(G);                                              /* k = # of rows = dimension
*/
srank=j(k,n,0);
do j=1 to n;
  srank[,j]=Rn(n,k,j,G);
end;
Srank=Srank`;
/*create OSR from srank [colname={SR1,SR2}];
append from srank;*/
finish score2;

/*--------------------------------------------------------------------------------*/
/*-> Srart Parameter estimation of signed-ranks means and covarioance matrix <-*/
/*--------------------------------------------------------------------------------*/
    NumSim=100000 ;
        nn=5;
        pp=5;
        RESULT_SR_MEAN = j(NumSim,pp,.);
        RESULT_SR_COV = j(pp,pp,0);
        call randseed(12345);  /* set seed for RANDGEN */
        X =  j(nn,pp,.);       /* initialize X matrix with n rows and p columns (n x p)
*/

do iii=1 to NumSim;
        call RANDGEN(X,'NORMAL'); /* fill X values from normal distribution default
parameters */
        *print X ;
        run score2;
        *print x Srank;
        /*--------------------------------------------------------------------------------*/
        /*-> calculate the Signed-rank statistic SR                                     <-*/
        /*--------------------------------------------------------------------------------*/
        SR_MEAN=Srank[:,];                          /* compute the column means of SR
matrix */
        *print SR_MEAN;
        RESULT_SR_MEAN[iii,]=SR_MEAN; /* save as the ith row of means matrix */
        COV_SR=cov (Srank);                  /* compute the covariance matrix of X matrix
*/
        RESULT_SR_COV = RESULT_SR_COV + COV_SR;
end;
        SR_D_BAR_MEAN_EST=RESULT_SR_MEAN[:,];/* calc. the samples means */
        COV_EST = RESULT_SR_COV / (NumSim*nn) ;      /* Calc. the average cov matrix */
        print   COV_EST;
        print   SR_D_BAR_MEAN_EST;

/*--------------------------------------------------------------------------------*/;
/**  END Estimatation of the covariance matrix and mean from 100,000 samples   */
/*--------------------------------------------------------------------------------*/;

SIMS=10000 ;
varNames = {"lambda" "kkk" "hhh" "Qt" "RL2"};
Wm =  j(1,pp,.);       /* initialize Wm matrix with 1 row and p columns (1 x 2 ) */
W =   j(1,pp,.);       /* initialize W matrix with 1 row and p columns (1 x 2 ) */
X =  j(nn,pp,.);       /* initialize X matrix with n rows and p columns (5 x 2) */
RL2=.   ;
RESULT=j(SIMS,5,.);    /* Result*/
call randseed(12345);/* set seed for RANDGEN */

/*---------------------------------------------------------------------*/
/*>>>  Start SRMEWMA simulation to find UCL for ARL0=200,500, &1000 <<< */
/*---------------------------------------------------------------------*/
```

```
do ll= 5 to 5;                              /* (1) Start iteration counter "do" loop */
       if ll=1  then lambda=0.01;
       if ll=2  then lambda=0.02;
       if ll=3  then lambda=0.03;
       if ll=4  then lambda=0.05 ;
       if ll=5  then lambda=0.1 ;
       if ll=6  then lambda=0.2 ;
       if ll=7  then lambda=0.5 ;

       do hhh= 19. TO 19.3  BY 0.1  ;               /* (2) Test UCL limits from 7 to 40
for each lambda */
         do kkk = 1 to SIMS   ;                                    /* (3) DO 10,000
ITERATIONS */
              do mmm= 1 to 10000   until (flag=0) ;   /* (4) generate up to 15000 sample
or until T > h */
               call RANDGEN(X,'NORMAL');     /* fill XX Matrix w/values from normal
dist. default parms */
               run score2;
               SR_MEAN=Srank[:,];                                  /*print SR_MEAN;*/
               if mmm=1 then flag=1;
               if mmm=1 then W=SR_D_BAR_MEAN_EST;           /*W={0 0}*/
               Wm = (lambda * SR_MEAN)+ (1-lambda)* W;/* compute the W vectors recrusive
SRMEWMA structure */
               W=Wm ;                                       /* save W lag vectors for
next vector in series */
               COV_Qt=(lambda/(2-lambda)) *
            (1-(1-lambda)**(2*mmm) )* COV_EST;     /*compute the covariance matrix of w
matrix */
               Qt =  W * SOLVE(COV_Qt,W`);                  /* compute the SRMEWMA
charting statistic Qt  */
                     if (Qt > hhh & flag=1) then  do;      /* Check if Qt > UCL   */
                     RL2=mmm    ;
                     R = lambda || kkk || hhh || Qt|| RL2;
                     RESULT[kkk,]=R;
                            flag=0;
                     end;
              end;                                                 /* (4) close/end  (m)
subgroup "do" loop */
              *print X Srank SR_MEAN lambda   KKK   hhh COV_Qt W  Qt  RL2;
         end;                        /* (3) close/end 10K iteration loop */
      append from RESULT;            /*Append 10000 iteration for eack lambda and UCL
into SAS data set */
       end;                                                        /* (2) close/end UCL
loop */
end;                                                               /* (1) close/end
lambda loop */

proc means data=out.SRMEWMA_PhaseIUCL_Nl_P5_N5XSn   ;
class lambda HHH;
var RL2;
output out=averages mean=ARL std=SDRL;
run;

data averages ;
 keep Lambda UCL ARL STDERROR_ARL _FREQ_;
 set averages ;
 where _type_=3;
 STDERROR_ARL=sdrl/100;
 UCL=h;
run;

proc print data=averages    ;
run;
```

```
/*-----------------------------------------------------------------------*/
/*-----------------------------------------------------------------------*/
/* >>>>>>>>>>>>>>>>  SRMEWMA Phase II ARL Determination <<<<<<<<<<<< */
/*-----------------------------------------------------------------------*/
/*-- A doctoral dissertation supplemental SAS code by Jamil Zeinab --*/
/*-----------------------------------------------------------------------*/
/*-----------------------------------------------------------------------*/
/* 1. The IML SRMEWMA code generates 10,000 iterations of            */
/*    of up to 15,000 p-variate samples from a                       */
/*    sampling distribution and computes MEWMA                       */
/* 2. It creates a SRMEWMA chating statistic Qt                      */
/* 3. It calculates the UCLs for MEWMA  for which the                */
/*    in-control average run length is ARL_0 = 200, 500, & 1000.     */
/* 4. Several versions of this code will exist based on the sampling */
/*    criteria below (distribution, p-vartiates).                    */
/*-----------------------------------------------------------------------*/
/*                  >>>> Sampling criteria <<<<<                     */
/* Distribution: Multivariate Normal (Normal Marginals)              */
/* Number of qualirty variables: p=2, n=5                            */
/* Estimated Covariance matrix & mean vector, from 100,000 samples   */
/*-----------------------------------------------------------------------*/

/*-----------------------------------------------------------------------*/
/*  >>>>>>>>>>>>>>>>>>>    Notes for usage <<<<<<<<<<<<<<<<<<<              */
/* 1. Change the sampling distribution as needed, I used Normal, T,  */
/*    & Gamma.                                                       */
/* 2. Change the sampling size as desired.                           */
/* 3. Chamge the smoothing parameter "lambda" as desired.            */
/*-----------------------------------------------------------------------*/

dm 'output'  clear ;
dm 'log'  clear ;

options   mlogic mprint FULLSTIMER compress=yes nonumber THREADS CPUCOUNT=ACTUAL ;
libname out 'F:\ASRM2\SRM799_SIMULATION_WORK\PHASE_II\SRMEWMA\SRMEWMA_PHASEII_ARL_P2_N5';

/** -- to avoid the problem of filling SAS Log
  Window in display save the SAS log and listing to files;*/
proc printto  new
 log  =  'C:\SIMULATION_N5\SRMEWMA\SRMEWMA_PAHSEI_UCL_P5_N5\SRMEWMAP1N_P2N5_Xn.txt'  ;
 print = "C:\SIMULATION_N5\SRMEWMA\SRMEWMA_PAHSEI_UCL_P5_N5\Temp_LIST.txt";

RUN; title 'SRMEWMA ARL PHASE II Simulation - Normal for ARL0=200 , p=2, n=5';


/*------------------------------------------------------------------*/
/* >>>>>>>>> Oja Centered Signed-Rank Vectors <<<<<<<<<<<<<<<<<<< */
/*                                                                */
/* infile:   Specifies a name for the input SAS data set. The ith */
/*           observation is the ith observation vector. The jth   */
/*           variable is the jth element of the observation       */
/*           vector.                                              */
/* outfile:  Specifies a name for the output SAS data set.        */
/*           This data set will contain the Oja signed-rank       */
/*           vectors (Hettmansperger, Mottonen and Oja (1996).    */
/*           Affine invariant multivariate one-sample signed-rank */
/*           tests. Mathematics, University of Oulu).             */
/*------------------------------------------------------------------*/
/*------------------------------------------------------------------*/
/*------------------------------------------------------------------*/
/* Courtesy of:                                                           */
/*          Jyrki Mottonen, PhD                                   */
/*          Department of Mathematical Sciences,                  */
/*          University of Oulu,                                   */
/*          FIN-90570 Oulu,                                       */
/*          Finland                                               */
/*                                                                */
/*          E-mail: Jyrki.Mottonen@oulu.fi                        */
/*------------------------------------------------------------------*/
```

```
/*-----------------------------------------------------------------*/
/** >>>>>>>>>>>>>>>>> Module "Rn" <<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<**/
/*-----------------------------------------------------------------*/
proc iml;


start Rn(n,k,i,X); /* Module "Rn" gives the signed-rank */
p=1:k;   s=j(k,1,0);   cnt=0;
do until(lc=0);
  cnt=cnt+1;
  s=s+Qp(k,i,p,X);
  run nextp(n,k,p,lc);   /* take next p */
end;
return(s/(cnt*2**k)); /* cnt = N_p = "n choose k" */
/**/print s;
finish Rn;
/*-----------------------------------------------------------------*/
/** >>>>>>>>>>>>>>>>>>>>>>> Module "nextp" <<<<<<<<<<<<<<<<<<<<<<<**/
/*-----------------------------------------------------------------*/
start nextp(N,k,p,lc);        /* Module "nextp" gives the next k-subset of an N-set  */
                                     /* lc=0 if the previous k-subset was the last k-
subset */
lc=k;
if(p[k]<N)then do;
   p[k]=p[k]+1;
end;                   /* returns the p-set combination {1,2,3}.  */
else do;       /*for example. Order is not important */
  flag=0;
  do until((lc=0)|(flag=1));
    lc=lc-1;
    if(lc>0)then do;
      if(p[lc]<(N-k+lc))then flag=1;
    end;
  end;
  if(lc>0) then do;
    p[lc]=p[lc]+1;
    do i=(lc+1) to k;
      p[i]=p[i-1]+1;
    end;
  end;
end;
finish nextp;
/*-----------------------------------------------------------------*/
/** >>>>>>>>>>>>>>>>>>> Module "Qp" <<<<<<<<<<<<<<<<<<<<<<<<<<<<<**/
/*-----------------------------------------------------------------*/
start Qp(k,i,p,X);    /* Module "Qp" gives the vector (2.8) times 2**k */
s=j(k,1,0);
temp=2##(0:(k-1));            /* temp = (1 2 4 ... 2**(k-1)) */
do j=0 to 2**k-1;
  a=2*(band(temp,j)>0)-1;     /* a[m]=+1 or -1, m=1,...k */
  if(all(a#p-i))then do;      /* if i is in a#p, Spdp=0  */
    Y=repeat(a,k,1)#X[,p];
    s=s+Spdp(k,Y,X[,i]);
  end;
end;
return(s);
finish Qp;
/*-----------------------------------------------------------------*/
/** >>>>>>>>>>>>>>>>>>>>>>> Module "Spdp" <<<<<<<<<<<<<<<<<<<<<<<<**/
/*-----------------------------------------------------------------*/
/* Module "Spdp" computes the vector S_pa()d_pa in (2.8) Mottenen paper */
start Spdp(k,Y,xi);
W1=Y[,2:k]-repeat(Y[,1],1,k-1); /* (xi2-xi1,...,xik-xi1) */
W2=Y-repeat(xi,1,k);            /* (xi1-xi,...,xik-xi)   */
dp=j(k,1,0);
do j=1 to k;
  dp[j]=((-1)**j)*det(W1[remove(1:k,j),]);
end;
return(sign(det(W2))*dp);            /*sign function return 1,-1, or 0 */
finish Spdp;
```

```
/*--------------------------------------------------------------------*/
/** >>>>>>>>>>>>>>>>>>>>>>> Module "score" <<<<<<<<<<<<<<<<<<<<<<**/
/*--------------------------------------------------------------------*/
start score;                  /*print"Oja signed-rank vectors";*/
                                             /*print"Signed-Rank Vectors"; */
*use X;
*read all into X;
X=X`;
N=ncol(X);                    /* n = # of cols = # of observations */
k=nrow(X);                    /* k = # of rows = dimension         */
srank=j(k,N,0);
do j=1 to N;
  srank[,j]=Rn(n,k,j,X);
end;
Srank=Srank`;
/*create OSR from srank [colname={SR1,SR2}];
append from srank;*/
finish score;
/*--------------------------------------------------------------------*/
/** >>>>>>>>>>>>>>>>>>>>>>> Module "score" <<<<<<<<<<<<<<<<<<<<<<**/
/*--------------------------------------------------------------------*/
start score2;                 /*print"Oja signed-rank vectors";*/
                                             /*print"Signed-Rank Vectors"; */
*use X;
*read all into X;
G=X`;
n=ncol(G);                            /* n = # of cols = # of observations */
k=nrow(G);                            /* k = # of rows = dimension         */
srank=j(k,n,0);
do j=1 to n;
  srank[,j]=Rn(n,k,j,G);
end;
Srank=Srank`;
/*create OSR from srank [colname={SR1,SR2}];
append from srank;*/
finish score2;
/*--------------------------------------------------------------------*/
/** >>>>>>>>>>>>>>>>>>>>>>> Module "score" <<<<<<<<<<<<<<<<<<<<<<**/
/*--------------------------------------------------------------------*/
start score3;          /*print"Oja signed-rank vectors";*/
                                      /*print"Signed-Rank Vectors"; */
*use X;
*read all into X;
QQ=XX`;
n=ncol(QQ);                   /* n = # of cols = # of observations */
k=nrow(QQ);                   /* k = # of rows = dimension         */
srank=j(k,n,0);
do j=1 to n;
  srank[,j]=Rn(n,k,j,QQ);
end;
Srank=Srank`;
/*create OSR from srank [colname={SR1,SR2}];
append from srank;*/
finish score3;

/*---------------------------------------------------------------------------------*/
/*-> Srart Parameter estimation of signed-ranks means and covarioance matrix <-*/
/*---------------------------------------------------------------------------------*/
    NumSim=100000 ;
       nn=5;
       pp=2;
       RESULT_SR_MEAN = j(NumSim,pp,.);
       RESULT_SR_COV = j(pp,pp,0);
       var_x1= j (1,1,0);
       call randseed(12345);         /* set seed for RANDGEN */
       X =  j(nn,pp,.);                     /* initialize X matrix with n rows and p
columns (n x p)*/

do iii=1 to NumSim;
```

```
        call RANDGEN(X,'NORMAL');     /* fill X values from normal distribution default
param. */
        *print X ;
        run score2;
        *print x Srank;
        /*----------------------------------------------------*/
        /*-> calculate the Signed-rank statistic SR          <-*/
        /*----------------------------------------------------*/
        SR_MEAN=Srank[:,];                    /* compute the column means of SR matrix */
        *print SR_MEAN;
        RESULT_SR_MEAN[iii,]=SR_MEAN; /* save as the ith row of means matrix */
        COV_SR=cov (Srank);                  /* compute the covariance matrix of X matrix
*/
        RESULT_SR_COV = RESULT_SR_COV + COV_SR;
end;
        SR_D_BAR_MEAN_EST=RESULT_SR_MEAN[:,];/* calculate the samples means */
        COV_EST = RESULT_SR_COV/(NumSim*nn) ;/* Calculate teh average covariance matrix */
        var_X1= COV_EST[1,1];
        print   COV_EST;
                print   var_x1;

        print   SR_D_BAR_MEAN_EST;

/*----------------------------------------------------------------------------*/;
/**  END Estimatation of the covariance matrix and mean from 100,000 samples   */
/*----------------------------------------------------------------------------*/;

SIMS=10000 ;
varNames = {"lambda" "delta" "kkk" "hhh" "Qt" "RL2"};
create out.SRMEWMA_200_PhaseII_ARL_N_P2 var varNames;
Wm =  j(1,pp,.);      /* initialize Wm matrix with 1 row and p columns (1 x 2 ) */
W =   j(1,pp,.);      /* initialize W matrix with 1 row and p columns (1 x 2 ) */
XX =  j(nn,pp,.);     /* initialize X matrix with n rows and p columns (5 x 2) */
XX1= j(nn,1,.);
XX2= j(nn,1,.);
RL2=.   ;
RESULT=j(SIMS,6,.);            /* Result*/
call randseed(12345);  /* set seed for RANDGEN */

/*-----------------------------------------------------------------------*/
/*>>>  Start SRMEWMA simulation to find UCL for ARL0=200,500, &1000 <<< */
/*-----------------------------------------------------------------------*/

do ll= 1 to 7;                          /* (1) Start iteration counter "do" loop */
        if ll=1  then lambda=0.01;
        if ll=2  then lambda=0.02;
        if ll=3  then lambda=0.03;
        if ll=4  then lambda=0.05 ;
        if ll=5  then lambda=0.1 ;
        if ll=6  then lambda=0.2 ;
        if ll=7  then lambda=0.5 ;
        if ll=1  then hhh=5.3;
        if ll=2  then hhh=6.3;
        if ll=3  then hhh=6.9;
        if ll=4  then hhh=7.7;
        if ll=5  then hhh=8.85;
        if ll=6  then hhh=9.9;
        if ll=7  then hhh=10.9;
        do ss= 1 to 5;                  /* (2) Test shift in first variable for each lambda
*/
                if ss=1 then Delta=0.25 * sqrt(var_x1);
                if ss=2 then Delta=0.5 *  sqrt(var_x1);
                if ss=3 then Delta=1.0 *  sqrt(var_x1);
                if ss=4 then Delta=1.5 *  sqrt(var_x1);
                if ss=5 then Delta=2.5 *  sqrt(var_x1);

          do kkk = 1 to SIMS ; /* (3) DO 10,000 ITERATIONS */
                do mmm= 1 to 15000   until (flag=0) ; /* (4) generate up to 15000 sample or
until T > h */
                 if mmm=1 then flag=1;
```

```
                      call RANDGEN(XX1,'NORMAL'); /* fill x1 values from normal dist.
default params */
               call RANDGEN(XX2,'NORMAL'); /* fill x2 values from normal dist. default
params */
                      XX1=XX1+Delta;
                      XX[,1]=XX1;
                      XX[,2]=XX2;           /* fill XX Matrix w/values from normal dist.
default parameters */

                run score3;
                SR_MEAN=Srank[:,];           /*print SR_MEAN;*/

                if mmm=1 then W=SR_D_BAR_MEAN_EST;                 /*W={0 0}*/
                Wm = (lambda * SR_MEAN)+ (1-lambda)* W;
                W=Wm ;                                      /* save W lag vectors for
next vector in series */
                COV_Qt=(lambda/(2-lambda)) *
                (1-(1-lambda)**(2*mmm) )* COV_EST;          /*compute the covariance
matrix of w matrix */
                Qt =  W * SOLVE(COV_Qt,W`);                 /* compute the SRMEWMA
charting statistic Qt  */
                      if (Qt > hhh & flag=1) then  do;      /* Check if Qt > UCL   */
                      RL2=mmm    ;
                      R = lambda || delta || kkk || hhh || Qt|| RL2  ;
                      RESULT[kkk,]=R;
                             flag=0;
                      end;
               end;                                  /* (4) close/end  (m) subgroup "do"
loop */
               *print XX Srank SR_MEAN lambda   KKK   hhh COV_Qt W  Qt  RL2;

          end;                                       /* (3) close/end 10K iteration loop
*/
       append from RESULT;                  /*      Append 10000 iteration for eack
lambda and UCL into SAS data set */
       end;                                          /* (2) close/end UCL loop */
end;                                                 /* (1) close/end lambda loop */

proc means data=out.SRMEWMA_200_PhaseII_ARL_N_P2   ;
class lambda HHH delta ;
var RL2;
output out=averages mean=ARL std=SDRL;
run;

data averages2 ;
 keep Lambda UCL ARL sdrl _FREQ_;
 set averages ;
 where _type_=7;
  UCL=hhh;
run;

proc print data=averages2    ;
run;

/*-------------------------------------------------------------------*/
/*-------------------------------------------------------------------*/
/* >>>>>>>>  Hotellings T^2  Phase I UCL Determination <<<<<<<<<<<<< */
/*-------------------------------------------------------------------*/
/*-- A doctoral dissertation supplemental SAS code by Jamil Zeinab --*/
/*-------------------------------------------------------------------*/
/*-------------------------------------------------------------------*/
/* 1. The IML MEWMA code generates 10,000 iterations of              */
/*    of up to 15,000 p-variate samples from a                       */
/*    sampling distribution and computes MEWMA                       */
/* 2. It creates a MEWMA chating statistic T                         */
/* 3. It calculates the UCLs for MEWMA  for which the                */
/*    in-control average run length is ARL_0 = 200, 500, & 1000.     */
/* 4. Several versions of this code will exist based on the sampling */
/*    criteria below (distribution, p-vartiates).                    */
/*-------------------------------------------------------------------*/
```

```
/*                >>>> Sampling criteria <<<<<                       */
/* Distribution: Multivariate GAMMA (GAMMA Marginals)                */
/* Number of qualirty variables: p=2, n=5                            */
/* Estimated Covariance matrix & mean vector, from 100,000 samples   */
/*-------------------------------------------------------------------*/

/*-------------------------------------------------------------------*/
/*  >>>>>>>>>>>>>>>>>>>    Notes for usage <<<<<<<<<<<<<<<<<<<              */
/* 1. Change the sampling distribution as needed, I used Normal, T,  */
/*    & Gamma.                                                       */
/* 2. Change the sampling size as desired.                           */
/* 3. Change the smoothing parameter "lambda" as desired.            */
/*-------------------------------------------------------------------*/
dm 'output'  clear ;
dm 'log'  clear ;

options   mlogic mprint FULLSTIMER compress=yes nonumber THREADS CPUCOUNT=ACTUAL ;
libname out 'E:\ASRM2\SRM799_SIMULATION_WORK\PHASE_I\HOTELLINGS';

/* to avoid the problem of filling SAS Log */
/* Window in display save the SAS log and lis  to files */
proc printto  new
 log  =  'C:\SIMULATION_N5\MEWMA\MEWMA_PHASEI_UCL_P2_N5\MEWMAP1G_P2N5_Xn.txt'  ;
 print = "C:\SIMULATION_N5\MEWMA\MEWMA_PHASEI_UCL_P2_N5\Temp_LIST.txt";*/



/*-----------------------------------------------------------------------*/;
/** Clear old SAS data sets with the same name                          */;
/*-----------------------------------------------------------------------*/;
/*
proc datasets library=out;
 delete   HOT2_PhaseIUCL_NORMAL_P2 ;
Run;
*/
%macro hotellings (variates);
proc iml;

/*-----------------------------------------------------------------------*/;
/** START Estimate the covariance matrix and mean from 100,000 samples   */;
/*-----------------------------------------------------------------------*/;

       NumSim=100000 ;
       n=5;
       p=%eval(&variates);
       RESULT_MEAN = j(NumSim,p,.);
       RESULT_COV = j(p,p,0);
       call randseed(1);                            /* set seed for RANDGEN */
       XX =  j(n,p,.);                              /* initialize X matrix with n rows
and p columns (n x p)*/
       do iii=1 to NumSim;
              call RANDGEN(XX,'NORMAL');    /* fill X values from normal distribution
default parameter */
       X_MEAN=XX[:,];                       /* compute the column means of X matrix */
              RESULT_MEAN[iii,]=X_MEAN;     /* save as the ith row of means matrix */
              COV_X=cov (XX);                       /* compute the covariance matrix of X
matrix */
              RESULT_COV = RESULT_COV + COV_X;
       end;
       X_BAR_BAR_EST=RESULT_MEAN[:,];       /* calcu. the samples means */
       COV_EST = RESULT_COV / (NumSim);/* Calc. the avg covariance matrix to estimate
sample cov matrix */
       print   COV_EST;             /* This is the est. cov matrix of X samples */
       print   X_BAR_BAR_EST;       /* This is the est. mean vextor matrix of X samples,
use as initial Z value */

/*-----------------------------------------------------------------------------*/;
/**  END Estimatation of the covariance matrix and mean from 100,000 samples   */
/*-----------------------------------------------------------------------------*/;
```

```
        varNames = {"p" "K" "h" "T2" "RL"};
        create out.HOT2_PhaseIUCL_NORMAL_P2 var varNames;
        SIMS=500 ;
        Z =  j(1,p,.);              /* initialize Z matrix with 1 row and p columns (1 x 
2 ) */
     X =  j(n,p,.);            /* initialize X matrix with n rows and p columns (5 x 2) */
     RL=.      ;
        RESULT=j(SIMS,5,.);   /* Result*/
     call randseed(12345);    /* set seed for RANDGEN */

/*--------------------------------------------------------------------*/;
/**  Start Hotellings t simulation to find UCL for ARL0=200,500, &1000  */
/*--------------------------------------------------------------------*/

       do h= 10  to 21 by  .1;                               /* (2) Test UCL limits 
from 7 to 40 for each lambda */
              do k = 1 to SIMS   ;                                  /* (3) DO 
10,000 ITERATIONS */
                     do m= 1 to 15000   until (flag=0) ;          /* (4) generate up to 
15000 sample or until T > h */
                  if m=1 then flag=1;
              call RANDGEN(X,'NORMAL');                      /* fill x1 values from 
normal distribution default parameters */
              X_MEAN=X[:,];                                         /* compute the 
column means of X matrix */

                     if m=1 then Z=X_BAR_BAR_EST;                /* usew instead of 
Z={0 0} */
                     T_X_MEAN=t(X_MEAN);                                   /* 
compute the transpose of the  X_MEANS matrix */
                     INV_COV_EST=inv (COV_EST);
                     T2 =  ( X_MEAN * SOLVE(COV_EST,T_X_MEAN) ); /* compute the MEWMA 
charting statistic T**2 works as below*/
                     *T2 =   X_MEAN * INV_COV_EST * T_X_MEAN ;    /* compute the MEWMA 
charting statistic T**2 works as abiove^*/


                            if (T2 > h & flag=1) then  do;              /* start T > h 
condition test loop   */
                            RL=m     ;                                     /* set 
RL = number of subgroups or samples */
                                   R = p||K||h||T2||RL ;                     /* 
Concatinate the X_MEAN, Z, & T matrices into one matrix R */
                                   RESULT[k,]=R;
                     flag=0;
                     end;           /* >> close T > h condition test loop << */
              end;                         /* (4) close/end  (m) subgroup "do" loop */
       end;                                /* (3) close/end 10K iteration loop */
       append from RESULT;    /*     Append 10000 iteration for eack lambda and UCL into 
SAS data set */
  end;                                     /* (2) close/end UCL loop */
                                                  /* (1) close/end lambda loop */


title 'Hotellings T^2 UCL PHASE I Simulation - Standard Normal, p=%eval(&variates)';
proc means data=out.HOT2_PhaseIUCL_NORMAL_P%eval(&variates)  ;
proc means data=out.HOT2_PhaseIUCL_NORMAL_p2  ;

class  H;
var RL;
output out=averages mean=ARL std=SDRL;
run;

data averages ;
 keep Lambda UCL ARL sdrl _FREQ_;
 set averages ;
 where _type_=3;
  UCL=h;
run;
```

```
proc print data=averages    ;
run;

%mend Hotellings;
%hotellings(2);
%hotellings(3);
%hotellings(4);
%hotellings(5);


/*-------------------------------------------------------------------*/
/*-------------------------------------------------------------------*/
/* >>>>>>>>>  HOTELLING'S Phase II UCL Determination    <<<<<<<<<<<< */
/*-------------------------------------------------------------------*/
/*-- A doctoral dissertation supplemental SAS code by Jamil Zeinab --*/
/*-------------------------------------------------------------------*/
/*-------------------------------------------------------------------*/
/* 1. The IML code generates 10,000 iterations of                    */
/*    of up to 15,000 p-variate samples from a                       */
/*    sampling distribution and computes Hotelling's chi square      */
/* 2. It creates a Hotelling's chi square chating statistic T^2      */
/* 3. It calculates the out of control ARL for each combination of   */
/*    variates, shift, and UCL obtaioned from Phase I for ARL0=200,  */
/*    500, & 1000.                                                   */
/* 4. Several versions of this code will exist based on the sampling */
/*    criteria below (distribution, p-vartiates).                    */
/*-------------------------------------------------------------------*/
/*                  >>>> Sampling criteria <<<<<                     */
/* Distribution: Multivariate Normal (Normal Marginals)              */
/* Number of qualirty variables: p=2, n=5                            */
/* Estimated Covariance matrix & mean vector, from 100,000 samples   */
/*-------------------------------------------------------------------*/

/*-------------------------------------------------------------------*/
/*  >>>>>>>>>>>>>>>>>>>    Notes for usage <<<<<<<<<<<<<<<<<<              */
/* 1. Change the sampling distribution as needed, I used Normal, T,  */
/*    & Gamma.                                                       */
/* 2. Change the sampling size as desired.                           */
/* 3. Change the smoothing parameter "lambda" as desired.            */
/*-------------------------------------------------------------------*/


 DM 'LOG; CLEAR';
options   mlogic mprint SYMBOLGEN FULLSTIMER  nonumber THREADS CPUCOUNT=ACTUAL ;

/** -- to avoid the problem of filling SAS Log Window
in display save the SAS log and listing to files */
proc printto  new
 log  =  'C:\SIMULATION_N5\MEWMA\MEWMA_PHASE_II_UCL_P2_N5\MEWMAP2N_P2N5_Xn.txt'  ;
 print = "C:\SIMULATION_N5\MEWMA\MEWMA_PHASE_II_UCL_P2_N5\Temp_LIST.txt";

RUN;
/*-----------------------------------------------------------------------*/
/** Clear old SAS data sets with the same name                           */
/*-----------------------------------------------------------------------*/

 libname out 'f:\ASRM2\SRM799_SIMULATION_WORK\PHASE_II\HOTELLINGS';


%macro Hotphase2(arl0200,ucl1,arl0500,ucl2,arl01000,ucl3,variates);
proc iml;

title " Hotelling's UCL PHASE II Simulation - MV t(df=5) for: (P=&variates)";

/*-----------------------------------------------------------------------*/
/** START Estimate the covariance matrix and mean from 100,000 samples   */
/*-----------------------------------------------------------------------*/

       NumSim=10000 ;
       n=5;
```

```
        p=%eval(&variates);
        var_x1=0;
        RESULT_MEAN = j(NumSim,p,.);
        RESULT_COV = j(p,p,0);
        call randseed(12345);                          /* set seed for RANDGEN */
        XX =  j(n,p,.);                                        /* initialize X matrix with n
rows and p columns (n x p)    */
        do iii=1 to NumSim;
                call RANDGEN(XX,'T',5);       /* fill X values from normal distribution
default parameters */
        X_MEAN=XX[:,];                                 /* compute the column means of X
matrix */
                RESULT_MEAN[iii,]=X_MEAN;              /* save as the ith row of means
matrix */
                COV_X=cov (XX);                                /* compute the covariance
matrix of X matrix */
                RESULT_COV = RESULT_COV + COV_X;
        end;
        X_BAR_BAR_EST=RESULT_MEAN[:,];                 /* calculate the samples means */
        COV_EST = RESULT_COV / (NumSim *n);
        var_x1=COV_EST[1,1];
        print   result_cov COV_EST;   /* This is the est. cov. matrix of X samples */

        print   X_BAR_BAR_EST var_x1;/*The est. mean vextor matrix of X samples, use as
initial Z value */


/*---------------------------------------------------------------------------*/
/*   END Estimatation of the covariance matrix and mean from 100,000 samples   */
/*---------------------------------------------------------------------------*/

         varNames = {"P" "ARL0" "Delta" "H" "T2" "RL"};
         /* varNames = {"K" "h" "X1_BAR" "X2_BAR" "Z1" "Z2" "T" "RL"}; */
         create out.HHOT2_PII_ARL_Tdf5_P%eval(&variates) var varNames;
         SIMS=10000 ;
         Z =  j(1,p,.);        /* initialize Z matrix with 1 row and p columns (1 x 2)*/
     X =  j(n,p,.);    /* initialize X matrix with n rows and p columns (5 x 2)*/
         X1=  j(n,1,.);
         X2=  j(n,1,.);
     RL=.      ;
         RESULT=j(SIMS,6,.);   /* define the Result matrix */
     *call randseed(12345);   /* set seed for RANDGEN */

/*----------------------------------------------------------------------------*/
/*Start Hotellling's T^2 To Generate Phase II ARL values for ARL0=200,500, &1000 */
/*----------------------------------------------------------------------------*/

do l= 1 to 3;                 /* (1) for each ARL0 and UCL */
        IF l=1 THEN ARL0= %eval(&arl0200) ;
        IF l=2 THEN ARL0= %eval(&arl0500) ;
        IF l=3 THEN ARL0= %eval(&arl01000);

        if l=1  then H=&ucl1 ;
        if l=2  then H=&ucl2 ;
        if l=3  then H=&ucl3 ;

 do ss= 1 to 5;                                                /* (2) Test shift in
first variable  */
        if ss=1 then Delta= 0.25 * sqrt(var_x1);
        if ss=2 then Delta= 0.50 * sqrt(var_x1);
        if ss=3 then Delta= 1.00 * sqrt(var_x1);
        if ss=4 then Delta= 1.50 * sqrt(var_x1);
        if ss=5 then Delta= 2.50 * sqrt(var_x1);

   do k = 1 to SIMS   ;                                        /* (3) DO 10,000
ITERATIONS */
                do m= 1 to 15000   until (flag=0) ; /* (4) generate up to 15000 sample or
until T > h */
            if m=1 then flag=1;
        call RANDGEN(X1,'T',5); /* fill x1 values from normal dist. default parameters */
```

```
       call RANDGEN(X2,'T',5); /* fill x1 values from normal dist. default parameters */
               X1D=X1+delta;
               X[,1]=X1D;
               X[,2]=X2;

               X_MEAN=X[:,];                                /* compute the column means
of X matrix */
               X_MINUS=(X_MEAN) - (X_BAR_BAR_EST) ;
               T_X_MINUS=t (X_MINUS);
               T_X_MEAN=t(X_MEAN);                          /* compute the transpose of
the  X_MEANS matrix */
               INV_COV_EST=inv (COV_EST);
               *T2 =  ( X_MEAN * SOLVE(COV_EST,T_X_MEAN)            T2 =   X_MINUS *
INV_COV_EST * T_X_MINUS

                      if (T2 > h & flag=1) then  do;               /* start T > h
condition test loop   */
                      RL=m     ;                                   /* set RL = number of
subgroups or samples */
                              R = p||ARL0||delta||h||T2||RL ;
       RESULT[k,]=R;
                      flag=0;
               end;                  /* (5) >> close T > h condition test loop << */
   end;                                      /* (4) close/end  (m) subgroup "do" loop */
  end;                                       /* (3) close/end 10K "k" iteration loop */
       append from RESULT;           /*      Append 10000 iteration for eack lambda and
UCL into SAS data set */
 end;                                        /* (2) close/end "ss" or shift loop */
end;                                                 /* (1) close/end "i" UCL loop */



quit;

/****** Results Analysis **********/
  title " Hotelling's UCL PHASE II Simulation - t(df=5) for: (P=&variates)";


proc means data=out.HHOT2_PII_ARL_Tdf5_P%eval(&variates)  ;
class p ARL0 h Delta;
var RL;
output out=out.HHOT2_PII_ARL_Tdf5_P%eval(&variates)_AVG mean=ARL std=SDRL;
run;


proc print data=out.HHOT2_PII_ARL_Tdf5_P%eval(&variates)_AVG;
run;


data out.HHOT2_PII_ARL_Tdf5_P%eval(&variates)_AVG2 ;
 keep UCL Delta ARL sdrl _FREQ_;
 set out.HHOT2_PII_ARL_Tdf5_P%eval(&variates)_AVG ;
   UCL=h;
   where _type_=15;
run;

proc print data=out.HHOT2_PII_ARL_Tdf5_P%eval(&variates)_AVG2    ;
run;

 %mend;

 %Hotphase2(200,13.5,500,17.6,1000,21.5,2);
 %Hotphase2(200,16.6,500,21.1,1000,25.6,3);
 %Hotphase2(200,19.2,500,24.3,1000,29.2,4);
 %Hotphase2(200,21.6,500,27.0,1000,32.4,5);
```

APPENDIX B

A SIGNED-RANK NUMERICAL EXAMPLE

A Signed-Rank Numerical Example

The computations of the invariant signed-ranks are highly intensive and a stochastic algorithm is used to calculate the signed-rank estimates by sampling observation hyperplanes (Oja, 1983).

Let $\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}$ be a random sample from a continuous *3*-variate distribution. $\boldsymbol{X}$ is defined in Equation 124 as a (*3 x 3*) matrix of three variables and three observations. Hence, *k = 3* and *n = 3*, and

$$X = \begin{bmatrix} 6 & -10 & 12 \\ -7 & 13 & -11 \\ 5 & 7 & 15 \end{bmatrix}. \tag{1}$$

$$P = \{p = (i_1, i_2, i_3) \colon i_1 < i_2 < i_3 \leq n\} \tag{2}$$

be the set of $N_p = \binom{n}{k}$ different k-tuples of index set {1,2,3}. In this example, there is only one set of $N = \binom{3}{3} = \frac{3!}{3!} = 1$. Therefore, the set $P = \{p = (1,2,3)\}$, and the index $p \in P$ refer to a k-subset of the original observations. Recall the multivariate sign Equation 121 defined below as Equation 3.

$$\boldsymbol{Q}_p^+(\boldsymbol{x}) = 2^{-k} \sum_{a \in A} \boldsymbol{S_{pa}}(\boldsymbol{x}) \boldsymbol{d}_{pa} \, . \tag{3}$$

Since k=3,

$$\boldsymbol{Q}_p^+(\boldsymbol{X}) = \frac{1}{8} \sum_{a \in A} \boldsymbol{S_{pa}}(\boldsymbol{x}) \boldsymbol{d}_{pa}. \tag{4}$$

Also, the signed-rank function from Equation 123 for $N_P^{-1} = 1$is

$$\boldsymbol{R}_n^+(\boldsymbol{x}) = \sum_{p \in P} \boldsymbol{Q}_p^+(\boldsymbol{x}). \tag{5}$$

Substituting Equation 126 into Equation 127, we get the empirical signed-rank function

$$\boldsymbol{R}_n^+(\boldsymbol{x}) = \frac{1}{8}\sum_{\boldsymbol{a}\in\boldsymbol{A}} \boldsymbol{S}_{\boldsymbol{pa}}(\boldsymbol{x})\boldsymbol{d}_{\boldsymbol{pa}}. \tag{6}$$

The multivariate sign is an average of all possible vector set *A*, such that *A* is the set of 2^k possible vectors$(\pm 1\ ,\quad \pm 1,\quad \pm 1)$. Since k = 3, we have a set *A* with 8 possible vectors.

$$\begin{aligned} A = \{a_0 &= [-1 \quad -1 \quad -1],\\ a_1 &= [1 \quad -1 \quad -1],\\ a_2 &= [-1 \quad 1 \quad -1],\\ a_3 &= [1 \quad 1 \quad -1],\\ a_4 &= [-1 \quad -1 \quad 1],\\ a_5 &= [1 \quad -1 \quad 1],\\ a_6 &= [-1 \quad 1 \quad 1],\\ a_7 &= [1 \quad 1 \quad 1]\} \end{aligned} \tag{7}$$

Let

$$\boldsymbol{X}` = \begin{bmatrix} 6 & -7 & 5 \\ -10 & 13 & 7 \\ 12 & -11 & 15 \end{bmatrix}. \tag{8}$$

1. The signed-rank vector $\boldsymbol{R}_n^+(\boldsymbol{x})$ for $\boldsymbol{x}`_{\boldsymbol{1}}$

We start by calculating the sign vector $\boldsymbol{Q}_1^+(\boldsymbol{x})$ and signed-rank vector $\boldsymbol{R}_n^+(\boldsymbol{x})$ for $\boldsymbol{x}`_{\boldsymbol{1}} = \begin{bmatrix} 6 \\ -10 \\ 12 \end{bmatrix}$. We will calculate the sign vector $\boldsymbol{Q}_1^+(\boldsymbol{x})$ and signed-rank vector $\boldsymbol{R}_n^+(\boldsymbol{x})$ for the other vector components, $\boldsymbol{x}`_{\boldsymbol{2}} = \begin{bmatrix} -7 \\ 13 \\ -11 \end{bmatrix}$, and $\boldsymbol{x}`_{\boldsymbol{3}} = \begin{bmatrix} 5 \\ 7 \\ 15 \end{bmatrix}$ in the same manner.

Calculate all vectors $a_t p : t = 0,1,2,3,4,5,6,7$ and check if $i = 1 \in \boldsymbol{a_i p}$. If $i = 1 \in \boldsymbol{a_i p}$, then $\boldsymbol{S_{pa}}(\boldsymbol{x})\boldsymbol{d}_{pa} = 0$, so that

$$\boldsymbol{a_0 p} = [-1 \quad -1 \quad -1]\#[1 \quad 2 \quad 3] = [-1 \quad -2 \quad -3],$$

$$\boldsymbol{a_1 p} = [1 \quad -1 \quad -1]\#[1 \quad 2 \quad 3] = [\mathbf{1} \quad -2 \quad -3],$$

$$\boldsymbol{a_2 p} = [-1 \quad 1 \quad -1]\#[1 \quad 2 \quad 3] = [-1 \quad 2 \quad -3],$$

$$\boldsymbol{a_3 p} = [1 \quad 1 \quad -1]\#[1 \quad 2 \quad 3] = [\mathbf{1} \quad 2 \quad -3],$$

$$\mathbf{a_4}\boldsymbol{p} = [-1 \quad -1 \quad 1]\#[1 \quad 2 \quad 3] = [-1 \quad -2 \quad 3],$$

$$\boldsymbol{a_5 p} = [1 \quad -1 \quad 1]\#[1 \quad 2 \quad 3] = [\mathbf{1} \quad -2 \quad 3],$$

$$\boldsymbol{a_6 p} = [-1 \quad 1 \quad 1]\#[1 \quad 2 \quad 3] = [-1 \quad 2 \quad 3],$$

$$\boldsymbol{a_7 p} = [1 \quad 1 \quad 1]\#[1 \quad 2 \quad 3] = [\mathbf{1} \quad 2 \quad 3],$$

Note that $i = 1 \in \boldsymbol{a_i p}$ for $i = 1, 3, 5, \& \ 7$. Therefore, $\boldsymbol{Q}_p^+(\boldsymbol{x}) = \frac{1}{8}\sum_{a \in A} \boldsymbol{S_{pa}}(\boldsymbol{x})\boldsymbol{d}_{pa}$ is determined by vectors $\boldsymbol{a_0}, \boldsymbol{a_2}, \boldsymbol{a_4}, \& \ \boldsymbol{a_6}$ only. We now calculate the component $\boldsymbol{S_{pa}}(\boldsymbol{x})\boldsymbol{d}_{pa}$ for $\boldsymbol{a_0}, \boldsymbol{a_2}, \boldsymbol{a_4}, \& \ \boldsymbol{a_6}$ only.

Define

$$\boldsymbol{Y_i = a_i \# X`},$$

$$\boldsymbol{W_1} = (\boldsymbol{Y_i} = [\, \boldsymbol{y_2} \quad \boldsymbol{y_3}]) - [\, \boldsymbol{x_i} \quad \boldsymbol{x_i}] : \boldsymbol{i = 1, 2, \text{or } 3},$$

$$\boldsymbol{W_2 = Y_0} - [\boldsymbol{x_i} \quad \boldsymbol{x_i} \quad \boldsymbol{x_i}] : \boldsymbol{i = 1, 2, \text{or } 3}, \tag{9}$$

$$\boldsymbol{d_{pa}} = \begin{bmatrix} (-1)^1 \left| \boldsymbol{W_1} = \begin{bmatrix} \boldsymbol{w_2} \\ \boldsymbol{w_3} \end{bmatrix} \right| \\ (-1)^2 \left| \boldsymbol{W_1} = \begin{bmatrix} \boldsymbol{w_1} \\ \boldsymbol{w_3} \end{bmatrix} \right| \\ (-1)^3 \left| \boldsymbol{W_1} = \begin{bmatrix} \boldsymbol{w_1} \\ \boldsymbol{w_2} \end{bmatrix} \right| \end{bmatrix}, or$$

$$d_{pa} = \begin{bmatrix} -1 \left| W_1 = \begin{bmatrix} w_2 \\ w_3 \end{bmatrix} \right| \\ 1 \left| W_1 = \begin{bmatrix} w_1 \\ w_3 \end{bmatrix} \right| \\ -1 \left| W_1 = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right| \end{bmatrix}, \text{ and}$$

$$S_{pa}d_{pa} = sign(|W_2|d_{pa}).$$

We now calculate $W_1, W_2, d_{p0},$ and $S_{p0}d_{p0}$ as follows

$$Y_0 = [-1 \quad -1 \quad -1]\# \begin{bmatrix} 6 & -7 & 5 \\ -10 & 13 & 7 \\ 12 & -11 & 15 \end{bmatrix} = \begin{bmatrix} -6 & 7 & -5 \\ 10 & -13 & -7 \\ -12 & 11 & -15 \end{bmatrix},$$

we have

$$W_1 = (Y_0 = [\, y_2 \quad y_3]) - [\, x_i \quad x_i],$$

$$W_1 = \begin{bmatrix} 7 & -5 \\ -13 & -7 \\ 11 & -15 \end{bmatrix} - \begin{bmatrix} -6 & -6 \\ 10 & 10 \\ -12 & -12 \end{bmatrix} = \begin{bmatrix} 13 & 1 \\ -23 & -17 \\ 23 & -3 \end{bmatrix},$$

$$W_2 = Y_0 - [x_1 \quad x_1 \quad x_1],$$

$$W_2 = \begin{bmatrix} -6 & 7 & -5 \\ 10 & -13 & -7 \\ -12 & 11 & -15 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ -10 & -10 & -10 \\ 12 & 12 & 12 \end{bmatrix} = \begin{bmatrix} -12 & 1 & -11 \\ 20 & -3 & 3 \\ -24 & -1 & -27 \end{bmatrix},$$

$$\boldsymbol{d_{p0}} = \begin{bmatrix} -1\begin{vmatrix} -23 & -17 \\ 23 & -3 \end{vmatrix} \\ 1\begin{vmatrix} 13 & 1 \\ 23 & -3 \end{vmatrix} \\ -1\begin{vmatrix} 13 & 1 \\ -23 & -17 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -1(460) \\ 1(-62) \\ -1(-198) \end{bmatrix} = \begin{bmatrix} -460 \\ -62 \\ 198 \end{bmatrix},$$

and the sign vector is

$$\boldsymbol{S_{p0}d_{p0}} = sign(|\boldsymbol{W_2}|\boldsymbol{d_{p0}}) = sign\left(\left\|\begin{bmatrix} -12 & 1 & -11 \\ 20 & -3 & 3 \\ -24 & -1 & 27 \end{bmatrix}\right\| * \begin{bmatrix} -460 \\ -62 \\ 198 \end{bmatrix}\right) = \begin{bmatrix} -460 \\ -62 \\ 198 \end{bmatrix}.$$

Using the same formulas in equation (3.18), we calculate

$\boldsymbol{W_1}, \boldsymbol{W_2}, \boldsymbol{d_{p2}},$ and $\boldsymbol{S_{p2}d_{p2}}$ for $\mathbf{Y_2}, \boldsymbol{Y}_4,$ and $\boldsymbol{Y_6}$ as

$$\boldsymbol{Y_2} = [-1 \quad 1 \quad -1]\# \begin{bmatrix} 6 & -7 & 5 \\ -10 & 13 & 7 \\ 12 & -11 & 15 \end{bmatrix} = \begin{bmatrix} -6 & -7 & -5 \\ 10 & 13 & -7 \\ -12 & -11 & -15 \end{bmatrix},$$

$$\boldsymbol{W_1} = (\boldsymbol{Y_2} = [\,\boldsymbol{y_2} \quad \boldsymbol{y_3}]) - [\,\boldsymbol{x_i} \quad \boldsymbol{x_i}],$$

$$\boldsymbol{W_1} = \begin{bmatrix} -7 & -5 \\ 13 & -7 \\ -11 & -15 \end{bmatrix} - \begin{bmatrix} -6 & -6 \\ 10 & 10 \\ -12 & -12 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3 & -17 \\ 1 & -3 \end{bmatrix},$$

$$\boldsymbol{W_2} = \boldsymbol{Y_2} - [\boldsymbol{x_1} \quad \boldsymbol{x_1} \quad \boldsymbol{x_1}]$$

$$\boldsymbol{W_2} = \begin{bmatrix} -6 & -7 & -5 \\ 10 & 13 & -7 \\ -12 & -11 & -15 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ -10 & -10 & -10 \\ 12 & 12 & 12 \end{bmatrix} = \begin{bmatrix} -12 & -13 & -11 \\ 20 & 23 & 3 \\ -24 & -23 & -27 \end{bmatrix},$$

$$\boldsymbol{d_{p2}} = \begin{bmatrix} -1\begin{vmatrix} 3 & -17 \\ 1 & -3 \end{vmatrix} \\ 1\begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} \\ -1\begin{vmatrix} -1 & 1 \\ 3 & -17 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -1(8) \\ 1(2) \\ -1(14) \end{bmatrix} = \begin{bmatrix} -8 \\ 2 \\ -14 \end{bmatrix},$$

and the sign vector is

$$\boldsymbol{S_{p2}d_{p2}} = sign(|\boldsymbol{W_2}|\boldsymbol{d_{p2}}) = sign\left(\left|\begin{bmatrix} -12 & -13 & -11 \\ 20 & 23 & 3 \\ -24 & -23 & -27 \end{bmatrix}\right| * \begin{bmatrix} -8 \\ 2 \\ -14 \end{bmatrix}\right) = \begin{bmatrix} 8 \\ -2 \\ 14 \end{bmatrix}.$$

$$\boldsymbol{Y_4} = [-1 \quad -1 \quad 1] \# \begin{bmatrix} 6 & -7 & 5 \\ -10 & 13 & 7 \\ 12 & -11 & 15 \end{bmatrix} = \begin{bmatrix} -6 & 7 & 5 \\ 10 & -13 & 7 \\ -12 & 11 & 15 \end{bmatrix},$$

$$\boldsymbol{W_1} = (\boldsymbol{Y_4} = [\,\boldsymbol{y_2} \quad \boldsymbol{y_3}]) - [\,\boldsymbol{x_i} \quad \boldsymbol{x_i}],$$

$$\boldsymbol{W_1} = \begin{bmatrix} 7 & -5 \\ -13 & -7 \\ 11 & -15 \end{bmatrix} - \begin{bmatrix} -6 & -6 \\ 10 & 10 \\ -12 & -12 \end{bmatrix} = \begin{bmatrix} 13 & 11 \\ -23 & -3 \\ 23 & 27 \end{bmatrix},$$

$$\boldsymbol{W_2} = \boldsymbol{Y_4} - [\boldsymbol{x_1} \quad \boldsymbol{x_1} \quad \boldsymbol{x_1}],$$

$$\boldsymbol{W_2} = \begin{bmatrix} -6 & 7 & 5 \\ 10 & -13 & 7 \\ -12 & 11 & 15 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ -10 & -10 & -10 \\ 12 & 12 & 12 \end{bmatrix} = \begin{bmatrix} -12 & 1 & -1 \\ 20 & -3 & 17 \\ -24 & -1 & 3 \end{bmatrix},$$

$$\boldsymbol{d_{p4}} = \begin{bmatrix} -1\begin{vmatrix} -23 & -3 \\ 23 & 27 \end{vmatrix} \\ 1\begin{vmatrix} 13 & 11 \\ 23 & 27 \end{vmatrix} \\ -1\begin{vmatrix} 13 & 11 \\ -23 & -3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -1(-552) \\ 1(98) \\ -1(214) \end{bmatrix} = \begin{bmatrix} 552 \\ 98 \\ -214 \end{bmatrix},$$

and the sign vector is

$$\boldsymbol{S_{p4}d_{p4}} = sign(|\boldsymbol{W_2}|\boldsymbol{d_{p4}}) = sign\left(\left|\begin{bmatrix} -12 & 1 & -1 \\ 20 & -3 & 17 \\ -24 & -1 & 3 \end{bmatrix}\right| * \begin{bmatrix} 552 \\ 98 \\ -214 \end{bmatrix}\right) = \begin{bmatrix} -552 \\ -98 \\ 214 \end{bmatrix}$$

$$\boldsymbol{FOR}: \boldsymbol{Y_6} = [-1 \quad 1 \quad 1]\# \begin{bmatrix} 6 & -7 & 5 \\ -10 & 13 & 7 \\ 12 & -11 & 15 \end{bmatrix} = \begin{bmatrix} -6 & -7 & 5 \\ 10 & 13 & 7 \\ -12 & -11 & 15 \end{bmatrix},$$

$$\boldsymbol{W_1} = (\boldsymbol{Y_6} = [\,\boldsymbol{y_2} \quad \boldsymbol{y_3}]) - [\,\boldsymbol{x_i} \quad \boldsymbol{x_i}],$$

$$\boldsymbol{W_1} = \begin{bmatrix} -7 & -5 \\ 13 & -7 \\ -11 & -15 \end{bmatrix} - \begin{bmatrix} -6 & -6 \\ 10 & 10 \\ -12 & -12 \end{bmatrix} = \begin{bmatrix} -1 & 11 \\ 3 & -3 \\ 1 & 27 \end{bmatrix},$$

$$\boldsymbol{W_2} = \boldsymbol{Y_6} - [\boldsymbol{x_1} \quad \boldsymbol{x_1} \quad \boldsymbol{x_1}],$$

$$\boldsymbol{W_2} = \begin{bmatrix} -6 & -7 & 5 \\ 10 & 13 & 7 \\ -12 & -11 & 15 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ -10 & -10 & -10 \\ 12 & 12 & 12 \end{bmatrix} = \begin{bmatrix} -12 & -13 & -1 \\ 20 & 23 & 17 \\ -24 & -23 & 3 \end{bmatrix},$$

$$\boldsymbol{d_{p6}} = \begin{bmatrix} -1\begin{vmatrix} 3 & -3 \\ 1 & 27 \end{vmatrix} \\ 1\begin{vmatrix} -1 & 11 \\ 1 & 27 \end{vmatrix} \\ -1\begin{vmatrix} -1 & 11 \\ 3 & -3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -1(84) \\ 1(-38) \\ -1(-30) \end{bmatrix} = \begin{bmatrix} -84 \\ -38 \\ 30 \end{bmatrix},$$

and the sign vector is

$$\boldsymbol{S_{p6}d_{p6}} = sign(|\boldsymbol{W_2}|\boldsymbol{d_{p6}}) = sign\left(\left|\begin{bmatrix} -12 & -13 & -1 \\ 20 & 23 & 17 \\ -24 & -23 & 3 \end{bmatrix}\right| * \begin{bmatrix} -84 \\ -38 \\ 30 \end{bmatrix}\right) = \begin{bmatrix} -84 \\ -38 \\ 30 \end{bmatrix}.$$

Applying Equations 128 through 132, the signed-rank vector for the original $\boldsymbol{X`_1} = \begin{bmatrix} 6 \\ -10 \\ 12 \end{bmatrix}$ is

$$\boldsymbol{R}^{+}(\boldsymbol{x}`_{\mathbf{1}}) = \sum_{p\in P} \boldsymbol{Q}_{1}^{+}(\boldsymbol{x}) = \frac{1}{8}\sum_{a\in A} S_{pa}(\boldsymbol{x})\boldsymbol{d}_{pa}$$

$$\boldsymbol{R}^{+}(\boldsymbol{x}`_{\mathbf{1}}) = \frac{1}{8}\left(\boldsymbol{S}_{\boldsymbol{p0}}\boldsymbol{d}_{\boldsymbol{p0}} + \boldsymbol{S}_{\boldsymbol{p2}}\boldsymbol{d}_{\boldsymbol{p2}} + \boldsymbol{S}_{\boldsymbol{p4}}\boldsymbol{d}_{\boldsymbol{p4}} + \boldsymbol{S}_{\boldsymbol{p6}}\boldsymbol{d}_{\boldsymbol{p6}}\right)$$

$$\boldsymbol{R}^{+}(\boldsymbol{x}`_{\mathbf{1}}) = \frac{1}{8}\left(\begin{bmatrix}-460\\-62\\198\end{bmatrix} + \begin{bmatrix}8\\-2\\14\end{bmatrix} + \begin{bmatrix}-552\\-98\\214\end{bmatrix} + \begin{bmatrix}-84\\-38\\30\end{bmatrix}\right) = \begin{bmatrix}-136\\-25\\57\end{bmatrix}.$$

2. The signed-rank vector $\boldsymbol{R}_{n}^{+}(\boldsymbol{x})$ for $\boldsymbol{x}`_{\mathbf{2}}$

In the same manner, we calculate the sign vector $\boldsymbol{Q}_{2}^{+}(\boldsymbol{x})$ and signed-rank vector

$\boldsymbol{R}_{n}^{+}(\boldsymbol{x})$ for $\boldsymbol{x}`_{\mathbf{2}} = \begin{bmatrix}-7\\13\\-11\end{bmatrix}$.

Calculate all vectors $a_t p : t = 0,1,2,3,4,5,6,7$ and check if $i = 2 \in a_i p$. If $i = 1 \in \boldsymbol{a_i p}$, then $\boldsymbol{S_{pa}}(\boldsymbol{x})\boldsymbol{d}_{pa} = 0$, so that

$$\boldsymbol{a_0 p} = [-1 \quad -1 \quad -1]\#[1 \quad 2 \quad 3] = [-1 \quad -2 \quad -3],$$

$$\boldsymbol{a_1 p} = [1 \quad -1 \quad -1]\#[1 \quad 2 \quad 3] = [1 \quad -2 \quad -3],$$

$$\boldsymbol{a_2 p} = [-1 \quad 1 \quad -1]\#[1 \quad 2 \quad 3] = [-1 \quad \mathbf{2} \quad -3],$$

$$\boldsymbol{a_3 p} = [1 \quad 1 \quad -1]\#[1 \quad 2 \quad 3] = [1 \quad \mathbf{2} \quad -3],$$

$$\boldsymbol{a_4 p} = [-1 \quad -1 \quad 1]\#[1 \quad 2 \quad 3] = [-1 \quad -2 \quad 3],$$

$$\boldsymbol{a_5 p} = [1 \quad -1 \quad 1]\#[1 \quad 2 \quad 3] = [1 \quad -2 \quad 3],$$

$$\boldsymbol{a_6 p} = [-1 \quad 1 \quad 1]\#[1 \quad 2 \quad 3] = [-1 \quad \mathbf{2} \quad 3],$$

$$\boldsymbol{a_7 p} = [1 \quad 1 \quad 1]\#[1 \quad 2 \quad 3] = [1 \quad \mathbf{2} \quad 3],$$

Note that $i = 2 \in \boldsymbol{a_i p}$ for $i = 2, 3, 6, \& \ 7$. Therefore, $\boldsymbol{Q}_{p}^{+}(\boldsymbol{x}) = \frac{1}{8}\sum_{a\in A} \boldsymbol{S_{pa}}(\boldsymbol{x})\boldsymbol{d}_{pa}$ is determined by vectors $\boldsymbol{a_0}, \boldsymbol{a_1}, \boldsymbol{a_4}, \& \ \boldsymbol{a_5}$ only. We now calculate the component $\boldsymbol{S_{pa}}(\boldsymbol{x})\boldsymbol{d}_{pa}$ for $\boldsymbol{a_0}, \boldsymbol{a_1}, \boldsymbol{a_4}, \& \ \boldsymbol{a_5}$ only.

Again, using the same formulas in Equation 132, we calculate

$\boldsymbol{W_1}, \boldsymbol{W_2}, \boldsymbol{d_{p2}}$, and $\boldsymbol{S_{p2}}\boldsymbol{d_{p2}}$ for $\mathbf{Y_0}, \boldsymbol{Y}_1, \boldsymbol{Y}_4$and $\boldsymbol{Y_5}$:

$$\boldsymbol{Y_0} = [-1 \quad -1 \quad -1]\# \begin{bmatrix} 6 & -7 & 5 \\ -10 & 13 & 7 \\ 12 & -11 & 15 \end{bmatrix} = \begin{bmatrix} -6 & 7 & -5 \\ 10 & -13 & -7 \\ -12 & 11 & -15 \end{bmatrix},$$

$$\boldsymbol{S_{p0}}\boldsymbol{d_{p0}} = \begin{bmatrix} -460 \\ -62 \\ 198 \end{bmatrix},$$

$$\boldsymbol{F}\,\boldsymbol{Y_1} = [1 \quad -1 \quad -1]\# \begin{bmatrix} 6 & -7 & 5 \\ -10 & 13 & 7 \\ 12 & -11 & 15 \end{bmatrix} = \begin{bmatrix} 6 & 7 & -5 \\ -10 & -13 & -7 \\ 12 & 11 & -15 \end{bmatrix},$$

$$\boldsymbol{S_{p1}}\boldsymbol{d_{p1}} = \begin{bmatrix} 84 \\ 38 \\ -30 \end{bmatrix},$$

$$\boldsymbol{Y_4} = [-1 \quad -1 \quad 1]\# \begin{bmatrix} 6 & -7 & 5 \\ -10 & 13 & 7 \\ 12 & -11 & 15 \end{bmatrix} = \begin{bmatrix} -6 & 7 & 5 \\ 10 & -13 & 7 \\ -12 & 11 & 15 \end{bmatrix},$$

$$\boldsymbol{S_{p4}}\boldsymbol{d_{p4}} = \begin{bmatrix} -552 \\ -98 \\ 214 \end{bmatrix},$$

$$\boldsymbol{Y_5} = [1 \quad -1 \quad 1]\# \begin{bmatrix} 6 & -7 & 5 \\ -10 & 13 & 7 \\ 12 & -11 & 15 \end{bmatrix} = \begin{bmatrix} 6 & 7 & 5 \\ -10 & -13 & 7 \\ 12 & 11 & 15 \end{bmatrix}$$

$$\boldsymbol{S_{p5}}\boldsymbol{d_{p5}} = \begin{bmatrix} -8 \\ 2 \\ -14 \end{bmatrix}.$$

Applying Equations 128 through 132, the signed-rank vector for the original $\boldsymbol{X`_2} = \begin{bmatrix} -7 \\ 13 \\ -11 \end{bmatrix}$ is

$$\boldsymbol{R}^{+}(\boldsymbol{x`_2}) = \sum_{p \in P} \boldsymbol{Q}_2^{+}(\boldsymbol{x}) = \frac{1}{8} \sum_{a \in A} \boldsymbol{S_{pa}}(\boldsymbol{x})\boldsymbol{d}_{pa}$$

$$\boldsymbol{R}^{+}(\boldsymbol{X`_2}) = \frac{1}{8}\left(\boldsymbol{S_{p0}}\boldsymbol{d_{p0}} + \boldsymbol{S_{p1}}\boldsymbol{d_{p1}} + \boldsymbol{S_{p4}}\boldsymbol{d_{p4}} + \boldsymbol{S_{p5}}\boldsymbol{d_{P5}}\right)$$

$$\boldsymbol{R}^{+}(\boldsymbol{X}`_{\boldsymbol{2}}) = \frac{1}{8}\left(\begin{bmatrix}-460\\-62\\198\end{bmatrix} + \begin{bmatrix}84\\38\\-30\end{bmatrix} + \begin{bmatrix}-552\\-98\\214\end{bmatrix} + \begin{bmatrix}-8\\2\\-14\end{bmatrix}\right) = \begin{bmatrix}-117\\-15\\46\end{bmatrix}$$

3. The signed-rank vector $\boldsymbol{R}_n^{+}(\boldsymbol{x})$ for $\boldsymbol{x}`_{\boldsymbol{3}}$

And finally, in the same manner, we calculate the sign vector $\boldsymbol{Q}_3^{+}(\boldsymbol{x})$ and signed-rank vector $\boldsymbol{R}_n^{+}(\boldsymbol{x})$ for $\boldsymbol{x}`_{\boldsymbol{3}} = \begin{bmatrix}5\\7\\15\end{bmatrix}$.

Calculate all vectors $\boldsymbol{a_t p} : t = 0,1,2,3,4,5,6,7$ and check if $i = 2 \in a_i p$. If $i = 1 \in \boldsymbol{a_i p}$, then $\boldsymbol{S_{pa}}(\boldsymbol{x})\boldsymbol{d}_{pa} = 0$, so that

$$\boldsymbol{a_0 p} = [-1 \quad -1 \quad -1]\#[1 \quad 2 \quad 3] = [-1 \quad -2 \quad -3],$$

$$\boldsymbol{a_1 p} = [1 \quad -1 \quad -1]\#[1 \quad 2 \quad 3] = [1 \quad -2 \quad -3],$$

$$\boldsymbol{a_2 p} = [-1 \quad 1 \quad -1]\#[1 \quad 2 \quad 3] = [-1 \quad 2 \quad -3],$$

$$\boldsymbol{a_3 p} = [1 \quad 1 \quad -1]\#[1 \quad 2 \quad 3] = [1 \quad 2 \quad -3],$$

$$\boldsymbol{a_4 p} = [-1 \quad -1 \quad 1]\#[1 \quad 2 \quad 3] = [-1 \quad -2 \quad \boldsymbol{3}],$$

$$\boldsymbol{a_5 p} = [1 \quad -1 \quad 1]\#[1 \quad 2 \quad 3] = [1 \quad -2 \quad \boldsymbol{3}],$$

$$\boldsymbol{a_6 p} = [-1 \quad 1 \quad 1]\#[1 \quad 2 \quad 3] = [-1 \quad 2 \quad \boldsymbol{3}],$$

$$\boldsymbol{a_7 p} = [1 \quad 1 \quad 1]\#[1 \quad 2 \quad 3] = [1 \quad 2 \quad \boldsymbol{3}],$$

Note that $i = 3 \in \boldsymbol{a_i p}$ for $i = 4, 5, 6, and\ 7$. Therefore, $\boldsymbol{Q}_p^{+}(\boldsymbol{x}) = \frac{1}{8}\sum_{a \in A} \boldsymbol{S_{pa}}(\boldsymbol{x})\boldsymbol{d}_{pa}$ is determined by vectors $\boldsymbol{a_0}, \boldsymbol{a_1}, \boldsymbol{a_2}, \& \ \boldsymbol{a_3}$ only . We now calculate the component $\boldsymbol{S_{pa}}(\boldsymbol{x})\boldsymbol{d}_{pa}$ for $\boldsymbol{a_4}, \boldsymbol{a_5}, \boldsymbol{a_6}, \& \ \boldsymbol{a_7}$ only.

Again, using the same formulas in Equation132, we calculate

$\boldsymbol{W_1}, \boldsymbol{W_2}, \boldsymbol{d_{p2}}$, and $\boldsymbol{S_{p2} d_{p2}}$ for $\mathbf{Y_0}, \boldsymbol{Y}_1, \boldsymbol{Y}_2$ and $\boldsymbol{Y_3}$ as

$$\boldsymbol{Y_0} = [-1 \quad -1 \quad -1]\# \begin{bmatrix} 6 & -7 & 5 \\ -10 & 13 & 7 \\ 12 & -11 & 15 \end{bmatrix} = \begin{bmatrix} -6 & 7 & -5 \\ 10 & -13 & -7 \\ -12 & 11 & -15 \end{bmatrix},$$

$$\boldsymbol{S_{p0}d_{p0}} = \begin{bmatrix} -460 \\ -62 \\ 198 \end{bmatrix},$$

$$\boldsymbol{Y_1} = [1 \quad -1 \quad -1]\# \begin{bmatrix} 6 & -7 & 5 \\ -10 & 13 & 7 \\ 12 & -11 & 15 \end{bmatrix} = \begin{bmatrix} 6 & 7 & -5 \\ -10 & -13 & -7 \\ 12 & 11 & -15 \end{bmatrix},$$

$$\boldsymbol{S_{p1}d_{p1}} = \begin{bmatrix} 84 \\ 38 \\ -30 \end{bmatrix},$$

$$\boldsymbol{Y_2} = [-1 \quad 1 \quad -1]\# \begin{bmatrix} 6 & -7 & 5 \\ -10 & 13 & 7 \\ 12 & -11 & 15 \end{bmatrix} = \begin{bmatrix} -6 & -7 & -5 \\ 10 & 13 & -7 \\ -12 & -11 & -15 \end{bmatrix},$$

$$\boldsymbol{S_{p2}d_{p2}} = \begin{bmatrix} 8 \\ -2 \\ 14 \end{bmatrix},$$

$$\boldsymbol{Y_3} = [1 \quad 1 \quad -1]\# \begin{bmatrix} 6 & -7 & 5 \\ -10 & 13 & 7 \\ 12 & -11 & 15 \end{bmatrix} = \begin{bmatrix} 6 & -7 & -5 \\ -10 & 13 & -7 \\ 12 & -11 & -15 \end{bmatrix},$$

$$\boldsymbol{S_{p3}d_{p3}} = \begin{bmatrix} 552 \\ 98 \\ -214 \end{bmatrix}.$$

Applying Equations128 through132, the signed-rank vector for the original $\boldsymbol{x`_3} = \begin{bmatrix} 5 \\ 7 \\ 15 \end{bmatrix}$ is

$$\boldsymbol{R}^{+}(\boldsymbol{x`_3}) = \sum_{p \in P} \boldsymbol{Q}_3^{+}(\boldsymbol{x}) = \frac{1}{8} \sum_{a \in A} \boldsymbol{S_{pa}}(\boldsymbol{x}) \boldsymbol{d}_{pa}$$

$$\boldsymbol{R}^{+}(\boldsymbol{x`_3}) = \frac{1}{8}\left(\boldsymbol{S_{p0}d_{p0}} + \boldsymbol{S_{p1}d_{p1}} + \boldsymbol{S_{p2}d_{p2}} + \boldsymbol{S_{p2}d_{p3}}\right)$$

$$\boldsymbol{R}^{+}(\boldsymbol{x`_3}) = \frac{1}{8}\left(\begin{bmatrix} -460 \\ -62 \\ 198 \end{bmatrix} + \begin{bmatrix} 84 \\ 38 \\ -30 \end{bmatrix} + \begin{bmatrix} 8 \\ -2 \\ 14 \end{bmatrix} + \begin{bmatrix} 552 \\ 98 \\ -214 \end{bmatrix}\right) = \begin{bmatrix} 23 \\ 9 \\ -4 \end{bmatrix}$$

We now have $R_n^+ = [\boldsymbol{R}^+(\boldsymbol{x}`_{\boldsymbol{1}}) \quad \boldsymbol{R}^+(\boldsymbol{x}`_{\boldsymbol{2}}) \quad \boldsymbol{R}^+(\boldsymbol{x}`_{\boldsymbol{3}})] = \begin{bmatrix} -136 & -117 & 23 \\ -25 & -15 & 9 \\ 57 & 46 & -4 \end{bmatrix}$.

The calculated signed-rank vectors $\boldsymbol{R}^+(\boldsymbol{x}`_1), \boldsymbol{R}^+(\boldsymbol{x}`_2)$, and $\boldsymbol{R}^+(\boldsymbol{x}`_{\boldsymbol{3}})$ were generated for the transposed $\boldsymbol{X}$ or $\boldsymbol{X}` = \begin{bmatrix} 6 & -7 & 5 \\ -10 & 13 & 7 \\ 12 & -11 & 15 \end{bmatrix}$, and the signed-rank vectors or matrix is then transposed to give the final signed-ranks matrix

$$\boldsymbol{R}_{\boldsymbol{n}}^+ = (\boldsymbol{R}_{\boldsymbol{n}}^+)` = \begin{bmatrix} -136 & -25 & 57 \\ -117 & -15 & 46 \\ 23 & 9 & -4 \end{bmatrix}.$$

The above result is identical to the signed-rank vectors or matrix obtained by applying the SAS code and interactive matrix language (IML) routines for the computation of the purpose of calculating the Oja invariant signed-rank vectors in Appendix A.

APPENDIX C

PHASE I UPPER CONTROL LIMIT SIMULATION RESULTS

Table 57

The Upper Control Limits of the Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 200 from the Multivariate Normal Distribution

	p											
	2			3			4			5		
λ	h_1	$ARL_0\cong$	SDRL	h_1	$ARL_0\cong$	SDRL	h_1	$ARL_0\cong$	SDRL	h_1	$ARL_0\cong$	SDRL
0.01	5.30	193	288	7.20	196.	295	8.8	201	301	10.4	200	289
0.02	6.20	197	245	8.20	200	244	9.9	199	246	11.6	200	241
0.03	6.80	194	223	8.80	193	220	10.7	203	232	12.3	195	222
0.05	7.70	201	214	9.70	198	214	11.6	193	209	13.4	200	218
0.10	8.80	201	205	11.0	209	214	12.9	200	203	14.7	199	201
0.20	9.70	202	203	12.0	204	207	13.9	199	199	15.8	199	197
0.50	10.40	198	200	12.6	195	194	14.7	201	201	16.6	197	195

Table 58

The Upper Control Limits of the Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 500 from the Multivariate Normal Distribution

	p											
	2			3			4			5		
λ	h_1	$ARL_0\cong$	SDRL	h_1	$ARL_0\cong$	SDRL	h_1	$ARL_0\cong$	SDRL	h_1	$ARL_0\cong$	SDRL
0.01	7.0	505	625	9.0	504	635	10.8	500	623	12.5	499	625
0.02	8.1	494	551	10.2	488	54	12.1	492	546	14.0	504	560
0.03	8.8	501	537	11.0	495	521	13.0	503	530	14.8	494	520
0.05	9.7	502	501	11.9	483	496	14.0	506	523	15.9	499	514
0.10	10.8	498	505	13.0	473	475	15.2	505	513	17.1	488	489
0.20	11.6	482	488	14.0	498	497	16.2	501	500	18.1	485	485
0.50	12.3	504	504	14.6	490	485	16.8	496	502	18.8	495	493

Table 59

The Upper Control Limits of the Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 1,000 from the Multivariate Normal Distribution

	p											
	2			3			4			5		
λ	h_1	$ARL_0 \cong$	SDRL	h_1	$ARL_0 \cong$	SDRL	h_1	$ARL_0 \cong$	SDRL	h_1	$ARL_0 \cong$	SDRL
0.01	8.3	967	1,100	10.6	1,027	1,184	12.5	1,010	1,150	14.3	1,001	1,128
0.02	9.6	976	1,030	11.8	942	1,002	13.9	962	1,011	15.8	996	1,049
0.03	10.3	971	999	12.6	955	972	14.8	1,006	1,029	16.7	996	1,008
0.05	11.2	968	996	13.6	995	1,004	15.8	1,004	1,032	17.7	980	1,015
0.10	12.3	981	993	14.8	1,014	1,022	16.9	973	978	18.9	974	959
0.20	13.1	987	973	15.6	994	992	17.8	1,006	1,003	19.8	974	965
0.50	13.7	1,011	1,012	16.2	1,014	1,019	18.3	985	975	20.4	985	983

Table 60

The Upper Control Limits of the Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 200 from the Multivariate $t_p(5)$-Distribution

	p											
	2			3			4			5		
λ	h_1	$ARL_0 \cong$	SDRL	h_1	$ARL_0 \cong$	SDRL	h_1	$ARL_0 \cong$	SDRL	h_1	$ARL_0 \cong$	SDRL
0.01	5.3	200	290	7.1	199	294	8.8	192	280	10.4	192	280
0.02	6.2	198	246	8.2	200	247	10.0	201	247	11.6	196	248
0.03	6.8	195	224	8.9	201	234	10.8	201	233	12.5	200	229
0.05	7.7	197	212	9.9	202	219	11.8	200	214	13.6	194	214
0.10	9.0	202	208	11.3	199	206	13.2	196	204	15.2	201	209
0.20	10.3	198	198	12.7	200	198	14.8	198	200	16.8	200	200
0.50	12.3	200	200	15.0	198	199	17.5	199	197	19.7	197	194

Table 61

The Upper Control Limits of the Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 500 from the Multivariate t_p(5)-Distribution

	p											
	2			3			4			5		
λ	h_1	ARL_0≅	SDRL	h_1	ARL_0≅	SDRL	h_1	ARL_0≅	SDRL	h_1	ARL_0≅	SDRL
0.01	6.9	493	605	9.0	507	634	10.8	488	617	12.6	496	619
0.02	8.1	488	544	10.3	500	553	12.2	491	545	14.1	501	562
0.03	8.9	492	528	11.1	492	434	13.2	498	547	15.0	496	534
0.05	9.8	474	486	12.2	491	510	14.2	482	504	16.2	494	519
0.10	11.2	489	496	13.7	493	500	15.9	504	521	17.8	490	498
0.20	12.7	486	491	15.4	492	490	17.7	501	501	19.8	503	507
0.50	15.7	500	501	18.8	507	510	21.5	493	494	23.9	486	489

Table 62

The Upper Control Limits of the Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 1,000 from the Multivariate t_p(5)-Distribution

	p											
	2			3			4			5		
λ	h_1	ARL_0≅	SDRL	h_1	ARL_0≅	SDRL	h_1	ARL_0≅	SDRL	h_1	ARL_0≅	SDRL
0.01	8.3	956	1,091	10.6	998	1,161	12.5	983	1,145	14.4	1,001	1,152
0.02	9.7	975	1,047	12.0	985	1,076	14.0	971	1,029	16.0	1,007	1,068
0.03	10.5	1,006	1,054	12.9	1,007	1,060	15.0	1,005	1,078	16.9	983	1,013
0.05	11.5	984	1,010	14.0	1002	1,038	16.2	1,002	1,022	18.1	1,017	1,058
0.10	13.0	984	982	15.6	1,006	1,026	17.9	1,002	1,019	19.9	992	995
0.20	14.8	995	1,022	17.6	1,003	1,007	20.0	995	1,013	22.2	984	986
0.50	18.8	997	993	22.1	964	966	25.3	985	1,002	27.9	996	1,090

Table 63

The Upper Control Limits of the Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length $\approx 200,\ 500, and\ 1{,}000$ *from the Gamma₂ (α=3,β=1) Distribution with* $\rho_{12} = 0.5$

	IC ARL=200			IC ARL=500			IC ARL=1,000		
λ	h_1	$ARL_0 \cong$	SDRL	h_1	$ARL_0 \cong$	SDRL	h_1	$ARL_0 \cong$	SDRL
0.01	4,124.1	200.05	215.23	4,209.5	499.98	485.80	4,268.4	1,002	959.10
0.02	2,116.6	200.1	201.53	2,175.5	500.45	480.18	2,211.7	1,002	993.84
0.03	1,442.9	200.5	195.80	1.486.7	500.1	480.08	1,515.7	997.5	993.66
0.05	896.6	200.0	192.00	928.5	500.3	490.50	948.8	1,000	995.40
0.10	476.8	200.2	193.90	498.3	501.0	499.80	512.0	1,007	1,003.3
0.20	257.8	200.2	196.60	272.6	498.0	503.80	282.8	1,001	993.05
0.50	116.1	200.7	202.30	125.8	503.7	502.16	132.7	999	982.17

Table 64

The Upper Control Limits of the Signed-Rank Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 200 *from the Multivariate Normal Distribution*

	p											
	2			3			4			5		
λ	L	$ARL_0 \cong$	SDRL	L	$ARL_0 \cong$	SDRL	L	$ARL_0 \cong$	SDRL	L	$ARL_0 \cong$	SDRL
0.01	5.30	196	287	7.30	204	307	9.00	202	305	10.60	199	301
0.02	6.30	206	253	8.30	197	257	10.30	201	263	12.60	202	273
0.03	6.90	193	233	9.10	200	239	11.45	202	256	14.10	200	248
0.05	7.70	198	213	10.20	201	221	13.05	200	230	16.90	201	231
0.10	8.85	197	200	12.00	202	214	16.25	200	218	22.80	199	216
0.20	9.90	196	197	14.45	200	206	21.45	200	210	33.50	201	208
0.50	10.90	201	203	19.10	200	197	33.50	200	200	58.30	200	200

Table 65

The Upper Control Limits of the Signed-Rank Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 500 from the Multivariate Normal Distribution

	p											
	2			3			4			5		
λ	*L*	ARL_0≅	SDRL	*L*	ARL_0≅	SDRL	*L*	ARL_0≅	SDRL	*L*	ARL_0≅	SDRL
0.01	6.90	496	599	9.10	487	634	11.15	494	659	13.30	495	669
0.02	8.10	492	556	10.50	497	568	12.90	494	585	15.90	500	615
0.03	8.85	499	539	11.40	502	558	14.20	498	563	18.05	501	575
0.05	9.75	494	517	12.65	500	533	16.30	498	541	21.90	503	556
0.10	10.95	502	507	14.70	497	520	20.50	499	522	30.60	501	527
0.20	11.95	491	484	17.60	497	505	27.65	500	515	46.90	501	507
0.50	12.75	482	479	23.80	504	500	44.50	500	505	84.00	493	497

Table 66

The Upper Control Limits of the Signed-Rank Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 1,000 from the Multivariate Normal Distribution

	p											
	2			3			4			5		
λ	*L*	ARL_0≅	SDRL	*L*	ARL_0≅	SDRL	*L*	ARL_0≅	SDRL	*L*	ARL_0≅	SDRL
0.01	8.40	1,007	1,151	10.70	996	1,180	13.00	988	1,187	15.55	1,005	1,269
0.02	9.60	985	1,047	12.20	993	1,019	15.00	988	1,121	18.60	1,010	1,178
0.03	10.40	990	1,024	13.20	996	1,056	16.50	997	1,096	21.30	998	1,103
0.05	11.35	1,003	1,033	14.20	1,004	1,035	18.85	1,000	1,078	26.10	1,006	1,092
0.10	12.50	995	1,007	16.90	993	1,010	23.95	1,001	1,053	37.50	998	1,046
0.20	13.50	992	982	20.25	992	998	32.80	992	1,020	58.40	983	997
0.50	14.20	967	960	27.40	994	1,019	53.60	989	989	107.9	983	973

Table 67

The Upper Control Limits of the Signed-Rank Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 200 from the Multivariate t_p(5)- Distribution

	p											
	2			3			4			5		
λ	*L*	ARL_0≅	SDRL	*L*	ARL_0≅	SDRL	*L*	ARL_0≅	SDRL	*L*	ARL_0≅	SDRL
0.01	5.35	203	300	7.10	196	283	8.75	199	296	10.30	200	297
0.02	6.30	201	248	8.35	200	253	10.35	198	252	12.69	201	262
0.03	6.90	197	226	9.25	200	241	11.70	199	240	14.70	201	243
0.05	7.85	200	221	10.70	200	227	14.00	201	229	18.60	202	231
0.10	9.20	201	208	13.30	200	215	19.20	201	211	27.55	199	209
0.20	10.70	199	205	17.65	200	206	28.30	199	206	43.80	200	208
0.50	13.40	200	204	27.40	200	203	49.80	200	199	81.90	198	201

Table 68

The Upper Control Limits of the Signed-Rank Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 500 from the Multivariate t_p(5)- Distribution

	p											
	2			3			4			5		
λ	*L*	ARL_0 ≅	SDRL	*L*	ARL_0 ≅	SDRL	*L*	ARL_0 ≅	SDRL	*L*	ARL_0 ≅	SDRL
0.01	6.95	495	609	9.20	504	650	11.30	501	667	13.70	498	663
0.02	8.20	500	558	10.75	498	579	13.62	502	604	17.60	497	589
0.03	8.95	497	539	11.95	503	564	15.60	506	578	20.90	497	563
0.05	10.00	501	527	13.80	501	546	19.20	503	541	27.60	502	548
0.10	11.55	493	500	17.50	500	516	27.60	501	521	43.40	498	519
0.20	13.50	496	504	24.15	495	496	43.00	501	505	72.90	499	507
0.50	17.50	501	501	40.10	496	493	79.00	497	493	142.30	500	500

Table 69

The Upper Control Limits of the Signed-Rank Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length $\approx 1{,}000$ *from the Multivariate* $t_p(5)$*- Distribution*

	p											
	2			3			4			5		
λ	*L*	$ARL_0\cong$	SDRL	*L*	$ARL_0\cong$	SDRL	*L*	$ARL_0\cong$	SDRL	*L*	$ARL_0\cong$	SDRL
0.01	8.40	991	1,132	10.90	997	1,179	13.50	998	1,246	16.90	996	1,238
0.02	9.80	1,010	1,087	12.75	997	1,093	16.45	992	1,132	22.20	1,000	1,152
0.03	10.60	998	1,042	14.15	1,006	1,082	19.05	992	1,099	27.20	995	1,091
0.05	11.75	993	1,020	16.40	998	1,076	23.95	999	1,041	37.30	996	1,059
0.10	13.50	1,007	1,021	21.20	982	1,016	35.50	992	1,007	61.20	1,000	1,010
0.20	15.80	997	1,015	30.20	996	1,002	58.00	988	984	106.9	1,000	1,009
0.50	21.15	981	972	52.20	999	985	111.1	997	987	212.10	1,006	1,006

Table 70

The Upper Control Limits of the Signed-Rank Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length $\approx 200,\ 500,$ *and* $1{,}000$ *from the Gamma*$_2$ *(α=3,β=1) Distribution with* $\rho_{12} = 0.5$

	IC ARL=200			IC ARL=500			IC ARL=1,000		
λ	*L*	$ARL_0\cong$	SDRL	*L*	$ARL_0\cong$	SDRL	*L*	$ARL_0\cong$	SDRL
0.01	*1,366.3*	201	236.48	*1419.9*	499.4	513.19	*1,460.0*	994.0	988.60
0.02	*723.1*	200	209.37	*761.7*	494.7	494.26	*798.1*	999.4	980.70
0.03	*505.6*	200	200.50	*537.4*	498.5	495.26	*559.3*	999.4	966.63
0.05	*328.5*	200	196.52	*352.7*	500	500.16	*369.2*	1,001	1,002.63
0.10	*190.3*	200	199.56	*207.7*	500	497.38	*219.4*	997.0	990.75
0.20	*116.2*	200	196.41	*129.3*	500	509.81	*138.7*	997.0	994.06
0.50	*65.9*	201	200.60	*76.7*	502	502.64	*84.5*	1,000	997.33

Table 71

The Upper Control Limits (h_2) of the Hotelling's χ^2 That Achieved an In-Control Average Run Length ≈ $200,\ 500,\ and\ 1{,}000$ under p-variates Multivariate Normal Distribution

p	ARL_0 200	500	1,000
2	10.59	12.42	13.88
3	12.83	14.79	16.26
4	14.86	16.92	18.46
5	16.74	18.90	20.51

Table 72

The Upper Control Limits(h_2) of the Hotelling's χ^2 That Achieved an In-Control Average Run Length ≈ $200,\ 500, and\ 1{,}000$ under p-variates Multivariate $t_p(5)$-Distribution

	IC ARL=200			IC ARL=500			IC ARL=1,000		
p	h_2	ARL_0≅	SDRL	h_2	ARL_0≅	SDRL	h_2	ARL_0≅	SDRL
2	13.50	200	196.61	17.60	488	483.73	21.50	992	986.65
3	16.60	200	196.61	21.10	494	493.82	25.60	998	978.42
4	19.20	201	202.63	24.30	500	501.81	29.20	988	984.37
5	21.60	201	198.70	27.00	494	493.68	32.40	1,000	990.61

Table 73

The Upper Control Limits (h_2) of the Hotelling's χ^2 That Achieved an In-Control Average Run Length $\approx$ 200, 500, *and* 1,000 *under p-variates Multivariate Gamma$_2$(α=3,β=1) Distribution*

	IC ARL=200			IC ARL=500			IC ARL=1,000		
p	h_2	$ARL_0\cong$	SDRL	h_2	$ARL_0\cong$	SDRL	h_2	$ARL_0\cong$	SDRL
2	25.30	201	197.98	34.82	503	505.74	42.7	993	1,000.55
3	15.50	200	202.82	18.95	501	507.26	21.6	995	983.80
4	17.93	200	199.38	21.5	500	501.74	24.28	1,000	1,000.13
5	20.00	200	197.66	23.6	500	491.81	26.45	999	1,011.04

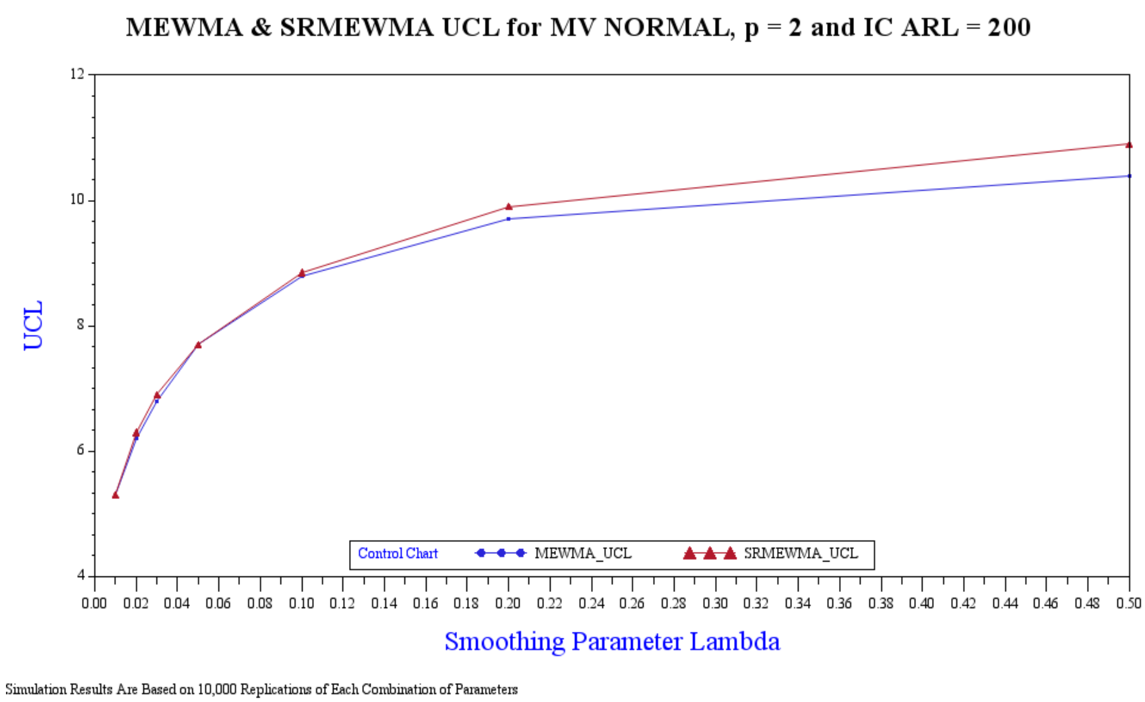

Figure 18. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate normal, p = 2 and in-control average run length = 200.

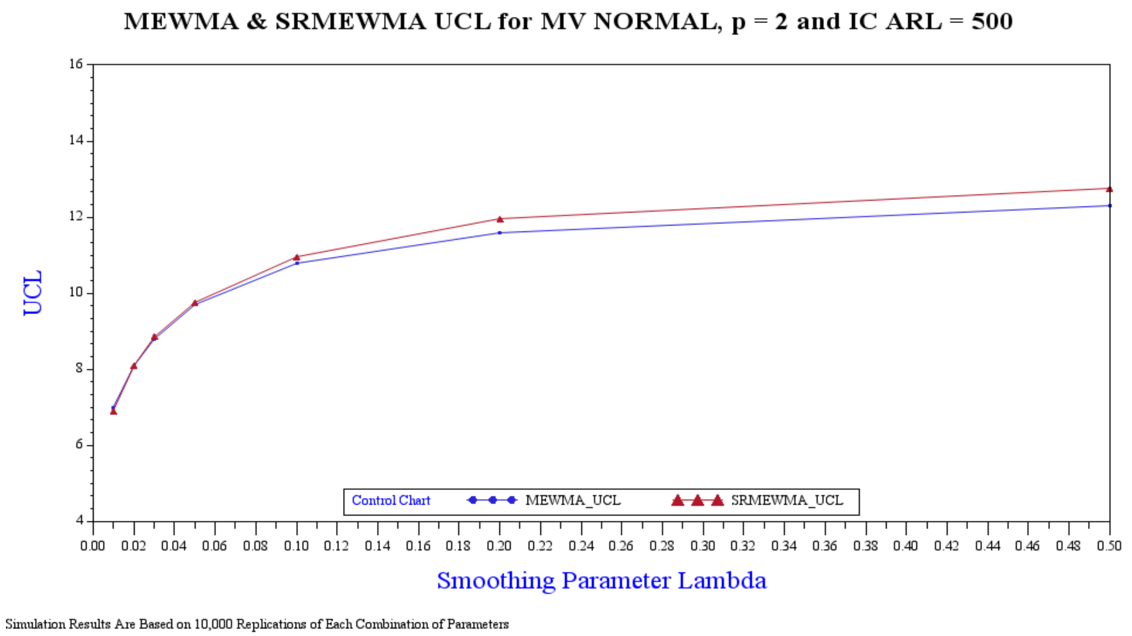

Figure 19. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate normal, *p* = 2 and in-control average run length = 500.

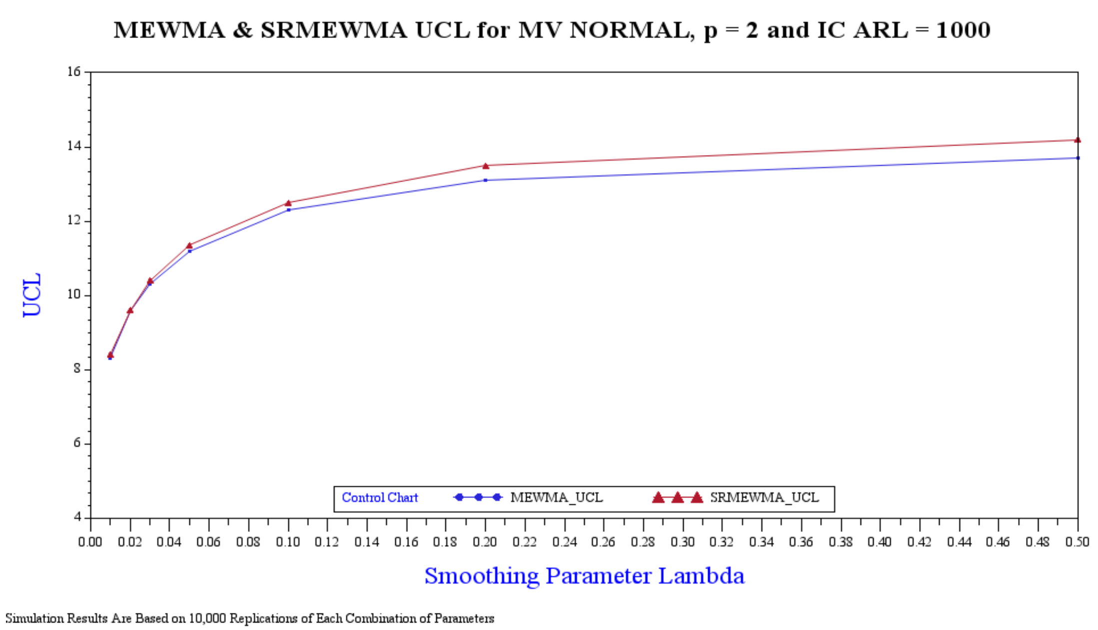

Figure 20. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate normal, *p* = 2 and in-control average run length = 1,000.

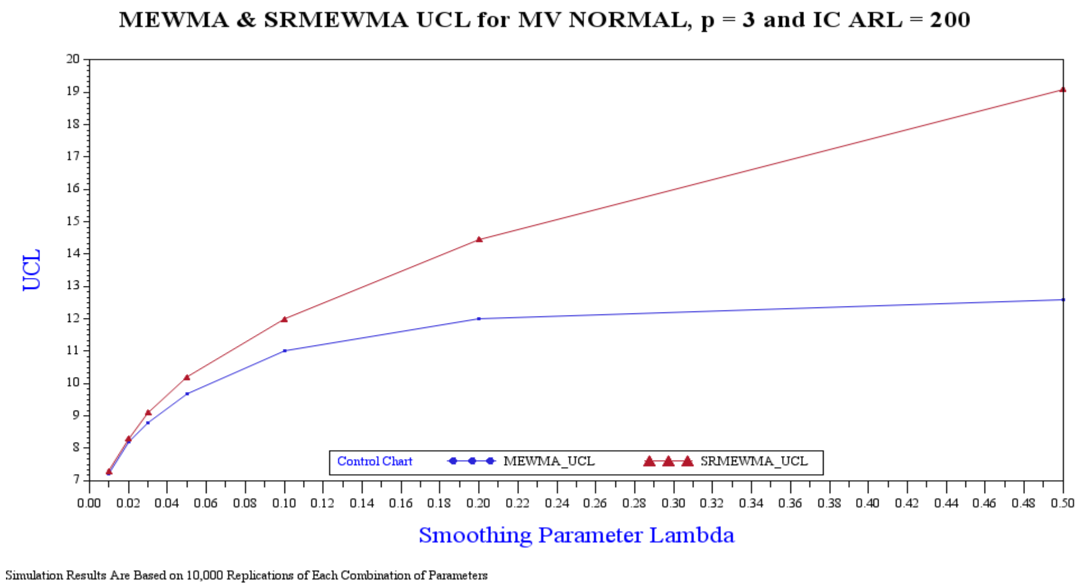

Figure 21. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate normal, p = 3 and in-control average run length = 200.

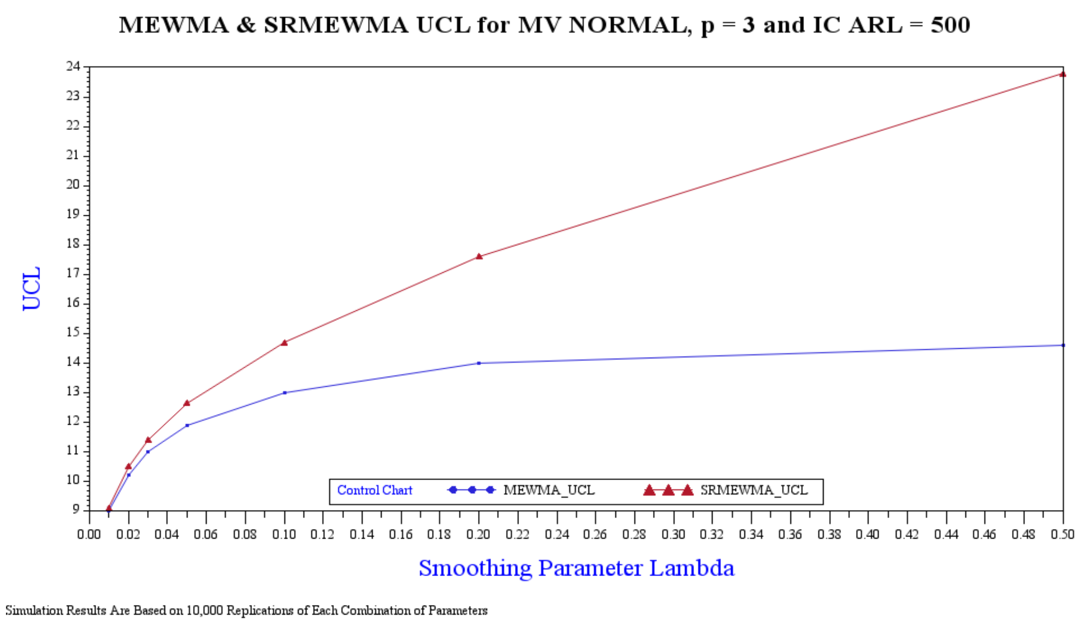

Figure 22. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate normal, p = 3 and in-control average run length = 500.

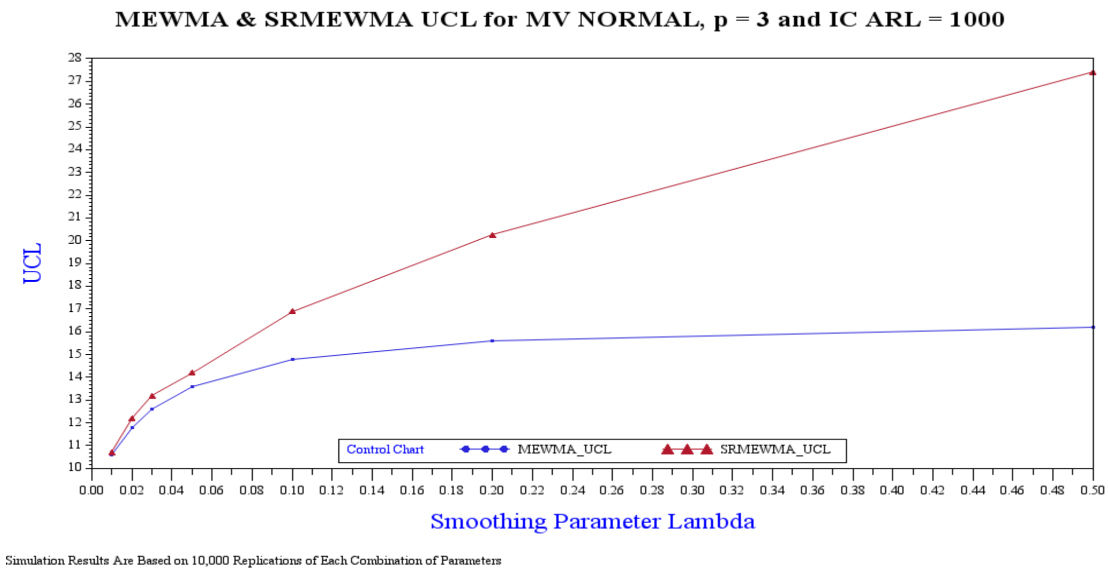

Figure 23. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate normal, p = 3 and in-control average run length = 1,000.

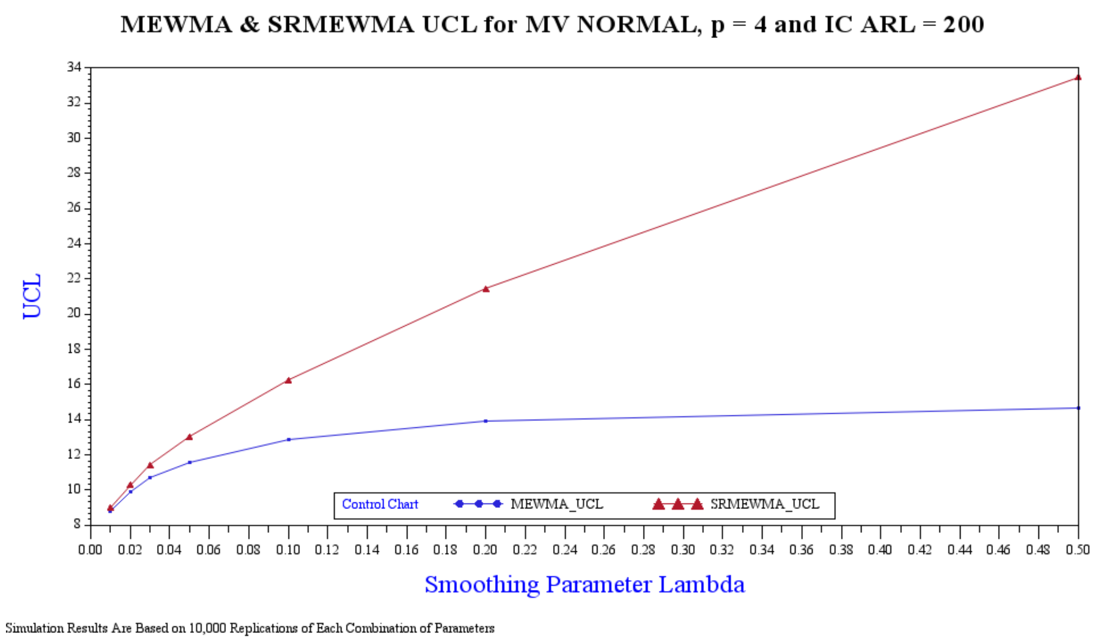

Figure 24. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate normal, p = 4 and in-control average run length = 200.

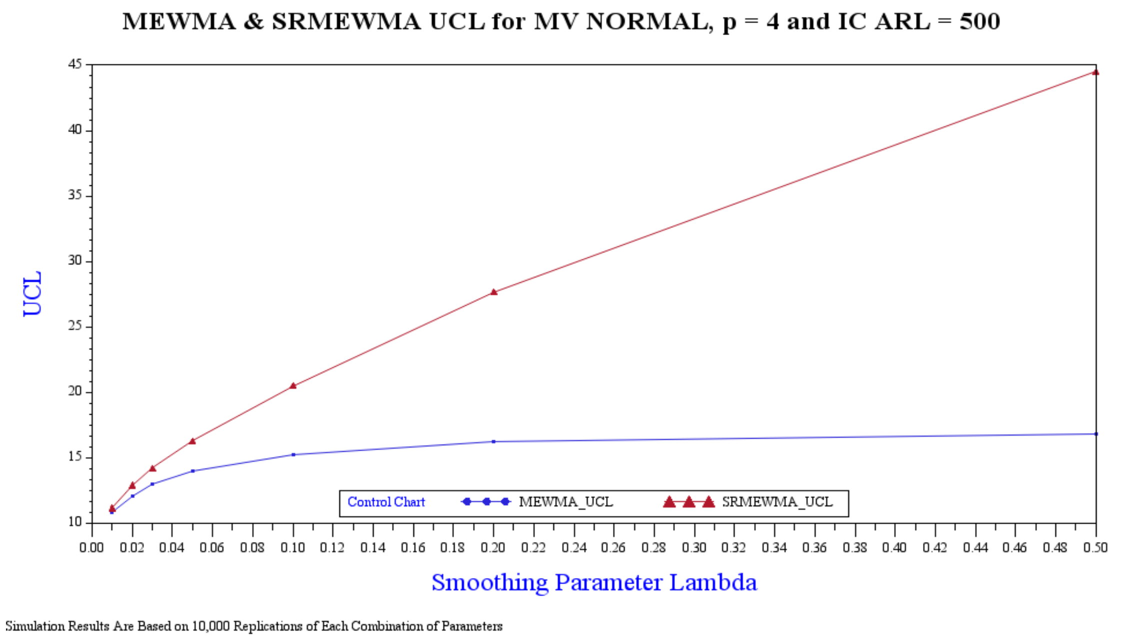

Figure 25. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate normal, p = 4 and in-control average run length = 500.

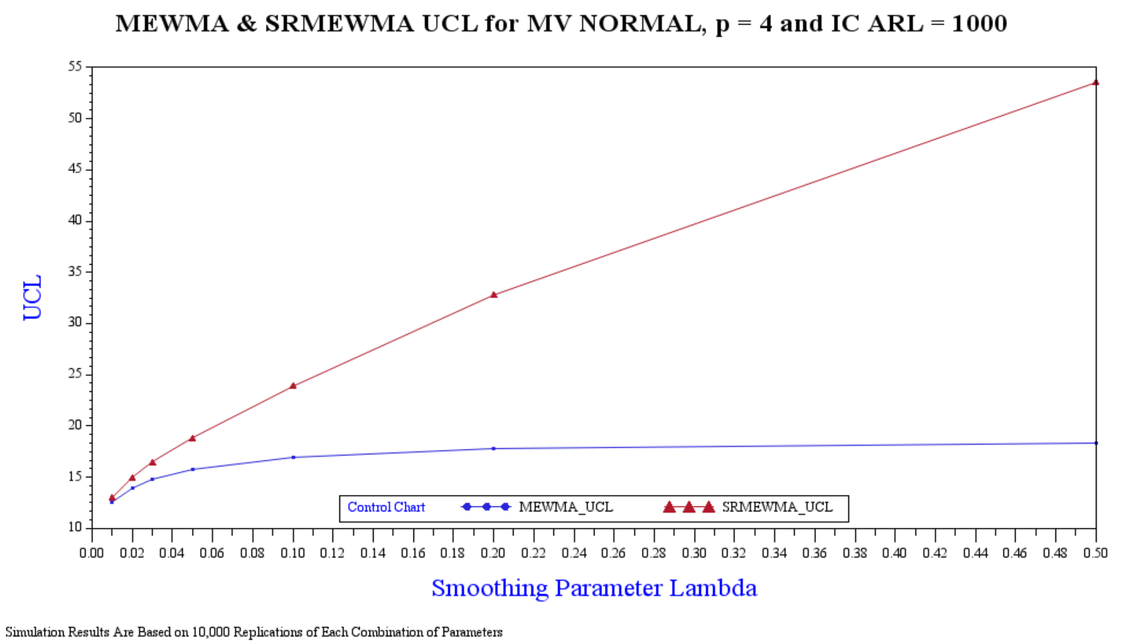

Figure 26. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate normal, p = 4 and in-control average run length = 1,000.

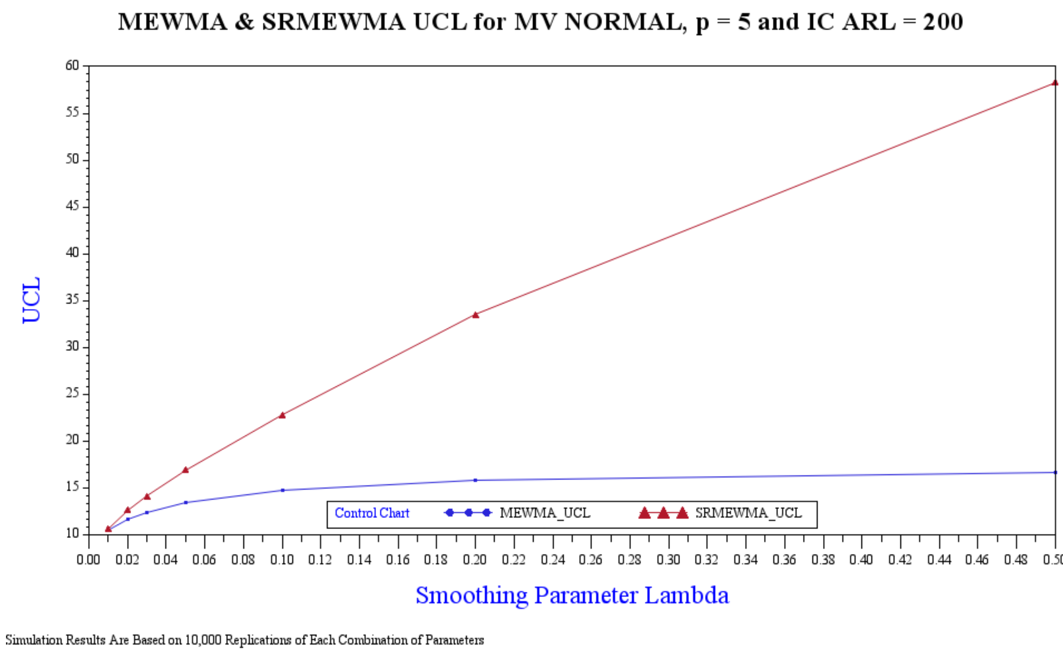

Figure 27. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate normal, p = 5 and in-control average run length = 200.

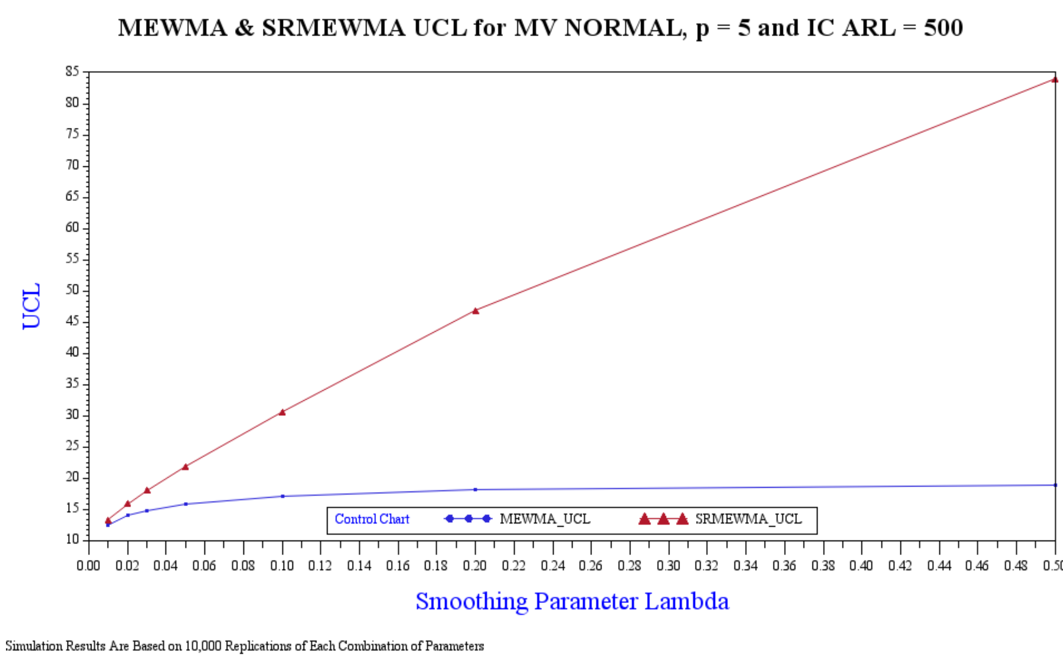

Figure 28. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate normal, p = 5 and in-control average run length = 500.

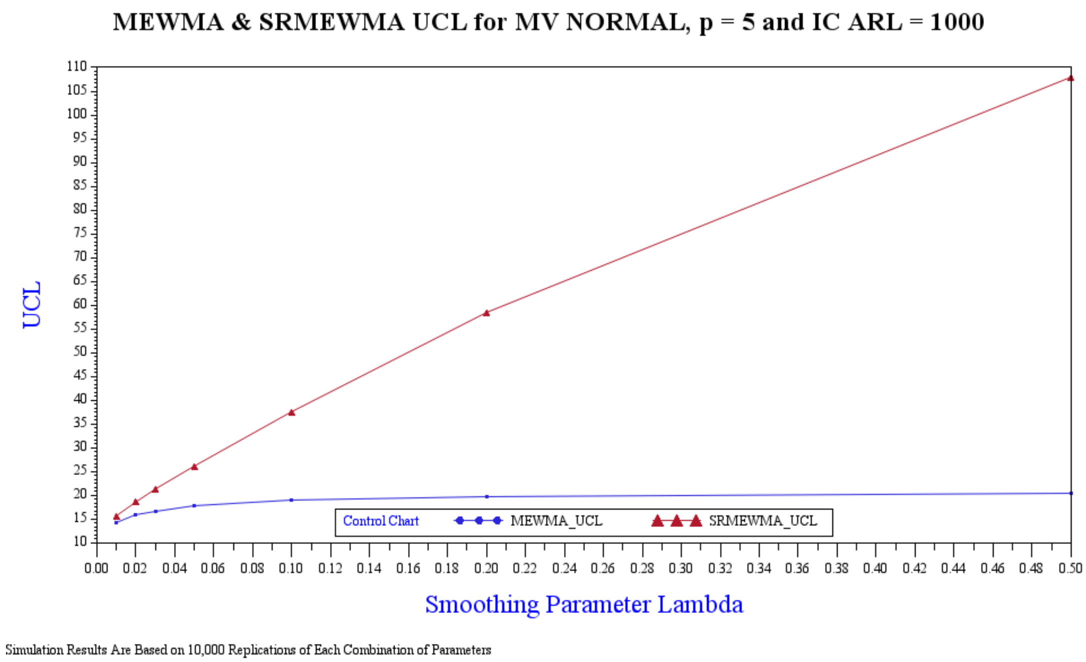

Figure 29. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate normal, p = 5 and in-control average run length = 1,000.

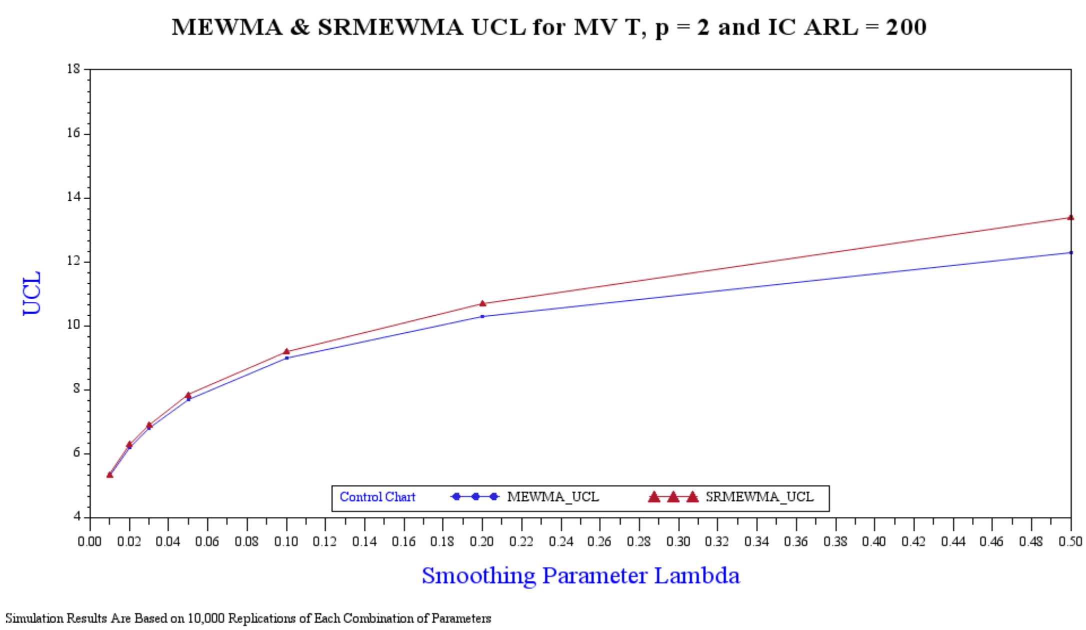

Figure 30. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate t_p (df = 5), p = 2 and in-control average run length = 200.

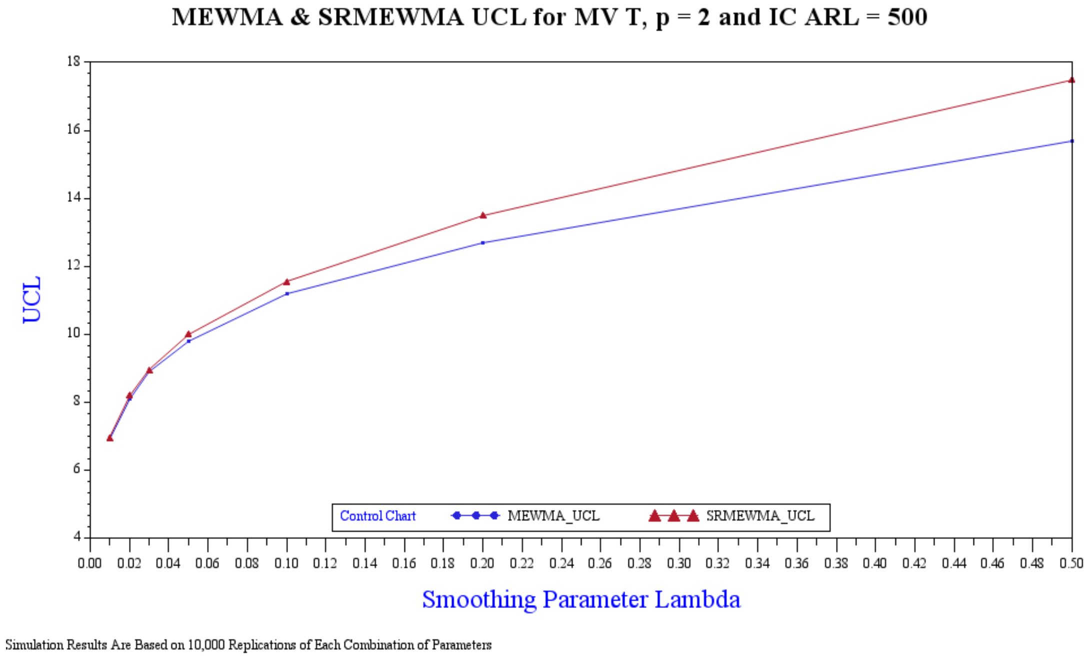

Figure 31. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate t_p (df = 5), p = 2 and in-control average run length = 500.

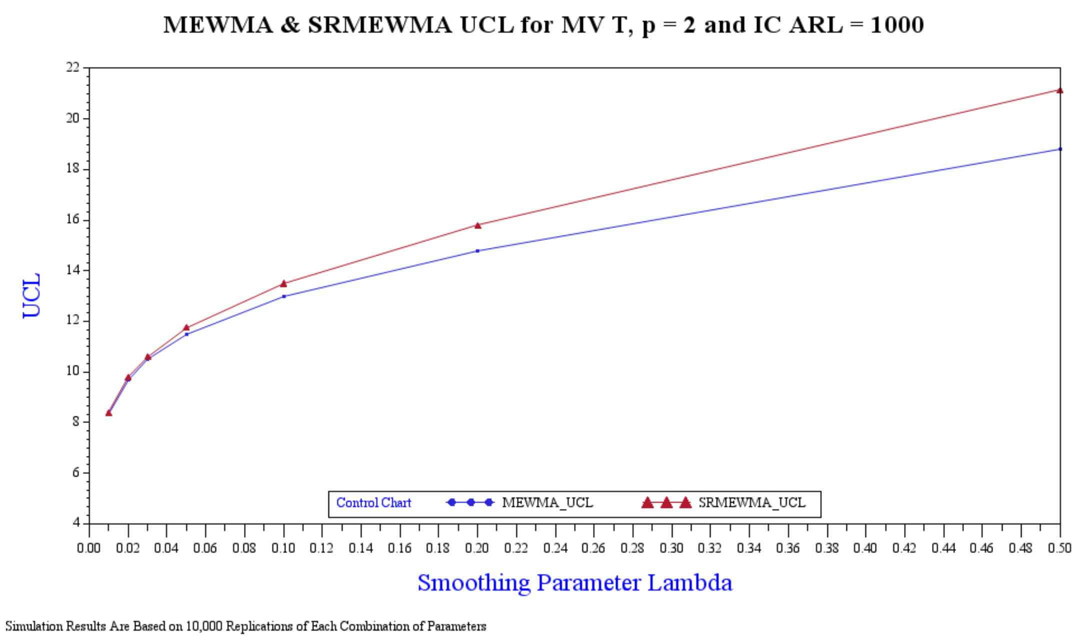

Figure 32. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate t_p (df = 5), p = 2 and in-control average run length = 1,000.

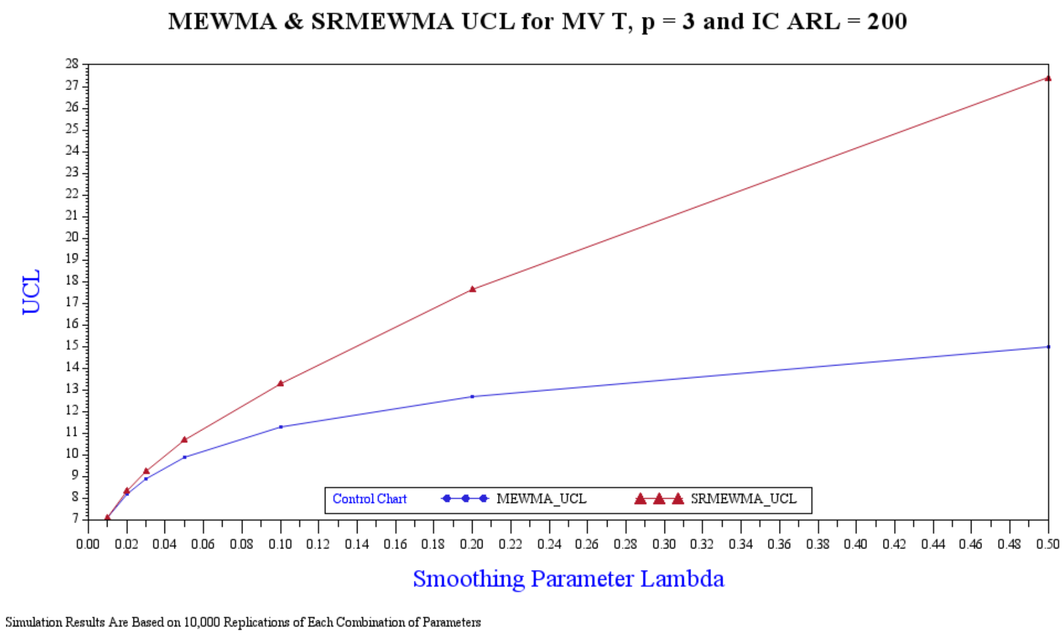

Figure 33. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate t_p (df = 5), p = 3 and in-control average run length = 200.

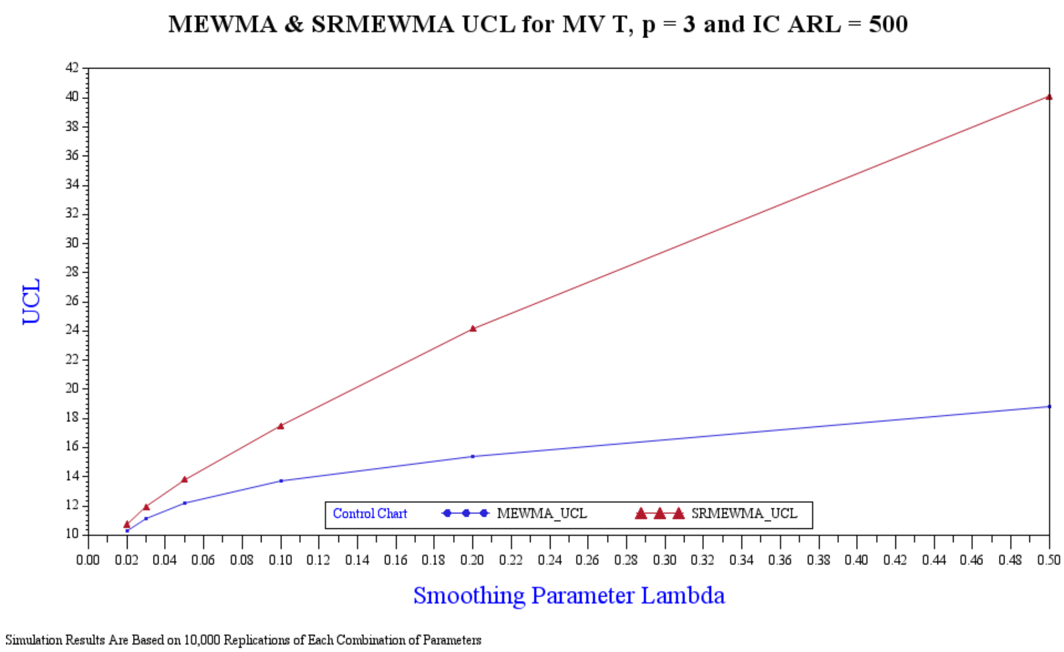

Figure 34. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate t_p (df = 5), p = 3 and in-control average run length = 500.

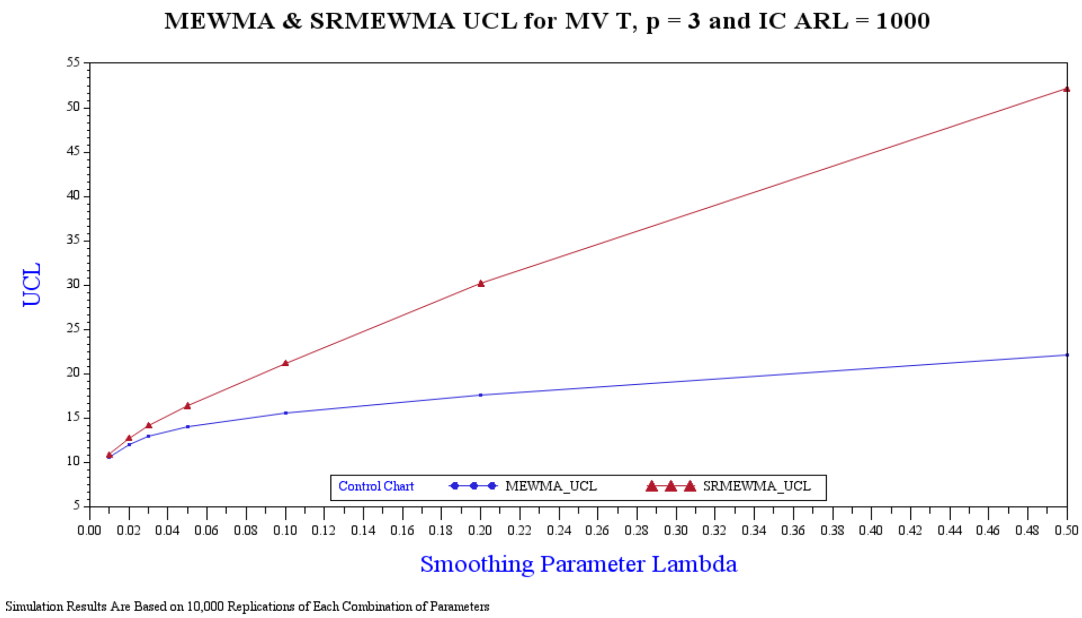

Figure 35. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate t_p (df = 5), p = 3 and in-control average run length = 1,000.

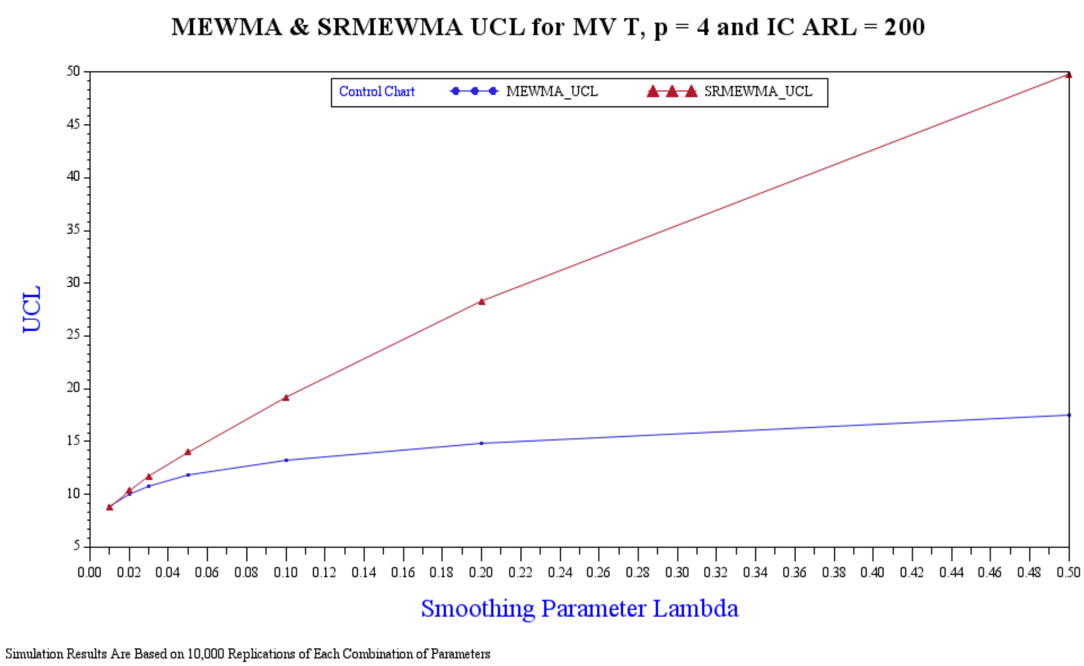

Figure 36. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate t_p (df = 5), p = 4 and in-control average run length = 200.

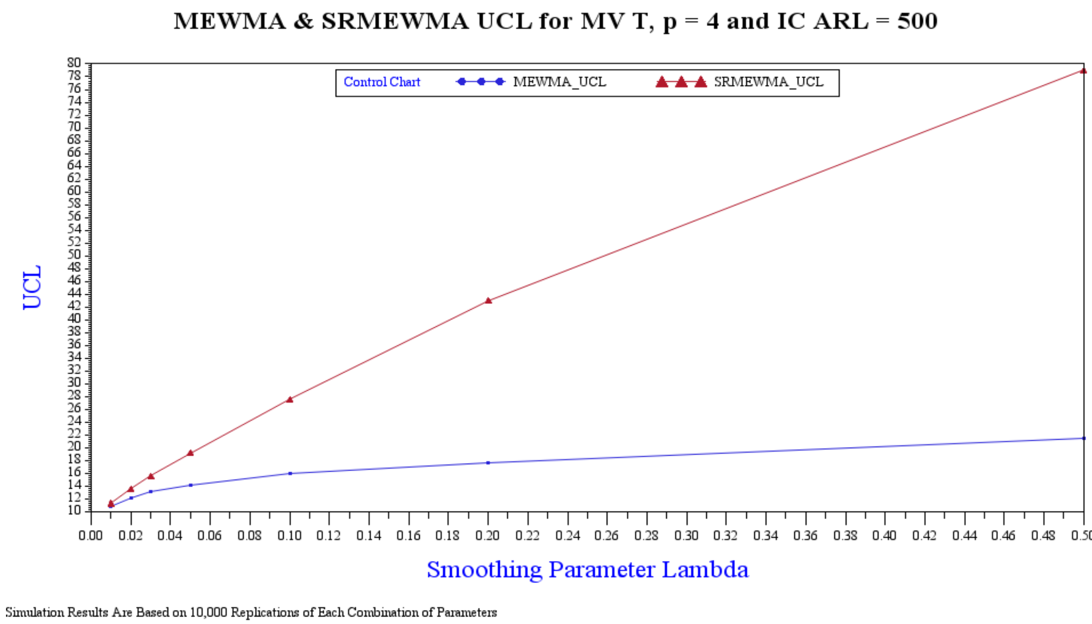

Figure 37. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate t_p (df = 5), p = 4 and in-control average run length = 500.

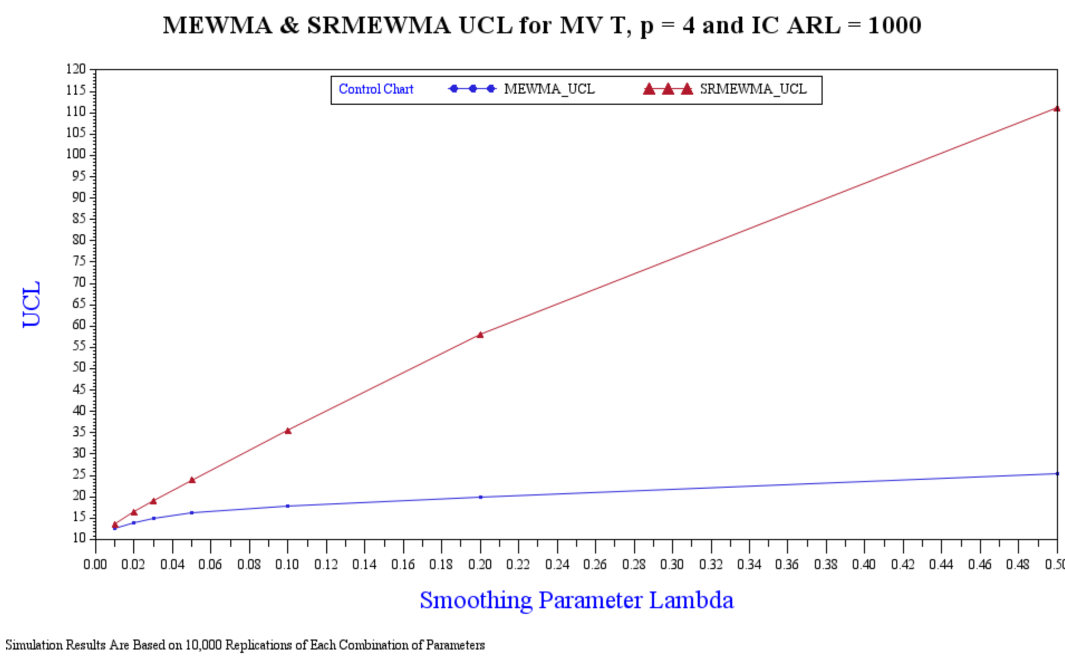

Figure 38. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate t_p (df = 5), p = 4 and in-control average run length = 1,000.

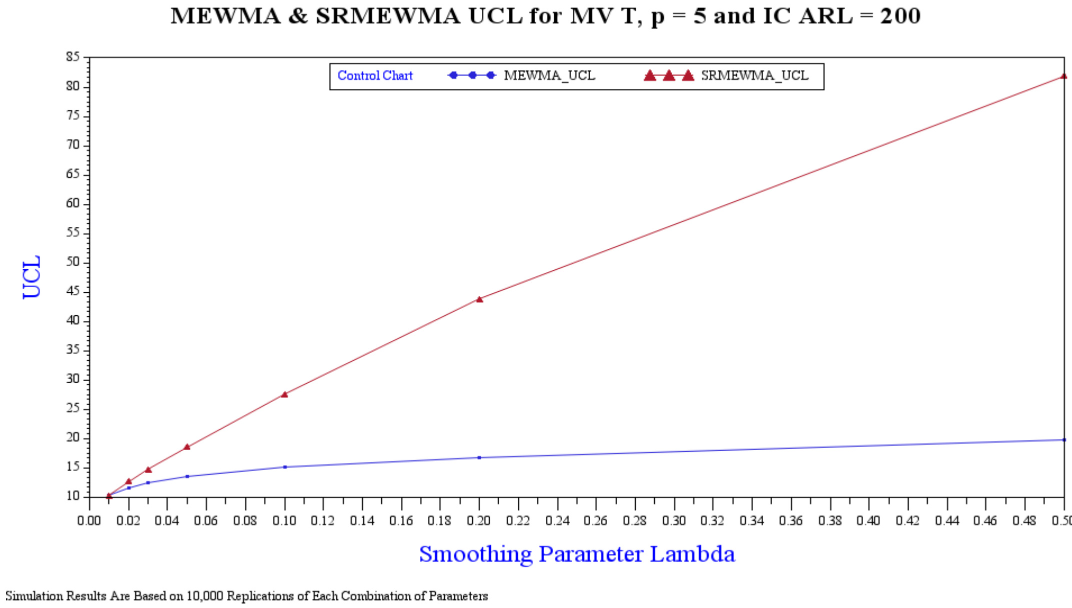

Figure 39. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate t_p (df = 5), p = 5 and in-control average run length = 200.

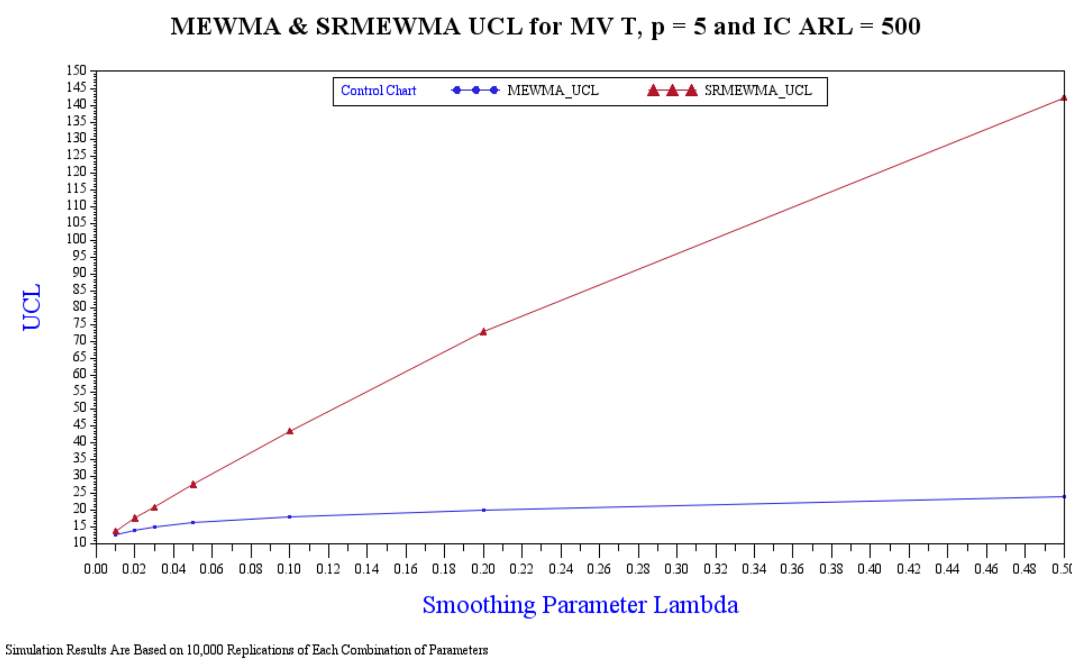

Figure 40. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate t_p (df = 5), p = 5 and in-control average run length = 500.

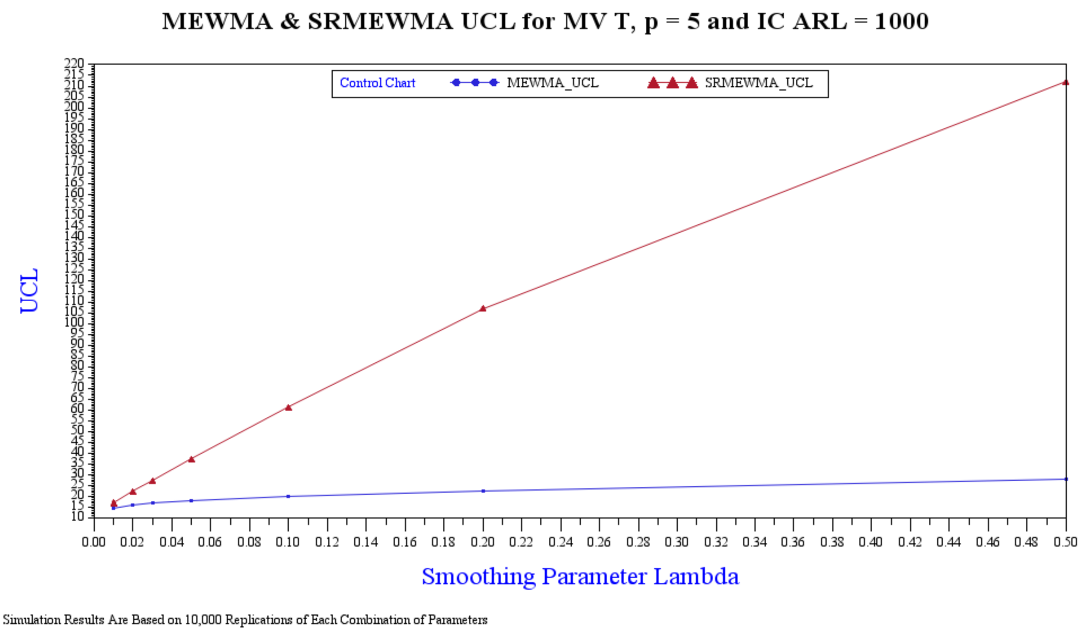

Figure 41. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate t_p (df = 5), p = 5 and in-control average run length = 1,000.

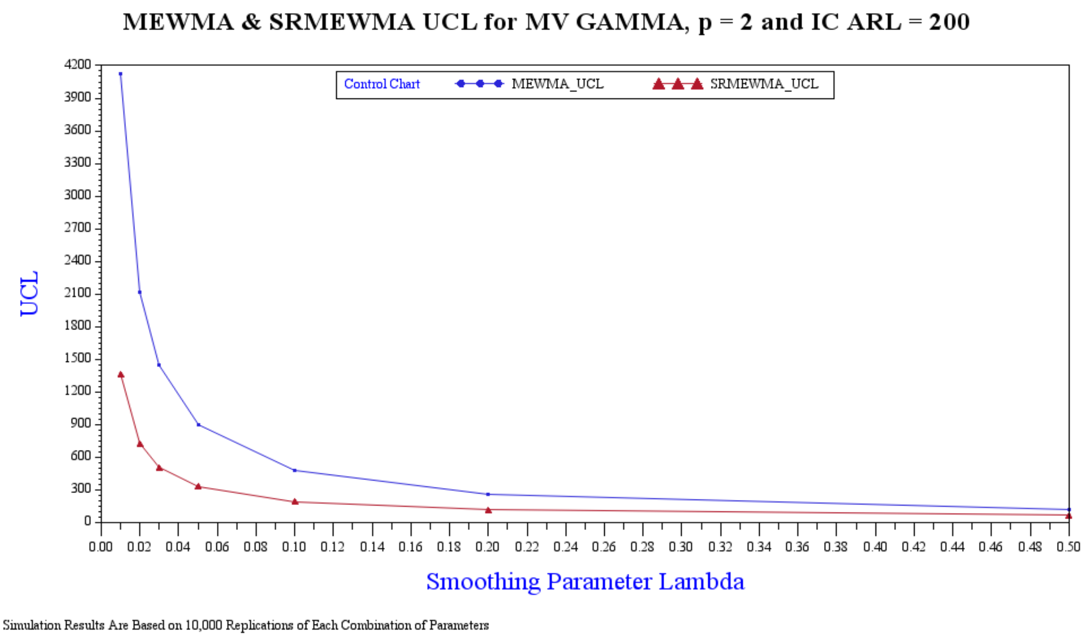

Figure 42. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate gamma$_2$ (α = 3, β = 1), p = 2 and in-control average run length = 200.

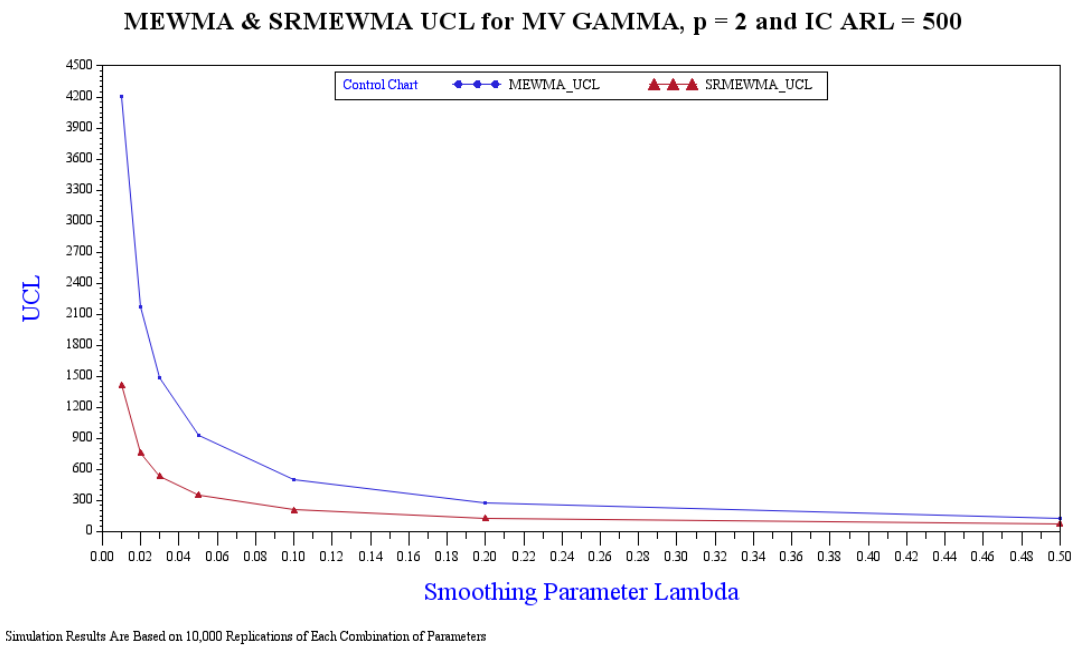

Figure 43. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate gamma$_2$ (α = *3*, β = *1*), *p* = 2 and in-control average run length = 500.

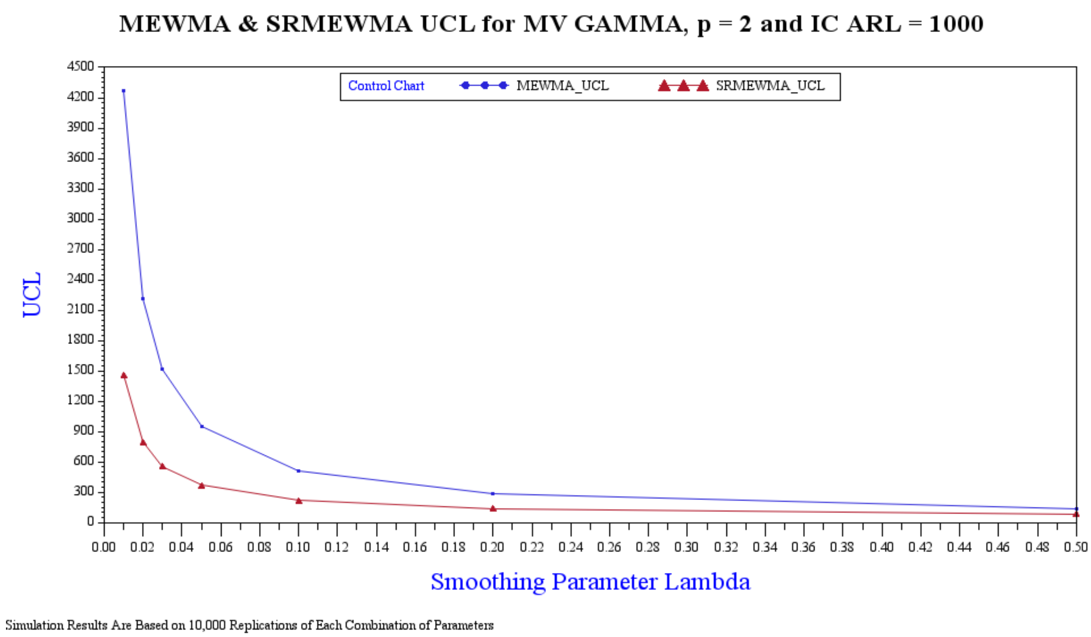

Figure 44. Multivariate exponentially weighted moving average and signed-rank multivariate exponentially weighted moving average upper control limit for multivariate gamma$_2$ (α = *3*, β = *1*), *p* = 2 and in-control average run length = 1,000.

APPENDIX D

PHASE II AVERAGE RUN LENGTH PERFORMANCE SIMULATION RESULTS

Table 74

The Upper Control Limits and Average Run Length Values of the Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 200 from the Multivariate Normal Distribution

			λ						
p			0.01	0.02	0.03	0.05	0.1	0.2	0.5
2	δ	h_1	5.30	6.20	6.80	7.7	8.8	9.7	10.40
	0.00		193	197	194	201	201	202	198
			(288)	(245)	(223)	(214)	(205)	(203)	(200)
	0.25		35.91	45.42	49.77	59.51	74.91	94.88	133.82
			(42.77)	(76.75)	(48.67)	(57.43)	(71.68)	(92.43)	(131.22)
	0.50		13.06	16.12	18.19	20.77	25.17	33.84	62.70
			(13.57)	(14.56)	(15.51)	(16.87)	(20.79)	(30.71)	(61.51)
	1.00		4.39	5.33	5.92	6.73	7.78	8.94	15.37
			(3.75)	(4.02)	(4.30)	(4.61)	(5.13)	(6.31)	(13.55)
	1.50		2.44	2.92	3.16	3.58	4.01	4.46	6.06
			(1.67)	(1.92)	(2.02)	(2.17)	(2.34)	(2.56)	(4.44)
	2.50		1.32	1.48	1.57	1.71	1.87	2.04	2.25
			(0.60)	(0.72)	(0.76)	(0.83)	(0.92)	(0.99)	(1.20-)
3	δ	h_1	7.20	8.20	8.80	9.70	11.00	12.00	12.60
	0.00		196	200	193	198	209	204	195
			(295)	(244)	(220)	(214)	(214)	(207)	(195)
	0.25		42.86	51.46	56.38	65.63	83.18	110.95	142.83
			(48.77)	(52.06)	(54.32)	(62.80)	(80.16)	(109.87)	(143.54)
	0.50		15.78	19.23	21.01	23.17	29.07	40.67	73.98
			(15.30)	(16.49)	(17.36)	(18.70)	(24.04)	(36.61)	(72.18)
	1.00		5.51	6.48	6.97	7.68	8.73	10.49	18.70
			(4.30)	(4.67)	(4.83)	(5.06)	(5.70)	(7.56)	(16.91)
	1.50		3.06	3.47	3.69	4.04	4.49	5.06	7.02
			(2.03)	(2.16)	(2.30)	(2.40)	(2.59)	(2.95)	(5.31)
	2.50		1.60	1.79	1.79	1.92	2.07	2.23	2.45
			(0.78)	(0.83)	(0.88)	(0.94)	(0.99)	(1.06)	(1.317)
4	δ	h_1	8.80	9.90	10.70	11.60	12.90	13.90	14.70
	0.00		201	199	203	193	200	199	201
			(301)	(246)	(232)	(209)	(203)	(199)	(201)
	0.25		44.09	55.30	61.93	71.22	89.81	116.74	156.31
			(52.03)	(55.12)	(60.20)	(67.50)	(86.98)	(115.13)	(15.30)
	0.50		16.06	20.26	22.11	24.94	31.44	43.76	84.06
			(16.39)	(17.69)	(18.33)	(20.02)	(25.71)	(40.22)	(83.19)
	1.00		5.42	6.69	7.48	8.28	9.57	11.34	21.64
			(4.48)	(5.00)	(5.24)	(5.51)	(6.19)	(8.07)	(19.52)
	1.50		2.99	3.58	3.95	4.27	4.86	5.33	8.06
			(2.10)	(2.32)	(2.44)	(2.56)	(2.77)	(3.11)	(6.31)
	2.50		1.53	1.75	1.87	2.01	2.18	2.35	2.70
			(0.76)	(0.88)	(0.93)	(0.98)	(1.05)	(1.12)	(1.47)
5	δ	h_1	10.40	11.60	12.30	13.40	14.70	15.8	16.6
	0.00		200	200	195	200	199	199	197
			(289)	(241)	(222)	(218)	(201)	(197)	(185)
	0.25		49.80	59.12	66.05	77.00	95.78	123.33	160.37
			(55.27)	(58.57)	(64.05)	(74.76)	(93.40)	(122.15)	(160.06)
	0.50		18.98	21.94	24.03	27.23	34.32	49.28	92.99
			(17.64)	(18.42)	(19.63)	(21.39)	(29.07)	(45.03)	(92.07)
	1.00		6.65	7.66	8.11	8.91	10.05	12.40	25.05
			(4.99)	(5.35)	(5.52)	(5.83)	(6.43)	(8.74)	(23.05)
	1.50		3.69	4.03	4.24	4.69	5.12	5.77	9.08
			(2.37)	(2.49)	(2.57)	(2.69)	(2.83)	(3.33)	(7.31)
	2.50		1.83	1.94	2.04	2.18	12.34	2.50	2.86
			(0.91)	(0.95)	(1.00)	(1.05)	(1.12)	(1.18)	(1.56)

Note. Standard deviation of run length is in parentheses.

Table 75

The Upper Control Limits and Average Run Length Values of the Signed-Rank Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 200 from the Multivariate Normal Distribution

			λ						
p			0.01	0.02	0.03	0.05	0.1	0.2	0.5
2	δ	*L*	5.30	6.30	6.90	7.70	8.85	9.90	10.90
	0.00		196	206	193	198	197	196	201
			(287)	(253)	(233)	(213)	(200)	(197)	(203)
	0.25		38.65	46.86	51.45	59.11	74.92	97.13	141.65
			(43.62)	(47.19)	(49.27)	(57.49)	(73.52)	(93.66)	(142.57)
	0.50		13.98	16.93	18.72	20.52	25.62	35.10	68.02
			(13.59)	(14.81)	(15.57)	(16.60)	(21.27)	(32.32)	(66.90)
	1.00		4.96	5.71	6.21	6.88	7.81	9.36	16.82
			(3.84)	(4.17)	(4.47)	(4.71)	(5.14)	(6.62)	(15.41)
	1.50		2.73	3.07	3.32	3.61	4.07	4.60	6.57
			(1.76)	(1.94)	(2.06)	(2.16)	(2.36)	(2.68)	(4.99)
	2.50		1.43	1.56	1.63	1.74	1.90	2.08	2.34
			(0.66)	(0.74)	(0.79)	(0.84)	(0.90)	(0.98)	(1.21)
3		*L*	7.30	8.30	9.10	10.20	12.00	14.45	19.10
	0.00		204	197	200	201	202	200	200
			(307)	(257)	(239)	(221)	(214)	(206)	(197)
	0.25		40.40	50.11	56.84	69.70	96.00	136.97	180.71
			(49.26)	(53.77)	(59.74)	(69.97)	(96.21)	(138.34)	(181.89)
	0.50		14.19	18.15	20.79	24.29	33.60	61.03	132.92
			(15.12)	(16.79)	(17.95)	(20.31)	(29.27)	(59.21)	(133.35)
	1.00		4.66	5.93	6.79	7.84	9.74	14.53	51.80
			(4.02)	(4.53)	(4.85)	(5.22)	(6.37)	(11.05)	(49.62)
	1.50		2.47	3.21	3.60	4.12	4.92	6.32	19.03
			(1.71)	(2.08)	(2.22)	(2.41)	(2.71)	(3.65)	(17.17)
	2.50		1.26	1.55	1.73	1.92	2.21	2.63	4.38
			(0.57)	(0.73)	(0.82)	(0.88)	(1.00)	(1.15)	(2.62)
4		*L*	9.00	10.30	11.45	13.05	16.25	21.45	33.50
	0.00		202	201	202	200	200	200	200
			(305)	(263)	(256)	(230)	(218)	(210)	(200)
	0.25		46.90	56.33	67.32	84.7	123.19	171.09	196.13
			(54.48)	(61.76)	(71.02)	(87.25)	(131.13)	(174.35)	(195.50)
	0.50		17.09	21.13	24.61	30.43	50.77	102.06	174.16
			(16.60)	(18.66)	(21.07)	(25.94)	(47.12)	(102.95)	(176.10)
	1.00		6.08	7.33	8.20	9.62	13.10	28.11	114.80
			(4.55)	(5.13)	(5.49)	(6.23)	(8.60)	(24.39)	(114.95)
	1.50		3.37	3.89	4.33	4.97	6.37	10.13	63.05
			(2.04)	(2.26)	(2.46)	(2.72)	(3.36)	(6.30)	(61.71)
	2.50		1.67	1.88	2.05	2.28	2.75	3.65	14.35
			(0.73)	(0.81)	(0.87)	(0.97)	(1.16)	(1.51)	(11.92)
5		*L*	10.60	12.60	14.10	16.90	22.80	33.50	58.30
	0.00		199	202	200	201	199	201	200
			(301)	(273)	(248)	(231)	(216)	(208)	(200)
	0.25		51.67	67.46	81.28	111.61	156.56	187.51	194.70
			(59.17)	(73.35)	(88.24)	(120.36)	(169.48)	(194.42)	(198.98)
	0.50		19.86	25.44	30.00	41.03	82.16	147.13	191.50
			(18.21)	(22.09)	(25.34)	(36.27)	(80.09)	(147.88)	(189.93)
	1.00		7.29	8.70	9.86	12.33	20.78	67.24	161.04
			(5.03)	(5.86)	(6.39)	(7.66)	(14.85)	(65.52)	(161.80)
	1.50		4.08	4.73	5.18	6.28	9.02	23.43	124.96
			(2.22)	(2.52)	(2.78)	(3.21)	(4.59)	(18.56)	(124.57)
	2.50		1.99	2.25	2.43	2.81	3.66	5.79	55.24
			(0.76)	(0.98)	(0.96)	(1.11)	(1.41)	(2.30)	(52.50)

Note. Standard deviation of run length is in parentheses.

Table 76

The Upper Control Limits and Average Run Length Values of the Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 500 from the Multivariate Normal Distribution

			λ						
p	δ		0.01	0.02	0.03	0.05	0.1	0.2	0.5
2		h_1	7.00	8.10	8.80	9.70	10.80	11.60	12.30
	0.00		505	494	501	502	498	482	504
			(625)	(551)	(537)	(501)	(505)	(488)	(504)
	0.25		58.44	71.61	81.70	98.18	135.56	192.24	312.80
			(58.30)	(64.31)	(72.82)	(89.98)	(130.63)	(187.02)	(311.50)
	0.50		19.22	23.18	25.57	29.10	36.58	54.61	126.77
			(16.96)	(18.18)	(19.01)	(21.38)	(30.27)	(50.51)	(124.78)
	1.00		5.93	7.24	7.81	8.62	9.77	11.82	24.44
			(4.85)	(4.95)	(5.16)	(5.50)	(6.19)	(8.41)	(22.060)
	1.50		3.06	3.72	4.02	4.40	4.90	5.45	8.13
			(2.07)	(2.29)	(2.40)	(2.48)	(2.69)	(3.07)	(6.19)
	2.50		1.53	1.72	1.84	2.01	2.17	2.33	2.65
			(0.74)	(0.83)	(0.90)	(0.96)	(1.02)	(1.08)	(1.43)
3		h_1	9.00	10.20	11.00	11.90	13.00	14.00	14.60
	0.00		504	488	495	483	473	498	490
			(635)	(541)	(521)	(496)	(475)	(497)	(485)
	0.25		67.25	82.04	93.13	111.13	154.84	228.20	340.06
			(64.37)	(73.04)	(82.56)	(10.69)	(149.68)	(224.68)	(342.13)
	0.50		22.73	26.91	29.42	33.09	42.25	66.77	153.04
			(18.91)	(19.66)	(21.33)	(24.54)	(35.25)	(62.48)	(15.85)
	1.00		7.29	8.53	9.03	9.76	10.85	13.75	30.68
			(5.06)	(5.57)	(5.62)	(5.89)	(6.68)	(9.67)	(28.86)
	1.50		3.85	4.30	4.64	4.95	5.41	6.06	9.58
			(2.37)	(2.51)	(2.62)	(2.70)	(2.89)	(3.44)	(7.70)
	2.50		1.84	2.01	2.10	2.22	2.38	2.55	2.91
			(0.92)	(0.97)	(1.00)	(1.06)	(1.10)	(1.17)	(1.60)
4		h_1	10.80	12.10	13.00	14.00	15.20	16.20	16.80
	0.00		500	492	503	506	505	501	496
			(623)	(546)	(530)	(523)	(513)	(500)	(502)
	0.25		71.92	88.82	102.08	127.39	176.99	2.54.83	365.86
			(69.64)	(79.56)	(91.42)	(119.16)	(169.92)	(248.83)	(362.06)
	0.50		23.80	28.44	31.96	36.44	48.13	78.83	177.91
			(20.28)	(21.60)	(23.37)	(27.36)	(39.80)	(74.97)	(177.33)
	1.00		7.37	8.74	9.58	10.52	12.17	15.12	36.42
			(5.44)	(5.79)	(6.09)	(6.33)	(7.37)	(10.81)	(34.68)
	1.50		3.82	4.50	4.90	5.34	5.83	6.70	11.34
			(2.51)	(2.69)	(2.83)	(2.93)	(3.19)	(3.82)	(9.30)
	2.50		1.86	2.06	2.20	2.37	2.56	2.72	3.18
			(0.93)	(1.01)	(1.07)	(1.11)	(1.18)	(1.25)	(1.74)
5		h_1	12.50	14.00	14.80	15.90	17.10	18.10	18.80
	0.00		499	504	494	499	488	485	495
			(625)	(560)	(520)	(514)	(489)	(485)	(493)
	0.25		74.75	98.27	111.00	136.86	187.48	269.60	377.46
			(74.12)	(86.67)	(102.35)	(129.26)	(178.73)	(263.70)	(372.21)
	0.50		26.87	31.66	34.47	39.28	53.20	87.29	199.40
			(21.24)	(23.50)	(24.52)	(29.06)	(45.89)	(82.18)	(196.75)
	1.00		8.17	9.99	10.72	11.50	12.98	16.61	43.11
			(5.84)	(6.21)	(6.50)	(6.77)	(7.90)	(12.24)	(40.68)
	1.50		4.55	5.03	5.36	5.79	6.24	7.12	12.91
			(2.73)	(2.83)	(3.00)	(3.10)	(3.26)	(4.01)	(10.88)
	2.50		2.16	2.32	2.41	2.55	2.70	2.87	3.44
			(1.03)	(1.11)	(1.14)	(1.19)	(1.23)	(1.12)	(1.93)

Note. Standard deviation of run length is in parentheses.

Table 77

The Upper Control Limits and Average Run Length Values of the Signed-Rank Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 500 from the Multivariate Normal Distribution

			λ						
p	δ		0.01	0.02	0.03	0.05	0.1	0.2	0.5
2		*L*	6.90	8.10	8.85	9.75	10.95	11.95	12.75
	0.00		496	492	499	494	502	491	482
			(599)	(556)	(539)	(517)	(507)	(484)	(479)
	0.25		60.51	72.30	82.34	99.10	139.49	205.31	315.56
			(58.07)	(63.57)	(73.68)	(91.71)	(134.16)	(200.77)	(309.19)
	0.50		20.04	23.60	25.73	29.07	37.75	58.31	130.24
			(16.58)	(17.89)	(19.28)	(21.90)	(30.95)	(52.92)	(125.96)
	1.00		6.32	7.42	8.21	8.81	9.95	12.32	26.05
			(4.61)	(4.93)	(5.31)	(5.47)	(6.24)	(8.74)	(23.92)
	1.50		3.36	3.83	4.18	4.51	4.93	5.60	8.78
			(2.09)	(2.26)	(2.44)	(2.58)	(2.67)	(3.19)	(6.95)
	2.50		1.65	1.81	1.91	2.06	2.22	2.40	2.73
			(0.80)	(0.89)	(0.91)	(0.96)	(1.02)	(1.10)	(1.46)
3		*L*	9.10	10.50	11.40	12.65	14.70	17.60	23.80
	0.00		487	497	502	500	497	497	504
			(634)	(568)	(558)	(533)	(520)	(505)	(500)
	0.25		63.45	83.17	97.48	124.44	197.21	309.43	455.59
			(66.68)	(79.85)	(91.36)	(121.37)	(193.48)	(311.71)	(456.01)
	0.50		19.94	25.64	29.45	35.21	53.98	121.00	325.60
			(18.20)	(20.57)	(22.79)	(27.02)	(47.88)	(119.55)	(323.42)
	1.00		6.20	8.00	8.97	10.24	12.89	21.95	118.43
			(4.89)	(5.53)	(5.80)	(6.35)	(8.05)	(17.33)	(117.12)
	1.50		3.15	4.06	4.52	5.09	6.13	8.15	38.87
			(2.15)	(2.47)	(2.61)	(2.80)	(3.22)	(4.72)	(36.49)
	2.50		1.44	1.84	2.03	2.28	2.64	3.13	6.49
			(0.70)	(0.88)	(0.94)	(1.05)	(1.14)	(1.33)	(4.48)
4		*L*	11.15	12.90	14.20	16.30	20.50	27.65	44.50
	0.00		494	494	498	498	499	500	500
			(659)	(585)	(563)	(541)	(522)	(515)	(505)
	0.25		73.54	95.05	116.70	168.17	292.27	414.18	494.08
			(73.97)	(92.36)	(115.55)	(173.42)	(306.87)	(424.22)	(501.77)
	0.50		24.34	30.61	35.19	47.32	98.37	244.56	440.20
			(21.19)	(24.02)	(27.19)	(37.85)	(94.88)	(245.11)	(438.85)
	1.00		8.03	9.53	10.63	12.84	19.00	57.38	304.36
			(5.53)	(6.24)	(6.68)	(7.65)	(12.31)	(52.69)	(304.84)
	1.50		4.17	4.98	5.46	6.32	8.34	16.53	165.90
			(2.44)	(2.80)	(2.93)	(3.30)	(4.21)	(11.35)	(162.85)
	2.50		1.96	2.25	2.44	2.75	3.41	4.75	35.37
			(0.86)	(0.98)	(1.05)	(1.14)	(1.37)	(1.86)	(33.16)
5		*L*	13.30	15.90	18.05	21.90	30.60	46.90	84.00
	0.00		495	500	501	503	501	501	493
			(669)	(615)	(575)	(556)	(527)	(507)	(497)
	0.25		83.83	119.63	157.31	251.28	382.48	472.85	484.76
			(83.77)	(121.54)	(160.08)	(270.44)	(399.29)	(483.38)	(488.94)
	0.50		28.92	37.00	45.54	74.17	204.96	391.20	472.17
			(24.10)	(29.02)	(36.24)	(66.29)	(205.60)	(396.67)	(476.26)
	1.00		9.47	11.58	13.31	17.39	38.14	180.66	417.38
			(6.15)	(7.13)	(7.97)	(10.00)	(29.00)	(181.51)	(409.12)
	1.50		5.11	5.95	6.78	8.39	13.26	61.01	331.98
			(2.71)	(3.12)	(3.46)	(4.02)	(6.60)	(55.38)	(330.77)
	2.50		*2.42*	*2.70*	*2.96*	*3.55*	*4.90*	*9.38*	*168.45*
			(0.94)	*(1.06)*	*(1.15)*	*(1.35)*	*(1.75)*	*(4.04)*	*(165.21)*

Note. Standard deviation of run length is in parentheses.

Table 78

The Upper Control Limits and Average Run Length Values of the Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 1,000 from the Multivariate Normal Distribution

			λ						
p	δ		0.01	0.02	0.03	0.05	0.1	0.2	0.5
2		h_1	8.30	9.60	10.30	11.20	12.30	13.10	13.70
	0.00		967	976	971	968	981	987	1,011
			(1,100)	(1,030)	(999)	(996)	(993)	(973)	(1,012)
	0.25		78.74	96.42	112.73	138.55	212.96	338.80	592.36
			(69.76)	(82.69)	(96.39)	(125.68)	(203.08)	(343.19)	(588.96)
	0.50		24.17	29.03	31.34	35.93	48.32	80.86	220.45
			(19.26)	(20.94)	(22.06)	(25.79)	(39.95)	(76.00)	(215.62)
	1.00		7.07	8.72	9.31	10.25	11.49	14.38	35.33
			(5.07)	(5.62)	(2.71)	(6.10)	(6.83)	(10.18)	(33.42)
	1.50		3.63	4.20	4.62	5.07	5.53	6.18	10.19
			(2.30)	(2.48)	(2.63)	(2.76)	(2.94)	(3.45)	(8.04)
	2.50		1.69	1.95	2.08	2.25	2.42	2.55	2.94
			(0.84)	(0.95)	(0.99)	(1.07)	(1.11)	(1.15)	(1.59)
3		h_1	10.60	11.80	12.60	13.60	14.80	15.60	16.20
	0.00		1,027	942	955	995	1,014	994	1,014
			(1,184)	(1,002)	(972)	(1,004)	(1,022)	(992)	(1,019)
	0.25		92.87	111.06	127.78	168.78	267.60	405.19	662.42
			(78.04)	(93.48)	(111.44)	(156.50)	(263.21)	(402.90)	(656.69)
	0.50		29.02	33.67	35.84	41.28	58.42	101.07	281.23
			(21.38)	(23.43)	(24.70)	(29.69)	(48.81)	(96.38)	(283.71)
	1.00		8.94	9.95	10.53	11.47	13.09	16.76	46.64
			(5.77)	(6.01)	(6.21)	(6.61)	(7.70)	(12.13)	(43.84)
	1.50		4.57	4.98	5.36	5.75	6.22	6.91	12.62
			(2.65)	(2.75)	(2.89)	(2.98)	(3.18)	(3.84)	(10.48)
	2.50		2.11	2.24	2.37	2.49	2.65	2.82	3.31
			(1.02)	(1.06)	(1.10)	(1.13)	(1.18)	(1.26)	(1.83)
4		h_1	12.50	13.90	14.80	15.80	16.90	17080	18.30
	0.00		1,010	962	1,006	1,004	973	1,006	985
			(1,150)	(1,011)	(1,029)	(1,032)	(978)	(1,003)	(975)
	0.25		99.18	122.45	147.80	192.27	291.72	449.13	693.38
			(85.03)	(102.24)	(128.73)	(179.86)	(281.78)	(446.28)	(691.43)
	0.50		30.94	36.25	39.72	45.86	64.79	120.75	312.13
			(23.43)	(24.93)	(27.01)	(33.01)	(55.75)	(116.88)	(310.32)
	1.00		9.11	10.50	11.48	12.44	13.96	18.81	55.72
			(6.16)	(6.40)	(6.67)	(7.13)	(8.47)	(13.74)	(53.39)
	1.50		4.52	5.23	5.74	6.10	6.63	7.59	14.48
			(2.80)	(2.94)	(3.11)	(3.13)	(3.37)	(4.26)	(12.26)
	2.50		2.04	2.34	2.49	2.65	2.83	2.99	3.55
			(1.04)	(1.12)	(1.17)	(1.22)	(1.26)	(1.33)	(1.99)
5		h_1	14.30	15.80	16.70	17.70	18.90	19.80	20.40
	0.00		1,001	996	996	980	974	974	985
			(1,128)	(1,049)	(1,008)	(1,015)	(959)	(965)	(983)
	0.25		110.24	135.81	161.01	209.37	320.54	482.92	732.91
			(90.26)	(113.43)	(142.45)	(195.36)	(308.27)	(485.13)	(730.76)
	0.50		34.37	39.31	42.41	50.25	73.18	135.84	355.85
			(24.51)	(26.44)	(28.42)	(37.24)	(61.71)	(131.21)	(355.10)
	1.00		10.44	11.78	12.41	13.37	15.13	20.98	67.22
			(6.40)	(6.86)	(6.96)	(7.41)	(8.91)	(5.79)	(64.83)
	1.50		5.34	5.92	6.18	6.54	7.09	8.07	17.16
			(2.96)	(3.12)	(3.21)	(3.33)	(3.58)	(4.58)	(14.96)
	2.50		2.44	2.60	2.71	2.82	2.98	3.16	3.88
			(1.14)	(1.20)	(1.24)	(1.26)	(1.32)	(1.40)	(2.23)

Note. Standard deviation of run length is in parentheses.

Table 79

The Upper Control Limits and Average Run Length Values of the Signed-Rank Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 1,000 from the Multivariate Normal Distribution

			λ						
p	δ		0.01	0.02	0.03	0.05	0.1	0.2	0.5
2		*L*	8.40	9.60	10.40	11.35	12.50	13.50	14.20
	0.00		1,007	985	990	1,003	995	992	967
			(1,151)	(1,047)	(1,024)	(1,033)	(1,007)	(982)	(960)
	0.25		83.21	96.92	113.27	145.00	225.56	356.92	593.27
			(71.13)	(80.26)	(97.61)	(132.17)	(222.12)	(362.23)	(595.55)
	0.50		25.51	29.46	32.28	37.22	49.83	87.85	227.82
			(19.38)	(20.76)	(22.32)	(26.89)	(40.86)	(83.30)	(229.10)
	1.00		7.77	8.92	9.50	10.34	11.66	15.19	38.08
			(5.20)	(5.60)	(5.77)	(6.02)	(6.90)	(10.75)	(35.63)
	1.50		3.95	4.48	4.77	5.16	5.66	6.40	11.02
			(2.37)	(2.53)	(2.62)	(2.77)	(2.97)	(3.56)	(9.11)
	2.50		1.89	2.04	2.16	2.31	2.46	2.64	3.10
			(0.91)	(0.96)	(1.00)	(1.07)	(1.11)	(1.18)	(1.68)
3		*L*	10.70	12.20	13.20	14.20	16.90	20.25	27.40
	0.00		996	993	996	1,004	993	992	994
			(1,180)	(1,019)	(1,056)	(1,035)	(1,010)	(998)	(1,019)
	0.25		**87.34**	113.84	140.38	179.71	353.79	606.84	875.22
			(81.08)	(102.33)	(127.29)	(170.82)	(356.41)	(609.62)	(888.06)
	0.50		**26.26**	**32.44**	36.90	43.56	81.58	208.82	607.44
			(21.75)	**(23.88)**	(26.94)	(33.21)	(74.49)	(204.87)	(623.17)
	1.00		**7.61**	**9.64**	10.64	11.81	15.66	30.78	219.68
			(5.63)	**(6.18)**	(6.49)	(6.94)	(9.71)	(24.83)	(218.42)
	1.50		**3.78**	**4.77**	**5.29**	5.77	7.11	10.11	67.05
			(2.46)	**(2.76)**	**(2.90)**	(3.00)	(3.60)	(5.87)	(66.02)
	2.50		**1.67**	**2.11**	**2.30**	2.50	2.94	3.58	9.06
			(0.83)	**(0.99)**	**(1.05)**	(1.11)	(1.26)	(1.48)	(6.66)
4		*L*	13.00	15.00	16.50	18.85	23.95	32.80	53.60
	0.00		988	988	997	1,000	1,001	992	989
			(1,187)	(1,121)	(1,096)	(1,078)	(1,053)	(1,020)	(989)
	0.25		100.12	136.53	181.43	290.42	554.93	819.76	937.25
			(92.87)	(126.91)	(179.26)	(297.15)	(563.57)	(843.67)	(936.07)
	0.50		31.82	38.37	45.67	63.16	168.09	489.67	849.45
			(24.77)	(28.17)	(34.27)	(50.95)	(163.65)	(491.70)	(843.00)
	1.00		9.61	11.43	12.97	15.41	25.00	106.08	586.70
			(6.41)	(6.91)	(7.59)	(8.74)	(16.09)	(99.79)	(575.21)
	1.50		4.84	5.74	6.35	7.36	10.10	24.93	324.48
			(2.75)	(3.07)	(3.30)	(3.69)	(4.85)	(18.69)	(323.49)
	2.50		2.22	2.54	2.80	3.15	3.93	5.79	74.59
			(0.97)	(1.10)	(1.17)	(1.28)	(1.52)	(2.27)	(71.30)
5		*L*	15.55	18.60	21.30	26.10	37.50	58.40	107.90
	0.00		1,005	1,010	998	1,006	998	983	983
			(1,269)	(1,178)	(1,103)	(1,092)	(1,046)	(997)	(973)
	0.25		116.80	180.93	269.54	472.39	785.62	920.76	965.83
			(108.83)	(174.33)	(283.60)	(498.75)	(813.27)	(928.77)	(958.74)
	0.50		36.81	47.84	62.88	118.70	412.80	774.33	940.47
			(28.26)	(34.53)	(47.81)	(108.19)	(427.53)	(780.33)	(946.45)
	1.00		11.61	14.12	16.60	22.45	66.30	384.64	834.64
			(7.14)	(8.28)	(9.30)	(12.38)	(55.39)	(391.47)	(824.48)
	1.50		6.00	7.06	8.11	10.10	18.22	132.42	695.32
			(3.12)	(3.51)	(3.88)	(4.63)	(9.24)	(127.31)	(691.56)
	2.50		2.70	3.10	3.48	4.15	6.03	14.32	369..98
			(1.06)	(1.21)	(1.33)	(1.56)	(2.07)	(7.42)	(363.46)

Note. Standard deviation of run length is in parentheses.

Table 80

The Upper Control Limits and Average Run Length Values of the Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 200 from the Multivariate $t_p(5)$- Distribution

			λ						
p			0.01	0.02	0.03	0.05	0.1	0.2	0.5
2	δ	h_1	5.30	6.20	6.80	7.70	9.00	10.30	12.30
	0.00		200	198	195	197	202	198	200
			(290.00)	(246)	(244)	(212)	(208)	(198)	(200)
	0.25		38.31	45.63	50.12	59.11	77.35	105.85	160.7
			(43.17)	(46.42)	(49.19)	(55.92)	(74.18)	(104.85)	(159.49)
	0.50		13.71	16.67	18.54	20.80	26.06	38.20	93.25
			(13.38)	(14.55)	(15.41)	(16.79)	(21.37)	(35.52)	(90.85)
	1.00		4.79	5.62	6.13	6.93	8.04	9.80	24.24
			(3.77)	(4.11)	(4.33)	(4.63)	(5.27)	(6.90)	(22.49)
	1.50		2.66	2.99	3.30	3.62	4.11	4.76	8.38
			(1.76)	(1.90)	(2.04)	(2.15)	(2.33)	(2.73)	(6.53)
	2.50		1.40	1.50	1.60	1.73	1.93	2.12	2.64
			(0.65)	(0.70)	(0.77)	(0.83)	(0.92)	(0.99)	(1.38)
3		h_1	7.10	8.20	8.90	9.90	11.30	12.70	15.00
	0.00		199	200	201	202	199	200	198
			(294)	(247)	(234)	(219)	(206)	(198)	(199)
	0.25		41.48	51.54	56.16	67.63	88.96	118.18	166.60
			(47.57)	(51.58)	(55.02)	(65.77)	(86.71)	(116.92)	(163.88)
	0.50		15.19	18.82	20.69	23.58	30.25	45.71	106.47
			(14.65)	(16.23)	(17.30)	(19.25)	(25.09)	(42.39)	(104.71)
	1.00		5.26	6.31	6.90	7.70	9.11	11.37	30.70
			(4.12)	(4.63)	(4.80)	(5.03)	(5.84)	(8.15)	(29.04)
	1.50		2.89	3.32	3.60	4.07	4.61	5.42	10.46
			(1.93)	(2.13)	(2.20)	(2.40)	(2.63)	(3.10)	(8.64)
	2.50		1.50	1.65	1.78	1.89	2.12	3.34	3.02
			(0.72)	(0.80)	(0.87)	(0.91)	(1.01)	(1.08)	(1.63)
4		h_1	8.80	10.00	10.80	11.80	13.20	14.80	17.50
	0.00		192	201	201	200	196	198	199
			(280)	(247)	(233)	(214)	(204)	(200)	(197)
	0.25		47.11	57.57	64.36	72.97	92.65	125.48	176.02
			(52.38)	(56.97)	(62.77)	(71.38)	(92.28)	(127.11)	(171.98)
	0.50		18.48	21.37	23.24	26.13	32.21	51.51	120.17
			(16.80)	(18.29)	(19.00)	(20.85)	(28.33)	(47.89)	(121.42)
	1.00		6.68	7.31	7.80	8.56	9.87	12.76	38.23
			(4.84)	(5.10)	(5.33)	(5.54)	(6.36)	(9.23)	(36.96)
	1.50		3.70	3.93	4.17	4.53	5.02	5.88	12.68
			(2.22)	(2.36)	(2.47)	(2.61)	(2.78)	(3.37)	(10.68)
	2.50		1.89	1.94	1.99	2.10	2.28	2.52	3.41
			(0.88)	(0.94)	(0.95)	(0.99)	(1.06)	(1.16)	(1.90)
5		h_1	10.40	11.60	12.50	13.60	15.20	16.80	19.70
	0.00		192	196	200	194	201	200	197
			(280)	(248)	(229)	(214)	(209)	(200)	(194)
	0.25		47.31	58.18	65.21	77.10	102.13	130.92	176.93
			(55.68)	(60.50)	(64.69)	(75.39)	(102.23)	(12.85)	(175.00)
	0.50		17.92	21.83	24.04	27.63	36.92	57.13	125.52
			(17.23)	(18.83)	(19.98)	(22.63)	(31.70)	(54.52)	(123.27)
	1.00		6.15	7.31	8.09	8.92	10.56	13.97	44.35
			(4.91)	(5.38)	(5.56)	(5.84)	(6.74)	(10.09)	(43.08)
	1.50		3.40	3.91	4.30	4.75	5.33	6.27	14.62
			(2.30)	(2.46)	(2.62)	(2.76)	(2.92)	(3.64)	(12.65)
	2.50		1.70	1.89	2.02	2.19	2.40	2.65	3.72
			(0.86)	(0.93)	(0.99)	(1.04)	(1.13)	(1.25)	(2.14)

Note. Standard deviation of run length is in parentheses.

Table 81

The Upper Control Limits and Average Run Length Values of the Signed-Rank Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 200 from the Multivariate $t_p(5)$- Distribution

			λ						
p			0.01	0.02	0.03	0.05	0.1	0.2	0.5
2	δ	*L*	5.35	6.30	6.90	7.85	9.20	10.70	13.40
	0.00		230	201	197	200	201	199	200
			(300)	(248)	(226)	(221)	(208)	(205)	(204)
	0.25		38.91	47.36	51.34	60.82	80.56	112.22	164.98
			(43.33)	(47.58)	(49.45)	(59.76)	(77.97)	(112.97)	(164.07)
	0.50		14.57	17.05	18.72	21.49	27.12	41.68	101.41
			(13.90)	(14.92)	(15.79)	(17.22)	(22.43)	(38.00)	(101.40)
	1.00		4.95	5.80	6.28	7.03	8.35	10.63	29.88
			(3.71)	(4.14)	(4.32)	(4.74)	(5.36)	(7.57)	(27.75)
	1.50		2.84	3.17	3.33	3.72	4.24	4.98	10.16
			(1.79)	(1.94)	(2.00)	(2.17)	(2.40)	(2.80)	(8.33)
	2.50		1.46	1.56	1.63	1.78	1.99	2.19	2.91
			(0.68)	(0.74)	(0.76)	(0.84)	(0.92)	(1.00)	(1.53)
3		*L*	7.10	8.35	9.25	10.70	13.30	17.65	2.40
	0.00		196	200	200	200	200	200	200
			(283)	(253)	(241)	(227)	(215)	(206)	(203)
	0.25		43.22	53.22	61.27	79.69	115.66	163.46	188.95
			(49.03)	(55.90)	(61.59)	(81.16)	(118.31)	(167.63)	(190.11)
	0.50		16.26	19.71	22.38	27.18	43.25	91.76	168.17
			(15.06)	(17.08)	(18.44)	(22.02)	(38.55)	(90.89)	(166.17)
	1.00		6.64	6.75	7.42	8.72	11.35	22.39	105.31
			(4.00)	(4.63)	(4.95)	(5.52)	(7.08)	(18.19)	(103.6)
	1.50		3.12	3.59	3.94	4.51	5.60	8.48	53.01
			(1.82)	(2.06)	(2.22)	(2.40)	(2.91)	(4.92)	(51.02)
	2.50		1.58	1.75	1.87	2.08	2.45	3.18	9.74
			(0.69)	(0.76)	(0.81)	(0.89)	(1.03)	(1.28)	(7.29)
4		*L*	8.75	10.35	11.70	14.00	19.20	28.30	49.80
	0.00		199	198	199	201	201	199	200
			(296)	(252)	(240)	(229)	(211)	(206)	(199)
	0.25		45.22	60.28	75.31	105.20	155.38	191.56	196.88
			(53.34)	(64.29)	(79.62)	(113.01)	(164.26)	(192.02)	(197.67)
	0.50		16.21	22.00	26.73	36.33	75.55	146.52	188.51
			(15.70)	(18.81)	(21.96)	(30.93)	(74.50)	(148.49)	(186.32)
	1.00		5.40	7.29	8.59	10.71	17.85	58.89	160.96
			(4.02)	(4.88)	(5.54)	(6.47)	(11.50)	(56.08)	(162.93)
	1.50		2.77	3.79	4.38	5.36	7.91	18.75	122.07
			(1.68)	(2.08)	(2.32)	(2.76)	(3.86)	(13.60)	(122.30)
	2.50		1.24	1.76	1.99	2.41	3.20	4.94	49.29
			(0.52)	(0.71)	(0.79)	(0.95)	(1.21)	(1.85)	(48.18)
5		*L*	10.30	12.69	14.70	18.60	27.55	43.80	81.90
	0.00		200	201	201	202	199	200	198
			(297)	(262)	(243)	(231)	(209)	(208)	(201)
	0.25		54.42	76.16	96.06	134.85	178.49	190.74	198.43
			(59.37)	(78.89)	(104.47)	(150.44)	(185.49)	(196.67)	(197.72)
	0.50		20.86	28.08	34.70	54.74	120.79	176.05	196.04
			(18.07)	(22.45)	(28.02)	(48.94)	(124.97)	(178.26)	(194.71)
	1.00		7.14	9.27	11.01	14.86	32.98	111.64	183.36
			(4.56)	(5.66)	(6.36)	(8.46)	(25.04)	(110.42)	(184.40)
	1.50		3.85	4.81	5.65	7.17	11.98	51.88	162.28
			(1.96)	(2.37)	(2.71)	(3.32)	(5.61)	(47.46)	(160.67)
	2.50		1.82	2.19	2.53	3.10	4.46	8.53	107.64
			(0.63)	(0.80)	(0.91)	(1.10)	(1.53)	(3.37)	(109.46)

Note. Standard deviation of run length is in parentheses.

Table 82

The Upper Control Limits and Average Run Length Values of the Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 500 from the Multivariate $t_p(5)$- Distribution

			λ						
p	δ		0.01	0.02	0.03	0.05	0.1	0.2	0.5
2		h_1	6.90	8.10	8.90	9.80	11.20	12.70	15.70
	0.00		493	488	492	474	489	486	500
			(605)	(544)	(528)	(486)	(496)	(491)	(501)
	0.25		59.92	72.08	82.81	100.95	143.73	233.51	407.26
			(58.31)	(65.25)	(74.17)	(92.18)	(141.69)	(233.07)	(407.61)
	0.50		19.46	23.58	26.44	29.77	39.81	69.90	237.60
			(169.61)	(18.03)	(19.40)	(22.28)	(33.39)	(66.05)	(235.31)
	1.00		6.25	7.45	8.05	8.85	10.29	13.61	54.41
			(4.53)	(4.94)	(5.21)	(5.45)	(6.28)	(9.81)	(52.46)
	1.50		3.29	3.84	4.14	4.47	5.14	6.00	15.00
			(2.04)	(2.29)	(2.38)	(2.51)	(2.73)	(3.33)	(12.80)
	2.50		1.61	1.80	1.93	2.04	2.26	2.50	3.51
			(0.79)	(0.86)	(0.94)	(0.96)	(1.03)	(1.13)	(1.95)
3		h_1	9.00	10.30	11.10	12.20	13.70	15.40	18.80
	0.00		507	500	492	491	493	492	507
			(634)	(553)	(434)	(510)	(500)	(490)	(510)
	0.25		67.16	81.91	93.94	117.14	174.66	277.41	424.80
			(64.62)	(74.27)	(85.08)	(108.68)	(171.45)	(273.43)	(431.93)
	0.50		22.31	26.54	29.03	34.05	46.57	89.49	274.92
			(18.73)	(20.10)	(21.65)	(25.79)	(39.43)	(85.61)	(274.27)
	1.00		7.16	8.33	9.07	9.92	11.67	16.34	75.43
			(5.04)	(5.54)	(5.81)	(6.06)	(7.24)	(12.12)	(72.71)
	1.50		3.67	4.19	4.53	5.07	5.68	6.82	20.00
			(2.29)	(2.50)	(2.59)	(2.83)	(3.00)	(3.91)	(17.63)
	2.50		1.73	1.94	2.07	2.26	2.49	2.77	4.08
			(0.85)	(0.94)	(0.98)	(1.06)	(1.13)	(1.23)	(2.34)
4		h_1	10.80	12.20	13.20	14.20	15.90	17.70	21.50
	0.00		488	491	498	482	504	501	493
			(617)	(545)	(498)	(504)	(521)	(501)	(494)
	0.25		76.08	92.29	105.13	129.50	193.91	293.52	435.97
			(72.07)	(83.13)	(94.55)	(122.91)	(192.41)	(295.04)	(431.24)
	0.50		26.12	30.06	32.91	36.89	53.60	102.60	304.80
			(20.48)	(22.02)	(23.67)	(27.27)	(45.68)	(99.35)	(305.58)
	1.00		8.63	9.38	10.16	10.91	12.95	18.45	93.32
			(5.64)	(5.89)	(6.17)	(6.50)	(7.97)	(13.90)	(91.87)
	1.50		4.58	4.91	5.16	5.54	6.18	7.39	24.75
			(2.58)	(2.73)	(2.80)	(2.95)	(3.26)	(4.14)	(22.33)
	2.50		2.19	2.26	2.35	2.46	2.69	2.98	4.67
			(1.00)	(1.06)	(1.09)	(1.13)	(1.20)	(1.33)	(2.82)
5		h_1	12.60	14.10	15.0	16.20	17.80	19.80	23.90
	0.00		496	501	496	494	490	503	486
			(619)	(562)	(534)	(519)	(498)	(507)	(489)
	0.25		78.90	96.73	111.39	142.65	203.74	308.14	437.01
			(76.32)	(88.77)	(101.80)	(135.24)	(199.17)	(306.56)	(435.84)
	0.50		26.14	30.69	34.73	40.29	56.97	115.58	323.66
			(21.87)	(23.33)	(25.18)	(30.48)	(48.46)	(112.66)	(326.35)
	1.00		8.31	9.79	10.41	11.58	13.60	20.43	112.13
			(5.98)	(6.34)	(6.52)	(6.96)	(8.40)	(15.71)	(109.40)
	1.50		4.32	4.93	5.31	5.84	6.53	8.00	30.13
			(2.68)	(2.85)	(2.99)	(3.14)	(3.39)	(4.57)	(27.82)
	2.50		2.00	2.25	2.39	2.56	2.81	3.168	5.18
			(0.99)	(1.09)	(1.14)	(1.19)	(1.27)	(1.41)	(3.26)

Note. Standard deviation of run length is in parentheses.

Table 83

The Upper Control Limits and Average Run Length Values of the Signed-Rank Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 500 from the Multivariate $t_p(5)$- Distribution

			λ						
p	δ		0.01	0.02	0.03	0.05	0.1	0.2	0.5
2		*L*	6.95	8.20	8.95	10.00	11.55	13.50	17.50
	0.00		495	500	497	501	493	496	501
			(609)	(558)	(539)	(527)	(500)	(504)	(501)
	0.25		60.36	73.90	83.53	104.71	157.01	259.49	424.97
			(57.74)	(65.40)	(76.69)	(97.00)	(152.40)	(258.56)	(416.26)
	0.50		20.36	23.97	26.25	30.67	42.35	82.33	273.16
			(17.03)	(18.15)	(19.83)	(23.10)	(35.54)	(79.53)	(272.22)
	1.00		6.56	7.57	8.26	9.19	10.76	15.34	73.58
			(4.49)	(5.03)	(5.26)	(5.56)	(6.64)	(11.22)	(71.69)
	1.50		3.49	3.94	4.27	4.63	5.29	6.47	20.80
			(2.07)	(2.29)	(2.42)	(2.50)	(2.78)	(3.63)	(18.90)
	2.50		1.69	1.86	1.95	2.09	2.34	2.65	4.08
			(0.79)	(0.87)	(0.91)	(0.95)	(1.04)	(1.15)	(2.31)
3		*L*	9.20	10.75	11.95	13.80	17.50	24.15	40.10
	0.00		504	498	503	501	500	495	496
			(650)	(579)	(564)	(546)	(516)	(496)	(493)
	0.25		69.70	87.95	109.47	155.02	275.10	414.57	482.74
			(67.86)	(81.83)	(104.44)	(154.19)	(282.80)	(421.05)	(480.92)
	0.50		23.32	28.06	32.29	41.94	85.82	248.76	441.43
			(19.20)	(21.45)	(23.93)	(33.17)	(81.15)	(246.24)	(438.39)
	1.00		7.51	8.90	9.86	11.70	16.68	51.99	309.48
			(5.00)	(5.58)	(5.97)	(6.62)	(10.24)	(47.41)	(306.82)
	1.50		3.97	4.58	5.01	5.80	7.51	14.17	176.74
			(2.25)	(2.48)	(2.66)	(2.96)	(3.68)	(8.92)	(174.33)
	2.50		1.87	2.08	2.28	2.55	3.11	4.33	32.54
			(0.81)	(0.88)	(0.96)	(1.06)	(1.24)	(1.65)	(30.00)
4		*L*	11.30	13.62	15.60	19.20	27.60	43.00	79.00
	0.00		501	502	506	503	501	501	497
			(667)	(604)	(578)	(541)	(521)	(505)	(493)
	0.25		76.00	109.75	153.05	244.85	412.40	483.72	488.96
			(76.59)	(108.63)	(154.96)	(256.84)	(425.25)	(493.73)	(494.61)
	0.50		24.50	33.43	42.21	69.86	212.10	400.25	474.16
			(20.77)	(25.47)	(32.29)	(60.68)	(214.56)	(402.21)	(471.84)
	1.00		7.46	10.15	12.15	15.97	35.00	191.11	430.84
			(5.16)	(6.15)	(7.06)	(8.77)	(25.36)	(187.82)	(430.54)
	1.50		3.74	5.08	6.03	7.62	12.36	60.62	348.80
			(2.19)	(2.67)	(2.97)	(3.65)	(5.64)	(53.05)	(348.90)
	2.50		1.64	2.21	2.60	3.19	4.51	8.45	184.92
			(0.72)	(0.90)	(1.02)	(1.21)	(1.60)	(3.30)	(185.17)
5		*L*	13.70	17.60	20.90	27.60	43.40	72.90	142.30
	0.00		498	497	497	502	498	499	500
			(663)	(589)	(563)	(548)	(519)	(507)	(500)
	0.25		92.82	158.47	232.12	350.48	451.09	484.18	466.93
			(91.76)	(160.15)	(245.54)	(372.37)	(466.83)	(492.87)	(487.79)
	0.50		31.80	45.37	64.41	136.87	335.31	444.15	490.88
			(24.63)	(33.31)	(49.44)	(131.07)	(348.58)	(449.58)	(492.93)
	1.00		9.93	13.41	16.68	25.01	103.93	339.13	170.90
			(6.00)	(7.48)	(8.83)	(13.30)	(96.48)	(338.08)	(467.27)
	1.50		5.17	6.67	8.07	11.00	24.74	196.52	434.44
			(2.54)	(3.13)	(3.64)	(4.67)	(13.55)	(191.60)	(436.35)
	2.50		2.27	2.91	3.42	4.46	7.09	29.54	337.41
			(0.85)	(1.03)	(1.21)	(1.48)	(2.20)	(21.34)	(338.54)

Note. Standard deviation of run length is in parentheses.

Table 84

The Upper Control Limits and Average Run Length Values of the Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 1,000 from the Multivariate $t_p(5)$- Distribution

			λ						
p	δ		0.01	0.02	0.03	0.05	0.1	0.2	0.5
2		h_1	8.30	9.70	10.50	11.50	13.0	14.80	18.80
	0.00		956	975	1,006	984	984	995	997
			(1,091)	(1,047)	(1,054)	(1,010)	(982)	(1,022)	(993)
	0.25		79.61	100.02	116.35	146.30	248.20	451.94	829.65
			(69.69)	(84.86)	(100.40)	(135.15)	(243.19)	(456.02)	(836.81)
	0.50		25.20	29.59	32.30	36.77	54.37	118.96	513.42
			(19.46)	(20.84)	(22.60)	(26.68)	(45.16)	(114.87)	(518.66)
	1.00		7.60	8.94	9.67	10.58	12.31	17.93	113.52
			(5.14)	(5.54)	(5.80)	(6.24)	(7.41)	(13.24)	(110.85)
	1.50		3.86	4.51	4.81	5.23	5.88	7.21	26.22
			(2.30)	(2.53)	(2.63)	(2.76)	(3.05)	(4.01)	(23.53)
	2.50		1.81	2.03	2.15	2.31	2.53	2.83	4.47
			(0.86)	(0.96)	(1.02)	(1.06)	(1.13)	(1.22)	(2.57)
3		h_1	10.60	12.00	12.90	14.00	15.60	17.60	22.10
	0.00		998	985	1,007	1,002	1,006	1,003	964
			(1,161)	(1,076)	(1,060)	(1,038)	(1,026)	(1,007)	(966)
	0.25		93.00	111.44	135.13	179.27	306.13	524.51	847.64
			(78.46)	(95.51)	(118.27)	(167.31)	(300.75)	(521.30)	(842.48)
	0.50		28.39	33.15	36.84	43.33	66.519	152.54	576.62
			(21.24)	(23.46)	(25.03)	(31.16)	(56.76)	(146.63)	(567.46)
	1.00		8.58	9.88	10.73	11.79	13.88	21.76	159.32
			(5.60)	(6.15)	(6.41)	(6.74)	(8.18)	(16.63)	(160.23)
	1.50		4.33	5.00	2.35	5.79	6.56	8.11	37.17
			(2.54)	(2.77)	(2.88)	(3.01)	(3.36)	(4.53)	(34.24)
	2.50		1.97	2.19	2.36	2.54	2.78	3.16	5.39
			(0.95)	(1.03)	(1.09)	(1.14)	(1.22)	(1.35)	(3.33)
4		h_1	12.50	14.00	15.00	16.20	17.90	20.00	25.30
	0.00		983	971	1,005	1,002	1,022	995	985
			(1,145)	(1,029)	(1,078)	(1,022)	(1,019)	(1,013)	(1,002)
	0.25		103.37	125.64	150.27	202.22	343.43	576.40	914.15
			(86.73)	(106.06)	(134.04)	(187.93)	(334.60)	(574.35)	(912.51)
	0.50		33.15	36.67	40.73	47.81	76.44	183.41	662.16
			(23.36)	(25.02)	(28.10)	(35.19)	(66.79)	(180.18)	(664.80)
	1.00		10.33	11.20	11.96	13.03	15.34	25.41	211.76
			(6.19)	(6.57)	(6.73)	(7.26)	(9.23)	(20.08)	(209.37)
	1.50		5.29	5.63	5.98	6.36	7.12	9.04	51.21
			(2.83)	(2.99)	(3.10)	(3.22)	(3.55)	(5.18)	(49.17)
	2.50		2.45	2.53	2.63	2.77	3.01	3.40	6.45
			(1.09)	(1.14)	(1.17)	(1.12)	(1.31)	(1.46)	(4.20)
5		h_1	14.40	16.00	16.90	18.10	19.90	22.20	27.90
	0.00		1,001	1,007	983	1,017	992	984	996
			(1,152)	(1,068)	(1,013)	(1,058)	(995)	(986)	(1,090)
	0.25		108.14	134.88	160.42	216.90	367.53	597.20	895.20
			(93.45)	(115.57)	(144.71)	(203.83)	(360.77)	(590.33)	(892.94)
	0.50		33.38	38.68	43.03	51.50	84.88	211.39	707.00
			(24.75)	(26.36)	(29.27)	(37.75)	(75.19)	(206.03)	(696.38)
	1.00		10.09	11.62	12.51	13.57	16.24	28.588	251.83
			(6.59)	(6.89)	(7.30)	(7.74)	(9.76)	(22.90)	(246.78)
	1.50		5.04	5.75	6.14	6.60	7.51	9.80	63.40
			(2.96)	(3.13)	(3.26)	(3.38)	(3.78)	(5.63)	(60.74)
	2.50		2.28	2.52	2.69	2.86	3.13	3.57	7.36
			(1.12)	(1.18)	(1.22)	(1.28)	(1.34)	(1.51)	(4.97)

Note. Standard deviation of run length is in parentheses.

Table 85

The Upper Control Limits and Average Run Length Values of the Signed-Rank Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 1,000 from the Multivariate $t_p(5)$ Distribution

			λ						
p	δ		0.01	0.02	0.03	0.05	0.1	0.2	0.5
2		*L*	8.40	9.80	10.60	11.75	13.50	15.80	21.15
	0.00		991	1,010	998	993	1,007	997	981
			(1,132)	(1,087)	(1,042)	(1,020)	(1,021)	(1,015)	(972)
	0.25		82.39	100.32	118.13	158.85	276.64	499.36	876.76
			(70.38)	(84.64)	(103.67)	(149.20)	(268.10)	(495.19)	(873.96)
	0.50		25.77	30.46	32.82	38.74	59.31	143.37	579.86
			(19.39)	(21.47)	(22.69)	(28.05)	(50.43)	(140.85)	(580.42)
	1.00		7.86	9.09	9.87	10.79	12.91	20.84	159.60
			(5.10)	(5.60)	(5.89)	(6.10)	(7.62)	(15.56)	(157.76)
	1.50		4.08	4.61	4.90	5.33	6.16	7.80	38.24
			(2.36)	(2.55)	(2.62)	(2.75)	(3.08)	(4.37)	(35.77)
	2.50		1.91	2.08	2.22	2.36	2.64	3.01	5.52
			(0.89)	(0.95)	(1.02)	(1.06)	(1.14)	(1.27)	(3.47)
3		*L*	10.90	12.75	14.15	16.40	21.20	30.20	52.20
	0.00		997	997	1,006	998	982	996	999
			(1,179)	(1,093)	(1,082)	(1,076)	(1,016)	(1,002)	(985)
	0.25		94.32	125.16	164.19	267.22	556.61	834.21	942.23
			(82.60)	(111.37)	(151.99)	(267.45)	(568.27)	(844.72)	(937.06)
	0.50		29.50	36.31	42.31	58.25	158.50	526.33	880.94
			(22.11)	(25.40)	(29.86)	(44.98)	(153.06)	(520.62)	(890.63)
	1.00		9.23	10.80	12.03	14.30	22.57	108.84	665.17
			(5.85)	(6.31)	(6.72)	(7.87)	(13.88)	(100.71)	(664.35)
	1.50		4.63	5.40	5.89	6.93	9.28	23.15	409.59
			(2.54)	(2.80)	(3.00)	(3.28)	(4.32)	(16.30)	(411.25)
	2.50		2.11	2.39	2.60	2.96	3.69	5.51	93.19
			(0.92)	(1.00)	(1.09)	(1.17)	(1.38)	(2.07)	(91.61)
4		*L*	13.50	16.45	19.05	23.95	35.50	58.00	111.10
	0.00		998	992	992	999	992	988	997
			(1,246)	(1,132)	(1,099)	(1,041)	(1,007)	(984)	(987)
	0.25		108.38	170.69	270.31	515.02	817.43	943.14	996.9
			(98.33)	(162.48)	(275.08)	(522.18)	(839.57)	(947.81)	(999.07)
	0.50		32.36	45.55	59.66	122.05	465.15	823.31	982.98
			(25.31)	(31.84)	(43.42)	(111.14)	(463.09)	(840.16)	(982.92)
	1.00		9.39	12.78	15.48	21.46	67.87	457.19	890.51
			(6.20)	(7.36)	(8.30)	(11.09)	(55.49)	(464.79)	(881.57)
	1.50		4.56	6.31	7.41	9.69	17.58	169.83	783.17
			(2.52)	(3.12)	(3.48)	(4.27)	(8.35)	(163.55)	(788.80)
	2.50		1.94	2.61	3.08	3.90	5.85	15.02	483.22
			(0.84)	(1.05)	(1.19)	(1.42)	(1.95)	(7.67)	(482.63)
5		*L*	16.90	22.20	27.20	37.30	61.20	106.90	212.10
	0.00		996	1,000	995	996	1,000	1,000	1,006
			(1,238)	(1,152)	(1,091)	(1,059)	(1,010)	(1,009)	(1,006)
	0.25		142.27	289.74	491.37	752.58	910.47	990.63	973.42
			(128.30)	(292.96)	(525.26)	(796.65)	(912.74)	(998.40)	(969.13)
	0.50		42.49	66.27	109.66	327.65	738.87	937.27	979.06
			(29.92)	(45.24)	(88.26)	(328.63)	(741.88)	(940.32)	(1,001.09)
	1.00		12.78	17.60	22.98	39.70	290.46	746.42	935.70
			(7.23)	(9.05)	(11.25)	(21.51)	(287.43)	(755.25)	(940.07)
	1.50		6.39	8.61	10.76	15.69	59.69	518.06	901.28
			(3.03)	(3.79)	(4.48)	(6.00)	(44.92)	(514.47)	(902.16)
	2.50		2.67	3.61	4.34	5.94	10.41	122.44	740.84
			(1.00)	(1.25)	(1.48)	(1.87)	(3.13)	(116.12)	(748.72)

Note. Standard deviation of run length is in parentheses.

Table 86

The Upper Control Limits and Average Run Length Values of the Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 200, 500, and 1,000 under Multivariate Gamma$_2$ (α=3, β=1) Distribution

			λ						
ARL_0	δ		0.01	0.02	0.03	0.05	0.1	0.2	0.5
200		h_1	4,124.1	2,116.6	1,442.9	896.6	476.8	257.8	116.1
	0.00		200	200	200	200	200	200	200
			(215.23)	(201.53)	(195.80)	(192.00)	(193.90)	(196.60)	(202.30)
	0.25		58.24	62.89	65.75	73.06	80.47	95.80	130.19
			(47.74)	(50.77)	(53.43)	(63.22)	(73.11)	(90.70)	(129.27)
	0.50		29.78	31.05	32.08	32.95	37.47	47.43	78.21
			(18.94)	(19.65)	(21.15)	(23.37)	(31.23)	(43.78)	(76.99)
	1.00		14.50	14.73	14.52	14.22	13.79	15.53	29.06
			(6.43)	(6.53)	(6.66)	(7.13)	(8.10)	(11.27)	(27.52)
	1.50		6.70	9.54	9.35	8.68	7.97	7.90	12.34
			(3.47)	(3.41)	(3.35)	(3.30)	(3.46)	(4.36)	(10.50)
	2.50		5.88	5.71	5.50	5.03	4.36	3.81	3.88
			(1.60)	(1.57)	(1.49)	(1.41)	(1.33)	(1.34)	(2.16)
500		h_1	4,209.5	2,175.5	1,486.7	928.5	498.3	272.6	125.8
	0.00		500	500	500	500	501	498	503
			(485.80)	(480.18)	(480.08)	(490.50)	(499.80)	(503.80)	(502.16)
	0.25		109.17	115.43	122.14	135.68	170.89	217.21	316.65
			(77.90)	(88.44)	(102.09)	(120.26)	(161.73)	(213.21)	(318.75)
	0.50		50.59	49.42	48.56	51.75	63.85	93.26	177.47
			(26.06)	(28.52)	(30.40)	(37.01)	(53.76)	(86.92)	(173.15)
	1.00		23.49	21.62	20.25	18.91	19.18	24.13	58.86
			(8.42)	(8.29)	(8.46)	(9.03)	(11.23)	(18.92)	(56.08)
	1.50		15.42	13.69	12.49	11.28	10.24	10.55	21.04
			(4.47)	(4.28)	(4.03)	(3.97)	(4.42)	(6.14)	(18.51)
	2.50		9.14	7.98	7.15	6.30	5.23	4.53	5.16
			(2.00)	(1.84)	(1.73)	(1.60)	(1.50)	(1.57)	(3.21)
1,000		h_1	4,268.4	2,211.7	1,515.7	948.8	512.0	282.8	132.7
	0.00		1,002	1,002	998	1,000	1,007	1,001	999
			(959.10)	(993.84)	(993.66)	(995.40)	(1,003.3)	(993.05)	(982.17)
	0.25		159.02	171.30	185.61	216.14	288.32	411.96	601.31
			(109.35)	(135.49)	(157.16)	(197.23)	(278.74)	(404.02)	(603.94)
	0.50		67.21	63.78	64.41	70.43	93.85	156.50	331.37
			(32.22)	(36.21)	(40.64)	(52.68)	(83.42)	(149.59)	(327.96)
	1.00		29.91	25.95	24.17	22.71	23.47	33.68	98.39
			(9.71)	(9.43)	(9.80)	(10.55)	(14.41)	(27.83)	(95.80)
	1.50		19.28	16.15	14.70	13.00	11.75	12.90	32.58
			(4.96)	(4.55)	(4.52)	(4.43)	(4.96)	(7.76)	(30.41)
	2.50		11.37	9.36	8.33	7.08	5.18	5.11	6.53
			(2.23)	(1.99)	(1.88)	(1.72)	(1.59)	(1.78)	(4.28)

Note. Standard deviation of run length is in parentheses.

Table 87

The Upper Control Limits and Average Run Length Values of the Signed-Rank Multivariate Exponentially Weighted Moving Average That Achieved an In-Control Average Run Length ≈ 200, 500, and 1,000 under Multivariate Gamma$_2$ (α=3, β=1) Distribution

			λ						
ARL$_0$	δ		0.01	0.02	0.03	0.05	0.1	0.2	0.5
200		*L*	1,366.3	723.1	505.6	328.5	190.3	116.2	65.9
	0.00		201	200	200	200	200	200	201
			(236.48)	(209.37)	(200.50)	(196.52)	(199.56)	(196.41)	(200.60)
	0.25		10.46	11.40	11.53	11.72	12.38	15.24	30.93
			(4.60)	(5.10)	(5.28)	(5.81)	(7.25)	(11.33)	(29.17)
	0.50		5.24	5.61	5.56	5.40	5.18	5.18	8.04
			(1.60)	(1.68)	(1.71)	(1.77)	(1.93)	(2.42)	(6.23)
	1.00		2.97	2.92	2.87	2.74	2.51	2.24	2.14
			(0.59)	(0.61)	(0.61)	(0.62)	(0.61)	(0.62)	(0.85)
	1.50		2.00	2.05	2.02	1.96	1.83	1.60	1.26
			(0.27)	(0.32)	(0.29)	(0.28)	(0.38)	(0.49)	(0.44)
	2.50		1.21	1.34	1.26	1.01	1.00	1.00	1.00
			(0.41)	(0.47)	(0.44)	(0.31)	(0.528)	(0)	(0)
500		*L*	1,419.9	761.7	537.4	352.7	207.7	129.3	76.7
	0.00		499	494	498	500	500	500	502
			(513.19)	(494.26)	(495.26)	(500.16)	(497.38)	(509.81)	(502.64)
	0.25		17.28	16.79	16.44	16.15	17.48	23.76	66.31
			(6.17)	(6.56)	(6.90)	(7.60)	(10.46)	(19.03)	(64.08)
	0.50		8.43	7.95	7.53	7.05	6.52	6.69	13.71
			(2.10)	(2.05)	(2.04)	(2.13)	(2.3)	(3.25)	(11.65)
	1.00		4.33	4.03	3.79	3.44	3.00	2.65	2.73
			(0.74)	(0.72)	(0.71)	(0.70)	(0.68)	(0.71)	(1.17)
	1.50		3.02	2.82	2.65	2.37	2.04	1.85	1.53
			(0.43)	(0.43)	(0.48)	(0.49)	(0.33)	(0.37)	(0.51)
	2.50		1.99	1.97	1.94	1.77	1.25	1.00	1.00
			(0.05)	(0.14)	(0.22)	(0.42)	(0.43)	(.045)	(0.01)
1,000		*L*	1,460.0	798.1	559.3	369.2	219.4	138.7	84.5
	0.00		994	1,000	1,000	1,001	997	997	1,000
			(988.60)	(980.70)	(966.63)	(1,002.63)	(990.75)	(994.06)	(997.33)
	0.25		22.55	22.22	20.09	19.64	21.79	34.21	115.95
			(7.39)	(7.90)	(7.98)	(9.27)	(13.54)	(28.56)	(115.52)
	0.50		10.88	10.23	8.96	8.15	7.50	7.96	20.60
			(2.40)	(2.41)	(2.32)	(2.33)	(2.67)	(3.89)	(18.14)
	1.00		5.49	5.06	4.42	3.90	3.33	2.93	3.27
			(0.84)	(0.83)	(0.78)	(0.75)	(0.74)	(0.76)	(1.51)
	1.50		3.97	3.48	3.03	2.71	2.25	1.97	1.70
			(0.50)	(0.53)	(0.43)	(0.47)	(0.45)	(0.35)	(0.51)
	2.50		2.42	2.08	1.99	1.95	1.58	1.05	1.00
			(0.49)	(0.28)	(0.07)	(0.22)	(0.49)	(0.23)	(0.00)

Note. Standard deviation of run length is in parentheses.

Table 88

Average Run Length Values of the Hotelling's T^2 That Achieved an In-Control Average Run Length ≈ 200 under p-variates Multivariate Normal Distribution

	p							
	2		3		4		5	
δ	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL
0.00	200.00		200.00		200.00		200.00	
0.25	162.11	160.46	175.80	174.36	182.42	180.29	186.44	186.26
0.5	112.45	113.48	130.19	129.89	140.16	140.78	144.74	147.05
1.0	41.50	41.28	53.42	53.31	60.59	60.09	68.62	69.67
1.5	15.67	15.24	20.77	19.84	25.18	24.87	28.94	28.21
2.5	3.61	3.08	4.46	3.95	5.27	4.72	6.03	5.58

Table 89

Average Run Length Values of the Hotelling's T^2 That Achieved an In-Control Average Run Length ≈ 500 under p-variates Multivariate Normal Distribution

	p							
	2		3		4		5	
δ	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL
0.00	500.00		500.00		500.00		500.00	
0.25	395.29	397.41	413.52	420.55	444.34	443.09	461.89	463.25
0.5	254.13	253.84	307.15	303.17	330.17	331.60	347.66	352.88
1.0	83.13	82.97	111.56	110.77	130.83	128.88	148.05	148.71
1.5	28.05	27.53	38.56	38.09	47.14	46.54	56.08	55.04
2.5	5.14	4.57	6.57	6.07	7.90	7.39	9.19	8.51

Table 90

Average Run Length Values of the Hotelling's T^2 That Achieved an In-Control Average Run Length ≈ 1,000 under p-variates Multivariate Normal Distribution

	p							
	2		3		4		5	
δ	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL
0.00	1,000.00		1,000.00		1,000.00		1,000.00	
0.25	771.20	764.14	837.75	846.95	881.11	876.92	898.83	910.71
0.5	474.75	483.32	577.60	578.22	640.94	641.13	669.04	674.43
1.0	140.49	138.42	196.35	196.55	231.75	230.24	269.50	273.11
1.5	43.55	43.93	63.09	63.10	77.30	77.85	92.68	91.56
2.5	6.88	6.44	9.24	8.83	11.35	10.89	13.31	12.91

Table 91

Average Run Length Values of the Hotelling's T^2 That Achieved an In-Control Average Run Length ≈ 200 under p-variates Multivariate $t_p(5)$- Distribution

	p							
	2		3		4		5	
δ	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL
0.00	200.00	196.61	200.00	196.61	201.00	202.63	201.00	198.70
0.25	173.44	170.68	183.13	180.89	185.73	184.26	189.25	187.77
0.5	142.94	142.15	159.21	159.41	166.28	163.48	176.03	174.21
1.0	76.65	75.09	100.52	101.41	110.14	110.60	120.05	120.40
1.5	34.06	33.18	48.77	47.86	59.15	58.04	67.38	67.27
2.5	6.78	6.23	10.04	9.52	12.70	12.42	15.88	15.19

Table 92

Average Run Length Values of the Hotelling's T^2 That Achieved an In-Control Average Run Length ≈ 500 under p-variates Multivariate $t_p(5)$- Distribution

	p							
	2		3		4		5	
δ	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL
0.00	488.00	483.73	494.00	493.82	500.00	501.81	494.00	493.63
0.25	439.56	436.29	449.27	449.74	464.87	463.89	477.49	483.03
0.5	375.94	377.00	393.81	393.33	435.11	434.70	445.17	444.86
1.0	212.54	211.93	265.61	264.29	301.30	297.92	325.54	323.34
1.5	96.17	95.01	135.90	134.77	174.03	171.95	194.43	192.73
2.5	16.08	15.62	25.48	25.35	37.06	36.35	45.63	45.19

Table 93

Average Run Length Values of the Hotelling's T^2 That Achieved an In-Control Average Run Length ≈ 1,000 under p-variates Multivariate $t_p(5)$- Distribution

	p							
	2		3		4		5	
δ	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL
0.00	992.00	986.65	998.00	978.42	988.00	984.37	1,000.00	990.61
0.25	880.93	881.01	927.34	916.77	952.96	951.14	950.61	967.43
0.5	774.23	769.71	844.34	855.33	864.67	877.62	921.47	936.77
1.0	467.87	464.36	578.61	576.05	657.06	649.00	713.52	707.90
1.5	230.38	225.60	323.24	325.44	411.02	403.19	478.71	488.31
2.5	38.36	38.12	65.96	66.47	94.77	92.68	124.12	122.95

Table 94

Average Run Length Values of the Hotelling's T^2 That Achieved an In-Control Average Run Length ≈ 200 under p-variates Multivariate Gamma$_p$(α=3,β=1) Distribution

	p							
	2		3		4		5	
δ	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL
0.00	201.00	197.98	200.00	202.82	200.00	199.38	200.00	197.66
0.25	146.77	148.19	159.24	158.20	169.50	170.27	171.74	171.20
0.50	92.89	93.04	120.46	119.94	134.60	136.27	138.17	136.37
1.00	30.32	29.92	60.17	60.77	73.02	72.75	76.62	77.76
1.50	8.73	8.33	28.90	28.29	35.66	35.68	39.57	38.52
2.50	1.05	0.22	7.31	6.93	8.97	8.47	10.07	9.45

Table 95

Average Run Length Values of the Hotelling's T^2 That Achieved an In-Control Average Run Length ≈ 500 under p-variates Multivariate Gamma$_p$(α=3,β=1) Distribution

	p							
	2		3		4		5	
δ	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL
0.00	503.00	505.74	501.00	507.26	500.00	501.74	500.00	491.81
0.25	363.96	363.06	397.45	395.36	412.64	417.49	412.11	410.32
0.50	234.19	236.49	285.51	284.53	325.10	321.41	331.05	328.59
1.00	76.57	77.11	139.67	138.28	166.91	166.25	177.37	175.75
1.50	21.76	21.00	61.99	62.36	78.13	77.70	84.48	84.08
2.50	1.65	1.04	13.38	12.86	17.01	16.44	19.02	18.44

Table 96

Average Run Length Values of the Hotelling's T^2 That Achieved an In-Control Average Run Length ≈ 1,000 under p-variates Multivariate Gamma$_p$(α=3,β=1) Distribution

	p							
	2		3		4		5	
δ	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL
0.00	993.00	1,000.55	995.00	983.80	1,000.00	1,000.13	999.00	1,011.04
0.25	731.07	737.23	771.44	767.91	837.93	847.06	828.34	822.73
0.50	453.92	455.09	559.94	557.73	636.27	638.27	632.96	636.14
1.00	148.01	148.15	261.96	262.37	312.41	315.38	336.51	336.29
1.50	43.76	43.17	113.08	112.54	139.09	140.24	153.77	152.90
2.50	3.20	2.65	21.73	20.98	27.85	27.17	31.48	31.31

APPENDIX E

AVERAGE RUN LENGTH COMPARISONS

Table 97

Average Run Length Comparisons for p = 2, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 200 from the Multivariate Normal Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	6.30	6.20	6.90	6.80	7.70	7.7	10.59
	0.00		206 (253)	197 (245)	193 (233)	194 (223)	198 (213)	201 (214)	200
	0.25		46.86 (47.19)	45.42 (76.75)	51.45 (49.27)	49.77 (48.67)	59.11 (57.49)	59.51 (57.43)	162.11 (160.46)
	0.50		16.93 (14.81)	16.12 (14.56)	18.72 (15.57)	18.19 (15.51)	20.52 (16.60)	20.77 (16.87)	112.45 (113.48
	1.00		5.71 (4.17)	5.33 (4.02)	6.21 (4.47)	5.92 (4.30)	6.88 (4.71)	6.73 (4.61)	41.50 (41.28
	1.50		3.07 (1.94)	2.92 (1.92)	3.32 (2.06)	3.16 (2.02)	3.61 (2.16)	3.58 (2.17)	15.67 (15.42)
	2.50		1.56 (0.74)	1.48 (0.72)	1.63 (0.79)	1.57 (0.76)	1.74 (0.84)	1.71 (0.83)	3.61 (3.08)

Note. Standard deviation of run length is in parentheses.

Table 98

Average Run Length Comparisons for p = 2, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 200 from the Multivariate Normal Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	8.85	8.8	9.90	9.7	10.90	10.40	10.59
	0.00		197 (200)	201 (205)	196 (197)	202 (203)	201 (203)	198 (200)	200
	0.25		74.92 (73.52)	74.91 (71.68)	97.13 (93.66)	94.88 (92.43)	141.65 (142.57)	133.82 (131.22)	162.11 (160.46)
	0.50		25.62 (21.27)	25.17 (20.79)	35.10 (32.32)	33.84 (30.71)	68.02 (66.90)	62.70 (61.51)	112.45 (113.48
	1.00		7.81 (5.14)	7.78 (5.13)	9.36 (6.62)	8.94 (6.31)	16.82 (15.41)	15.37 (13.55)	41.50 (41.28
	1.50		4.07 (2.36)	4.01 (2.34)	4.60 (2.68)	4.46 (2.56)	6.57 (4.99)	6.06 (4.44)	15.67 (15.42)
	2.50		1.90 (0.90)	1.87 (0.92)	2.08 (0.98)	2.04 (0.99)	2.34 (1.21)	2.25 (1.20)	3.61 (3.08)

Note. Standard deviation of run length is in parentheses.

Table 99

Average Run Length Comparisons for p = 3, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 200 *from the Multivariate Normal Distribution*

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
3	δ	*UCL*	8.30	8.20	9.10	8.80	10.20	9.70	12.83
	0.00		197 (257)	200 (244)	200 (239)	193 (220)	201 (221)	198 (214)	200
	0.25		50.11 (53.77)	51.46 (52.06)	56.84 (59.74)	56.38 (54.32)	69.70 (69.97)	65.63 (62.80)	175.80 (174.36)
	0.50		18.15 (16.79)	19.23 (16.49)	20.79 (17.95)	21.01 (17.36)	24.29 (20.31)	23.17 (18.70)	130.19 (129.89)
	1.00		5.93 (4.53)	6.48 (4.67)	6.79 (4.85)	6.97 (4.83)	7.84 (5.22)	7.68 (5.06)	53.42 (53.31)
	1.50		3.21 (2.08)	3.47 (2.16)	3.60 (2.22)	3.69 (2.30)	4.12 (2.41)	4.04 (2.40)	20.77 (19.84)
	2.50		1.55 (0.73)	1.79 (0.83)	1.73 (0.82)	1.79 (0.88)	1.92 (0.88)	1.92 (0.94)	4.46 (3.95)

Note. Standard deviation of run length is in parentheses.

Table 100

Average Run Length Comparisons for p = 3, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 200 *from the Multivariate Normal Distribution*

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
3	δ	*UCL*	12.00	11.00	14.45	12.00	19.10	12.60	12.83
	0.00		202 (214)	209 (214)	200 (206)	204 (207)	200 (197)	195 (195)	200
	0.25		96.00 (96.21)	83.18 (80.16)	136.97 (138.34)	110.95 (109.87)	180.71 (181.89)	142.83 (143.54)	175.80 (174.36)
	0.50		33.60 (29.27)	29.07 (24.04)	61.03 (59.21)	40.67 (36.61)	132.92 (133.35)	73.98 (72.18)	130.19 (129.89)
	1.00		9.74 (6.37)	8.73 (5.70)	14.53 (11.05)	10.49 (7.56)	51.80 (49.62)	18.70 (16.91)	53.42 (53.31)
	1.50		4.92 (2.71)	4.49 (2.59)	6.32 (3.65)	5.06 (2.95)	19.03 (17.17)	7.02 (5.31)	20.77 (19.84)
	2.50		2.21 (1.00)	2.07 (0.99)	2.63 (1.15)	2.23 (1.06)	4.38 (2.62)	2.45 (1.317)	4.46 (3.95)

Note. Standard deviation of run length is in parentheses.

Table 101

Average Run Length Comparisons for p = 4, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 200 from the Multivariate Normal Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
4	δ	*UCL*	10.30	9.90	11.45	10.70	13.05	11.60	14.86
	0.00		201 (263)	199 (246)	202 (256)	203 (232)	200 (230)	193 (209)	200
	0.25		56.33 (61.76)	55.30 (55.12)	67.32 (71.02)	61.93 (60.20)	84.7 (87.25)	71.22 (67.50)	182.42 (180.29)
	0.50		21.13 (18.66)	20.26 (17.69)	24.61 (21.07)	22.11 (18.33)	30.43 (25.94)	24.94 (20.02)	140.16 (140.78)
	1.00		7.33 (5.13)	6.69 (5.00)	8.20 (5.49)	7.48 (5.24)	9.62 (6.23)	8.28 (5.51)	60.59 (60.09)
	1.50		3.89 (2.26)	3.58 (2.32)	4.33 (2.46)	3.95 (2.44)	4.97 (2.72)	4.27 (2.56)	25.18 (24.87)
	2.50		1.88 (0.81)	1.75 (0.88)	2.05 (0.87)	1.87 (0.93)	2.28 (0.97)	2.01 (0.98)	5.27 (4.72)

Note. Standard deviation of run length is in parentheses.

Table 102

Average Run Length Comparisons for p = 4, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 200 from the Multivariate Normal Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
4	δ	*UCL*	16.25	12.90	21.45	13.90	33.50	14.70	14.86
	0.00		200 (218)	200 (203)	200 (210)	199 (199)	200 (200)	201 (201)	200
	0.25		123.19 (131.13)	89.81 (86.98)	171.09 (174.35)	116.74 (115.13)	196.13 (195.50)	156.31 (15.30)	182.42 (180.29)
	0.50		50.77 (47.12)	31.44 (25.71)	102.06 (102.95)	43.76 (40.22)	174.16 (176.10)	84.06 (83.19)	140.16 (140.78)
	1.00		13.10 (8.60)	9.57 (6.19)	28.11 (24.39)	11.34 (8.07)	114.80 (114.95)	21.64 (19.52)	60.59 (60.09)
	1.50		6.37 (3.36)	4.86 (2.77)	10.13 (6.30)	5.33 (3.11)	63.05 (61.71)	8.06 (6.31)	25.18 (24.87)
	2.50		2.75 (1.16)	2.18 (1.05)	3.65 (1.51)	2.35 (1.12)	14.35 (11.92)	2.70 (1.47)	5.27 (4.72)

Note. Standard deviation of run length is in parentheses.

Table 103

Average Run Length Comparisons for p = 5, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 200 from the Multivariate Normal Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
5	δ	*UCL*	12.60	11.60	14.10	12.30	16.90	13.40	16.74
	0.00		202 (273)	200 (241)	200 (248)	195 (222)	201 (231)	200 (218)	200
	0.25		67.46 (73.35)	59.12 (58.57)	81.28 (88.24)	66.05 (64.05)	111.61 (120.36)	77.00 (74.76)	186.44 (186.26)
	0.50		25.44 (22.09)	21.94 (18.42)	30.00 (25.34)	24.03 (19.63)	41.03 (36.27)	27.23 (21.39)	144.74 (147.05)
	1.00		8.70 (5.86)	7.66 (5.35)	9.86 (6.39)	8.11 (5.52)	12.33 (7.66)	8.91 (5.83)	68.62 (69.67)
	1.50		4.73 (2.52)	4.03 (2.49)	5.18 (2.78)	4.24 (2.57)	6.28 (3.21)	4.69 (2.69)	28.94 (28.21)
	2.50		2.25 (0.98)	1.94 (0.95)	2.43 (0.96)	2.04 (1.00)	2.81 (1.11)	2.18 (1.05)	6.03 (5.58)

Note. Standard deviation of run length is in parentheses.

Table 104

Average Run Length Comparisons for p = 5, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 200 from the Multivariate Normal Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
5	δ	*UCL*	22.80	14.70	33.50	15.8	58.30	16.6	16.74
	0.00		199 (216)	199 (201)	201 (208)	199 (197)	200 (200)	197 (185)	200
	0.25		156.56 (169.48)	95.78 (93.40)	187.51 (194.42)	123.33 (122.15)	194.70 (198.98)	160.37 (160.06)	186.44 (186.26)
	0.50		82.16 (80.09)	34.32 (29.07)	147.13 (147.88)	49.28 (45.03)	191.50 (189.93)	92.99 (92.07)	144.74 (147.05)
	1.00		20.78 (14.85)	10.05 (6.43)	67.24 (65.52)	12.40 (8.74)	161.04 (161.80)	25.05 (23.05)	68.62 (69.67)
	1.50		9.02 (4.59)	5.12 (2.83)	23.43 (18.56)	5.77 (3.33)	124.96 (124.57)	9.08 (7.31)	28.94 (28.21)
	2.50		3.66 (1.41)	12.34 (1.12)	5.79 (2.30)	2.50 (1.18)	55.24 (52.50)	2.86 (1.56)	6.03 (5.58)

Note. Standard deviation of run length is in parentheses.

Table 105

Average Run Length Comparisons for p = 2, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 500 from the Multivariate Normal Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	8.10	8.10	8.85	8.80	9.75	9.70	12.42
	0.00		492 (556)	494 (551)	499 (539)	501 (537)	494 (517)	502 (501)	500
	0.25		72.30 (63.57)	71.61 (64.31)	82.34 (73.68)	81.70 (72.82)	99.10 (91.71)	98.18 (89.98)	395.29 (397.41)
	0.50		23.60 (17.89)	23.18 (18.18)	25.73 (19.28)	25.57 (19.01)	29.07 (21.90)	29.10 (21.38)	254.13 (253.84)
	1.00		7.42 (4.93)	7.24 (4.95)	8.21 (5.31)	7.81 (5.16)	8.81 (5.47)	8.62 (5.50)	83.13 (82.97)
	1.50		3.83 (2.26)	3.72 (2.29)	4.18 (2.44)	4.02 (2.40)	4.51 (2.58)	4.40 (2.48)	28.05 (27.53)
	2.50		1.81 (0.89)	1.72 (0.83)	1.91 (0.91)	1.84 (0.90)	2.06 (0.96)	2.01 (0.96)	5.14 (4.57)

Note. Standard deviation of run length is in parentheses.

Table 106

Average Run Length Comparisons for p = 2, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 500 from the Multivariate Normal Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	10.95	10.80	11.95	11.60	12.75	12.30	12.42
	0.00		502 (507)	498 (505)	491 (484)	482 (488)	482 (479)	504 (504)	500
	0.25		139.49 (134.16)	135.56 (130.63)	205.31 (200.77)	192.24 (187.02)	315.56 (309.19)	312.80 (311.50)	395.29 (397.41)
	0.50		37.75 (30.95)	36.58 (30.27)	58.31 (52.92)	54.61 (50.51)	130.24 (125.96)	126.77 (124.78)	254.13 (253.84)
	1.00		9.95 (6.24)	9.77 (6.19)	12.32 (8.74)	11.82 (8.41)	26.05 (23.92)	24.44 (22.060)	83.13 (82.97)
	1.50		4.93 (2.67)	4.90 (2.69)	5.60 (3.19)	5.45 (3.07)	8.78 (6.95)	8.13 (6.19)	28.05 (27.53)
	2.50		2.22 (1.02)	2.17 (1.02)	2.40 (1.10)	2.33 (1.08)	2.73 (1.46)	2.65 (1.43)	5.14 (4.57)

Note. Standard deviation of run length is in parentheses.

Table 107

Average Run Length Comparisons for p = 3, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 500 from the Multivariate Normal Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
3	δ	*UCL*	10.50	10.20	11.40	11.00	12.65	11.90	14.79
	0.00		497 (568)	488 (541)	502 (558)	495 (521)	500 (533)	483 (496)	500
	0.25		83.17 (79.85)	82.04 (73.04)	97.48 (91.36)	93.13 (82.56)	124.44 (121.37)	111.13 (10.69)	413.52 (420.55)
	0.50		25.64 (20.57)	26.91 (19.66)	29.45 (22.79)	29.42 (21.33)	35.21 (27.02)	33.09 (24.54)	307.15 (303.17)
	1.00		8.00 (5.53)	8.53 (5.57)	8.97 (5.80)	9.03 (5.62)	10.24 (6.35)	9.76 (5.89)	111.56 (110.77)
	1.50		4.06 (2.47)	4.30 (2.51)	4.52 (2.61)	4.64 (2.62)	5.09 (2.80)	4.95 (2.70)	38.56 (38.09)
	2.50		1.84 (0.88)	2.01 (0.97)	2.03 (0.94)	2.10 (1.00)	2.28 (1.05)	2.22 (1.06)	6.57 (6.07)

Note. Standard deviation of run length is in parentheses.

Table 108

Average Run Length Comparisons for p = 3, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 500 from the Multivariate Normal Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
3	δ	*UCL*	14.70	13.00	17.60	14.00	23.80	14.60	14.79
	0.00		497 (520)	473 (475)	497 (505)	498 (497)	504 (500)	490 (485)	500
	0.25		197.21 (193.48)	154.84 (149.68)	309.43 (311.71)	228.20 (224.68)	455.59 (456.01)	340.06 (342.13)	413.52 (420.55)
	0.50		53.98 (47.88)	42.25 (35.25)	121.00 (119.55)	66.77 (62.48)	325.60 (323.42)	153.04 (15.85)	307.15 (303.17)
	1.00		12.89 (8.05)	10.85 (6.68)	21.95 (17.33)	13.75 (9.67)	118.43 (117.12)	30.68 (28.86)	111.56 (110.77)
	1.50		6.13 (3.22)	5.41 (2.89)	8.15 (4.72)	6.06 (3.44)	38.87 (36.49)	9.58 (7.70)	38.56 (38.09)
	2.50		2.64 (1.14)	2.38 (1.10)	3.13 (1.33)	2.55 (1.17)	6.49 (4.48)	2.91 (1.60)	6.57 (6.07)

Note. Standard deviation of run length is in parentheses.

Table 109

Average Run Length Comparisons for p = 4, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 500 from the Multivariate Normal Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
4	δ	*UCL*	12.90	12.10	14.20	13.00	16.30	14.00	16.92
	0.00		494 (585)	492 (546)	498 (563)	503 (530)	498 (541)	506 (523)	500
	0.25		95.05 (92.36)	88.82 (79.56)	116.70 (115.55)	102.08 (91.42)	168.17 (173.42)	127.39 (119.16)	444.34 (443.09)
	0.50		30.61 (24.02)	28.44 (21.60)	35.19 (27.19)	31.96 (23.37)	47.32 (37.85)	36.44 (27.36)	330.17 (331.60)
	1.00		9.53 (6.24)	8.74 (5.79)	10.63 (6.68)	9.58 (6.09)	12.84 (7.65)	10.52 (6.33)	130.83 (128.88)
	1.50		4.98 (2.80)	4.50 (2.69)	5.46 (2.93)	4.90 (2.83)	6.32 (3.30)	5.34 (2.93)	47.14 (46.54)
	2.50		2.25 (0.98)	2.06 (1.01)	2.44 (1.05)	2.20 (1.07)	2.75 (1.14)	2.37 (1.11)	7.90 (7.39)

Note. Standard deviation of run length is in parentheses.

Table 110

Average Run Length Comparisons for p = 4, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 500 from the Multivariate Normal Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
4	δ	*UCL*	20.50	15.20	27.65	16.20	44.50	16.80	16.92
	0.00		499 (522)	505 (513)	500 (515)	501 (500)	500 (505)	496 (502)	500
	0.25		292.27 (306.87)	176.99 (169.92)	414.18 (424.22)	2.54.83 (248.83)	494.08 (501.77)	365.86 (362.06)	444.34 (443.09)
	0.50		98.37 (94.88)	48.13 (39.80)	244.56 (245.11)	78.83 (74.97)	440.20 (438.85)	177.91 (177.33)	330.17 (331.60)
	1.00		19.00 (12.31)	12.17 (7.37)	57.38 (52.69)	15.12 (10.81)	304.36 (304.84)	36.42 (34.68)	130.83 (128.88)
	1.50		8.34 (4.21)	5.83 (3.19)	16.53 (11.35)	6.70 (3.82)	165.90 (162.85)	11.34 (9.30)	47.14 (46.54)
	2.50		3.41 (1.37)	2.56 (1.18)	4.75 (1.86)	2.72 (1.25)	35.37 (33.16)	3.18 (1.74)	7.90 (7.39)

Note. Standard deviation of run length is in parentheses.

Table 111

Average Run Length Comparisons for p = 5, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 500 from the Multivariate Normal Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
5	δ	*UCL*	15.90	14.00	18.05	14.80	21.90	15.90	18.90
	0.00		500 (615)	504 (560)	501 (575)	494 (520)	503 (556)	499 (514)	500
	0.25		119.63 (121.54)	98.27 (86.67)	157.31 (160.08)	111.00 (102.35)	251.28 (270.44)	136.86 (129.26)	461.89 (463.25)
	0.50		37.00 (29.02)	31.66 (23.50)	45.54 (36.24)	34.47 (24.52)	74.17 (66.29)	39.28 (29.06)	347.66 (352.88)
	1.00		11.58 (7.13)	9.99 (6.21)	13.31 (7.97)	10.72 (6.50)	17.39 (10.00)	11.50 (6.77)	148.05 (148.71)
	1.50		5.95 (3.12)	5.03 (2.83)	6.78 (3.46)	5.36 (3.00)	8.39 (4.02)	5.79 (3.10)	56.08 (55.04)
	2.50		2.70 (1.06)	2.32 (1.11)	2.96 (1.15)	2.41 (1.14)	3.55 (1.35)	2.55 (1.19)	9.19 (8.51)

Note. Standard deviation of run length is in parentheses.

Table 112

Average Run Length Comparisons for p = 5, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 500 from the Multivariate Normal Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
5	δ	*UCL*	30.60	17.10	46.90	18.10	84.00	18.80	18.90
	0.00		501 (527)	488 (489)	501 (507)	485 (485)	493 (497)	495 (493)	500
	0.25		382.48 (399.29)	187.48 (178.73)	472.85 (483.38)	269.60 (263.70)	484.76 (488.94)	377.46 (372.21)	461.89 (463.25)
	0.50		204.96 (205.60)	53.20 (45.89)	391.20 (396.67)	87.29 (82.18)	472.17 (476.26)	199.40 (196.75)	347.66 (352.88)
	1.00		38.14 (29.00)	12.98 (7.90)	180.66 (181.51)	16.61 (12.24)	417.38 (409.12)	43.11 (40.68)	148.05 (148.71)
	1.50		13.26 (6.60)	6.24 (3.26)	61.01 (55.38)	7.12 (4.01)	331.98 (330.77)	12.91 (10.88)	56.08 (55.04)
	2.50		4.90 (1.75)	2.70 (1.23)	9.38 (4.04)	2.87 (1.12)	168.45 (165.21)	3.44 (1.93)	9.19 (8.51)

Note. Standard deviation of run length is in parentheses.

Table 113

Average Run Length Comparisons for p = 2, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 1,000 from the Multivariate Normal Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	9.60	9.60	10.40	10.30	11.35	11.20	13.88
	0.00		985 (1,047)	976 (1,030)	990 (1,024)	971 (999)	1,003 (1,033)	968 (996)	1,000
	0.25		96.92 (80.26)	96.42 (82.69)	113.27 (97.61)	112.73 (96.39)	145.00 (132.17)	138.55 (125.68)	771.20 (764.14)
	0.50		29.46 (20.76)	29.03 (20.94)	32.28 (22.32)	31.34 (22.06)	37.22 (26.89)	35.93 (25.79)	474.75 (483.32)
	1.00		8.92 (5.60)	8.72 (5.62)	9.50 (5.77)	9.31 (2.71)	10.34 (6.02)	10.25 (6.10)	140.49 (138.42)
	1.50		4.48 (2.53)	4.20 (2.48)	4.77 (2.62)	4.62 (2.63)	5.16 (2.77)	5.07 (2.76)	43.55 (43.93)
	2.50		2.04 (0.96)	1.95 (0.95)	2.16 (1.00)	2.08 (0.99)	2.31 (1.07)	2.25 (1.07)	6.88 (6.44)

Note. Standard deviation of run length is in parentheses.

Table 114

Average Run Length Comparisons for p = 2, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 1,000 from the Multivariate Normal Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	12.50	12.30	13.50	13.10	14.20	13.70	13.88
	0.00		995 (1,007)	981 (993)	992 (982)	987 (973)	967 (960)	1,011 (1,012)	1,000
	0.25		225.56 (222.12)	212.96 (203.08)	356.92 (362.23)	338.80 (343.19)	593.27 (595.55)	592.36 (588.96)	771.20 (764.14)
	0.50		49.83 (40.86)	48.32 (39.95)	87.85 (83.30)	80.86 (76.00)	227.82 (229.10)	220.45 (215.62)	474.75 (483.32)
	1.00		11.66 (6.90)	11.49 (6.83)	15.19 (10.75)	14.38 (10.18)	38.08 (35.63)	35.33 (33.42)	140.49 (138.42)
	1.50		5.66 (2.97)	5.53 (2.94)	6.40 (3.56)	6.18 (3.45)	11.02 (9.11)	10.19 (8.04)	43.55 (43.93)
	2.50		2.46 (1.11)	2.42 (1.11)	2.64 (1.18)	2.55 (1.15)	3.10 (1.68)	2.94 (1.59)	6.88 (6.44)

Note. Standard deviation of run length is in parentheses.

Table 115

Average Run Length Comparisons for p = 3, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 1,000 from the Multivariate Normal Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
3	δ	*UCL*	12.20	11.80	13.20	12.60	14.20	13.60	16.26
	0.00		993 (1,019)	942 (1,002)	996 (1,056)	955 (972)	1,004 (1,035)	995 (1,004)	1,000
	0.25		113.84 (102.33)	111.06 (93.48)	140.38 (127.29)	127.78 (111.44)	179.71 (170.82)	168.78 (156.50)	837.75 (846.95)
	0.50		**32.44** **(23.88)**	33.67 (23.43)	36.90 (26.94)	35.84 (24.70)	43.56 (33.21)	41.28 (29.69)	577.60 (578.22)
	1.00		**9.64** **(6.18)**	9.95 (6.01)	10.64 (6.49)	10.53 (6.21)	11.81 (6.94)	11.47 (6.61)	196.35 (196.55)
	1.50		**4.77** **(2.76)**	4.98 (2.75)	**5.29** **(2.90)**	5.36 (2.89)	5.77 (3.00)	5.75 (2.98)	63.09 (63.10)
	2.50		**2.11** **(0.99)**	2.24 (1.06)	**2.30** **(1.05)**	2.37 (1.10)	2.50 (1.11)	2.49 (1.13)	9.24 (8.83)

Note. Standard deviation of run length is in parentheses.

Table 116

Average Run Length Comparisons for p = 3, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 1,000 from the Multivariate Normal Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
3	δ	*UCL*	16.90	14.80	20.25	15.60	27.40	16.20	16.26
	0.00		993 (1,010)	1,014 (1,022)	992 (998)	994 (992)	994 (1,019)	1,014 (1,019)	1,000
	0.25		353.79 (356.41)	267.60 (263.21)	606.84 (609.62)	405.19 (402.90)	875.22 (888.06)	662.42 (656.69)	837.75 (846.95)
	0.50		81.58 (74.49)	58.42 (48.81)	208.82 (204.87)	101.07 (96.38)	607.44 (623.17)	281.23 (283.71)	577.60 (578.22)
	1.00		15.66 (9.71)	13.09 (7.70)	30.78 (24.83)	16.76 (12.13)	219.68 (218.42)	46.64 (43.84)	196.35 (196.55)
	1.50		7.11 (3.60)	6.22 (3.18)	10.11 (5.87)	6.91 (3.84)	67.05 (66.02)	12.62 (10.48)	63.09 (63.10)
	2.50		2.94 (1.26)	2.65 (1.18)	3.58 (1.48)	2.82 (1.26)	9.06 (6.66)	3.31 (1.83)	9.24 (8.83)

Note. Standard deviation of run length is in parentheses.

Table 117

Average Run Length Comparisons for p = 4, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 1,000 from the Multivariate Normal Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
4	δ	*UCL*	15.00	13.90	16.50	14.80	18.85	15.80	18.46
	0.00		988 (1,121)	962 (1,011)	997 (1,096)	1,006 (1,029)	1,000 (1,078)	1,004 (1,032)	1,000
	0.25		136.53 (126.91)	122.45 (102.24)	181.43 (179.26)	147.80 (128.73)	290.42 (297.15)	192.27 (179.86)	881.11 (876.92)
	0.50		38.37 (28.17)	36.25 (24.93)	45.67 (34.27)	39.72 (27.01)	63.16 (50.95)	45.86 (33.01)	640.94 (641.13)
	1.00		11.43 (6.91)	10.50 (6.40)	12.97 (7.59)	11.48 (6.67)	15.41 (8.74)	12.44 (7.13)	231.75 (230.24)
	1.50		5.74 (3.07)	5.23 (2.94)	6.35 (3.30)	5.74 (3.11)	7.36 (3.69)	6.10 (3.13)	77.30 (77.85)
	2.50		2.54 (1.10)	2.34 (1.12)	2.80 (1.17)	2.49 (1.17)	3.15 (1.28)	2.65 (1.22)	11.35 (10.89)

Note. Standard deviation of run length is in parentheses.

Table 118

Average Run Length Comparisons for p = 4, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 1,000 from the Multivariate Normal Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
4	δ	*UCL*	23.95	16.90	32.80	17.80	53.60	18.30	18.46
	0.00		1,001 (1,053)	973 (978)	992 (1,020)	1,006 (1,003)	989 (989)	985 (975)	1,000
	0.25		554.93 (563.57)	291.72 (281.78)	819.76 (843.67)	449.13 (446.28)	937.25 (936.07)	693.38 (691.43)	881.11 (876.92)
	0.50		168.09 (163.65)	64.79 (55.75)	489.67 (491.70)	120.75 (116.88)	849.45 (843.00)	312.13 (310.32)	640.94 (641.13)
	1.00		25.00 (16.09)	13.96 (8.47)	106.08 (99.79)	18.81 (13.74)	586.70 (575.21)	55.72 (53.39)	231.75 (230.24)
	1.50		10.10 (4.85)	6.63 (3.37)	24.93 (18.69)	7.59 (4.26)	324.48 (323.49)	14.48 (12.26)	77.30 (77.85)
	2.50		3.93 (1.52)	2.83 (1.26)	5.79 (2.27)	2.99 (1.33)	74.59 (71.30)	3.55 (1.99)	11.35 (10.89)

Note. Standard deviation of run length is in parentheses.

Table 119

Average Run Length Comparisons for p = 5, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 1,000 from the Multivariate Normal Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
5	δ	*UCL*	18.60	15.80	21.30	16.70	26.10	17.70	20.51
	0.00		1,010 (1,178)	996 (1,049)	998 (1,103)	996 (1,008)	1,006 (1,092)	980 (1,015)	1,000
	0.25		180.93 (174.33)	135.81 (113.43)	269.54 (283.60)	161.01 (142.45)	472.39 (498.75)	209.37 (195.36)	898.83 (910.71)
	0.50		47.84 (34.53)	39.31 (26.44)	62.88 (47.81)	42.41 (28.42)	118.70 (108.19)	50.25 (37.24)	669.04 (674.43)
	1.00		14.12 (8.28)	11.78 (6.86)	16.60 (9.30)	12.41 (6.96)	22.45 (12.38)	13.37 (7.41)	269.50 (273.11)
	1.50		7.06 (3.51)	5.92 (3.12)	8.11 (3.88)	6.18 (3.21)	10.10 (4.63)	6.54 (3.33)	92.68 (91.56)
	2.50		3.10 (1.21)	2.60 (1.20)	3.48 (1.33)	2.71 (1.24)	4.15 (1.56)	2.82 (1.26)	13.31 (12.91)

Note. Standard deviation of run length is in parentheses.

Table 120

Average Run Length Comparisons for p = 5, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 1,000 from the Multivariate Normal Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
5	δ	*UCL*	37.50	18.90	58.40	19.80	107.90	20.40	20.51
	0.00		998 (1,046)	974 (959)	983 (997)	974 (965)	983 (973)	985 (983)	1,000
	0.25		785.62 (813.27)	320.54 (308.27)	920.76 (928.77)	482.92 (485.13)	965.83 (958.74)	732.91 (730.76)	898.83 (910.71)
	0.50		412.80 (427.53)	73.18 (61.71)	774.33 (780.33)	135.84 (131.21)	940.47 (946.45)	355.85 (355.10)	669.04 (674.43)
	1.00		66.30 (55.39)	15.13 (8.91)	384.64 (391.47)	20.98 (5.79)	834.64 (824.48)	67.22 (64.83)	269.50 (273.11)
	1.50		18.22 (9.24)	7.09 (3.58)	132.42 (127.31)	8.07 (4.58)	695.32 (691.56)	17.16 (14.96)	92.68 (91.56)
	2.50		6.03 (2.07)	2.98 (1.32)	14.32 (7.42)	3.16 (1.40)	369..98 (363.46)	3.88 (2.23)	13.31 (12.91)

Note. Standard deviation of run length is in parentheses.

Table 121

Average Run Length Comparisons for p = 2, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 200 from the Multivariate $t_p(5)$ Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	6.30	6.20	6.90	6.80	7.85	7.70	13.5
	0.00		201 (248)	198 (246)	197 (226)	195 (244)	200 (221)	197 (212)	200 (196.61)
	0.25		47.36 (47.58)	45.63 (46.42)	51.34 (49.45)	50.12 (49.19)	60.82 (59.76)	59.11 (55.92)	173.44 (170.68)
	0.50		17.05 (14.92)	16.67 (14.55)	18.72 (15.79)	18.54 (15.41)	21.49 (17.22)	20.80 (16.79)	142.94 (142.15)
	1.00		5.80 (4.14)	5.62 (4.11)	6.28 (4.32)	6.13 (4.33)	7.03 (4.74)	6.93 (4.63)	76.65 (75.09)
	1.50		3.17 (1.94)	2.99 (1.90)	3.33 (2.00)	3.30 (2.04)	3.72 (2.17)	3.62 (2.15)	34.06 (33.18)
	2.50		1.56 (0.74)	1.50 (0.70)	1.63 (0.76)	1.60 (0.77)	1.78 (0.84)	1.73 (0.83)	6.78 (6.23)

Note. Standard deviation of run length is in parentheses.

Table 122

Average Run Length Comparisons for p = 2, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 200 from the Multivariate $t_p(5)$ Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	9.20	9.00	10.70	10.30	13.40	12.30	13.5
	0.00		201 (208)	202 (208)	199 (205)	198 (198)	200 (204)	200 (200)	200 (196.61)
	0.25		80.56 (77.97)	77.35 (74.18)	112.22 (112.97)	105.85 (104.85)	164.98 (164.07)	160.7 (159.49)	173.44 (170.68)
	0.50		27.12 (22.43)	26.06 (21.37)	41.68 (38.00)	38.20 (35.52)	101.41 (101.40)	93.25 (90.85)	142.94 (142.15)
	1.00		8.35 (5.36)	8.04 (5.27)	10.63 (7.57)	9.80 (6.90)	29.88 (27.75)	24.24 (22.49)	76.65 (75.09)
	1.50		4.24 (2.40)	4.11 (2.33)	4.98 (2.80)	4.76 (2.73)	10.16 (8.33)	8.38 (6.53)	34.06 (33.18)
	2.50		1.99 (0.92)	1.93 (0.92)	2.19 (1.00)	2.12 (0.99)	2.91 (1.53)	2.64 (1.38)	6.78 (6.23)

Note. Standard deviation of run length is in parentheses.

Table 123

Average Run Length Comparisons for p = 3, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 200 from the Multivariate $t_p(5)$ Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
3	δ	*UCL*	8.35	8.20	9.25	8.90	10.70	9.90	16.6
	0.00		200 (253)	200 (247)	200 (241)	201 (234)	200 (227)	202 (219)	200 (196.61)
	0.25		53.22 (55.90)	51.54 (51.58)	61.27 (61.59)	56.16 (55.02)	79.69 (81.16)	67.63 (65.77)	183.13 (180.89)
	0.50		19.71 (17.08)	18.82 (16.23)	22.38 (18.44)	20.69 (17.30)	27.18 (22.02)	23.58 (19.25)	159.21 (159.41)
	1.00		6.75 (4.63)	6.31 (4.63)	7.42 (4.95)	6.90 (4.80)	8.72 (5.52)	7.70 (5.03)	100.52 (101.41)
	1.50		3.59 (2.06)	3.32 (2.13)	3.94 (2.22)	3.60 (2.20)	4.51 (2.40)	4.07 (2.40)	48.77 (47.86)
	2.50		1.75 (0.76)	1.65 (0.80)	1.87 (0.81)	1.78 (0.87)	2.08 (0.89)	1.89 (0.91)	10.04 (9.52)

Note. Standard deviation of run length is in parentheses.

Table 124

Average Run Length Comparisons for p = 3, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 200 from the Multivariate $t_p(5)$ Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
3	δ	*UCL*	13.30	11.30	17.65	12.70	2.40	15.00	16.6
	0.00		200 (215)	199 (206)	200 (206)	200 (198)	200 (203)	198 (199)	200 196.61)
	0.25		115.66 (118.31)	88.96 (86.71)	163.46 (167.63)	118.18 (116.92)	188.95 (190.11)	166.60 (163.88)	183.13 (180.89)
	0.50		43.25 (38.55)	30.25 (25.09)	91.76 (90.89)	45.71 (42.39)	168.17 (166.17)	106.47 (104.71)	159.21 (159.41)
	1.00		11.35 (7.08)	9.11 (5.84)	22.39 (18.19)	11.37 (8.15)	105.31 (103.6)	30.70 (29.04)	100.52 (101.41)
	1.50		5.60 (2.91)	4.61 (2.63)	8.48 (4.92)	5.42 (3.10)	53.01 (51.02)	10.46 (8.64)	48.77 (47.86)
	2.50		2.45 (1.03)	2.12 (1.01)	3.18 (1.28)	3.34 (1.08)	9.74 (7.29)	3.02 (1.63)	10.04 (9.52)

Note. Standard deviation of run length is in parentheses.

Table 125

Average Run Length Comparisons for p = 4, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 200 from the Multivariate t_p(5) Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
4	δ	*UCL*	10.35	10.00	11.70	10.80	14.00	11.80	19.2
	0.00		198 (252)	201 (247)	199 (240)	201 (233)	201 (229)	200 (214)	201 (202.62)
	0.25		60.28 (64.29)	57.57 (56.97)	75.31 (79.62)	64.36 (62.77)	105.20 (113.01)	72.97 (71.38)	185.73 (184.26)
	0.50		22.00 (18.81)	21.37 (18.29)	26.73 (21.96)	23.24 (19.00)	36.33 (30.93)	26.13 (20.85)	166.28 (163.48)
	1.00		7.29 (4.88)	7.31 (5.10)	8.59 (5.54)	7.80 (5.33)	10.71 (6.47)	8.56 (5.54)	101.41 (110.60)
	1.50		3.79 (2.08)	3.93 (2.36)	4.38 (2.32)	4.17 (2.47)	5.36 (2.76)	4.53 (2.61)	59.15 (58.04)
	2.50		1.76 (0.71)	1.94 (0.94)	1.99 (0.79)	1.99 (0.95)	2.41 (0.95)	2.10 (0.99)	12.70 (12.42)

Note. Standard deviation of run length is in parentheses.

Table 126

Average Run Length Comparisons for p = 4, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 200 from the Multivariate t_p(5) Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
4	δ	*UCL*	19.20	13.20	28.30	14.80	49.80	17.50	19.2
	0.00		201 (211)	196 (204)	199 (206)	198 (200)	200 (199)	199 (197)	201 (202.62)
	0.25		155.38 (164.26)	92.65 (92.28)	191.56 (192.02)	125.48 (127.11)	196.88 (197.67)	176.02 (171.98)	185.73 (184.26)
	0.50		75.55 (74.50)	32.21 (28.33)	146.52 (148.49)	51.51 (47.89)	188.51 (186.32)	120.17 (121.42)	166.28 (163.48)
	1.00		17.85 (11.50)	9.87 (6.36)	58.89 (56.08)	12.76 (9.23)	160.96 (162.93)	38.23 (36.96)	101.41 (110.60)
	1.50		7.91 (3.86)	5.02 (2.78)	18.75 (13.60)	5.88 (3.37)	122.07 (122.30)	12.68 (10.68)	59.15 (58.04)
	2.50		3.20 (1.21)	2.28 (1.06)	4.94 (1.85)	2.52 (1.16)	49.29 (48.18)	3.41 (1.90)	12.70 (12.42)

Note. Standard deviation of run length is in parentheses.

Table 127

Average Run Length Comparisons for p = 5, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 200 from the Multivariate t_p(5) Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
5	δ	*UCL*	12.69	11.60	14.70	12.50	18.60	13.60	21.60
	0.00		201 (262)	196 (248)	201 (243)	200 (229)	202 (231)	194 (214)	201 (198.70)
	0.25		76.16 (78.89)	58.18 (60.50)	96.06 (104.47)	65.21 (64.69)	134.85 (150.44)	77.10 (75.39)	189.25 (187.77)
	0.50		28.08 (22.45)	21.83 (18.83)	34.70 (28.02)	24.04 (19.98)	54.74 (48.94)	27.63 (22.63)	176.03 (174.21)
	1.00		9.27 (5.66)	7.31 (5.38)	11.01 (6.36)	8.09 (5.56)	14.86 (8.46)	8.92 (5.84)	120.05 (120.40)
	1.50		4.81 (2.37)	3.91 (2.46)	5.65 (2.71)	4.30 (2.62)	7.17 (3.32)	4.75 (2.76)	67.38 (67.27)
	2.50		2.19 (0.80)	1.89 (0.93)	2.53 (0.91)	2.02 (0.99)	3.10 (1.10)	2.19 (1.04)	15.88 (15.19)

Note. Standard deviation of run length is in parentheses.

Table 128

Average Run Length Comparisons for p = 5, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 200 from the Multivariate t_p(5) Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
5	δ	*UCL*	27.55	15.20	43.80	16.80	81.90	19.70	21.60
	0.00		199 (209)	201 (209)	200 (208)	200 (200)	198 (201)	197 (194)	201 (198.70)
	0.25		178.49 (185.49)	102.13 (102.23)	190.74 (196.67)	130.92 (12.85)	198.43 (197.72)	176.93 (175.00)	189.25 (187.77)
	0.50		120.79 (124.97)	36.92 (31.70)	176.05 (178.26)	57.13 (54.52)	196.04 (194.71)	125.52 (123.27)	176.03 (174.21)
	1.00		32.98 (25.04)	10.56 (6.74)	111.64 (110.42)	13.97 (10.09)	183.36 (184.40)	44.35 (43.08)	120.05 (120.40)
	1.50		11.98 (5.61)	5.33 (2.92)	51.88 (47.46)	6.27 (3.64)	162.28 (160.67)	14.62 (12.65)	67.38 (67.27)
	2.50		4.46 (1.53)	2.40 (1.13)	8.53 (3.37)	2.65 (1.25)	107.64 (109.46)	3.72 (2.14)	15.88 (15.19)

Note. Standard deviation of run length is in parentheses.

Table 129

Average Run Length Comparisons for p = 2, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 500 from the Multivariate $t_p(5)$ Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	8.20	8.10	8.95	8.90	10.00	9.80	17.60
	0.00		500 (558)	488 (544)	497 (539)	492 (528)	501 (527)	474 (486)	488 (183.73)
	0.25		73.90 (65.40)	72.08 (65.25)	83.53 (76.69)	82.81 (74.17)	104.71 (97.00)	100.95 (92.18)	439.56 (436.29)
	0.50		23.97 (18.15)	23.58 (18.03)	26.25 (19.83)	26.44 (19.40)	30.67 (23.10)	29.77 (22.28)	375.94 (377.00)
	1.00		7.57 (5.03)	7.45 (4.94)	8.26 (5.26)	8.05 (5.21)	9.19 (5.56)	8.85 (5.45)	212.54 (211.93)
	1.50		3.94 (2.29)	3.84 (2.29)	4.27 (2.42)	4.14 (2.38)	4.63 (2.50)	4.47 (2.51)	96.17 (95.01)
	2.50		1.86 (0.87)	1.80 (0.86)	1.95 (0.91)	1.93 (0.94)	2.09 (0.95)	2.04 (0.96)	16.08 (15.62)

Note. Standard deviation of run length is in parentheses.

Table 130

Average Run Length Comparisons for p = 2, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 500 from the Multivariate $t_p(5)$ Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	11.55	11.20	13.50	12.70	17.50	15.70	17.60
	0.00		493 (500)	489 (496)	496 (504)	486 (491)	501 (501)	500 (501)	488 (183.73)
	0.25		157.01 (152.40)	143.73 (141.69)	259.49 (258.56)	233.51 (233.07)	424.97 (416.26)	407.26 (407.61)	439.56 (436.29)
	0.50		42.35 (35.54)	39.81 (33.39)	82.33 (79.53)	69.90 (66.05)	273.16 (272.22)	237.60 (235.31)	375.94 (377.00)
	1.00		10.76 (6.64)	10.29 (6.28)	15.34 (11.22)	13.61 (9.81)	73.58 (71.69)	54.41 (52.46)	212.54 (211.93)
	1.50		5.29 (2.78)	5.14 (2.73)	6.47 (3.63)	6.00 (3.33)	20.80 (18.90)	15.00 (12.80)	96.17 (95.01)
	2.50		2.34 (1.04)	2.26 (1.03)	2.65 (1.15)	2.50 (1.13)	4.08 (2.31)	3.51 (1.95)	16.08 (15.62)

Note. Standard deviation of run length is in parentheses.

Table 131

Average Run Length Comparisons for p = 3, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 500 from the Multivariate t_p (5) Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
3	δ	*UCL*	10.75	10.30	11.95	11.10	13.80	12.20	21.1
	0.00		498 (579)	500 (553)	503 (564)	492 (434)	501 (546)	491 (510)	494 (493.82)
	0.25		87.95 (81.83)	81.91 (74.27)	109.47 (104.44)	93.94 (85.08)	155.02 (154.19)	117.14 (108.68)	449.27 (449.74)
	0.50		28.06 (21.45)	26.54 (20.10)	32.29 (23.93)	29.03 (21.65)	41.94 (33.17)	34.05 (25.79)	393.81 (393.33)
	1.00		8.90 (5.58)	8.33 (5.54)	9.86 (5.97)	9.07 (5.81)	11.70 (6.62)	9.92 (6.06)	265.61 (264.29)
	1.50		4.58 (2.48)	4.19 (2.50)	5.01 (2.66)	4.53 (2.59)	5.80 (2.96)	5.07 (2.83)	135.90 (134.77)
	2.50		2.08 (0.88)	1.94 (0.94)	2.28 (0.96)	2.07 (0.98)	2.55 (1.06)	2.26 (1.06)	25.48 (25.35)

Note. Standard deviation of run length is in parentheses.

Table 132

Average Run Length Comparisons for p = 3, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 500 from the Multivariate t_p(5) Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
3	δ	*UCL*	17.50	13.70	24.15	15.40	40.10	18.80	21.1
	0.00		500 (516)	493 (500)	495 (496)	492 (490)	496 (493)	507 (510)	494 (493.82)
	0.25		275.10 (282.80)	174.66 (171.45)	414.57 (421.05)	277.41 (273.43)	482.74 (480.92)	424.80 (431.93)	449.27 (449.74)
	0.50		85.82 (81.15)	46.57 (39.43)	248.76 (246.24)	89.49 (85.61)	441.43 (438.39)	274.92 (274.27)	393.81 (393.33)
	1.00		16.68 (10.24)	11.67 (7.24)	51.99 (47.41)	16.34 (12.12)	309.48 (306.82)	75.43 (72.71)	265.61 (264.29)
	1.50		7.51 (3.68)	5.68 (3.00)	14.17 (8.92)	6.82 (3.91)	176.74 (174.33)	20.00 (17.63)	135.90 (134.77)
	2.50		3.11 (1.24)	2.49 (1.13)	4.33 (1.65)	2.77 (1.23)	32.54 (30.00)	4.08 (2.34)	25.48 (25.35)

Note. Standard deviation of run length is in parentheses.

Table 133

Average Run Length Comparisons for p = 4, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 500 from the Multivariate t_p (5) Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
4	δ	*UCL*	13.62	12.20	15.60	13.20	19.20	14.20	24.30
	0.00		502 (604)	491 (545)	506 (578)	498 (498)	503 (541)	482 (504)	500 (501.81)
	0.25		109.75 (108.63)	92.29 (83.13)	153.05 (154.96)	105.13 (94.55)	244.85 (256.84)	129.50 (122.91)	464.87 (463.89)
	0.50		33.43 (25.47)	30.06 (22.02)	42.21 (32.29)	32.91 (23.67)	69.86 (60.68)	36.89 (27.27)	435.11 (434.70)
	1.00		10.15 (6.15)	9.38 (5.89)	12.15 (7.06)	10.16 (6.17)	15.97 (8.77)	10.91 (6.50)	301.30 (297.92)
	1.50		5.08 (2.67)	4.91 (2.73)	6.03 (2.97)	5.16 (2.80)	7.62 (3.65)	5.54 (2.95)	174.03 (171.95)
	2.50		2.21 (0.90)	2.26 (1.06)	2.60 (1.02)	2.35 (1.09)	3.19 (1.21)	2.46 (1.13)	37.06 (36.35)

Note. Standard deviation of run length is in parentheses.

Table 134

Average Run Length Comparisons for p = 4, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 500 from the Multivariate t_p(5) Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
4	δ	*UCL*	27.60	15.90	43.00	17.70	79.00	21.50	24.30
	0.00		501 (521)	504 (521)	501 (505)	501 (501)	497 (493)	493 (494)	500 (501.81)
	0.25		412.40 (425.25)	193.91 (192.41)	483.72 (493.73)	293.52 (295.04)	488.96 (494.61)	435.97 (431.24)	464.87 (463.89)
	0.50		212.10 (214.56)	53.60 (45.68)	400.25 (402.21)	102.60 (99.35)	474.16 (471.84)	304.80 (305.58)	435.11 (434.70)
	1.00		35.00 (25.36)	12.95 (7.97)	191.11 (187.82)	18.45 (13.90)	430.84 (430.54)	93.32 (91.87)	301.30 (297.92)
	1.50		12.36 (5.64)	6.18 (3.26)	60.62 (53.05)	7.39 (4.14)	348.80 (348.90)	24.75 (22.33)	174.03 (171.95)
	2.50		4.51 (1.60)	2.69 (1.20)	8.45 (3.30)	2.98 (1.33)	184.92 (185.17)	4.67 (2.82)	37.06 (36.35)

Note. Standard deviation of run length is in parentheses.

Table 135

Average Run Length Comparisons for p = 5, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 500 from the Multivariate $t_p(5)$ Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
5	δ	*UCL*	17.60	14.10	20.90	15.0	27.60	16.20	27.00
	0.00		497 (589)	501 (562)	497 (563)	496 (534)	502 (548)	494 (519)	494 (493.63)
	0.25		158.47 (160.15)	96.73 (88.77)	232.12 (245.54)	111.39 (101.80)	350.48 (372.37)	142.65 (135.24)	477.49 (483.03)
	0.50		45.37 (33.31)	30.69 (23.33)	64.41 (49.44)	34.73 (25.18)	136.87 (131.07)	40.29 (30.48)	445.17 (444.86)
	1.00		13.41 (7.48)	9.79 (6.34)	16.68 (8.83)	10.41 (6.52)	25.01 (13.30)	11.58 (6.96)	325.54 (323.34)
	1.50		6.67 (3.13)	4.93 (2.85)	8.07 (3.64)	5.31 (2.99)	11.00 (4.67)	5.84 (3.14)	194.43 (192.73)
	2.50		2.91 (1.03)	2.25 (1.09)	3.42 (1.21)	2.39 (1.14)	4.46 (1.48)	2.56 (1.19)	45.63 (45.19)

Note. Standard deviation of run length is in parentheses.

Table 136

Average Run Length Comparisons for p = 5, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 500 from the Multivariate $t_p(5)$ Distribution

			λ						
p			0.1		.2		.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
5	δ	*UCL*	43.40	17.80	72.90	19.80	142.30	23.90	27.00
	0.00		498 (519)	490 (498)	499 (507)	503 (507)	500 (500)	486 (489)	494 (493.63)
	0.25		451.09 (466.83)	203.74 (199.17)	484.18 (492.87)	308.14 (306.56)	466.93 (487.79)	437.01 (435.84)	477.49 (483.03)
	0.50		335.31 (348.58)	56.97 (48.46)	444.15 (449.58)	115.58 (112.66)	490.88 (492.93)	323.66 (326.35)	445.17 (444.86)
	1.00		103.93 (96.48)	13.60 (8.40)	339.13 (338.08)	20.43 (15.71)	170.90 (467.27)	112.13 (109.40)	325.54 (323.34)
	1.50		24.74 (13.55)	6.53 (3.39)	196.52 (191.60)	8.00 (4.57)	434.44 (436.35)	30.13 (27.82)	194.43 (192.73)
	2.50		7.09 (2.20)	2.81 (1.27)	29.54 (21.34)	3.168 (1.41)	337.41 (338.54)	5.18 (3.26)	45.63 (45.19)

Note. Standard deviation of run length is in parentheses.

Table 137

Average Run Length Comparisons for p = 2, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 1,000 from the Multivariate t_p(5) Distribution

p			λ						
			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	9.80	9.70	10.60	10.50	11.75	11.50	21.50
	0.00		1,010 (1,087)	975 (1,047)	998 (1,042)	1,006 (1,054)	993 (1,020)	984 (1,010)	992 (986.65)
	0.25		100.32 (84.64)	100.02 (84.86)	118.13 (103.67)	116.35 (100.40)	158.85 (149.20)	146.30 (135.15)	880.93 (881.01)
	0.50		30.46 (21.47)	29.59 (20.84)	32.82 (22.69)	32.30 (22.60)	38.74 (28.05)	36.77 (26.68)	774.23 (769.71)
	1.00		9.09 (5.60)	8.94 (5.54)	9.87 (5.89)	9.67 (5.80)	10.79 (6.10)	10.58 (6.24)	467.87 (464.36)
	1.50		4.61 (2.55)	4.51 (2.53)	4.90 (2.62)	4.81 (2.63)	5.33 (2.75)	5.23 (2.76)	230.38 (225.60)
	2.50		2.08 (0.95)	2.03 (0.96)	2.22 (1.02)	2.15 (1.02)	2.36 (1.06)	2.31 (1.06)	38.36 (38.12)

Note. Standard deviation of run length is in parentheses.

Table 138

Average Run Length Comparisons for p = 2, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 1,000 from the Multivariate t_p(5) Distribution

p			λ						
			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	13.50	13.0	15.80	14.80	21.15	18.80	21.50
	0.00		1,007 (1,021)	984 (982)	997 (1,015)	995 (1,022)	981 (972)	997 (993)	992 (986.65)
	0.25		276.64 (268.10)	248.20 (243.19)	499.36 (495.19)	451.94 (456.02)	876.76 (873.96)	829.65 (836.81)	880.93 (881.01)
	0.50		59.31 (50.43)	54.37 (45.16)	143.37 (140.85)	118.96 (114.87)	579.86 (580.42)	513.42 (518.66)	774.23 (769.71)
	1.00		12.91 (7.62)	12.31 (7.41)	20.84 (15.56)	17.93 (13.24)	159.60 (157.76)	113.52 (110.85)	467.87 (464.36)
	1.50		6.16 (3.08)	5.88 (3.05)	7.80 (4.37)	7.21 (4.01)	38.24 (35.77)	26.22 (23.53)	230.38 (225.60)
	2.50		2.64 (1.14)	2.53 (1.13)	3.01 (1.27)	2.83 (1.22)	5.52 (3.47)	4.47 (2.57)	38.36 (38.12)

Note. Standard deviation of run length is in parentheses.

Table 139

Average Run Length Comparisons for p = 3, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 1,000 from the Multivariate $t_p(5)$ Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
3	δ	*UCL*	12.75	12.00	14.15	12.90	16.40	14.00	25.60
	0.00		997 (1,093)	985 (1,076)	1,006 (1,082)	1,007 (1,060)	998 (1,076)	1,002 (1,038)	998 (978.42)
	0.25		125.16 (111.37)	111.44 (95.51)	164.19 (151.99)	135.13 (118.27)	267.22 (267.45)	179.27 (167.31)	927.3 (916.77)
	0.50		36.31 (25.40)	33.15 (23.46)	42.31 (29.86)	36.84 (25.03)	58.25 (44.98)	43.33 (31.16)	844.34 (855.33)
	1.00		10.80 (6.31)	9.88 (6.15)	12.03 (6.72)	10.73 (6.41)	14.30 (7.87)	11.79 (6.74)	578.61 (576.05)
	1.50		5.40 (2.80)	5.00 (2.77)	5.89 (3.00)	2.35 (2.88)	6.93 (3.28)	5.79 (3.01)	323.24 (325.44)
	2.50		2.39 (1.00)	2.19 (1.03)	2.60 (1.09)	2.36 (1.09)	2.96 (1.17)	2.54 (1.14)	65.96 (66.47)

Note. Standard deviation of run length is in parentheses.

Table 140

Average Run Length Comparisons for p = 3, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 1,000 from the Multivariate $t_p(5)$ Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
3	δ	*UCL*	21.20	15.60	30.20	17.60	52.20	22.10	25.60
	0.00		982 (1,016)	1,006 (1,026)	996 (1,002)	1,003 (1,007)	999 (985)	964 (966)	998 (978.42)
	0.25		556.61 (568.27)	306.13 (300.75)	834.21 (844.72)	524.51 (521.30)	942.23 (937.06)	847.64 (842.48)	927.3 (916.77)
	0.50		158.50 (153.06)	66.519 (56.76)	526.33 (520.62)	152.54 (146.63)	880.94 (890.63)	576.62 (567.46)	844.34 (855.33)
	1.00		22.57 (13.88)	13.88 (8.18)	108.84 (100.71)	21.76 (16.63)	665.17 (664.35)	159.32 (160.23)	578.61 (576.05)
	1.50		9.28 (4.32)	6.56 (3.36)	23.15 (16.30)	8.11 (4.53)	409.59 (411.25)	37.17 (34.24)	323.24 (325.44)
	2.50		3.69 (1.38)	2.78 (1.22)	5.51 (2.07)	3.16 (1.35)	93.19 (91.61)	5.39 (3.33)	65.96 (66.47)

Note. Standard deviation of run length is in parentheses.

Table 141

Average Run Length Comparisons for p = 4, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 1,000 from the Multivariate t_p(5) Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
4	δ	*UCL*	16.45	14.00	19.05	15.00	23.95	16.20	29.20
	0.00		992 (1,132)	971 (1,029)	992 (1,099)	1,005 (1,078)	999 (1,041)	1,002 (1,022)	988 (984.37)
	0.25		170.69 (162.48)	125.64 (106.06)	270.31 (275.08)	150.27 (134.04)	515.02 (522.18)	202.22 (187.93)	952.96 (951.14)
	0.50		45.55 (31.84)	36.67 (25.02)	59.66 (43.42)	40.73 (28.10)	122.05 (111.14)	47.81 (35.19)	864.67 (877.62)
	1.00		12.78 (7.36)	11.20 (6.57)	15.48 (8.30)	11.96 (6.73)	21.46 (11.09)	13.03 (7.26)	657.06 (649.00)
	1.50		6.31 (3.12)	5.63 (2.99)	7.41 (3.48)	5.98 (3.10)	9.69 (4.27)	6.36 (3.22)	411.02 (403.19)
	2.50		2.61 (1.05)	2.53 (1.14)	3.08 (1.19)	2.63 (1.17)	3.90 (1.42)	2.77 (1.12)	94.77 (92.68)

Note. Standard deviation of run length is in parentheses.

Table 142

Average Run Length Comparisons for p = 4, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 1,000 from the Multivariate t_p(5) Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
4	δ	*UCL*	35.50	17.90	58.00	20.00	111.10	25.30	29.20
	0.00		992 (1,007)	1,022 (1,019)	988 (984)	995 (1,013)	997 (987)	985 (1,002)	988 (984.37)
	0.25		817.43 (839.57)	343.43 (334.60)	943.14 (947.81)	576.40 (574.35)	996.9 (999.07)	914.15 (912.51)	952.96 (951.14)
	0.50		465.15 (463.09)	76.44 (66.79)	823.31 (840.16)	183.41 (180.18)	982.98 (982.92)	662.16 (664.80)	864.67 (877.62)
	1.00		67.87 (55.49)	15.34 (9.23)	457.19 (464.79)	25.41 (20.08)	890.51 (881.57)	211.76 (209.37)	657.06 (649.00)
	1.50		17.58 (8.35)	7.12 (3.55)	169.83 (163.55)	9.04 (5.18)	783.17 (788.80)	51.21 (49.17)	411.02 (403.19)
	2.50		5.85 (1.95)	3.01 (1.31)	15.02 (7.67)	3.40 (1.46)	483.22 (482.63)	6.45 (4.20)	94.77 (92.68)

Note. Standard deviation of run length is in parentheses.

Table 143

Average Run Length Comparisons for p = 5, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 1,000 from the Multivariate $t_p(5)$ Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
5	δ	*UCL*	22.20	16.00	27.20	16.90	37.30	18.10	32.40
	0.00		1,000 (1,152)	1,007 (1,068)	995 (1,091)	983 (1,013)	996 (1,059)	1,017 (1,058)	1,000 (990.61)
	0.25		289.74 (292.96)	134.88 (115.57)	491.37 (525.26)	160.42 (144.71)	752.58 (796.65)	216.90 (203.83)	950.61 (967.43)
	0.50		66.27 (45.24)	38.68 (26.36)	109.66 (88.26)	43.03 (29.27)	327.65 (328.63)	51.50 (37.75)	921.47 (936.77)
	1.00		17.60 (9.05)	11.62 (6.89)	22.98 (11.25)	12.51 (7.30)	39.70 (21.51)	13.57 (7.74)	713.52 (707.90)
	1.50		8.61 (3.79)	5.75 (3.13)	10.76 (4.48)	6.14 (3.26)	15.69 (6.00)	6.60 (3.38)	478.71 (488.31)
	2.50		3.61 (1.25)	2.52 (1.18)	4.34 (1.48)	2.69 (1.22)	5.94 (1.87)	2.86 (1.28)	124.12 (122.95)

Note. Standard deviation of run length is in parentheses.

Table 144

Average Run Length Comparisons for p = 5, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 1,000 from the Multivariate $t_p(5)$ Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
5	δ	*UCL*	61.20	19.90	106.90	22.20	212.10	27.90	32.40
	0.00		1,000 (1,010)	992 (995)	1,000 (1,009)	984 (986)	1,006 (1,006)	996 (1,090)	1,000 (990.61)
	0.25		910.47 (912.74)	367.53 (360.77)	990.63 (998.40)	597.20 (590.33)	973.42 (969.13)	895.20 (892.94)	950.61 (967.43)
	0.50		738.87 (741.88)	84.88 (75.19)	937.27 (940.32)	211.39 (206.03)	979.06 (1,001.09)	707.00 (696.38)	921.47 (936.77)
	1.00		290.46 (287.43)	16.24 (9.76)	746.42 (755.25)	28.588 (22.90)	935.70 (940.07)	251.83 (246.78)	713.52 (707.90)
	1.50		59.69 (44.92)	7.51 (3.78)	518.06 (514.47)	9.80 (5.63)	901.28 (902.16)	63.40 (60.74)	478.71 (488.31)
	2.50		10.41 (3.13)	3.13 (1.34)	122.44 (116.12)	3.57 (1.51)	740.84 (748.72)	7.36 (4.97)	124.12 (122.95)

Note. Standard deviation of run length is in parentheses.

Table 145

Average Run Length Comparisons for p = 2, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 200 with Multivariate Gamma Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	723.1	2,116.6	723.1	1,442.9	723.1	896.6	25.3
	0.00		200 (209.37)	200 (201.53)	200 (209.37)	200 (195.80)	200 (209.37)	200 (192.00)	201 (198.98)
	0.25		11.40 (5.10)	62.89 (50.77)	11.40 (5.10)	65.75 (53.43)	11.40 (5.10)	73.06 (63.22)	146.77 (148.19)
	0.50		5.61 (1.68)	31.05 (19.65)	5.61 (1.68)	32.08 (21.15)	5.61 (1.68)	32.95 (23.37)	92.89 (93.04)
	1.00		2.92 (0.61)	14.73 (6.53)	2.92 (0.61)	14.52 (6.66)	2.92 (0.61)	14.22 (7.13)	30.32 (29.92)
	1.50		2.05 (0.32)	9.54 (3.41)	2.05 (0.32)	9.35 (3.35)	2.05 (0.32)	8.68 (3.30)	8.73 (8.33)
	2.50		1.34 (0.47)	5.71 (1.57)	1.34 (0.47)	5.50 (1.49)	1.34 (0.47)	5.03 (1.41)	1.05 (0.22)

Note. Standard deviation of run length is in parentheses.

Table 146

Average Run Length Comparisons for p = 2, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 200 with Multivariate Gamma Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	190.3	2,116.6	116.2	1,442.9	65.9	896.6	25.3
	0.00		200 (199.56)	200 (201.53)	200 (196.41)	200 (195.80)	201 (200.60)	200 (192.00)	201 (198.98)
	0.25		12.38 (7.25)	62.89 (50.77)	15.24 (11.33)	65.75 (53.43)	30.93 (29.17)	73.06 (63.22)	146.77 (148.19)
	0.50		5.18 (1.93)	31.05 (19.65)	5.18 (2.42)	32.08 (21.15)	8.04 (6.23)	32.95 (23.37)	92.89 (93.04)
	1.00		2.51 (0.61)	14.73 (6.53)	2.24 (0.62)	14.52 (6.66)	2.14 (0.85)	14.22 (7.13)	30.32 (29.92)
	1.50		1.83 (0.38)	9.54 (3.41)	1.60 (0.49)	9.35 (3.35)	1.26 (0.44)	8.68 (3.30)	8.73 (8.33)
	2.50		1.00 (0.528)	5.71 (1.57)	1.00 (0)	5.50 (1.49)	1.00 (0)	5.03 (1.41)	1.05 (0.22)

Note. Standard deviation of run length is in parentheses.

Table 147

Average Run Length Comparisons for p = 2, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 500 with Multivariate Gamma Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	761.7	2,175.5	537.4	1,486.7	352.7	928.5	34.82
	0.00		494 (494.26)	500 (480.18)	498 (495.26)	500 (480.08)	500 (500.16)	500 (490.50)	503 (505.74)
	0.25		16.79 (6.56)	115.43 (88.44)	16.44 (6.90)	122.14 (102.09)	16.15 (7.60)	135.68 (120.26)	363.96 (363.06)
	0.50		7.95 (2.05)	49.42 (28.52)	7.53 (2.04)	48.56 (30.40)	7.05 (2.13)	51.75 (37.01)	234.19 (236.49)
	1.00		4.03 (0.72)	21.62 (8.29)	3.79 (0.71)	20.25 (8.46)	3.44 (0.70)	18.91 (9.03)	76.57 (77.11)
	1.50		2.82 (0.43)	13.69 (4.28)	2.65 (0.48)	12.49 (4.03)	2.37 (0.49)	11.28 (3.97)	21.76 (21.00)
	2.50		1.97 (0.14)	7.98 (1.84)	1.94 (0.22)	7.15 (1.73)	1.77 (0.42)	6.30 (1.60)	1.65 (1.04)

Note. Standard deviation of run length is in parentheses.

Table 148

Average Run Length Comparisons for p = 2, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 500 with Multivariate Gamma Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	207.7	498.3	129.3	272.6	76.7	125.8	34.82
	0.00		500 (497.38)	501 (499.80)	500 (509.81)	498 (503.80)	502 (502.64)	503 (502.16)	503 (505.74)
	0.25		17.48 (10.46)	170.89 (161.73)	23.76 (19.03)	217.21 (213.21)	66.31 (64.08)	316.65 (318.75)	363.96 (363.06)
	0.50		6.52 (2.3)	63.85 (53.76)	6.69 (3.25)	93.26 (86.92)	13.71 (11.65)	177.47 (173.15)	234.19 (236.49)
	1.00		3.00 (0.68)	19.18 (11.23)	2.65 (0.71)	24.13 (18.92)	2.73 (1.17)	58.86 (56.08)	76.57 (77.11)
	1.50		2.04 (0.33)	10.24 (4.42)	1.85 (0.37)	10.55 (6.14)	1.53 (0.51)	21.04 (18.51)	21.76 (21.00)
	2.50		1.25 (0.43)	5.23 (1.50)	1.00 (.045)	4.53 (1.57)	1.00 (0.01)	5.16 (3.21)	1.65 (1.04)

Note. Standard deviation of run length is in parentheses.

Table 149

Average Run Length Comparisons for p = 2, λ =.02, .03, and .05, and In-Control Average Run Length ≈ 1,000 with Multivariate Gamma Distribution

			λ						
p			.02		.03		.05		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	798.1	2,211.7	559.3	1,515.7	369.2	948.8	42.70
	0.00		1,000 (980.70)	1,002 (993.84)	1,000 (966.63)	998 (993.66)	1,001 (1,002.63)	1,000 (995.40)	993 (1,000.55)
	0.25		22.22 (7.90)	171.30 (135.49)	20.09 (7.98)	185.61 (157.16)	19.64 (9.27)	216.14 (197.23)	731.07 (737.23)
	0.50		10.23 (2.41)	63.78 (36.21)	8.96 (2.32)	64.41 (40.64)	8.15 (2.33)	70.43 (52.68)	453.92 (455.09)
	1.00		5.06 (0.83)	25.95 (9.43)	4.42 (0.78)	24.17 (9.80)	3.90 (0.75)	22.71 (10.55)	148.01 (148.15)
	1.50		3.48 (0.53)	16.15 (4.55)	3.03 (0.43)	14.70 (4.52)	2.71 (0.47)	13.00 (4.43)	43.76 (43.17)
	2.50		2.08 (0.28)	9.36 (1.99)	1.99 (0.07)	8.33 (1.88)	1.95 (0.22)	7.08 (1.72)	3.2 (2.65)

Note. Standard deviation of run length is in parentheses.

Table 150

Average Run Length Comparisons for p = 2, λ = 0.1, 0.2, and 0.5, and In-Control Average Run Length ≈ 1,000 with Multivariate Gamma Distribution

			λ						
p			0.1		0.2		0.5		
			SRMEWMA	MEWMA	SRMEWMA	MEWMA	SRMEWMA	MEWMA	Hotelling's T^2
2	δ	*UCL*	219.4	512.0	138.7	282.8	84.5	132.7	42.70
	0.00		997 (990.75)	1,007 (1,003.3)	997 (994.06)	1,001 (993.05)	1,000 (997.33)	999 (982.17)	993 (1,000.55)
	0.25		21.79 (13.54)	288.32 (278.74)	34.21 (28.56)	411.96 (404.02)	115.95 (115.52)	601.31 (603.94)	731.07 (737.23)
	0.50		7.50 (2.67)	93.85 (83.42)	7.96 (3.89)	156.50 (149.59)	20.60 (18.14)	331.37 (327.96)	453.92 (455.09)
	1.00		3.33 (0.74)	23.47 (14.41)	2.93 (0.76)	33.68 (27.83)	3.27 (1.51)	98.39 (95.80)	148.01 (148.15)
	1.50		2.25 (0.45)	11.75 (4.96)	1.97 (0.35)	12.90 (7.76)	1.70 (0.51)	32.58 (30.41)	43.76 (43.17)
	2.50		1.58 (0.49)	5.18 (1.59)	1.05 (0.23)	5.11 (1.78)	1.00 (0.00)	6.53 (4.28)	3.2 (2.65)

Note. Standard deviation of run length is in parentheses.

Table 151

The Aluminum Capacitor Data Set

obs.	Capacitance	Dissipation	Leakage	**obs.**	Capacitance	Dissipation	Leakage	**obs.**	Capacitance	Dissipation	Leakage	**obs.**	Capacitance	Dissipation	Leakage
1	443	5.81	21.5	**51**	443	4.29	14.9	**101**	454	4.47	18.6	**151**	449	4.63	31.5
2	448	4.53	25.3	**52**	441	5.39	26.8	**102**	444	4.11	24.1	**152**	450	4.15	19.3
3	443	4.23	33.7	**53**	443	4.43	15.3	**103**	465	3.95	45.9	**153**	442	4.72	28.2
4	446	4.65	17.6	**54**	440	5.62	25.3	**104**	444	3.42	25.5	**154**	463	4.32	18.6
5	439	3.65	20.7	**55**	444	4.19	18.4	**105**	444	3.98	17.9	**155**	446	5.62	19.8
6	435	3.98	18.7	**56**	474	5.34	28.4	**106**	449	4.14	19.6	**156**	446	4.98	23.9
7	447	4.17	19.5	**57**	469	4.25	21.2	**107**	450	4.09	21	**157**	445	3.95	19.5
8	454	4.45	21.8	**58**	459	4.25	29.5	**108**	454	3.85	16.5	**158**	446	5.67	20.5
9	445	5.39	20.8	**59**	443	5.39	24.5	**109**	443	4.47	36.7	**159**	448	4.41	31.2
10	443	4.39	18.7	**60**	449	4.1	24.5	**110**	441	5.19	21	**160**	449	4.25	16.5
11	442	4.67	31.1	**61**	443	4.29	20.4	**111**	449	4.67	27.9	**161**	434	3.49	17.9
12	445	4.55	31.2	**62**	447	4.29	15.5	**112**	447	3.56	15.9	**162**	469	4.75	31.4
13	446	4.41	29.6	**63**	466	4.85	26.6	**113**	437	4.47	21.5	**163**	439	4.49	15.3
14	448	4.65	37.7	**64**	449	4.63	33.1	**114**	459	4.52	26.4	**164**	448	4.33	27.9
15	446	4.32	18.2	**65**	456	4.36	14.7	**115**	445	4.25	16.5	**165**	456	3.47	27.2
16	446	6.01	19.5	**66**	445	3.95	25.9	**116**	456	4.13	20.9	**166**	493	4.12	19.5
17	459	4.54	16.5	**67**	442	5.68	23.5	**117**	485	3.96	27.5	**167**	447	3.95	21.2
18	441	5.39	25.7	**68**	453	4.36	17.7	**118**	463	4.5	16.7	**168**	446	4.39	30.6
19	439	5.39	20.8	**69**	441	4.34	18.3	**119**	465	4.97	26.8	**169**	449	3.98	20.4
20	439	4.23	17.3	**70**	465	3.93	19.3	**120**	449	4.26	20.9	**170**	453	4.45	18.6
21	454	4.47	16.5	**71**	473	4.32	29.6	**121**	446	4.30	30.9	**171**	456	4.36	15.3
22	440	4.25	23	**72**	447	4.36	15.4	**122**	439	5.55	23.2	**172**	449	4.35	18.9
23	440	4.69	31.3	**73**	449	4.36	19.5	**123**	457	4.53	25.9	**173**	459	4.19	27.5
24	445	4.25	22.4	**74**	456	3.67	25.6	**124**	446	4.39	30.6	**174**	449	4.05	25.6
25	469	3.85	23.6	**75**	449	4.77	24.9	**125**	447	4.25	15.2	**175**	446	5.63	21.9
26	447	4.87	27.5	**76**	449	4.27	28.6	**126**	443	4.63	17.2	**176**	445	4.35	18.3
27	463	3.49	20.9	**77**	446	4.37	30	**127**	434	4.27	15.9	**177**	443	4.69	26.5
28	457	4.55	19.5	**78**	477	4.25	26.5	**128**	447	4.22	24.7	**178**	437	4.45	15.7
29	438	6.32	19.8	**79**	445	5.39	18.7	**129**	470	4.83	30.5	**179**	441	4.56	35
30	449	6.76	22.9	**80**	454	3.85	17.2	**130**	445	3.95	16.5	**180**	447	3.47	19.5
31	440	4.82	19.2	**81**	445	4.25	20.7	**131**	446	4.59	36.7	**181**	440	4.08	20.5
32	446	4.74	24.7	**82**	439	4.17	17.1	**132**	439	5.93	25.7	**182**	439	5.73	24.3
33	445	4.25	17.6	**83**	445	5.61	23.9	**133**	445	4.95	33.5	**183**	436	3.92	17.4
34	439	4.94	26.4	**84**	449	4.56	31.1	**134**	466	4.58	28.6	**184**	440	4.52	15.4
35	463	4.64	27.6	**85**	450	4.65	30.7	**135**	459	4.09	17.7	**185**	446	5.62	19.9

36	471	4.91	30.1	**86**	449	3.59	23.9	**136**	440	4.98	21	**186**	439	4.47	16.7
37	448	4.65	18.5	**87**	457	3.49	25.9	**137**	452	4.44	37.6	**187**	445	4.32	19.5
38	445	5.23	20.8	**88**	442	4.52	31	**138**	446	4.24	15.6	**188**	439	4.27	20.1
39	469	4.35	17.5	**89**	451	3.85	19.5	**139**	441	5.33	19.6	**189**	442	4.11	21.3
40	453	4.25	18.2	**90**	444	5.8	21.9	**140**	444	4.5	30	**190**	442	4.98	29
41	434	3.93	16.5	**91**	441	5.17	21.7	**141**	467	3.92	18.5	**191**	438	3.83	14.7
42	459	4.37	18.5	**92**	471	4.65	31.3	**142**	468	4.58	34.8	**192**	453	4.27	17.3
43	447	3.95	24.7	**93**	448	4.56	46.5	**143**	445	4.74	20	**193**	448	4.93	17.5
44	445	4.27	17.5	**94**	440	5.98	23.9	**144**	446	4.55	21.9	**194**	447	4.39	19.8
45	446	4.99	28.6	**95**	440	4.67	25.9	**145**	472	4.81	27	**195**	447	4.15	30.3
46	441	5.31	19.8	**96**	441	4.87	32.8	**146**	436	5.63	29.1	**196**	447	4.52	17.5
47	435	4.23	14.4	**97**	443	5.98	21.9	**147**	457	4.17	17.4	**197**	465	4.37	24.7
48	450	4.04	20.8	**98**	448	4.13	19.9	**148**	448	4.32	29.6	**198**	447	4.47	17.2
49	442	4.49	17.9	**99**	470	4.71	27.5	**149**	485	3.45	25.7	**199**	443	4.73	31.3
50	451	4.37	17.2	**100**	447	4.15	16.7	**150**	450	4.67	29.5	**200**	456	4.37	16.7