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Unbiasedness of Prediction under Linex Loss Function in Autoregressive Moving Average Models

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UNBIASEDNESS OF PREDICTION UNDER LINEX LOSS FUNCTION IN AUTOREGRESSIVE MOVING AVERAGE MODELS

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ABSTRACT

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The asymmetric loss function is used in a situation where a positive error may be more serious than a negative error of the same magnitude or vice versa. One of the most commonly used asymmetric loss functions is the linex loss. The linex unbiased predictor has been developed and applied to real world applications. This study investigated how the linex unbiased prediction behaves when time series processes, AR(p), MA(q) and ARMA(p,q), parameters are unknown and being estimated, with different levels of variance, forecast step, shape parameter and series length. It started with deriving the predictor for each time series process, computing this predictor, and then discussing its properties.

Empirical studies of the behavior of this predictor were investigated by using the Monte Carlo simulation. The results of this study showed that, a simpler time series model produced values that were closer to the condition of linex unbiasedness than a complex model. The condition of linex unbiasedness was affected by the variance but not the sign of the linex loss function shape parameter. For any time series model and any condition, as series length increased, the condition of linex unbiasedness values approached zero. When the time series parameters are unknown, the prediction is asymptotically linex unbiased.

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CHAPTER I

INTRODUCTION

Background

Statistical theory and methods have a long and rich history of using sample statistics to make inferences about population parameters. For example, a random sample is taken from a target population, and the estimates of the parameters are obtained in order to describe the unknown parameters. It is reasonable to question the accuracy of these estimates in the parameter uncertainty and disturbances, that is, if the estimate is unbiased. Unbiasedness is usually a desirable property for estimators. Many results are available in the statistical inference for parameter estimates and hypothesis testing, such as Lehmann and Casella (1998) and Lehmann and Romano (2005).

However, in practice, the value of interest is not only parameter estimates. One often wants to predict an unknown random variable or an unknown future value, given the observed data or stochastic process. When a prediction is based on the most relevant data or the past observations in time, time series analysis are usually applied in order to forecast. Time series forecasts are used in many fields, such as weather forecast based on the temperature measured in successive hours, predicting stock share price based on the past share price on successive weeks, or project sales volume based on the recorded sales volume in successive months. Most

time series cannot be predicted exactly (i.e. deterministic), because the future values of any sample function cannot be exactly predicted from the observed past values, and time series is partially determined by the random noise (Babu, 2014). Thus, most time series are non-deterministic.

The unbiasedness property is sometimes desired in the context of prediction. To evaluate the usefulness of the prediction, one needs to specify the loss function. The loss is often a function of the difference between the predicted value and the actual value. Overprediction occurs when the predicted value is greater than the actual value; underprediction occurs when the actual value is greater than the predicted value. Symmetric loss functions are most commonly used, for example, mean squared error loss. However, in some practical situations, overprediction may be more serious than underprediction or vice versa. Many literatures have recognized that the use of symmetric loss may be unsuitable in some situations. Varian (1975) discussed problems about the value of real estate assessment. Varian (1975) stated that underassessment results in an approximately linear loss of revenue whereas overassessment often results in appeals with attendant, substantial litigation and other costs. Feynman (1987) indicated that the Space Shuttle Challenger disaster of 1986 was partially the result of overprediction of the average life of the solid-fuel rocket booster. When predicting the average life of components of a space ship or aircraft, overprediction is usually more serious than underprediction. Zellner (1986) also pointed out that an underprediction of the peak water level in a dam construction has more serious consequences than an overprediction. Similarly, J. Shao and Chow (1991) mentioned that an

overprediction of the safety risk of a drug may restrict the use of the product and reduce sales, while an underprediction may lead to potentially disastrous and costlier consequences. Because a symmetric loss function assigns equal weight to positive and negative prediction errors of the same magnitude, this does not always reflect actual gains and losses. Many researchers have discussed the inappropriate use of symmetric loss, and have given details on examples of the natural occurrence of asymmetric loss (Aitchison & Dunsmore, 1980; Berger, 1985; Ferguson, 1967; Granger, 1969; Harris, 1992).

Statement of the Problem

The majority of time series prediction literature have traditionally focused on symmetric loss functions and the prediction theory under such conditions are well established (Bao, 2007; Clements & Hendry, 1995; Cryer, Nankervis, & Savin, 1990; Dufour, 1984, 1985; Fuller & Hasza, 1980; Magnus & Pesaran, 1989, 1991; Malinvaud, 1975). The conventional time series predictions focus on the prediction error caused by estimation error, rather than the loss function different from the squared error loss (Patton & Timmermann, 2007). In recent decades, asymmetric loss functions have increasingly caught researchers' attention (Clatworthy, Peel, & Pope, 2012; Demetrescu, 2007; Granger, 1969; Varian, 1975; Wan, 1999; Wen & Levy, 2001; Zellner, 1986). Varian (1975) developed one of the most commonly used asymmetric loss functions, linex loss function (for linear-exponential), which is suitable for a case when positive error and negative error do not have the same magnitude.

It is natural to question the unbiasedness of the prediction in order to improve the prediction quality and accuracy. The unbiasedness of a predictor under asymmetric loss was characterized by a couple of researchers. Granger (1969) developed the general form of unconditional Gaussian process optimal predictor, under symmetric and asymmetric loss functions, Christoffersen and Diebold (1997) extended this result to conditional Gaussian processes under linex loss, and derived the linex unbiased predictor, and Xiao (2000) studied the linex unbiasedness in a prediction problem. The result of Christoffersen and Diebold (1997) has been used by many applied researchers, such as Batchelor and Peel (1998) who applied to the linex unbiased predictor to the Autoregressive Conditionally Heteroscedastic (ARCH) process, Ulu (2006) applied it to the Generalized Autoregressive Conditionally Heteroscedastic (GARCH) process, and Patton and Timmermann (2007) generalized it to optimal forecast properties.

In the studies of Patton and Timmermann (2007), the value of the parameters are assumed to be known. However, in practice, most of the parameters are unknown. Also, Patton and Timmermann (2007) showed that the property of ordinary unbiased prediction can be invalid under linex loss. Since the loss function is asymmetric, accessing linex unbiased prediction may be more suitable than accessing ordinary unbiased prediction. Patton and Timmermann (2007) did not evaluate the linex unbiasedness property.

The ARCH and GARCH are processes where the errors do not have constant variance. These models are useful in economic and financial studies. In contrast, the Autoregressive (AR), Moving Average (MA), and Autoregessive and Moving

Average (ARMA) processes have constant variance. These processes contain a large class of parsimonious time series models that are useful in describing a wide variety of time series encountered in practice (Wei, 1990). Thus, it is important to investigate the linex unbiasedness property, when applying linex unbiased predictor to the AR, MA and ARMA processes where the parameters are unknown.

Purpose of the Study

The current studies have established the condition of linex unbiased prediction when the parameters are known. However, when the parameters must be estimated, it is not clear that the same property holds. Since applied researchers use the linex loss function, it is important to determine whether or not the linex unbiased property will hold, if, the parameters of AR, MA and ARMA processes are estimated. This dissertation attempted to use Monte Carlo simulation results to inform practical researchers on how this theoretical construct of linex unbiased prediction behaves.

Research Questions

Q1 How does the condition of linex unbiasedness (CLU) behave when parameters of AR, MA, ARMA processes are unknown and being estimated?

Q2 How does the condition of linex unbiasedness differ with the changes in variance, σ^2, of the observed series is less than, equal to and greater than 1?

Q3 How does the condition of linex unbiasedness differ when the forecast step increases?

Q4 How does the condition of linex unbiasedness differ when the length of the observed series increases?

Q5 How does the condition of linex unbiasedness differ when the linex loss function shape parameter is positive or negative?

Q6 How does the risk of linex unbiased predictor change when parameters are estimated?

Summary

This dissertation developed general results of unbiased prediction of different stationary time series processes under linex loss. Time series models and time series processes were used interchangeably, and prediction and forecast were used interchangeably, too.

Chapter II reviewed time series processes, unbiased estimator and predictor, loss function, linex loss function, risk function and linex unbiasedness, and existing research that studies unbiased prediction under symmetric loss and asymmetric loss. Next, Chapter III described the methodology used in this study in order to investigate the research questions. Chapter IV presented the results of the research questions, including tables and figures. Chapter V discussed the results, limitations, and directions for future research.

CHAPTER II

REVIEW OF LITERATURE

This chapter reviewed the literature on the development of the unbiased prediction in univariate time series. The structure of this chapter is as follows. The Time Series section introduced the concepts of time series and the description of models that are relevant to this study which include autoregressive, moving average, autoregressive moving average, and autoregressive integrated moving average models.

The Unbiased Estimation section started with the definition of the unbiased estimator, loss function in estimation, linex loss function in estimation, risk function and unbiasedness in estimation and linex unbiased estimator. The Unbiased Prediction section began with the definition of the unbiased predictor, followed by loss function in prediction, linex loss function in prediction, risk function and unbiasedness in prediction and linex unbiased predictor.

The Unbiased Prediction for Time Series Models under Squared Error Loss Functions section began with the mean unbiased prediction for the autoregressive process of order 1, followed by the mean unbiased prediction for the autoregressive model of order p, error symmetry for the autoregressive moving average model of order p and q, the autoregressive integrated moving average model of order p, d and q, and the median unbiased prediction for the autoregressive process of order 1.

The Unbiased Prediction for Time Series Models under linex Loss Function section began with the unconditional Gaussian process under general asymmetric loss function, followed by the conditional Gaussian process under linex loss function. Finally, the important findings were summarized to address the need of this study.

Time Series

A time series is an ordered sequence of observations collected through time. Because the observations of a time series are naturally dependent or correlated, the statistical independence assumption is no longer applicable (Wei, 1990). The purpose of time series modeling is to develop an appropriate model which describes the structure of the series, and to use this model to predict the future values of the series.

Time series analysis is used in statistics, engineering, geophysics, meteorology, economics, finance, etc. There are two different domains in time series analysis. The approach that uses the autocorrelation and the autocovariance functions to evaluate a process according to the progression of its state with time is known as time domain; the approach that uses the sinusoidal wave to analyze a process according to its response for different frequencies is known as frequency domain (Wei, 1990). In this study, only univariate time series models in time domain are discussed.

A time series is essentially an example of stochastic processes. A stochastic process Z_t, $t \in T$ is a collection of random variables, where T is an index set for all of the possible values. When T represents time, the stochastic process is referred to as a time series (Woodward, Gray, & Elliott, 2012).

The stationary process is a special case of stochastic processes. A stochastic process is said to be strictly stationary if the joint probability distribution associated with m observations $z_{t_1}, z_{t_2}, \ldots, z_{t_m}$, made at any set of times $t_1, t_2, \ldots, t_m$, is the same for $z_{t_1+k}, z_{t_2+k}, \ldots, z_{t_m+k}$, made at times $t_1 + k, t_2 + k, \ldots, t_m + k$, (Box, Jenkins, & Reinsel, 1976). That is, the stochastic process has constant mean $\mu = E[Z_t]$, variance $\sigma^2 = Var(Z_t) = E[(Z_t - \mu)^2]$, and the joint probability distribution is the same for all times. However, the requirement of a strictly stationary process is difficult to establish mathematically. In fact, the distributions involved are unknown in most of the cases. Therefore, less restrictive notions of stationarity have been established (Woodward et al., 2012). A process is called weakly stationary (or covariance stationary, second-order stationary or stationary in the wide sense) if there is a constant mean, and the autocovariance function only depends on the time difference (i.e. the lag) $\gamma(k) = Cov[Z_t, Z_{t+k}]$. In the remaining context of this dissertation, unless specified otherwise, the term 'stationary' will refer to covariance stationary.

A stochastic process is said to be a normal or Gaussian process if all its finite joint probability distributions are normal. Because a normal distribution is uniquely characterized by its first two moments, strictly stationary and weakly stationary are equivalent for a Gaussian process (Wei, 1990). Most time series theories are based on the Gaussian process assumption, thus, the autocorrelation function and partial autocorrelation function are the fundamental tools in time series analysis.

Autocorrelation (acf) is the correlation of a time series with its own past and future value. Let Z_t be a stationary process, the acf between Z_t and Z_{t+k} is denoted by

$$\rho_k = \frac{Cov(Z_t, Z_{t+k})}{\sqrt{Var(Z_t)}\sqrt{Var(Z_{t+k})}} = \frac{\gamma_k}{\gamma_0},$$

where γ_k is the autocovariance function (acvf), which is the covariance between Z_t and Z_{t+k}. γ_k and ρ_k represent the covariance and correlation between Z_t and Z_{t+k} from the same process, separated only by k time lags.

The correlation between Z_t and Z_{t+k} after their mutual linear dependency on the variables on $Z_{t+1}, Z_{t+2}, \ldots, Z_{t+k-1}$ has been removed is called the partial autocorrelation function (pacf). The partial autocorrelation between Z_t and Z_{t+k} can be obtained as the regression coefficient associated with Z_t when one regresses Z_{t+k} on its k lagged variables $Z_{t+k-1}, Z_{t+k-2}, \ldots, Z_t$, see Wei (1990) for more details.

Autoregressive Process

The autoregressive process is a fundamental class of time series models. Let a_t be a sequence of uncorrelated random variables from a fixed distribution with mean zero and variance σ_a^2, i.e. a white noise process. A time series Z_t is said to be a zero mean autoregressive process or model of order p, denoted by AR(p), if there exist $\phi_1, \ldots, \phi_p \in \mathbb{R}$ with $\phi_p \neq 0$, and

$$Z_t = \phi_1 Z_{t-1} + \ldots + \phi_p Z_{t-p} + a_t, \qquad t = 1 \ldots T. \tag{1}$$

An AR process is describing the situation in which the value at time t depends on

its preceding values plus a random shock (Box et al., 1976), and is frequently used in econometrics. The pacf of an AR process cuts zero at lag p. This property is useful in identifying an AR model as a generating process for a time series.

An AR(p) process can be expressed by using the backward shift operator B. The B operator is defined as $B^j Z_t = Z_{t-j}$. Thus, an AR(p) process can be presented as

$$\Phi(B)Z_t = a_t,$$

where

$$\Phi(B) = (1 - \phi_1 B - \ldots - \phi_p B^p).$$

Moving Average Process

A moving average process is another essential approach for modeling time series. Let a_t be a white noise process with mean zero and variance σ_a^2. A process Z_t is said to be a zero mean moving average process or model of order q, denoted as MA(q), if

$$Z_t = a_t + \theta_1 a_{t-1} + \ldots + \theta_q a_{t-q}, \qquad t = 1 \ldots T, \tag{2}$$

where θ_i are constants. A MA process is describing the situation in which events produce an immediate effect that only lasts for short periods of time. The acf plot of a MA(q) process cuts off after lag q, hence acf can be used to identify if a given time series follows a MA process.

A MA(q) process can be represented in the B operator form,

$$Z_t = \Theta(B)a_t,$$

where

$$\Theta(B) = (1 + \theta_1 B + \ldots + \theta_q B^q).$$

A finite-order MA process is said to be invertible if it can be rewritten as an infinite-order AR process,

$$a_t = \frac{1}{\Theta(B)} Z_t = \pi(B) Z_t = Z_t - \pi Z_{t-1} - \pi Z_{t-2} \ldots,$$

where the roots of $\Theta(B) = 1 + \theta_1 B \ldots + \theta_q B^q = 0$ lie outside of the unit circle (a circle with radius of one). Invertibility implies that $1/\Theta(B)$ has a convergent series expression in powers of B. By convergent, it means that the AR coefficient decrease to 0 as the series goes back in time. Correspondingly, a finite-order AR process is said to be stationary if it can be rewritten as an infinite-order MA process,

$$Z_t = \frac{1}{\Phi(B)} a_t = \psi(B) a_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \ldots,$$

where the roots of $\Phi(B) = 1 - \phi_1 B \ldots - \phi_p B^p = 0$ lie outside of the unit circle.

Autoregressive Moving Average Process

Box et al. (1976) combined the autoregressive and the moving average processes. A stationary and invertible process can be expressed in either a moving

average form or an autoregressive form, but it may contain too many parameters. The mixed process contains fewer parameters than a pure AR or MA model itself. The parsimonious time series models are more efficient in estimation and often used in real practice. A mixed autoregressive moving average process which contains p AR terms and q MA terms is said to be an autoregressive moving average process or model of order (p,q), denoted by ARMA(p,q). For a zero mean process, it is given by

$$Z_t = \phi_1 Z_{t-1} + \ldots + \phi_p Z_{t-p} + a_t + \theta_1 a_{t-1} + \ldots + \theta_q a_{t-q}. \tag{3}$$

Using the backward shift operator B, an ARMA(p,q)process can be represented as

$$\Phi(B) Z_t = \Theta(B) a_t,$$

or where

$$\Phi(B) = 1 - \phi_1 B - \ldots - \phi_p B^p,$$

and

$$\Theta(B) = 1 + \theta_1 B + \ldots + \theta_q B^q.$$

For the process to be stationary, the roots of $\Phi(B) = 0$ must lie outside the unit circle. For the process to be invertible, the roots of $\Theta(B) = 0$ must lie outside the unit circle. It is assumed that $\Phi(B) = 0$ and $\Theta(B) = 0$ share no common roots.

Autoregressive Integrated Moving Average Process

The original key reference of autoregressive integrated moving average model is Box et al. (1976), and ARIMA models are sometimes called Box-Jenkins models. In practice, most of time series data are not stationary. When a time series is non-stationary, it should be transformed into a stationary series by considering relevant differences. The difference operator is denoted by $\nabla = 1 - B$. In general, dth degree of differencing can be written as

$$\nabla^d Z_t = (1 - B)^d Z_t.$$

Taking a proper degree of differencing can remove the trend and reduce a series to a stationary time series.

The process is said to be a zero mean autoregressive integrated moving average process or model of order (p,d,q), denoted by ARIMA(p,d,q), if $\nabla^d Z_t$ is an ARMA(p,q). The ARIMA model is giving by

$$\Phi(B)(1 - B)^d Z_t = \Theta(B) a_t,$$

or in compact form,

$$\Phi(B)\nabla^d Z_t = \Theta(B) a_t,$$

where the stationary AR operator $\Phi(B)$ and the invertible MA operator $\Theta(B)$ share no common factors.

Unbiased Estimation

Unbiased Estimator

In statistics, "bias" is a function which describes a statistical property. The bias of an estimator is the difference between the estimator's expected value and the true value of the parameter being estimated; if such difference is zero then the estimator is said to be a mean unbiased estimator. Let X be an observable random variable which does not depend on any unknown parameter θ in the parameter space Θ. When using a statistic $W(X)$ to estimate θ, $W(X)$ is said to be an unbiased estimator of θ if

$$E[W(X)] = \theta, \qquad \text{for all } \theta \in \Theta. \tag{4}$$

Besides the mean, bias can also be measured relative to the median. A number m is a median of X if $P(X \geq m) \geq \frac{1}{2}$ and $P(X \leq m) \geq \frac{1}{2}$. If X is continuous, m satisfies $\int_{-\infty}^{m} f(x)dx = \int_{m}^{\infty} f(x)dx = \frac{1}{2}$ (Casella & Berger, 1990). Then an estimator $\hat{m}$ is said to be median unbiased if

$$P[\hat{m} \leq m] = P[\hat{m} \geq m]. \tag{5}$$

Loss Function in Estimation

When an estimate differs from the true value of the parameter being estimated, one may consider the loss of such difference to be a function (Bain & Engelhardt, 1992). Let $W(X)$ be an estimator of θ, then a loss function is any real

valued function, $L(W(X), \theta)$, such that

$$L(W(X), \theta) \geq 0 \qquad \text{for every } X \text{ and } \theta, \tag{6}$$

and

$$L(W(X), \theta) = 0 \qquad \text{when } W(X) = \theta. \tag{7}$$

If $W(X)$ is close to θ, the loss is small; if $W(X)$ is far from θ, the loss is large. In general, the loss function increases as the distance between $W(X)$ and θ increases (Casella & Berger, 1990). The two commonly used loss functions are squared error (quadratic) loss

$$L(W(X); \theta) = (W(X) - \theta)^2, \tag{8}$$

and absolute error loss

$$L(W(X); \theta) = |W(X) - \theta|. \tag{9}$$

Squared error loss gives relatively more penalty for large discrepancies and absolute error loss gives relatively more penalty for small error loss (Casella & Berger, 1990). Both of these functions are symmetric loss functions.

Linex Loss Function
in Estimation

When the loss of overestimation is not equivalent to underestimation, the asymmetric loss function should be used. The asymmetric loss function accounts for the problem of overestimation and underestimation. A number of asymmetric loss functions have been developed, such as the linex loss function (Varian, 1975), the

linlin loss function (Granger, 1969), the Higgins Tsokos loss function (Camara & Tsokos, 1999), and the Blinex (Wen & Levy, 2001). Among all, the linex loss function is the most widely used.

The linex loss function was originally developed by Varian (1975) in the situation of real estate assessment. Varian (1975) explains that a greater loss is likely to be incurred from an overestimation than from an underestimation in an appraisal of a property (Gruber, 1990). The features of the linex loss functions are that the loss function should be linear for large negative errors, increasing for positive errors at a greater than linear rate, and increase monotonically for positive errors, or vice versa. The linex loss function is given by:

$$L(W(X), \theta) = e^{\alpha(W(X)-\theta)} - \alpha(W(X) - \theta) - 1. \tag{10}$$

The parameter α determines the shape of the loss function. When α is > 0, the linex loss function is approximately linear on the negative x-axis and approximately exponential on the positive x-axis; and vice-versa when α is < 0. The linex loss function is better in a situation where an overestimate or underestimate could have serious consequences. The linex loss function will be close to quadratic loss when α is small.

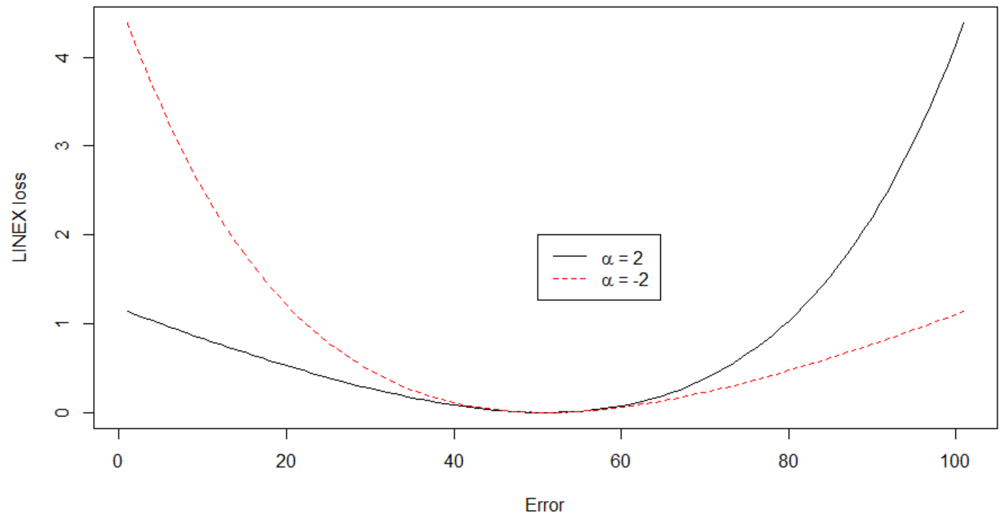

Figure 1. Plot of Linex Loss Function

Risk Function and Unbiasedness in Estimation

The quality of an estimator is measured by the risk function. The expected loss is called the risk function,

$$R(W, \theta) = E_\theta[L(W(X), \theta)]. \tag{11}$$

The concept of risk unbiasedness was first introduced by Lehmann (1951). An estimator $W(X)$ is said to be risk unbiased for θ if it satisfies

$$E[L(W(X), \theta)] \leq E[L(W(X), \theta')] \qquad \text{for all } \theta' \neq \theta. \tag{12}$$

Equation (12) stats that, on the average, $W(X)$ is at least as close to the true estimand θ as it is to any false estimand θ' (Lehmann & Casella, 1998).

When the squared error loss is taken, the corresponding risk function is mean squared error (MSE), and Equation (12) becomes

$$E[W(X)-\theta']^2 \geq E[W(X)-\theta]^2 \qquad \text{for all } \theta' \neq \theta. \tag{13}$$

The left side of Equation (13) is minimized by $\theta' = E[W(X)]$, thus, the risk unbiasedness reduces to $E[W(X)] = \theta$, which is mean unbiased (Lehmann & Casella, 1998). When the absolute loss is taken, the corresponding risk function is mean absolute deviation (MAD), and Equation (12) becomes

$$E[|W(X)-\theta'|] \geq E[|W(X)-\theta|] \qquad \text{for all } \theta' \neq \theta. \tag{14}$$

The left side of Equation (14) is minimized by any median of $W(X)$, therefore, it reduces to median $W(X) = \theta$. θ is a median of $W(X)$, which is median unbiased (Lehmann & Casella, 1998).

Linex Unbiased Estimator

Similarly, when the linex loss is taken, the corresponding linex risk function is given by

$$R(W,\theta) = E[L(W(X),\theta)] = E[e^{\alpha(W(X)-\theta)} - \alpha(W(X)-\theta) - 1]. \tag{15}$$

If $W(X)$ satisfies Equation (12), then $W(X)$ is said to be the linex unbiased estimator of θ. If

$$E[e^{\alpha W(X)}] < \infty \qquad \text{for all } \theta,$$

then $W(X)$ is linex unbiased for θ if

$$E[e^{\alpha W(X)}] = e^{\alpha\theta}, \tag{16}$$

(Shafie & Noorbaloochi, 1995). If a linex unbiased estimator has the minimum linex risk among all linex unbiased estimators of a given estimated θ, then that estimator is said to be the best linex unbiased estimator of θ. Such best linex unbiased estimator can be obtained by applying the Rao-Blackwellization procedure, see Shafie and Noorbaloochi (1995).

Unbiased Prediction

Unbiased Predictor

Analogously, the bias of a predictor is the discrepancy between the predictor's expected value and the expected value of the variable being predicted. Let Y be a future real random variable that being predicted, $\delta(X)$ be a predictor used to predict Y, and the joint distribution of $\delta(X)$ and Y depends on an unknown parameter θ. When using $\delta(X)$ to predict the value of Y, $\delta(X)$ is said to be an unbiased predictor of Y if for any $\theta \in \Omega$

$$E_\theta[Y] = E_\theta[\delta(X)]. \tag{17}$$

On the other hand, a statistic $\delta(X)$ is called a median unbiased predictor of Y if for all $\theta \in \Omega$ (Takada, 1991)

$$P_\theta[Y \leq \delta(X)] = P_\theta[Y \geq \delta(X)]. \tag{18}$$

Loss Function in Prediction

The loss function in prediction cases is similar to estimation cases. Let $\delta(X)$ be a predictor of Y, then a loss function is any real valued function, $L(Y, \delta(X))$, such that

$$L(Y, \delta(X)) \geq 0 \qquad \text{for every } X \text{ and } Y, \tag{19}$$

and

$$L(Y, \delta(X)) = 0 \qquad \text{if } P[\delta(X) = Y] = 1.$$

How to choose an appropriate loss function depends on the nature of the prediction problem. For instance, when considering that, in the long run, the amount of over and underprediction will balance, so the predicted value will be correct on average, the squared error loss function is used,

$$L(Y, \delta(X)) = (Y - \delta(X))^2. \tag{20}$$

When considering not the amount but only the frequency of over and underprediction, the absolute error loss function is used,

$$L(Y, \delta(X)) = |Y - \delta(X)|. \tag{21}$$

Linex Loss Function
in Prediction

When an overprediction or underprediction could have serious consequences, the linex loss function is used. The linex loss function in prediction is given by

$$L(Y, \delta(X)) = e^{\alpha(Y-\delta(X))} - \alpha(Y - \delta(X)) - 1. \tag{22}$$

When $\alpha > 0$, underpredictions carry an approximately exponential penalty; while over-predictions carry an approximately linear penalty. When $\alpha < 0$ the penalty for over-predictions is approximately exponential while the penalty for under-predictions is approximately linear (Patton & Timmermann, 2010).

Risk Function and Unbiasedness
in Prediction

The predictor $\delta(X)$ is said to be risk unbiased for Y if for each θ

$$E[L(Y, \delta(X))] = \min_c E[L(Y + c, \delta(X))], \tag{23}$$

where c is a real number (Xiao, 2000).

The risk function in predictions is similar to estimation cases. The risk function measures the quality of the prediction. When the squared error loss is taken, then

$$E[L(Y, \delta(X))] = E[Y - \delta(X)]^2, \tag{24}$$

then the risk unbiasedness reduces to $E[\delta(X)] = E[Y]$. If

$$\delta(X) = E[Y|X],$$

$\delta(X)$ is risk unbiased because $E[\delta(X)] = E[Y]$. When the absolute loss is taken,

$$E[L(Y, \delta(X))] = E[|Y - \delta(X)|], \tag{25}$$

the risk unbiasedness reduces to

$$P[Y \leq \delta(X)|X] = P[Y \geq \delta(X)|X].$$

If $\delta(X)$ is conditional median of Y, then $\delta(X)$ is median unbiased for Y.

Linex Unbiased Predictor

The risk function with respect to the linex loss function in prediction is

$$R(Y, \delta(X)) = E[L(Y, \delta(X))] = E[e^{\alpha(Y-\delta(X))} - \alpha(Y - \delta(X)) - 1] \tag{26}$$

Xiao (2000) showed that if the loss function is convex in its first argument and certain regularity conditions hold, then Equation (23) is equivalent to

$$E[\frac{\partial L}{\partial y}(Y, \delta(X))] = 0$$

(Nayak & Qin, 2010). When the linex loss is taken, the risk unbiasedness reduces to

$$E[e^{\alpha(Y-\delta(X))}] = 1, \tag{27}$$

i.e. $\delta(X)$ is linex unbiased for Y.

Unbiased Prediction for Time Series under Squared Loss Function

Autoregressive of Order One

Malinvaud (1975) stated that for the zero mean first order autoregressive model, if the parameter is estimated by ordinary least square (OLS), then the h-step-ahead prediction error $E(Z_{T+h} - \tilde{Z}_{T+h})$ will be zero, i.e. $E(\tilde{Z}_{T+h}) = E(Z_{T+h})$, $\tilde{Z}_{T+h}$ is the unbiased predictor of Z_{T+h}, when the distribution of error a_t is symmetric. This AR(1) unbiased prediction was stated without being proved.

Fuller and Hasza (1980) extended the study of Malinvaud (1975) and studied the properties of predictors for both stationary and non-stationary AR(1) processes with unknown parameters. The AR(1) model is defined by

$$Z_t = \begin{cases} c + \phi_1 Z_{t-1} + a_t, & t = 1, 2, \ldots, T \\ Z_0 & t = 0, \end{cases} \tag{28}$$

where c is a constant, Z_0 is a random variable symmetrically distributed about the mean $\mu = (1 - \phi_1)^{-1} c$ with finite variance, and $\{a_t; t = 0, 1, \ldots, T\}$ is a sequence of independent and identically distributed (IID) $(0, \sigma^2)$ random variables,

symmetrically distributed and independent of Z_0. The h-step-ahead prediction is

$$\tilde{Z}_{T+h} = \hat{c} + \hat{\phi}_1 Z_{T+h-1}, \qquad h = 1, 2, 3, \ldots.$$

where the OLS estimators are

$$\hat{\phi}_1 = \left[\sum_{t=1}^{T} Z_{t-1}^2 - T^{-1}(\sum_{t=1}^{T} Z_{t-1})^2\right]^{-1} \times \left[\sum_{t=1}^{T} Z_t Z_{t-1} - T^{-1}\sum_{t=1}^{T} Z_{t-1}\sum_{t=1}^{T} Z_t\right],$$
$$\hat{c} = T^{-1}\left[\sum_{t=1}^{T} Z_t - \hat{\phi}_1\sum_{t=1}^{T} Z_{t-1}\right].$$

Fuller and Hasza (1980) showed that when $|\phi_1| < 1$, the predictor is unbiased, i.e.

$$E[Z_{T+h} - \tilde{Z}_{T+h}] = 0. \tag{29}$$

When $c = 0$, Z_0 has zero mean and finite variance, the predictor is unbiased for both non-stationary $\phi_1 > 1$ and stationary $\phi_1 < 1$ processes. When $c = 0$ and $\phi_1 = 1$, the OLS predictor is unbiased for all values of Z_0,

$$E[Z_{T+h} - \tilde{Z}_{T+h}|Z_0] = 0.$$

Fuller and Hasza (1980) conducted a Monte Carlo study of mean squared prediction error (MSPE). A sequence of NID(0,1) random variables was generated by using the method that Marsaglia, Ananthanarayanan, & Paul (1976) developed. For stationary processes, the first observation was generated as $Z_0 = (1 - \phi_1^2)^{-\frac{1}{2}}a_0$, and the remaining observations of the sample as $Z_t = \phi_1 Z_{t-1} + a_t$ where $t = 1, \ldots T$.

For non-stationary process, Z_0 was set to zero. The prediction error is $Z_{t+h} - \tilde{Z}_{T+h} = a_{T+h} + (c - \hat{c}) + (\phi_1 - \hat{\phi}_1)Z_T$, and the MSPE is $E[(Z_{T+h} - \tilde{Z}_{T+h})^2] = E[a_{T+1}^2] + E[(c - \hat{c}) + (\phi_1 - \hat{\phi}_1)Z_T]^2$. Three entries 10, 20 and 60 for T, three entries 1, 2, and 3 for h, thirteen entries -1.0, -0.9, -0.5, 0, 0.2, 0.5, 0.7, 0.9, 0.95, 0.99, 1.00, 1.02, and 1.05 for ϕ, and 1000 replicates. The behavior of Monte Carlo prediction MSPE was slightly larger than the theoretical approximation for $|\phi_1|$ close to, but less than one. The results were in reasonable agreement with the theoretically developed approximations. The study concluded that the biases in $\hat{\phi}_1$ will not induce biased prediction. The behavior of MSPE in AR(1) with and without an intercept were also studied, see Hoque, Magnus, and Pesaran (1988) and Magnus and Pesaran (1989).

Autoregressive of Order p

Dufour (1984) stated that the results of Fuller and Hasza (1980) were under the assumption that the "true" model and the estimated model are the same. Dufour (1984) proved that when the AR parameters are estimated by OLS, and the process Z_t is joint symmetric about a given constant mean, μ, even if the fitting order of the predictor is mis-specified (either lower or higher), the h-step-ahead prediction will still be unbiased. That is, the probability density function $f(a_1 - \mu, a_2 - \mu, \ldots) = f(\mu - a_1, \mu - a_2, \ldots)$. So the inaccurate estimated parameters are adopted for prediction, the distributions of the prediction error, $\tilde{Z}_{T+h} - Z_{T+h}$, will still have distributions symmetric about zero.

Dufour (1984) stated that when using an AR(p) model to predict Z_{T+h}, $h = 1, 2, \ldots,$ based on the observed Z_t, $t = -p+1, \ldots, 0, \ldots, T$, an AR(p) model is given by

$$Z_t = c + \sum_{k=1}^{p} \phi_k Z_{t-k} + a_t \qquad t = 1, \ldots, T. \tag{30}$$

If the coefficient vector $\boldsymbol{\beta} = (c, \boldsymbol{\phi}_k)'$, where $\boldsymbol{\phi}_k = (\phi_1, \ldots, \phi_p)'$ is estimated by OLS, then the estimator of $\boldsymbol{\beta}$ is given by

$$\hat{\boldsymbol{\beta}}_T = \left(\bar{\boldsymbol{Z}}'_T \bar{\boldsymbol{Z}}_T\right)^{-1} \bar{\boldsymbol{Z}}'_T \boldsymbol{Z}^T_T, \tag{31}$$

where

$$\begin{aligned}
\bar{\boldsymbol{Z}}_T &= \left[i_T, \boldsymbol{Z}_T\right], \\
\boldsymbol{Z}^T_t &= \left(Z_{t-T+1}, Z_{t-T+2}, \ldots, Z_t\right)', \qquad t \le T - p, \\
\boldsymbol{Z}_T &= \left[\boldsymbol{Z}^T_{T-1}, \boldsymbol{Z}^T_{T-2}, \ldots, \boldsymbol{Z}^T_{T-p}\right],
\end{aligned}$$

and $i_T = (1, 1, \ldots, 1)'$ is the T unit vector. Assume that $\hat{\boldsymbol{\beta}}_T$ exists with probability 1, then the prediction of Z_{T+h} is

$$\tilde{Z}_{T+h} = \hat{\phi}_{0,T} + \sum_{k=1}^{p} \hat{\phi}_{k,T} \tilde{Z}_T(h-k), \qquad h \ge 1, \tag{32}$$

where $\hat{\phi}_{k,T}$ $(k = 0, 1, \ldots, p)$ is the k^{th} component of $\hat{\boldsymbol{\beta}}_T$ and

$$\tilde{Z}_{T+h} = Z_{T+h}, \qquad \text{if } h \le 0.$$

Since each prediction error $e_{T+h} \equiv Z_{T+h} - \tilde{Z}_{T+h}, h \geq 1$, where $\tilde{Z}_{T+h}$ is given by Equations (31) and (32), has a distribution symmetric about zero,

$$E\left[Z_{T+h} - \tilde{Z}_{T+h}\right] = 0 \qquad (h = 1, \ldots)$$

i.e. $\tilde{Z}_{T+h}$ is unbiased predictor for Z_{T+h}. The unbiased prediction in vector AR case was also studied, see Dufour (1985).

Error Symmetry in Autoregressive Moving Average Process

Cryer et al. (1990) further generalized the result of Dufour (1984) to ARMA(p,q) models, where the parameters were estimated by maximum likelihood, unconditional least squares and conditional least squares. The study concluded that these estimators will produce prediction errors that are symmetrically distributed about 0.

Let Z_t be a fitted ARMA (p,q) model with no intercept term. The model fitted need not be the correct model as long as the strict stationary assumption holds. The log-likelihood function of the ARMA model can be written as

$$L(\boldsymbol{\phi}, \boldsymbol{\theta}, \sigma|\mathbf{Z}) = -\frac{T}{2}log|\Gamma| - \frac{1}{2}\mathbf{Z}'\Gamma^{-1}\mathbf{Z}, \tag{33}$$

where $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_t)'$ is the observed vector of data and Γ is the $T \times T$ covariance matrix of $\bar{\mathbf{Z}} = [\boldsymbol{Z}_1, \boldsymbol{Z}_2, \ldots, \boldsymbol{Z}_T]'$ for model (3). The elements of Γ are complicated functions of $\boldsymbol{\phi} = (\phi_1, \phi_2, \ldots, \phi_p)'$, $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_p)'$, and σ. σ is not described here because its explicit form need not be concerned, see Cryer (1986) for

details. The log-likelihood function given by (33) is correct when the ARMA model error term $\{a_t; t = 0, 1, \ldots, T\}$ is a sequence of IID $N(0, \sigma^2)$ random variables.

From (33) it follows that

$$L(\boldsymbol{\phi}, \boldsymbol{\theta}, \sigma|\mathbf{Z}) = L(\boldsymbol{\phi}, \boldsymbol{\theta}, \sigma| -\mathbf{Z}), \qquad \text{for all values of } \boldsymbol{Z}, \boldsymbol{\phi}, \boldsymbol{\theta} \text{ and } \sigma. \tag{34}$$

The result (34) indicates that when fitting the ARMA model, the maximum likelihood estimators of $\boldsymbol{\phi}, \boldsymbol{\theta}$ and σ are all even functions of the observation vector $\mathbf{Z}$. Also, the unconditional least squares estimators are obtained by minimizing the quadratic form on the right side of (33). Thus, the unconditional least squares estimators of $\boldsymbol{\phi}$ and $\boldsymbol{\theta}$ and the residual sum of squares are all even functions of the observation vector $\mathbf{Z}$.

The conditional least squares function can be expressed as

$$\begin{aligned} L(\boldsymbol{\phi}, \boldsymbol{\theta}|\boldsymbol{Z}) = \sum_{t=p+1}^{T} (&Z_t - \phi_1 Z_{t-1} - \phi_2 Z_{t-2} - \ldots \\ &-\phi_p Z_{t-p} + \theta_1 a_{t-1} + \ldots + \theta_q a_{t-q})^2, \end{aligned} \tag{35}$$

where $a_p = a_{p-1} = \ldots = a_{p-q+1} = 0$ and the remaining are obtained from the recursion,

$$a_t = Z_t - \phi_1 Z_{t-1} - \phi_2 Z_{t-2} - \ldots - \phi_p Z_{t-p} + \theta_1 a_{t-1} + \ldots + \theta_q a_{t-q}, \tag{36}$$

for $t = p+1, p+2, \ldots, T$. The Equation (35) can also be written as a sum of squared

errors. The errors are odd functions of the observed values. Hence, the conditional sum of squares function, $L(\boldsymbol{\phi}, \boldsymbol{\theta}|\boldsymbol{Z})$, determined through (35) and (36), satisfies

$$L(\boldsymbol{\phi}, \boldsymbol{\theta}|\mathbf{Z}) = L(\boldsymbol{\phi}, \boldsymbol{\theta}| - \mathbf{Z}), \qquad \text{for all values of } \boldsymbol{Z}, \boldsymbol{\phi}, \text{ and } \boldsymbol{\theta}.$$

Therefore, the conditional least squares estimators of ϕ and θ and the sum of squares errors are all even functions of the observation vector of Z.

The residuals,

$$\begin{aligned} \hat{a}_t = \tilde{Z}_t - \hat{\phi}_1 \tilde{Z}_{t-1} - \hat{\phi}_2 \tilde{Z}_{t-2} - \ldots - \\ \hat{\phi}_p \tilde{Z}_{t-p} - \hat{\theta}_1 \hat{a}_{t-1} - \ldots - \hat{\theta}_q \hat{a}_{t-q}, \end{aligned}$$

where $\tilde{Z}_t = Z_t, t = 1, \ldots, T$, obtained from fitting the ARMA model by maximum likelihood, conditional least squares, or unconditional least squares are odd functions of the observation vector. Thus, under the strict stationary assumption, the residuals are jointly distributed symmetrically about 0.

The h-step-ahead prediction error,

$$\begin{aligned} Z_{T+h} - \tilde{Z}_{T+h} = Z_{T+h} - \hat{\phi}_1 \tilde{Z}_{T+h-1} - \hat{\phi}_2 \tilde{Z}_{T+h-2} - \ldots \\ -\hat{\phi}_p \tilde{Z}_{T+h-p} + \hat{\theta}_1 \hat{a}_{T+h-1} + \ldots + \hat{\theta}_q \hat{a}_{T+h-q}, \end{aligned}$$

is an odd function of the vector $(Z_1, Z_2, \ldots, Z_T, Z_{T+h})'$. Thus, under the strict

stationary assumption, the prediction error for $h = 1, 2, \ldots, H$ are also jointly distributed symmetrically about 0.

For an ARIMA (p,d,q) model, it is assumed that the number of differences, d, to achieve stationarity is known. In this case the basic assumption is that the differenced series $\nabla^d Z_t$ satisfies the joint symmetry condition either about 0 or a constant c, depending on whether the intercept term is included in the model fitting or not. In such a way, the symmetry results still apply since the difference equation form of the model can be presented, in the no intercept term case, as

$$Z_t = \varphi_1 Z_{t-1} + \varphi_2 Z_{t-2} + \ldots + \varphi_{p+d} Z_{t-p} + d + a_t - \theta_1 a_{t-1} - \ldots - \theta_q a_{t-q},$$

where the φ coefficients are determined from the relationship $\varphi(B) = \phi(B)(1 - B)^d$. All results then follow as the ARMA(p,q) model with φ's replacing ϕ's. The intercept case is similar.

Median Unbiased Prediction for Autoregressive of Order One

So far, the unbiased prediction under the squared loss function in time series models discussed are mean unbiased prediction. This subsection discussed median unbiased prediction. Gospodinov (2002) stated that obtaining a completed conditional predictive distribution of a variable of interest Z would provide a very valuable distribution for policy analysis and business decisions. Computing the median of the predictive distribution would give researchers a more robust point predictor whereas some relevant quantiles would help researchers assess the

uncertainty associated with the point prediction. Gospodinov (2002) studied a method for conditional median unbiased prediction of nearly non-stationary zero mean AR (1) processes.

Let the AR(1) process be the same as Equation (28), where $|\phi| \leq 1$, and $\phi_T = 1 + \frac{c}{T}$, and where $c \leq 0$ is a finite constant. Let the true conditional mean denoted by

$$Z_{T+h|T} = E[Z_{T+h}|Z_T] = \phi^h Z_T,$$

and the estimated conditional mean denoted by

$$\tilde{Z}_{T+h|T} = \hat{\phi}^h Z_T,$$

where $\hat{\phi}$ is estimated by OLS. The normalized deviation of the OLS prediction from its true conditional mean $g_T = T^{-\frac{1}{2}}(Z_{T+h|T} - \tilde{Z}_{T+h|T})$ is a function of the data Z, the h-step-ahead prediction and the parameter c. Since the data Z and the h-step-ahead prediction are predetermined at the time of the prediction, g_T is parameterized as a function of the parameter c, denoted by $g(c)$. Let $G_T(x|c) = Pr\{g(c) \leq x|c\}$ denote the sampling distribution of $g(c)$. Suppose that the 0.5^{th} quantile $q_{0.5}(c)$ of the distribution $G_T(x|c)$, which is given by $Pr\{g(c) \leq q_{0.5}(c)|c\} = 0.5$, is uniquely defined and monotonically increasing in c. Then the median unbiased prediction is obtained by inverting the median function (Stock, 1991; Andrews, 1993)

$$\tilde{Z}^{MU}_{T+h|T} = \tilde{Z}^{OLS}_{T+h|T} - T^{\frac{1}{2}} q^{-1}_{0.5}(g_T),$$

such that

$$Pr\{Z_{T+h|T} \leq \tilde{Z}^{MU}_{T+h|T}\} = Pr\{Z_{T+h|T} \geq \tilde{Z}^{MU}_{T+h|T}\} = 0.5.$$

The median unbiased prediction possesses the impartiality property that the probability of underprediction is equal to the probability of overprediction.

Gospodinov (2002) conducted a small Monte Carlo simulation experiment on nearly non-stationary AR(1) with $T = 100, h = 10$, $\phi_1 = 0.9, 0.95, 0.98, 0.99$ and 5000 replicates. The results showed that the OLS prediction underpredicted significantly the true conditional mean for all parameters.

Unbiased Prediction for Time Series under Linex Loss Function

Unconditional Gaussian Process under General Asymmetric Loss Function

In practice, the loss functions are not likely to be quadratic, but asymmetric. Granger (1969) demonstrated that under unconditional Gaussian process, any optimal predictor, which has minimum variance, under any asymmetric loss function, will exhibit a constant bias, the size of which will depend on the parameters of the loss function, and the constant prediction-error variance (Batchelor & Peel, 1998).

Let $\{Z_T\}$ be a purely non-deterministic stationary sequence of continuous random variables. The h-step-ahead optimal point prediction of $Z_{T+h}|Z_T, Z_{T-1}, \ldots$, is determined by some function $\delta(Z_T, Z_{T-1}, \ldots)$, which minimizes the risk. Let $L(e)$ be the loss of prediction error of numeric magnitude e. $L(0) = 0$, $L(e)$ is monotonic increasing (non-decreasing) for $e > 0$, monotonic decreasing (non-increasing) for

$e < 0$, and differential at least twice almost everywhere. It follows that $L'(e) \geq 0$ when $e > 0$, and $L'(e) \leq 0$ when $e < 0$. When Z_T is a Gaussian process, the optimal predictor of Z_{T+h} have the form:

$$\delta = \sum_{j=0}^{\infty} \zeta_j Z_{T-j} + \eta_0,$$

where $\sum_{j=0}^{\infty} \zeta_j Z_{T-j}$ is a linear function of $Z_T, Z_{T-1}, \ldots$, and $\zeta_j s$ are estimated by OLS and fully determined by the covariance matrix of the process, and only η_0 is dependent upon the loss function. The unbiased δ is that which minimizes the risk

$$E[L(Z_{T+h} - \delta)|Z_T, Z_{T-1}, \ldots] = E_c[L(Z_{T+h} - \delta)] = \int_{-\infty}^{\infty} L(e - \eta)\bar{f}_c(e)de,$$

where $\delta = E_c[Z_{T+h}] + \eta$, $\bar{f}_c(e)$ is conditional distribution of $Z_{T+h} - E_c[Z_{T+h}]$ given $Z_T, Z_{T-1}, \ldots$ and is independent of $Z_T, Z_{T-1}, \ldots$, thus η_0 will also be independent of $Z_T, Z_{T-1}, \ldots$. And $\eta = \eta_0$ will be chosen to minimize the risk.

The general asymmetric loss function was also studied by other researchers. For example, Demetrescu (2007) discussed the optimal prediction interval under asymmetric loss, and McCullough (2000) studied bootstrap methods for optimal prediction with a general loss function.

Conditional Gaussian Process under Linex Loss Function

Christoffersen and Diebold (1997) extended the study of Granger (1969) and derived the optimal predictor for linex loss function under the assumption of conditional Gaussian.

As mentioned before, the linex function assumes that losses L depend on the prediction error. Let $\hat{Z}_{T+h}$ be a predictor for Z_{T+h} under linex loss, then

$$L(Z_{T+h} - \hat{Z}_{T+h}) = [e^{\alpha(Z_{T+h} - \hat{Z}_{T+h})} - \alpha(Z_{T+h} - \hat{Z}_{T+h}) - 1].$$

Thus, by minimizing

$$\min_{\hat{Z}_{T+h}} E[e^{\alpha(Z_{T+h} - \hat{Z}_{T+h})} - \alpha(Z_{T+h} - \hat{Z}_{T+h}) - 1], \tag{37}$$

one obtains the linex loss optimal predictor of Z_{T+h} under conditional normality,

$$\hat{Z}_{T+h} = \mu_{T+h|T} + (\frac{\alpha}{2})\sigma^2_{T+h|T}, \tag{38}$$

which is the conditional mean plus a conditional variance. Christoffersen and Diebold (1997) stated that because the conditional prediction-error variance may be time-varying, the unbiased predictor under asymmetric loss is not a conditional mean, but the conditional mean shifted by a time-varying adjustment that depends on the conditional variance.

Whether the errors occur from predictions being greater than or less than the actual values, the error term is always positive. The following is a proof of Equation (38) satisfies the condition of Equation (27). An unbiased prediction is

defined by minimizing the conditional expected loss (Patton & Timmermann, 2007),

$$\min_{\hat{Z}_{t+h}} E[e^{\alpha(Z_{T+h}-\hat{Z}_{T+h})} - \alpha(Z_{T+h} - \hat{Z}_{T+h}) - 1|Z_1, \ldots, Z_T]. \tag{39}$$

When minimizing Equation (39), one obtains

$$\begin{aligned}
\frac{d}{d\hat{Z}_{T+h}} e^{-\alpha\hat{Z}_{T+h}} E[e^{\alpha Z_{T+h}}|Z_1, \ldots, Z_T] + \alpha\hat{Z}_{T+h} - \alpha E[Z_{T+h}|Z_1, \ldots, Z_T] - 1 = 0,\\
-\alpha e^{-\alpha\hat{Z}_{T+h}} E[e^{\alpha Z_{T+h}}|Z_1, \ldots, Z_T] + \alpha = 0,\\
-\alpha E[e^{\alpha(Z_{T+h}-\hat{Z}_{T+h})}|Z_1, \ldots, Z_T] = -\alpha,\\
E[e^{\alpha(Z_{T+h}-\hat{Z}_{T+h})}] = 1,
\end{aligned}$$

which is the linex unbiased prediction Xiao (2000) developed. When Z_{T+h} is conditional Gaussian process, then

$$\begin{aligned}
E[e^{\alpha(Z_{T+h}-\hat{Z}_{T+h})}|Z_1 \ldots Z_t] &= 1,\\
e^{-\alpha\hat{Z}_{T+h}} E[e^{\alpha Z_{T+h}}|Z_1 \ldots Z_t] &= 1,\\
log(E[e^{\alpha Z_{T+h}}|Z_1 \ldots Z_t]) &= \alpha\hat{Z}_{T+h},\\
\text{and } E[e^{\alpha Z_{T+h}}|Z_1 \ldots Z_t] &= e^{\mu_{T+h|T} + \frac{\alpha}{2}\sigma^2_{T+h|T}}.
\end{aligned}$$

The linex unbiased predictor is the conditional mean plus the conditional error variance which depends on the loss function shape parameter α.

Christoffersen and Diebold (1997) indicated that the optimal prediction under asymmetric loss can be found by using the conditional prediction error variance as an additional regressor. The ideas were generalized by Batchelor and

Peel (1998), who estimated an Autoregressive Conditionally Heteroscedastic (ARCH)-in-mean process, and Ulu (2006), who used Monte Carlo simulation to generate a Generalized Autoregressive Conditionally Heteroscedastic (GARCH) process. Also, Patton and Timmermann (2007) used the linex loss as an example to couple with the non-linear data generating processes and showed that ordinary optimal prediction properties $E(Z_{T+h} - \hat{Z}_{T+h}) = 0$ can be invalid under asymmetric loss functions and may be misleading as a benchmark for the optimal prediction. The authors suggested the need to develop new and more general methods for prediction evaluation that are robust to deviations from squared error loss.

Summary

The empirical literature typically evaluated the prediction unbiasedness with the assumption that squared error loss adequately represents the prediction's objectives. Under the squared error loss function, predictions are easy to compute through least squares methods and have well established properties of unbiasedness (Diebold & Lopez, 1996). Inference about the unbiasedness of time series prediction under symmetric loss is easy, and can be based on the observable prediction errors which do not depend on any unknown parameters of the forecasters' loss function.

Indeed, many studies discussed parameter properties and estimation methods of unbiased prediction, but there are less intensive literatures that discussed the choice of a loss function different from squared error loss (Patton & Timmermann, 2007). Nevertheless, there is a growing body of literature on different aspects of prediction under asymmetric loss. The importance of using asymmetric loss functions to measure the loss has caught researchers' attention, and the linex

unbiased predictor has also been established. However, no current literature investigates if the linex unbiased prediction will still hold, when the parameters are being estimated. The goal of this dissertation was to fill out this gap by applying the linex unbiased predictor to stationary AR, MA, and ARMA processes, and examining the condition of linex unbiasedness of each process, along with different levels of variance, forecast step, series length and shape parameter.

CHAPTER III

METHODOLOGY

The purpose of this chapter was to describe the methodology used in this study in order to answer the research questions presented in Chapter I. First, describing how to compute the linex unbiased predictor in the Forecasting section. The Forecasting section defined, for each process, the time series forecasting model, conditional mean, and conditional prediction-error variance. Second, describing how to simulate the data with different values of variance, length of observed series, forecast steps, and the linex loss function shape parameters in the Simulation Procedure section. The Simulation Procedure section presented the step-by-step simulation methods and procedures used in this study. Finally, comparing the empirical risk with theoretical risk in the Relative Efficiency section. The Relative Efficiency section defined the theoretical risk, empirical risk, and the ratio of empirical risk to theoretical risk. All the computation and analysis were performed in R programming language 3.2.1 and the coding can be found in the Appendix E.

Forecasting

Recall, from the previous chapter, the linex unbiased predictor in the case of conditional Gaussian process, is the conditional mean, plus the conditional prediction-error variance which depends on the shape parameter of the linex loss

function (Christoffersen & Diebold, 1997),

$$\hat{Z}_{T+h} = \mu_{T+h|T} + (\frac{\alpha}{2})\sigma^2_{T+h|T}. \tag{40}$$

The prediction model, h-step-ahead conditional mean, and h-step-ahead conditional prediction-error variance of some time series models are presented in the following.

Autoregressive Process

In this study, the stationary AR(1), AR(2) and AR(3) processes with zero mean were investigated. The order of models were used in Fuller and Hasza (1980) and Q. Shao and Yang (2011). The h-step-ahead prediction for the AR(p) process is (Abraham, 1983)

$$\tilde{Z}_{T+h} = \phi_1 \tilde{Z}_{T+h-1} + \phi_2 \tilde{Z}_{T+h-2} + \ldots + \phi_p \tilde{Z}_{T+h-p}, \tag{41}$$

where $\tilde{Z}_{T+h} = \mu_{T+h|T}$.

The conditional mean for the h-step-ahead AR(1) is

$$\mu_{T+h|T} = \phi_1^h Z_T. \tag{42}$$

The general form of the conditional prediction-error variance is

$$\sigma^2_{T+h|T} = Var(e_{T+h|T}) = \sigma^2_a(1 + \psi_1^2 + \psi_2^2 + \ldots + \psi_{h-1}^2), \tag{43}$$

where $e_{T+h} = Z_{T+h} - \tilde{Z}_{T+h} = a_{T+h}$ and the ψ_i weights can be calculated by equating

coefficients in $(1-\phi_1 B-\phi_2 B^2)(1+\psi_1 B+\psi_2 B^2+\ldots)=1$, see Abraham (1983). Because h has three levels, the conditional prediction-error variance for AR(1) is

$$\sigma^2_{T+h|T}=\sigma^2_a(1+\psi_1^2+\psi_2^2), \tag{44}$$

where $\psi_1=\phi_1$ and $\psi_2=\phi_1^2$. For the AR(1), the conditional prediction-error variance can be simplified as (Abraham, 1983)

$$\sigma^2_{T+h|T}=\sigma^2_a\frac{1-\phi^{2h}}{1-\phi^2}.$$

The conditional mean for h-step-ahead AR(2) is (Abraham, 1983)

$$\mu_{T+h|T}=\phi_1(\tilde{Z}_{T+h-1})+\phi_2(\tilde{Z}_{T+h-2}), \tag{45}$$

and the conditional prediction-error variance can be calculated from (43) by substituting the ψ weights for the AR(2) process. The ψ weights are $\psi_j=\phi_1\psi_{j-1}+\phi_2\psi_{j-2}$. Thus,

$$\sigma^2_{T+h|T}=\begin{cases}\sigma^2_a & h=1\\ \sigma^2_a(1+\phi_1^2) & h=2\\ \sigma^2_a(1+\phi_1^2+(\phi_1^2+\phi_2)^2) & h=3\end{cases} \tag{46}$$

The conditional mean for the h-step-ahead AR(3) is (Abraham, 1983)

$$\mu_{T+h|T}=\phi_1\tilde{Z}_{T+h-1}+\phi_2\tilde{Z}_{T+h-2}+\phi_3\tilde{Z}_{T+h-3}. \tag{47}$$

The conditional prediction-error variance for AR(3) is the same as AR(2) since the forecast step (h-step-ahead) is up to three.

Moving Average Process

The invertible MA(1), MA(2) and MA(3) processes with zero mean were studied. The MA models are arbitrary chosen. The h-step-ahead prediction MA process is (Abraham, 1983)

$$\tilde{Z}_{T+h} = \tilde{a}_{T+h} + \theta_1 \tilde{a}_{T+h-1} + \theta_2 \tilde{a}_{T+h-2} + \ldots \theta_q \tilde{a}_{T+h-q}. \tag{48}$$

The conditional expectation of a_{T+h} given $Z_1 \ldots Z_T$ is given by (Abraham, 1983)

$$E[a_{T+h|T}] = \begin{cases} a_{T+h} & h \leq 0 \\ 0 & h > 0. \end{cases} \tag{49}$$

The conditional mean for h-step-ahead MA(1) is

$$\mu_{T+h|T} = \begin{cases} \mu + \theta_1 a_T & h = 1 \\ \mu & h > 1. \end{cases} \tag{50}$$

The conditional prediction-error variance can also be calculated from (43) by substituting the ψ weights for the MA process. The ψ weights are $\psi_1 = \theta_1$, $\psi_2 = \theta_2$ and $\psi_3 = \theta_3$ (Box et al., 1976). Thus, for MA(1)

$$\sigma^2_{T+h|T} = \begin{cases} \sigma^2_a & h = 1 \\ \sigma^2_a(1 + \theta_1^2) & h > 1. \end{cases} \tag{51}$$

The conditional mean for the h-step-ahead MA(2) is

$$\mu_{T+h|T} = \begin{cases} \mu + \theta_1 a_T + \theta_2 a_{T-1} & h = 1 \\ \mu + \theta_2 a_{T-1} & h = 2 \\ \mu & h \geq 3, \end{cases} \tag{52}$$

and the conditional prediction-error variance for MA(2) is

$$\sigma^2_{T+h|T} = \begin{cases} \sigma^2_a & h = 1 \\ \sigma^2_a(1 + \theta^2_1) & h = 2 \\ \sigma^2_a(1 + \theta^2_1 + \theta^2_2) & h \geq 3. \end{cases} \tag{53}$$

The conditional mean for the h-step-ahead MA(3) is

$$\mu_{T+h|T} = \begin{cases} \mu + \theta_1 a_T + \theta_2 a_{T-1} + \theta_3 a_{T-2} & h = 1 \\ \mu + \theta_2 a_T + \theta_3 a_{T-1} & h = 2 \\ \mu + \theta_3 a_T & h = 3 \\ \mu & h \geq 4, \end{cases} \tag{54}$$

and the conditional prediction-error variance for MA(3) is the same as MA(2) since the forecast step is up to three.

Autoregressive Moving Average Process

The stationary and invertible ARMA(1,1), ARMA(2,1), ARMA(1,2), and ARMA(2,2) with zero mean were considered here. The order of (p,q) were arbitrarily chosen. Because the ARMA process is a combination of AR and MA

processes, the following discussions were presented in ARMA(p,q) form instead of displaying the explicit form of each order of (p,q).

Because the process is stationary, it can be written in a moving average representation (Wei, 1990),

$$\begin{aligned} Z_t =& \psi(B)a_t \\ =& a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \ldots, \end{aligned} \tag{55}$$

where

$$\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j = \frac{\theta(B)}{\phi(B)} \tag{56}$$

and $\phi_0 = 1$.

The h-step-ahead prediction for the ARMA process is (Hamilton, 1994)

$$\tilde{Z}_{T+h} = \begin{cases} \phi_1 \tilde{Z}_{T+h-1} + \phi_2 \tilde{Z}_{T+h-2} + \ldots + \phi_p \tilde{Z}_{T+h-p} \\ +\theta_h \tilde{a}_T + \theta_{h+1} \tilde{a}_{T-1} + \ldots + \theta_q \tilde{a}_{T+h-q} \\ \qquad\qquad\qquad\qquad \text{for } h = 1, 2, \ldots, q \\ \phi_1 \tilde{Z}_{T+h-1} + \phi_2 \tilde{Z}_{T+h-2} + \ldots + \phi_p \tilde{Z}_{T+h-p} \\ \qquad\qquad\qquad\qquad \text{for } h = q+1, q+2, \ldots. \end{cases} \tag{57}$$

For $h > q$, the forecasts follow a p^{th} order difference equation governed solely by the autoregressive parameters.

The conditional mean for the h-step-ahead ARMA(p,q) is (Wei, 1990)

$$\mu_{T+h|T} = \psi_h a_T + \psi_{h+1} a_{T-1} + \psi_{h+2} a_{T-2} + \dots. \tag{58}$$

The conditional prediction-error variance can be calculated by using Equation (43), which is

$$\sigma^2_{T+h|T} = \begin{cases} \sigma_a^2 & h = 1 \\ \sigma_a^2(1 + (\phi_1 + \theta_1)^2) & h = 2 \end{cases} \tag{59}$$

And when $h = 3$, $\sigma^2_{T+h|T}$ for ARMA(1,1) is $\sigma_a^2(1 + (\phi_1 + \theta_1)^2) + (\phi_1\theta_1)^2$, ARMA(2,1) is $\sigma_a^2(1 + (\phi_1 + \theta_1)^2) + (\phi_1\theta_1 + \phi_2)^2$, ARMA(1,2) is $\sigma_a^2(1 + (\phi_1 + \theta_1)^2) + (\phi_1\theta_1 + \theta_2)^2$, and ARMA(2,2) is $\sigma_a^2(1 + (\phi_1 + \theta_1)^2) + (\phi_1\theta_1 + \phi_2 + \theta_2)^2$.

Simulation Procedure

The simulation scheme is displayed in Table 1. The time series processes AR(p), with $p = 1, 2$ and 3, MA(q), with $q = 1, 2$ and 3, and ARMA(p,q) with $(p, q) = (1, 1), (1, 2), (2, 1)$ and $(2, 2)$. The conditional prediction-error variance is proportional to σ^2, and the values for σ^2 were set to be $0.5, 1$ and 5. The value of $\sigma^2 = 1$ was used by Fuller and Hasza (1980) and Q. Shao and Yang (2011), and the value of $\sigma^2 = 0.5$ and $\sigma^2 = 5$ were arbitrarily chosen. This study considered small sample sizes (i.e. the observed series) such as $T = 15, 25$ and 50, and large sample sizes $T = 100, 200$, and 400. $T = 15, 25$ were studied by Magnus and Pesaran (1989), $T = 50$ was arbitrarily chosen, and $T = 100, 200$, and 400 were studied by Q. Shao and Yang (2011). Time series models need to be updated in time to maintain the

long term forecast accuracy, but this study aimed to explore the short term prediction results of the construct, only three prediction periods with $h = 1, 2$ and 3 were considered. The values for linex loss function shape parameter α were arbitrarily chosen. When $\alpha = 1$, loss is greater for positive error, and conversely when $\alpha = -1$. The replication for each procedure was 10,000 times.

Table 1

Simulation Scheme

AR(p)						MA(q)						ARMA(p,q)					
$\alpha=1$			$\alpha=-1$			$\alpha=1$			$\alpha=-1$			$\alpha=1$			$\alpha=-1$		
σ^2	T	h	σ^2	T	h	σ^2	T	h	σ^2	T	h	σ^2	T	h	σ^2	T	h
0.5	15	1,2,3	0.5	15	1,2,3	0.5	15	1,2,3	0.5	15	1,2,3	0.5	15	1,2,3	0.5	15	1,2,3
	25	1,2,3		25	1,2,3		25	1,2,3		25	1,2,3		25	1,2,3		25	1,2,3
	50	1,2,3		50	1,2,3		50	1,2,3		50	1,2,3		50	1,2,3		50	1,2,3
	100	1,2,3		100	1,2,3		100	1,2,3		100	1,2,3		100	1,2,3		100	1,2,3
	200	1,2,3		200	1,2,3		200	1,2,3		200	1,2,3		200	1,2,3		200	1,2,3
	400	1,2,3		400	1,2,3		400	1,2,3		400	1,2,3		400	1,2,3		400	1,2,3
1	15	1,2,3	1	15	1,2,3	1	15	1,2,3	1	15	1,2,3	1	15	1,2,3	1	15	1,2,3
	25	1,2,3		25	1,2,3		25	1,2,3		25	1,2,3		25	1,2,3		25	1,2,3
	50	1,2,3		50	1,2,3		50	1,2,3		50	1,2,3		50	1,2,3		50	1,2,3
	100	1,2,3		100	1,2,3		100	1,2,3		100	1,2,3		100	1,2,3		100	1,2,3
	200	1,2,3		200	1,2,3		200	1,2,3		200	1,2,3		200	1,2,3		200	1,2,3
	400	1,2,3		400	1,2,3		400	1,2,3		400	1,2,3		400	1,2,3		400	1,2,3
5	15	1,2,3	5	15	1,2,3	5	15	1,2,3	5	15	1,2,3	5	15	1,2,3	5	15	1,2,3
	25	1,2,3		25	1,2,3		25	1,2,3		25	1,2,3		25	1,2,3		25	1,2,3
	50	1,2,3		50	1,2,3		50	1,2,3		50	1,2,3		50	1,2,3		50	1,2,3
	100	1,2,3		100	1,2,3		100	1,2,3		100	1,2,3		100	1,2,3		100	1,2,3
	200	1,2,3		200	1,2,3		200	1,2,3		200	1,2,3		200	1,2,3		200	1,2,3
	400	1,2,3		400	1,2,3		400	1,2,3		400	1,2,3		400	1,2,3		400	1,2,3

Generating Time Series

Step one was data generation. Each time series process was simulated by using the arima.sim function in R, given the model, order, series length, values for parameters and standard deviation. Because the h-step-ahead forecast was up to three, in each time series data generation, $T + 3$ observations were generated as the values of Z_T and Z_{T+h}.

The value and choices of time series model coefficients can be found in Table 2. The parameters of AR(1) and MA(1) were adapted from Q. Shao and Yang (2011). The parameters of AR(1) and MA(1) were chosen as -0.8, -0.4, -0.2, 0.2, 0.4, and 0.8, so that the time series range from the relatively weakly to the relatively highly correlated. The parameters of the AR(2) and MA(2) were arbitrarily chosen such that the polynomial (denoted by $p(x)$, where x is a dummy variable) $p(x) = 1 + 0.1x - 0.3x^2$ has two real roots and $p(x) = 1 - 0.1x + 0.3x^2$ has two complex roots. The parameters of AR(3) and MA(3) were arbitrarily chosen such that the polynomial $p(x) = 1 + 0.1x - 0.3x^2 - 0.2x^3$ has two complex roots and one real roots, and $p(x) = 1 - 0.3x - 0.5x^2 - 0.1x^3$ has three real roots.

The parameters of the ARMA(p,q) were chosen by combining the parameters of $p = 1, 2, 3$ and $q = 1, 2, 3$, which satisfy the conditions of stationarity and invertiblity. The parameters of ARMA(1,1) were selected from the parameters of AR(1) and MA(1). The ARMA(2,1) parameters were selected from AR(3), such that the polynomials $p(x) = 1 - 0.2x - 0.3x^2$ and $p(x) = 1 + 0.2x - 0.3x^2$ each has two real roots. The ARMA(1,2) parameters were selected from MA(3) such that the polynomial $p(x) = 1 + 0.5x - 0.1x^2$ has two real roots, and $p(x) = 1 - 0.5x + 0.1x^2$

has two complex roots. The ARMA(2,2) parameters were selected from ARMA(2,1) and ARMA(1,2) with the combinations of polynomials have two real roots, and one real roots and one complex roots. Although the time series correlation and the roots of the parameters were not concerned in this study, the information of these properties was still provided.

Table 2

Parameter Settings

Model	ϕ_1	ϕ_2	ϕ_3	θ_1	θ_2	θ_3
AR(1)	-0.8					
	-0.4					
	-0.2					
	0.2					
	0.4					
	0.8					
AR(2)	0.1	-0.3				
	-0.1	0.3				
AR(3)	0.1	-0.3	-0.2			
	-0.3	-0.5	0.1			
MA(1)				-0.8		
				-0.4		
				-0.2		
				0.2		
				0.4		
				0.8		
MA(2)				0.1	-0.3	
				-0.1	0.3	
MA(3)				0.1	-0.3	-0.2
				-0.3	-0.5	0.1
ARMA(1,1)	0.4			-0.2		
	-0.4			-0.4		
ARMA(2,1)	-0.2	-0.3		0.1		
	0.2	-0.3		-0.1		
ARMA(1,2)	0.3			0.5	-0.1	
	0.3			-0.5	0.1	
ARMA(2,2)	-0.2	-0.3		0.5	-0.1	
	0.2	-0.3		-0.5	0.1	

Parameter Estimates

Step two was parameter estimation. Because the parameters were assumed to be unknown and need to be estimated, only T observations in each time series dataset generated in the previous step, were being used to fit a univariate time series model. This step was performed by using the Arima function in the forecast package in R with model and order specified (i.e. AR(p), MA(q) and ARMA(p,q)), zero mean, and unconditional maximum likelihood estimation (MLE) method selected. The unconditional log-likelihood function is (Wei, 1990)

$$lnL(\boldsymbol{\phi}, \mu, \boldsymbol{\theta}, \sigma_a^2) = -\frac{n}{2} ln 2\pi\sigma_a^2 - \frac{S(\boldsymbol{\phi}, \mu, \boldsymbol{\theta})}{2\sigma_a^2}, \tag{60}$$

where $S(\boldsymbol{\phi}, \mu, \boldsymbol{\theta})$ is the unconditional sum of squares function given by

$$S(\boldsymbol{\phi}, \mu, \boldsymbol{\theta}) = \sum_{t=-\infty}^{n} [E[a_t|\boldsymbol{\phi}, \mu, \boldsymbol{\theta}, \boldsymbol{Z}]]^2, \tag{61}$$

and $E[a_t|\boldsymbol{\phi}, \mu, \boldsymbol{\theta}, \boldsymbol{Z}]$ is the conditional expectation of a_t given $\boldsymbol{\phi}, \mu, \boldsymbol{\theta}$ and $\boldsymbol{Z}$.

H-step Ahead Prediction

Step three was h-step-ahead prediction. This step calculated the quantities required for of computing the estimated $\hat{Z}_{T+h}$. The detailed formulas were shown in the Forecasting section, and the calculation was performed by using the prediction function in R. The estimates of conditional mean $\hat{\mu}_{T+h|T}$ and conditional prediction-error variance $\hat{\sigma}^2_{T+h|T}$ were obtained in this step, which were the necessary elements for the next step.

Computing Linex Unbiased Predictor

Step four was computing the estimated linex unbiased predictor. The estimates of conditional mean and conditional prediction-error variance obtained in step three were used here to calculate the linex unbiased predictor using Equation (40). Based on the previous steps, the estimated linex unbiased predictor became

$$\hat{\hat{Z}}_{T+h} = \hat{\mu}_{T+h|T} + (\frac{\alpha}{2})\hat{\sigma}^2_{T+h|T}. \tag{62}$$

Condition of Linex Unbiasedness

Step five was evaluating the CLU values. Recall, in Chapter II, $\hat{\hat{Z}}_{T+h}$ was linex unbiased for Z_{T+h} if

$$E[e^{\alpha(Z_{T+h}-\hat{\hat{Z}}_{T+h})}] = 1. \tag{63}$$

The Z_{T+h} was obtained in step one, the $\hat{\hat{Z}}_{T+h}$ was obtained in step four, and the $e^(\alpha(Z_{T+h}) - \hat{\hat{Z}}_{T+h})$ was obtained in this step. In order to obtain the expected value of $e^{\alpha(Z_{T+h})-\hat{\hat{Z}}_{T+h}}$, the author first calculated

$$\frac{\sum_{i=1}^{nsim} e^{\alpha(Z^i_{T+h}-\hat{\hat{Z}}^i_{T+h})}}{\text{nsim}}, \tag{64}$$

where nsim is the number of simulations, and used the result from Equation (64) minus 1 to examine the condition of being linex unbiased (i.e. if the final value is zero), as the values of α, σ^2, T, and the order of p and q change. The results of Equation (64) minus 1 were called the CLU values.

Relative Efficiency

In symmetrical loss function cases, the researchers (Fuller & Hasza, 1980; Hoque et al., 1988; Magnus & Pesaran, 1989) investigated the prediction MSE. In this study, the linex risk was investigated. The linex risk of $\hat{\hat{Z}}_{T+h}$ was given by

$$R(Z_{T+h}, \hat{\hat{Z}}_{T+h}) = E[e^{\alpha(Z_{T+h} - \hat{\hat{Z}}_{T+h})} - \alpha(Z_{T+h} - \hat{\hat{Z}}_{T+h}) - 1].$$

According to Equation (63), the linex risk of $\hat{\hat{Z}}_{T+h}$ can be obtained by subtracting

$$\alpha \frac{\sum_{i=1}^{nsim}(Z^i_{T+h} - \hat{\hat{Z}}^i_{T+h})}{\text{nsim}}$$

from the results of step 5, Equation (64) minus 1.

The linex risk of $\hat{Z}_{t+h}$ was given by

$$\begin{aligned}
R(Z_{T+h}, \hat{Z}_{T+h}) &= E[L(Z_{T+h}, \hat{Z}_{T+h})] \\
&= E[e^{\alpha(Z_{T+h} - \hat{Z}_{T+h})} - \alpha(Z_{T+h} - \hat{Z}_{T+h}) - 1] \\
&= E[e^{\alpha(Z_{T+h} - \hat{Z}_{T+h})}] - \alpha(E[Z_{T+h} - \hat{Z}_{T+h}]) - 1 \\
&= -\alpha(E[Z_{T+h} - \hat{Z}_{T+h}]) \\
&= -\alpha(E[Z_{T+h} - \mu_{T+h|T} - \frac{\alpha}{2}\sigma^2_{T+h|T}]) \\
&= -\alpha(E[Z_{T+h}] - E[\mu_{T+h|T}] - \frac{\alpha}{2}\sigma^2_{T+h|T}) \\
&= -\alpha(E[Z_{T+h}] - E[E[Z_{T+h}|Z_1, Z_2, \ldots, Z_T]] - \frac{\alpha}{2}\sigma^2_{T+h|T}) \\
&= -\alpha(E[Z_{T+h}] - E[Z_{T+h}] - \frac{\alpha}{2}\sigma^2_{T+h|T})
\end{aligned}$$

$$
\begin{aligned}
&= -\alpha(-\frac{\alpha}{2}\sigma^2_{T+h|T}) \\
&= \frac{\alpha^2}{2}\sigma^2_{T+h|T}. \qquad (65)
\end{aligned}
$$

Since α was either 1 or -1, the linex risk of $\hat{Z}_{T+h}$ became $\frac{\sigma^2_{T+h|T}}{2}$. The ratio of linex risk of $\hat{\hat{Z}}_{T+h}$ to linex risk of $\hat{Z}_{T+h}$ were then obtained by

$$
\text{relative efficiency} = \frac{R(Z_{T+h}, \hat{\hat{Z}}_{T+h})}{R(Z_{T+h}, \hat{Z}_{T+h})}, \qquad (66)
$$

which was denoted by $re(R\hat{\hat{Z}}_{T+h}), R(\hat{Z}_{T+h}))$. The purpose was to understand how much risk will increase when the parameters of linex unbiased predictor are estimated.

CHAPTER IV

RESULTS

Simulations under three forecast steps, six series lengths, three variances and two shape parameters were performed on twenty-eight time series models. In order to efficiently generate such a large amount of data, the source code was written to simulate values of the CLU for three forecast steps of all twenty-eight time series models in each simulation, under the fixed series length, variance, and shape parameter. In other words, when running each simulation, the values of series length, variance, and shape parameter were modified in order to obtain the results for different conditions. The R code for the simulation can be found in Appendix E. However, series length $T = 15$ was insufficient and unable to be simulated with such a setting. Therefore, $T = 15$ was discarded in this research. Other than $T = 15$, the values of CLU of all other conditions were successfully simulated. Most of the CLU values are positive, and the negative values start to appear as the series length increases.

The primary purpose of this chapter was to answer the six research questions addressed in this dissertation. In the following six sections, each section provided a relevant narrative, tables and figures to answer a research question. The results were listed in tables for numeric values and visualized in figures to show overall patterns.

Research Question 1

For research question 1, the author investigated how the condition of linex unbiasedness(CLU) behaves when parameters of the AR, MA and ARMA processes are unknown and being estimated. Recall, Chapter III described the methodology of estimating the time series model parameters and computing the linex unbiased predictors. Thus, to answer this research question, the CLU values of the AR(p), MA(q) and ARMA(p,q) models were analyzed. The results of CLU values were graphically displayed in Figures 2 through 16. Each graph contains the results of all time series models, with all parameters, at the fixed variance and series length, all levels of h-step-ahead and shape parameters. The narratives follow the order of time series models displayed on the x axes in the figures.

For the AR(1), which had a U shape pattern, the CLU values were higher and further from zero when $\phi_1 = -0.8$ and 0.8, and smaller and closer to zero for other smaller parameters $-0.4, -0.2$, 0.2, 0.4. This pattern existed consistently when $\sigma^2 = 0.5, 1$ and 5, and can be easily observed when $T = 25$ and 50. The CLU values for AR(1) depended on ϕ_1 when $T = 25, 50$ and 100. The CLU values were closer to zero when ϕ_1 was small. But for the MA(1), higher θ_1 did not always have higher CLU values. The MA(1) did not have a consistent pattern as θ_1 increases or decreases. The CLU values for MA(1) did not depend on θ_1. See Figures 2, 3, 7, 8, 12, and 13.

For the AR(2), $(\phi_1, \phi_2) = (0.1, -0.3)$ and $(-0.1, 0.3)$, no consistent pattern was observed if one parameter set always had CLU values further from or closer to

zero than the other, as T and σ^2 change. For the AR(3), $(\phi_1, \phi_2, \phi_3) = (0.1, -0.3, -0.2)$ and $(-0.3, -0.5, 0.1)$ also did not show persistent patterns when comparing their CLU values across all levels of T and σ^2. For the AR(p) models, overall the CLU values increased as p increased, except when p= 1 and ϕ_1 was high and close to non-stationary process. Similar to the cases of AR(2) and AR(3), the two parameter sets of MA(2) and in MA(3) did not show a consistent pattern if one parameter set was always different from the other, across all levels of T and σ^2. The CLU values for MA(q) models also increased as q increased. However, as T increases to 200 and 400, all the CLU values started to even out and patterns started to vanish.

For the ARMA(p,q) models, the ARMA(1,1) had CLU values closer to zero than ARMA(2,1), ARMA(1,2) and ARMA(2,2). And the ARMA(2,2) had CLU values further from zero among all ARMA(p,q) models, when series length was small. The ARMA(2,1) and ARMA(1,2) also did not have a persistent pattern if one model always performed different from the other, across all levels of T and σ^2. As series length increased, the CLU values of all ARMA(p,q) models decreased into the same range, the ARMA(2,2) no longer had the highest CLU values, see Figures 6, 11 and 16.

Although the CLU values increased as p and q increased and as the time series model became more complex, the nearly non-stationary AR(1) was an exception. AR(1) $\phi_1 = -0.8$ and 0.8 at $h = 2$ and 3, which sometimes had the CLU values as high as AR(3), MA(3), and ARMA(2,1), ARMA (1,2) and ARMA(2,2) models. This pattern was strong at all σ^2 levels and when $T = 25, 50$ and 100. The

nearly non-stationary AR(1) and other more complex models required larger series length to produce the CLU values that are close to zero. Also, based on the figures provided for this research question, no significant difference was observed when $\alpha = 1$ versus when $\alpha = -1$.

All the patterns described above were not easily observed in the figures for $\sigma^2 = 5$, especially when $T = 25, 50$, and 100. The higher variance produced higher CLU values, and higher CLU values compressed the graphs. But the CLU values for $\sigma^2 = 5$ can be found in the Appendix A, Tables 23, 24, 25, 26 and 27. Appendix A contains tables of all the CLU values. Each table displays the numeric CLU values, of all time series models at fixed series length and variance, with all parameters, alpha levels and forecast steps. The numeric CLU values were rounded to three decimal places, so 0.000* indicates the value was not true zero and was less than ± 0.0001. From the tables, it can be seen that the overall CLU values became smaller and close to zero as the series length increased.

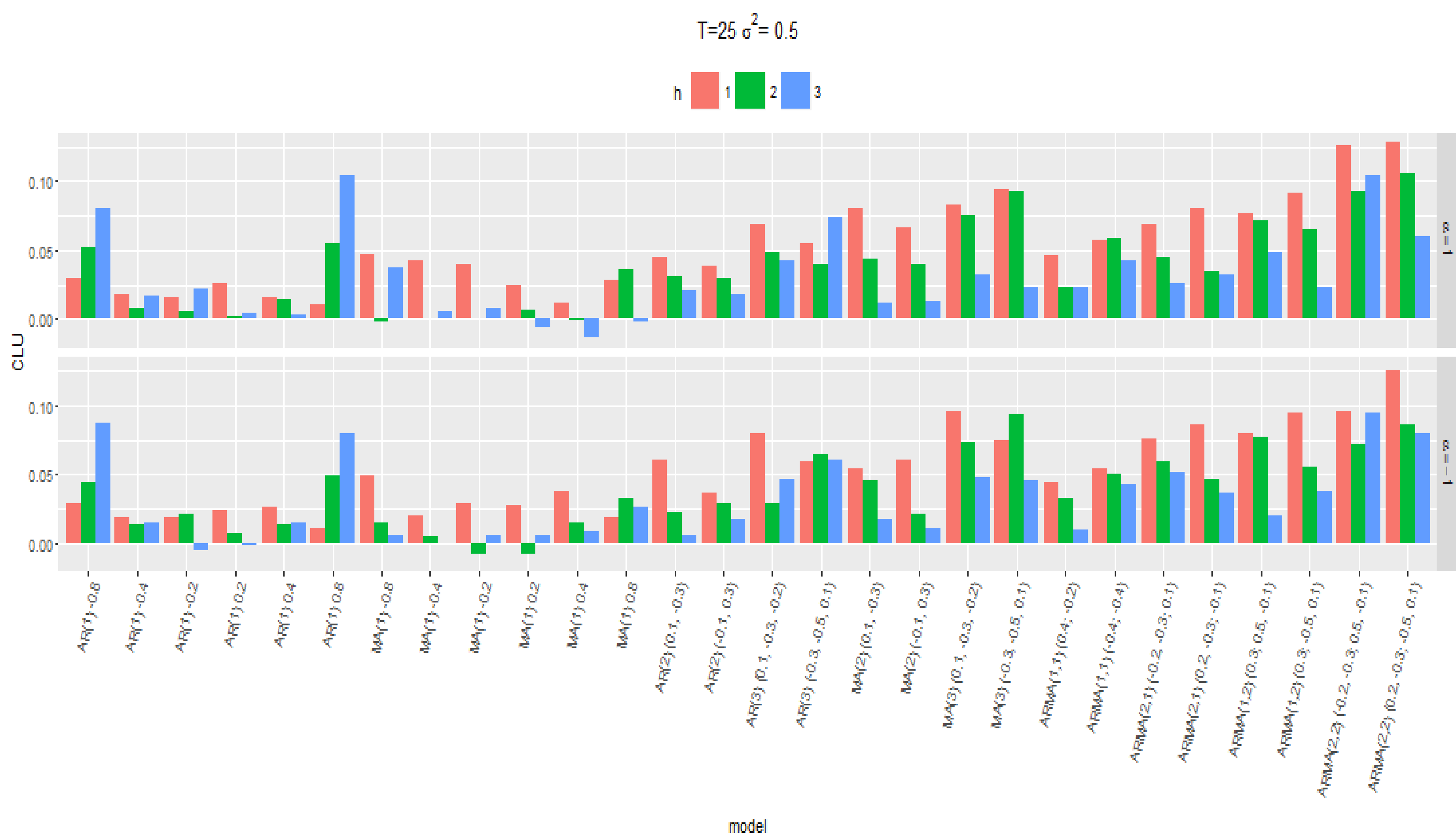

Figure 2. CLU by model at $T = 25$ and $\sigma^2 = 0.5$

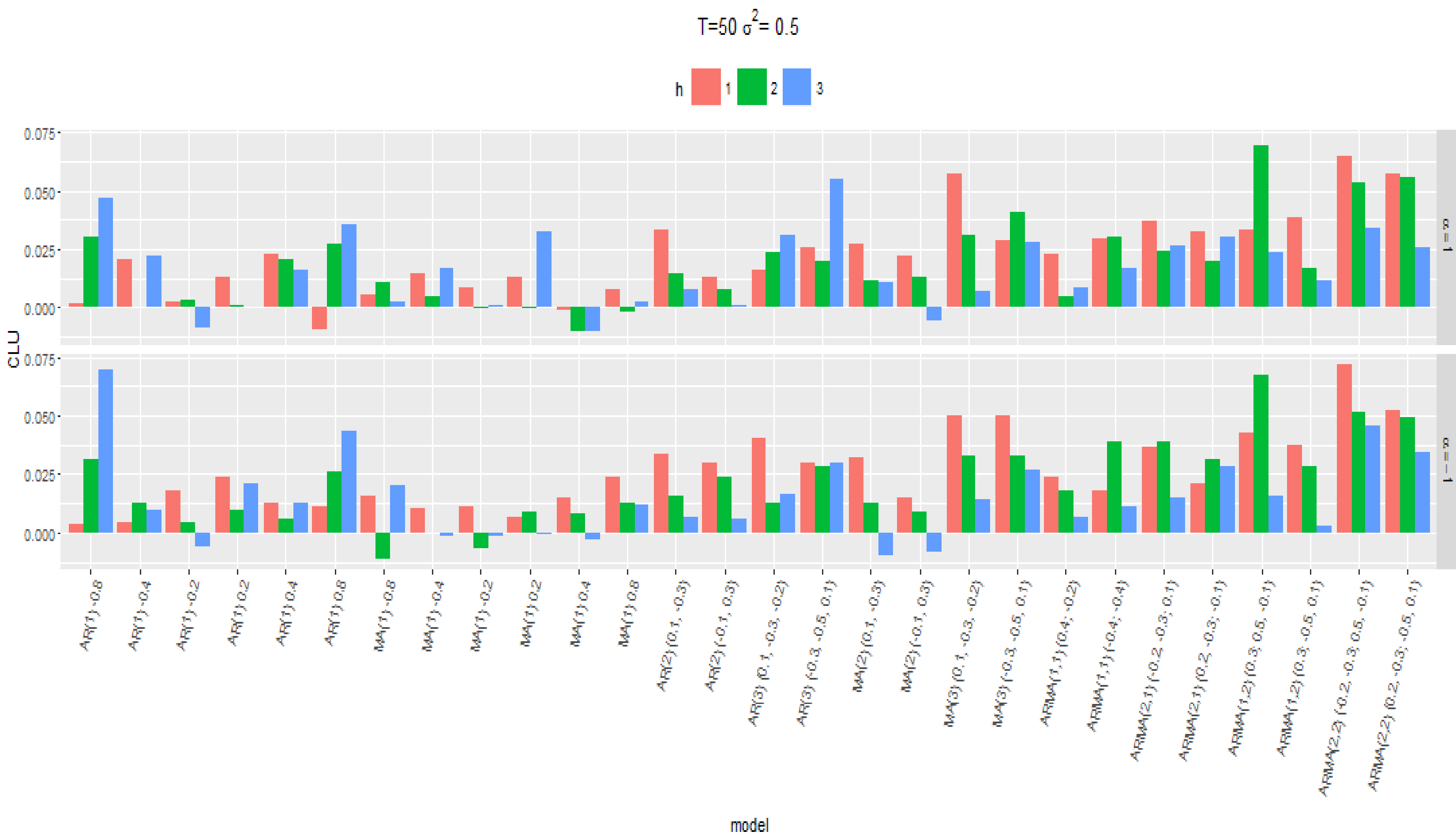

Figure 3. CLU by model at $T = 50$ and $\sigma^2 = 0.5$

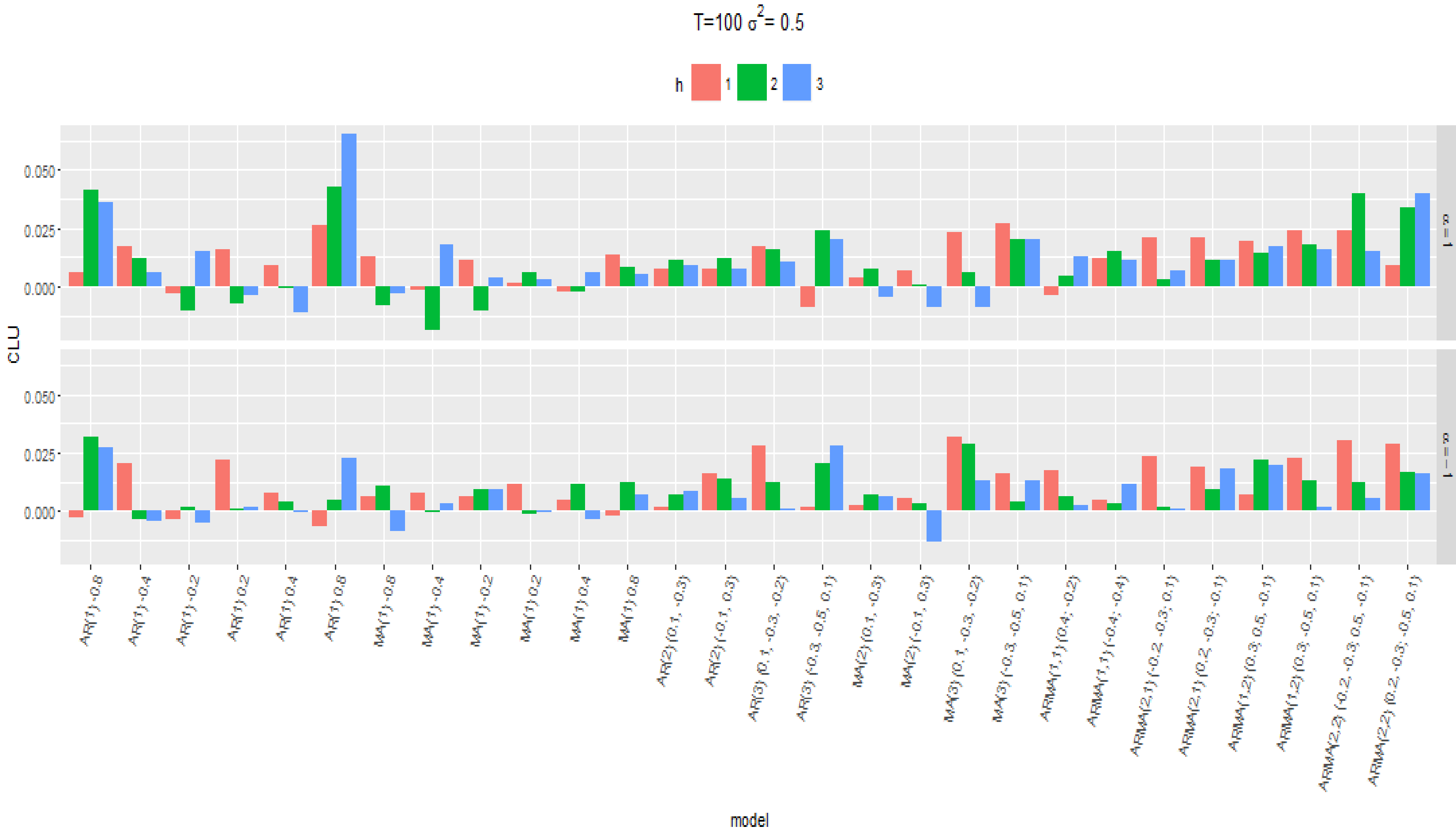

Figure 4. CLU by model at $T = 100$ and $\sigma^2 = 0.5$

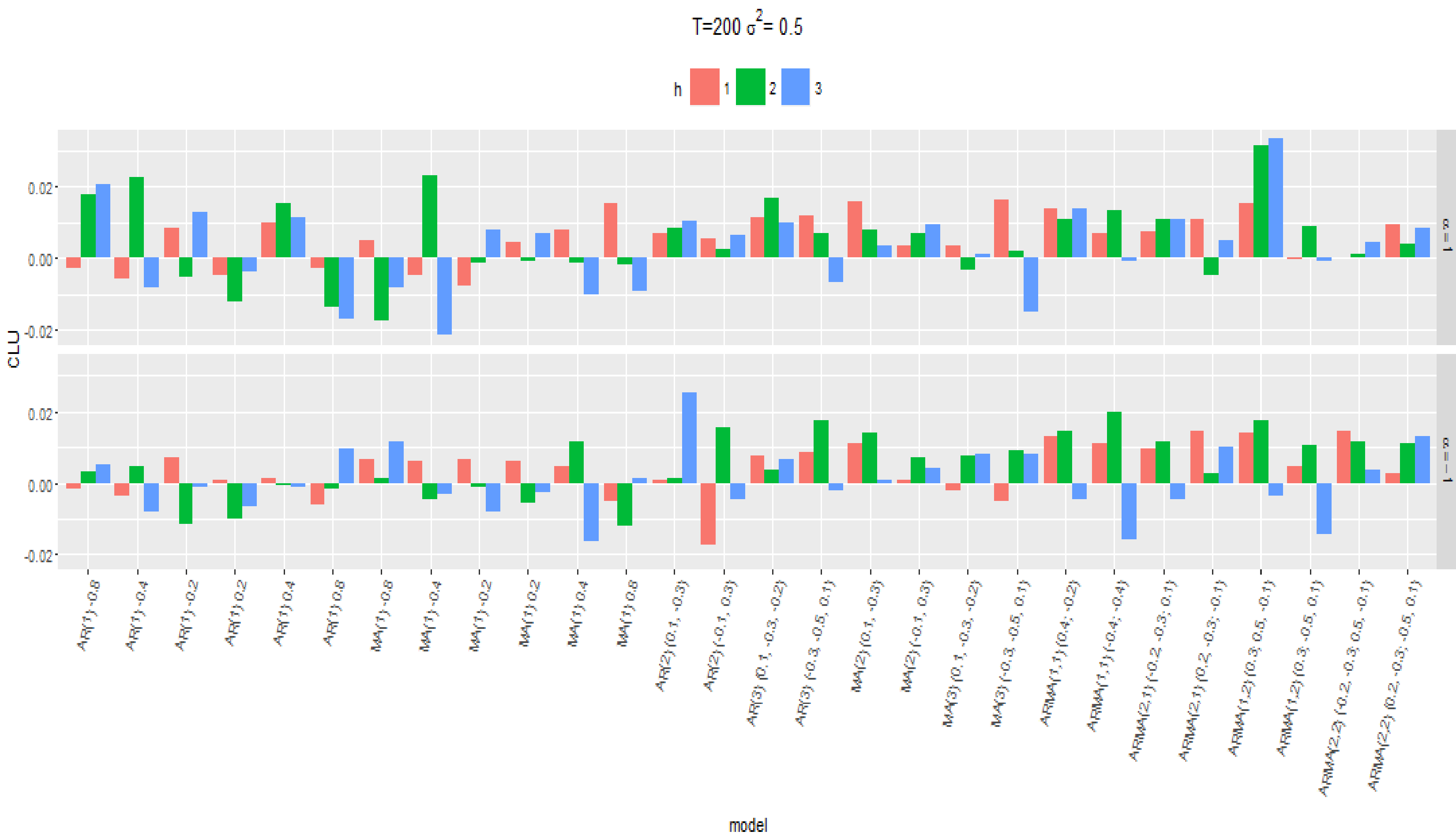

Figure 5. CLU by model at $T = 200$ and $\sigma^2 = 0.5$

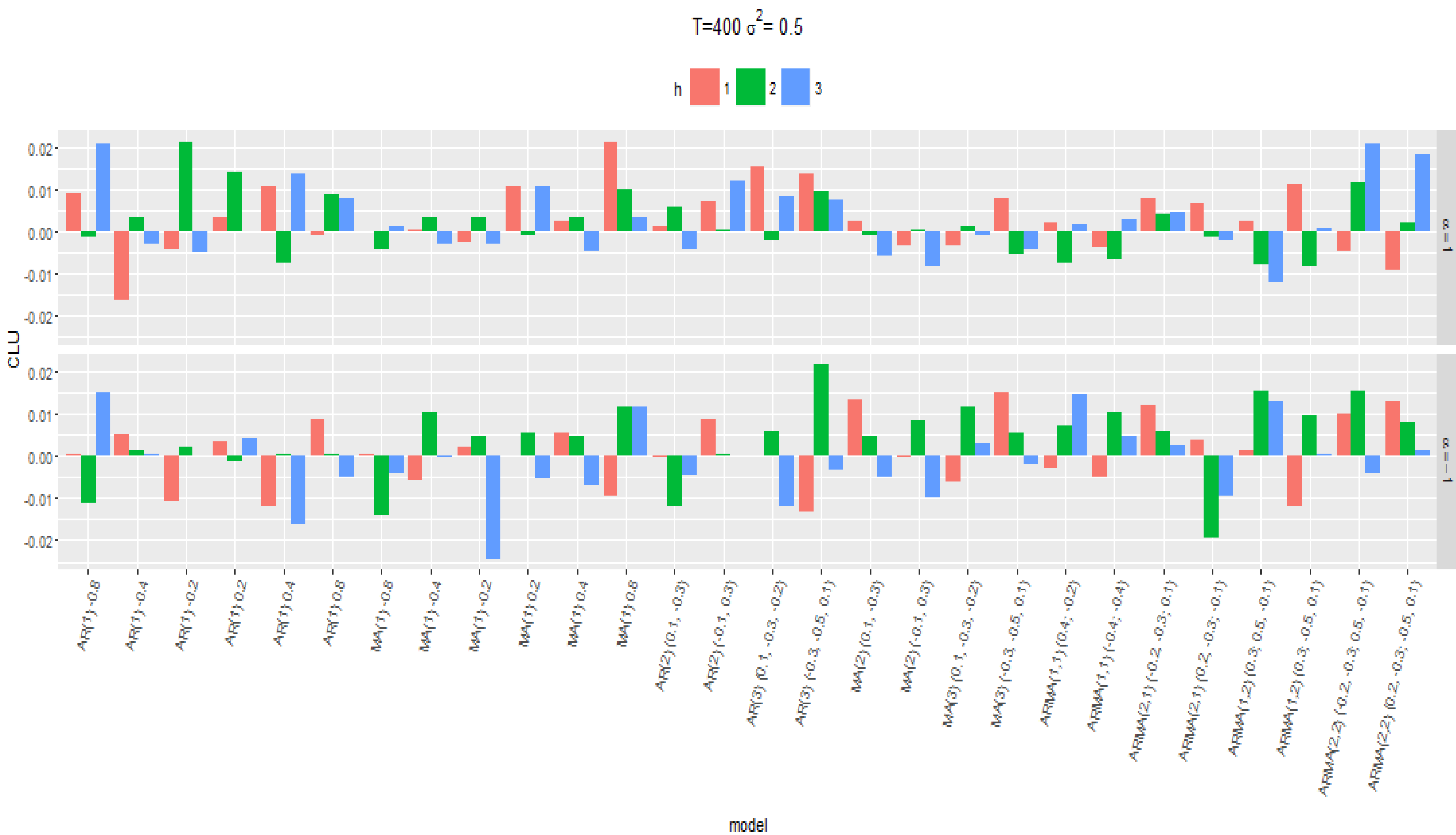

Figure 6. CLU by model at $T = 400$ and $\sigma^2 = 0.5$

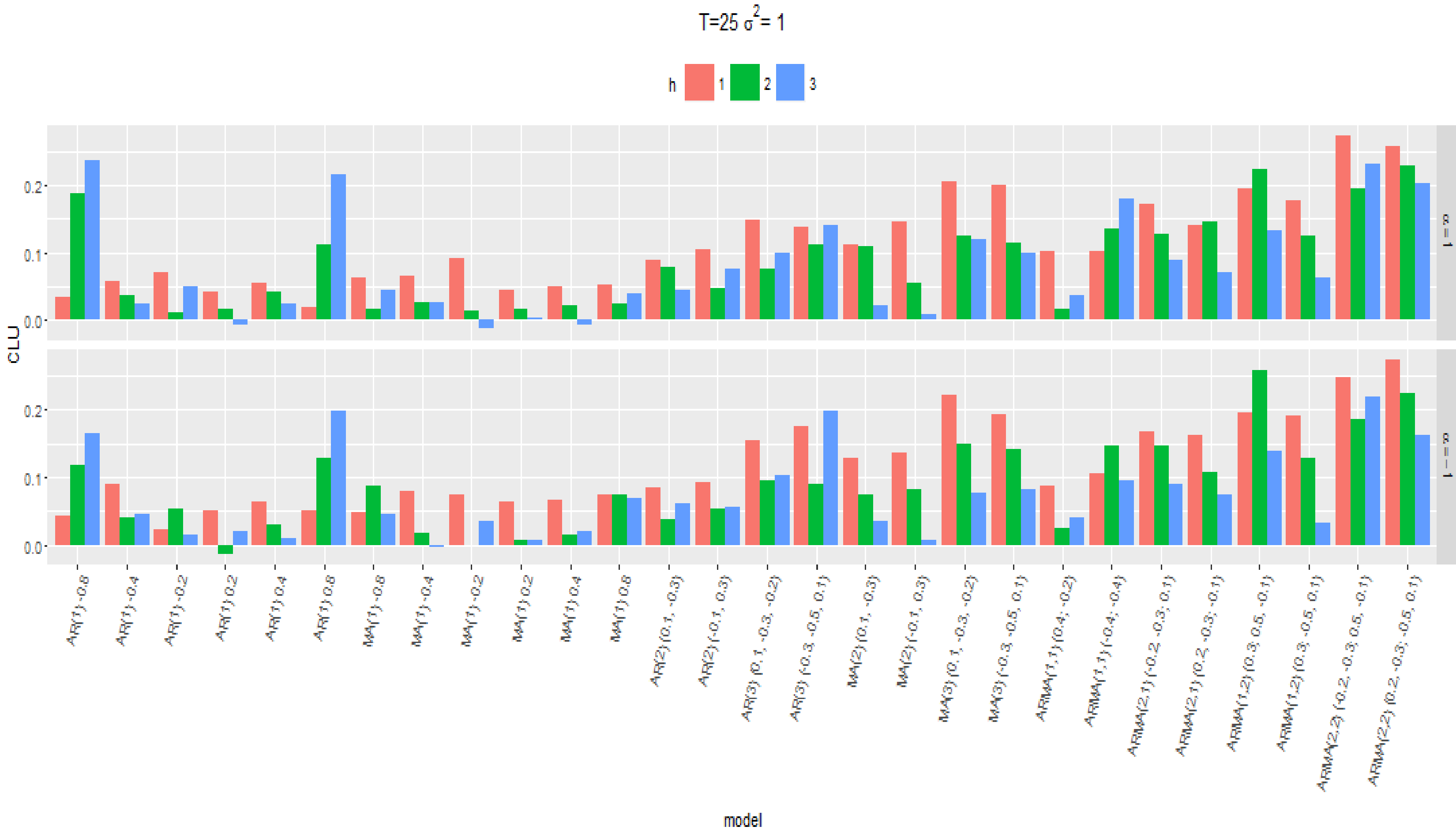

Figure 7. CLU by model at $T = 25$ and $\sigma^2 = 1$

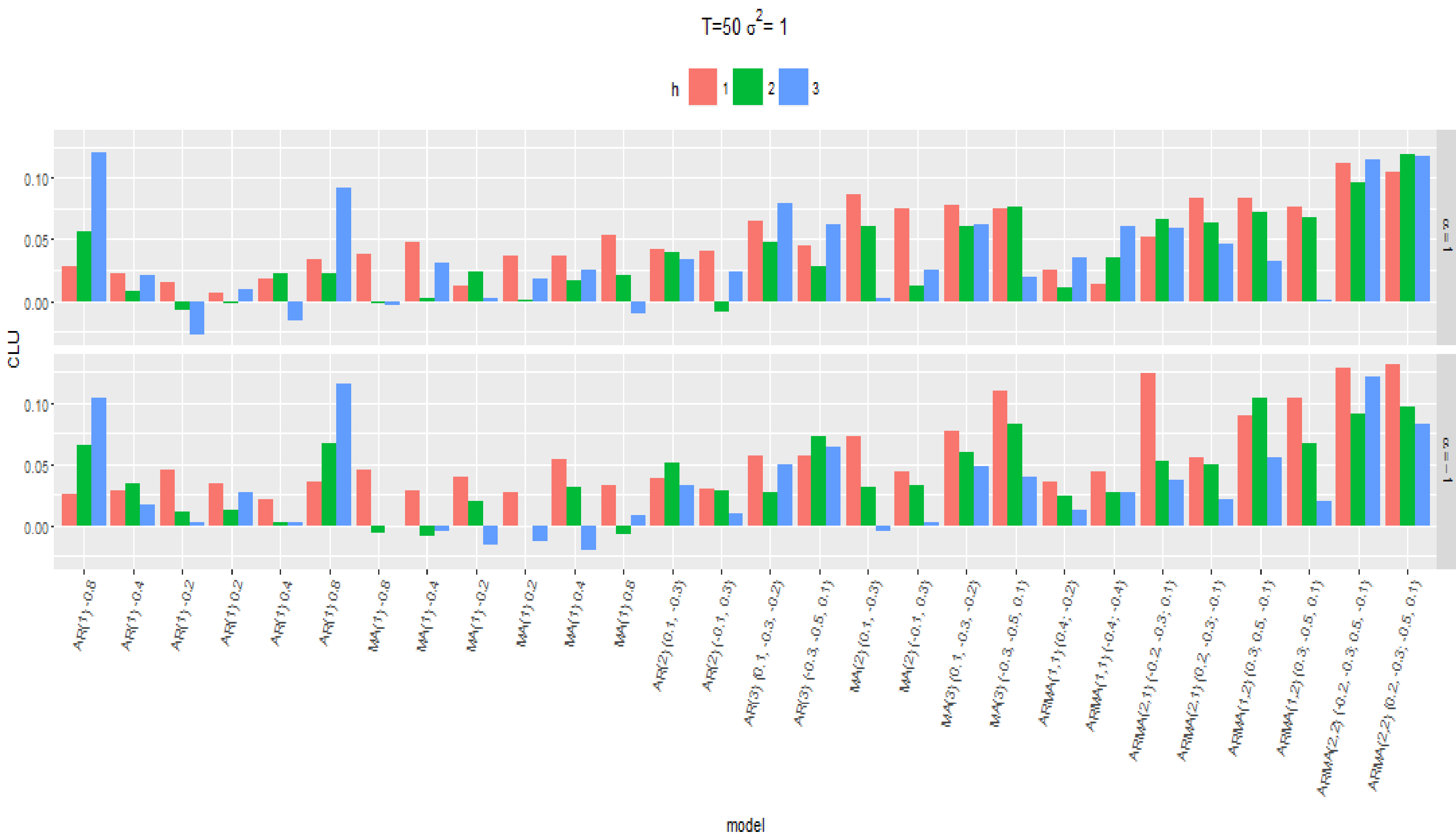

Figure 8. CLU by model at $T = 50$ and $\sigma^2 = 1$

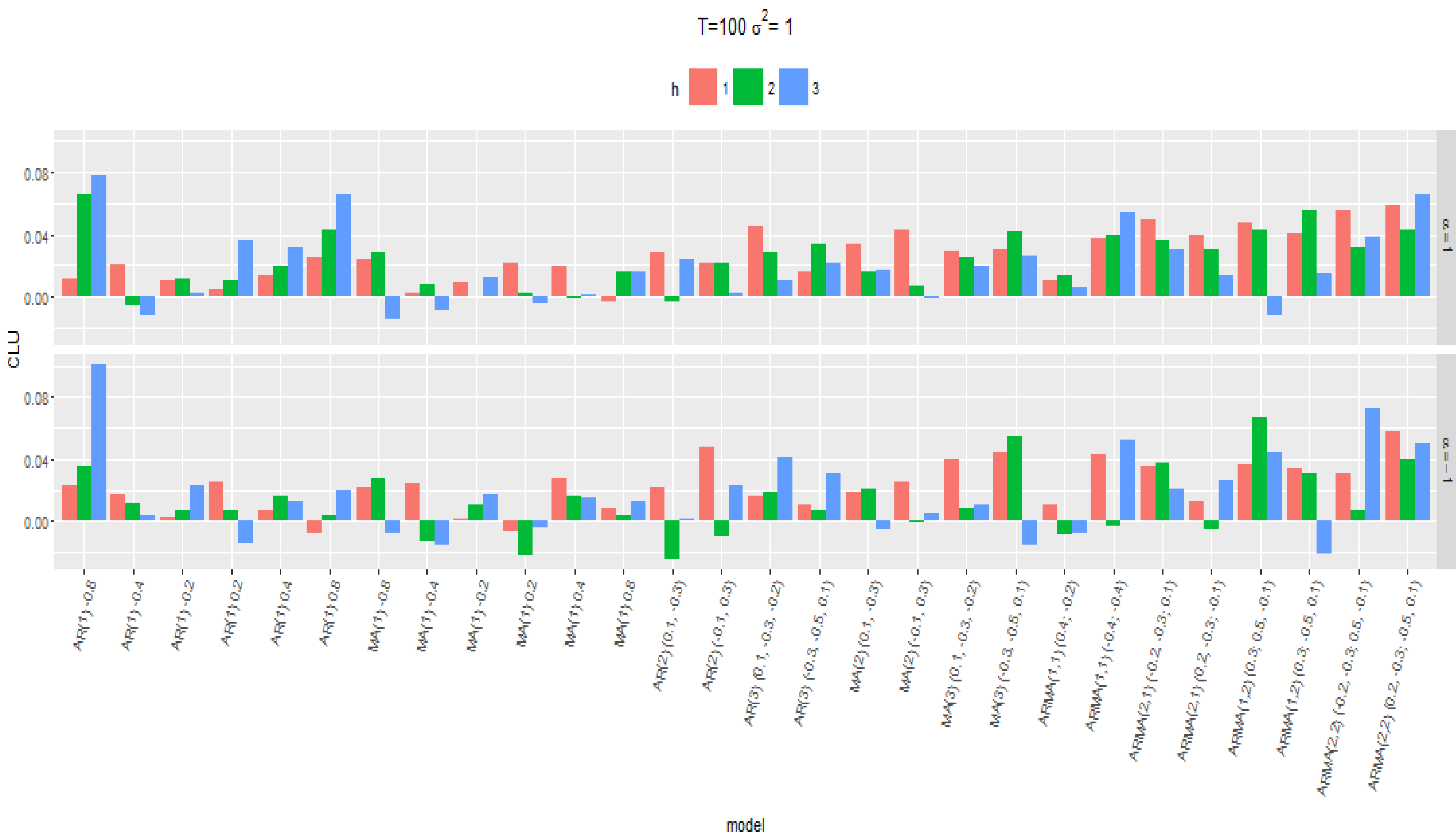

Figure 9. CLU by model at $T = 100$ and $\sigma^2 = 1$

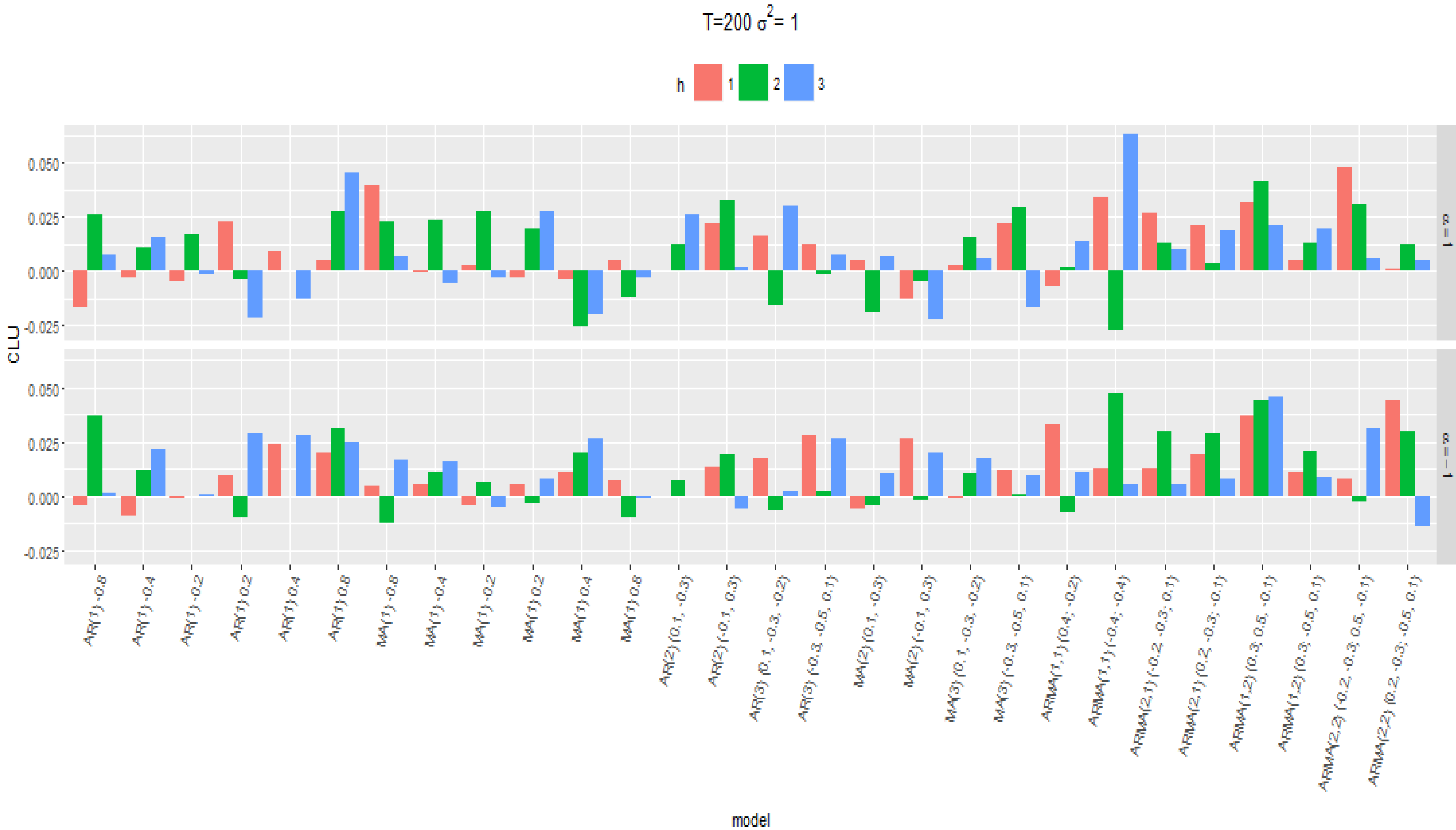

Figure 10. CLU by model at $T = 200$ and $\sigma^2 = 1$

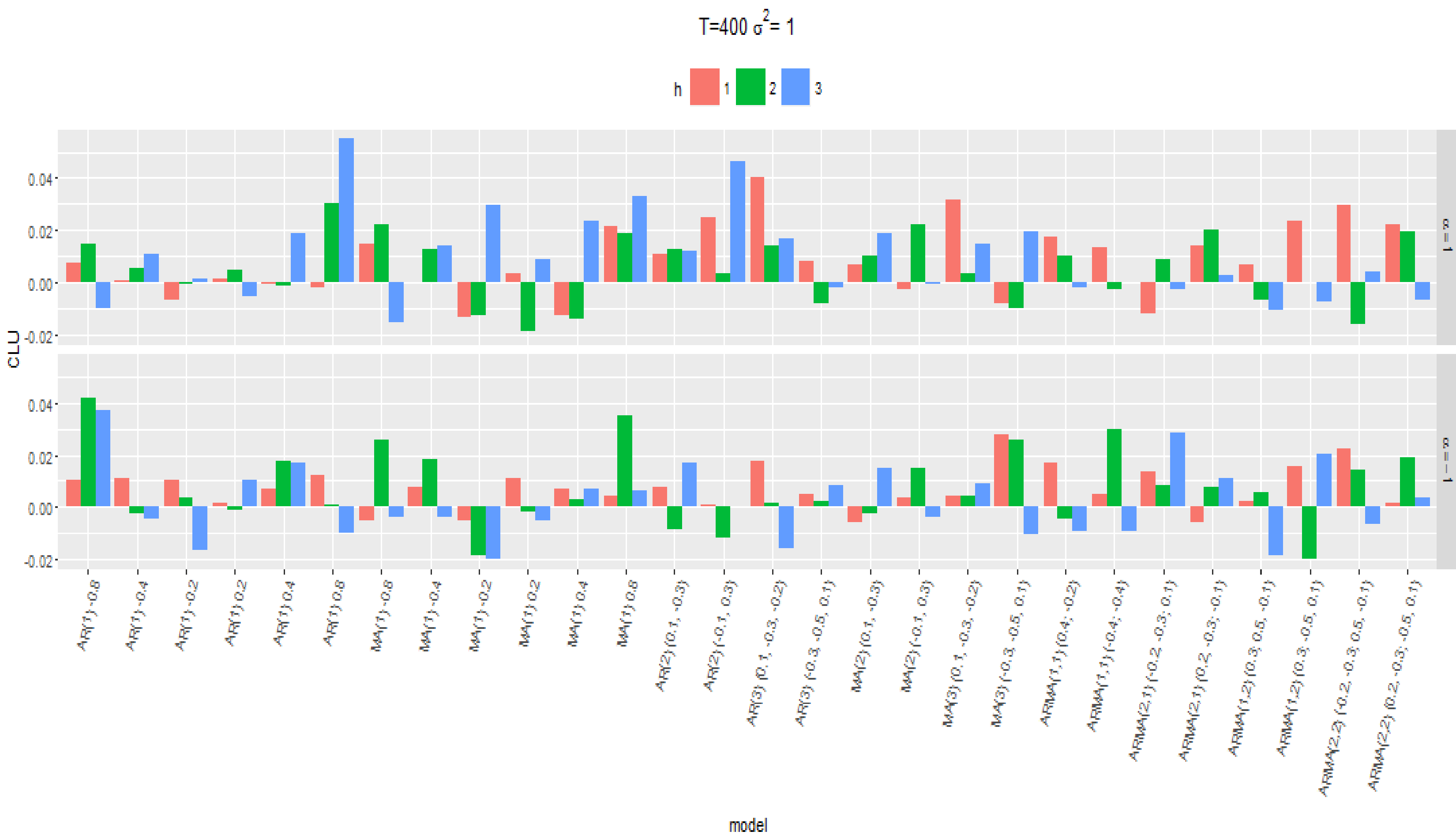

Figure 11. CLU by model at $T = 400$ and $\sigma^2 = 1$

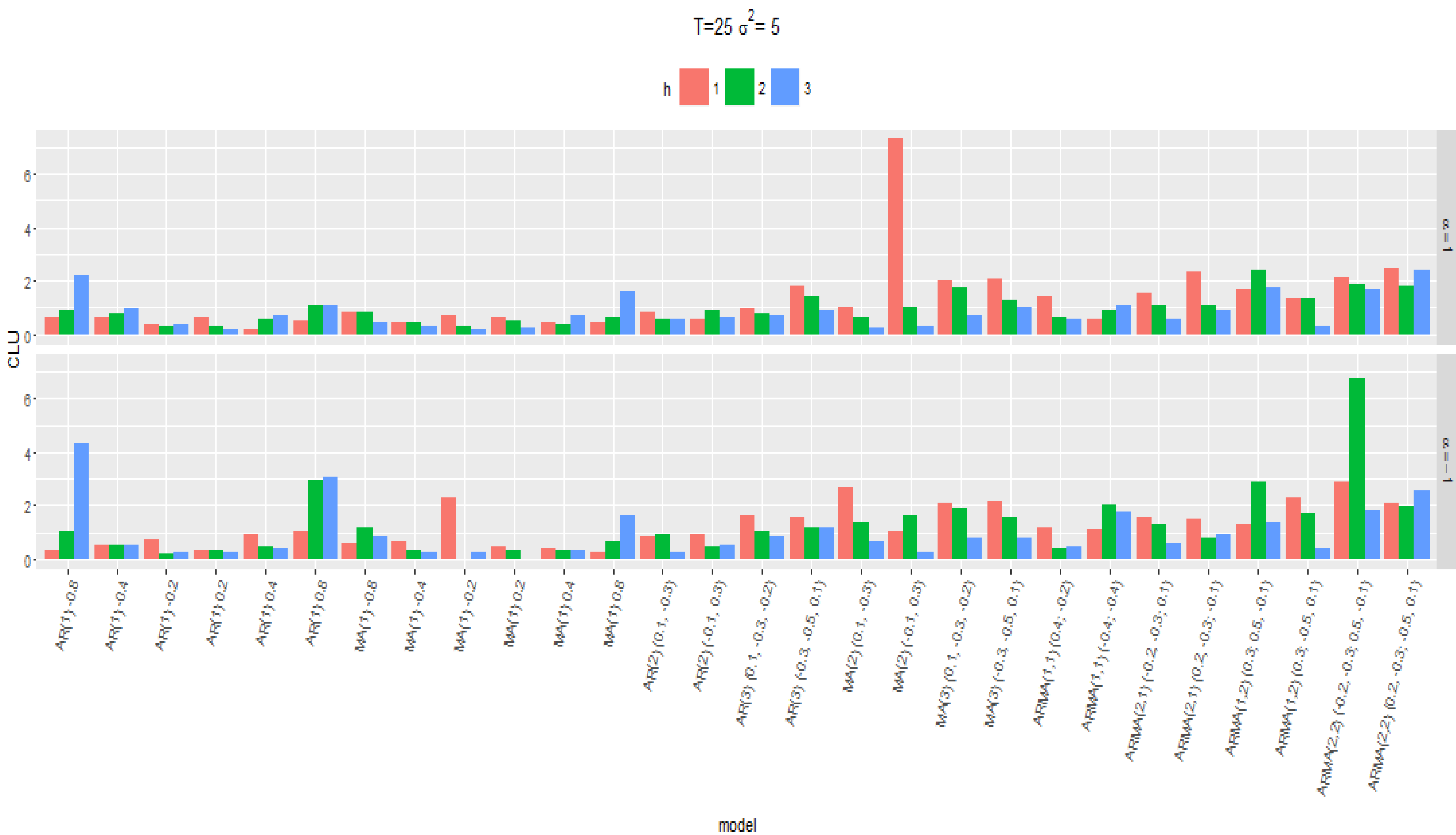

Figure 12. CLU by model at $T = 25$ and $\sigma^2 = 5$

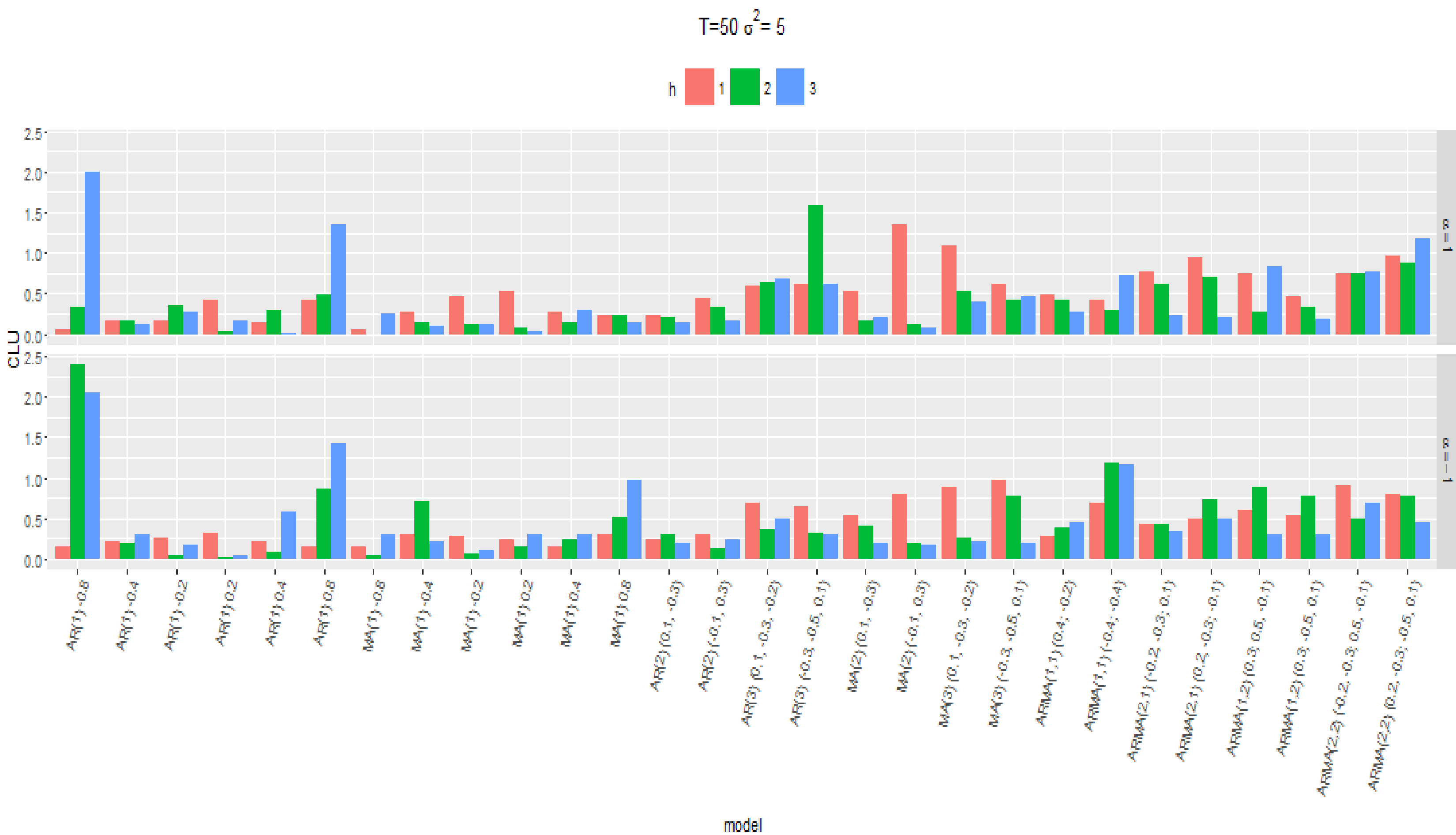

Figure 13. CLU by model at $T = 50$ and $\sigma^2 = 5$

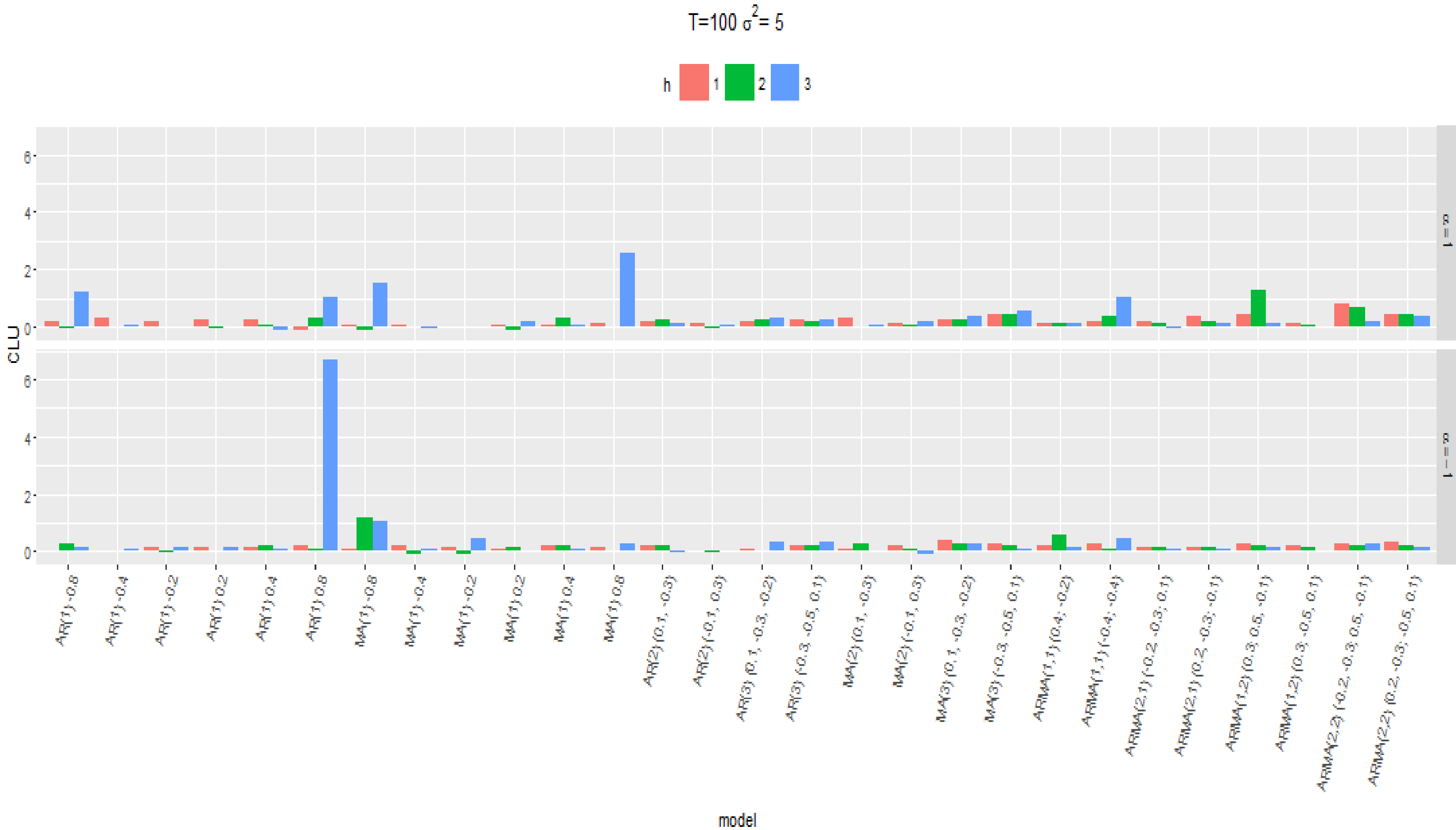

Figure 14. CLU by model at $T = 100$ and $\sigma^2 = 5$

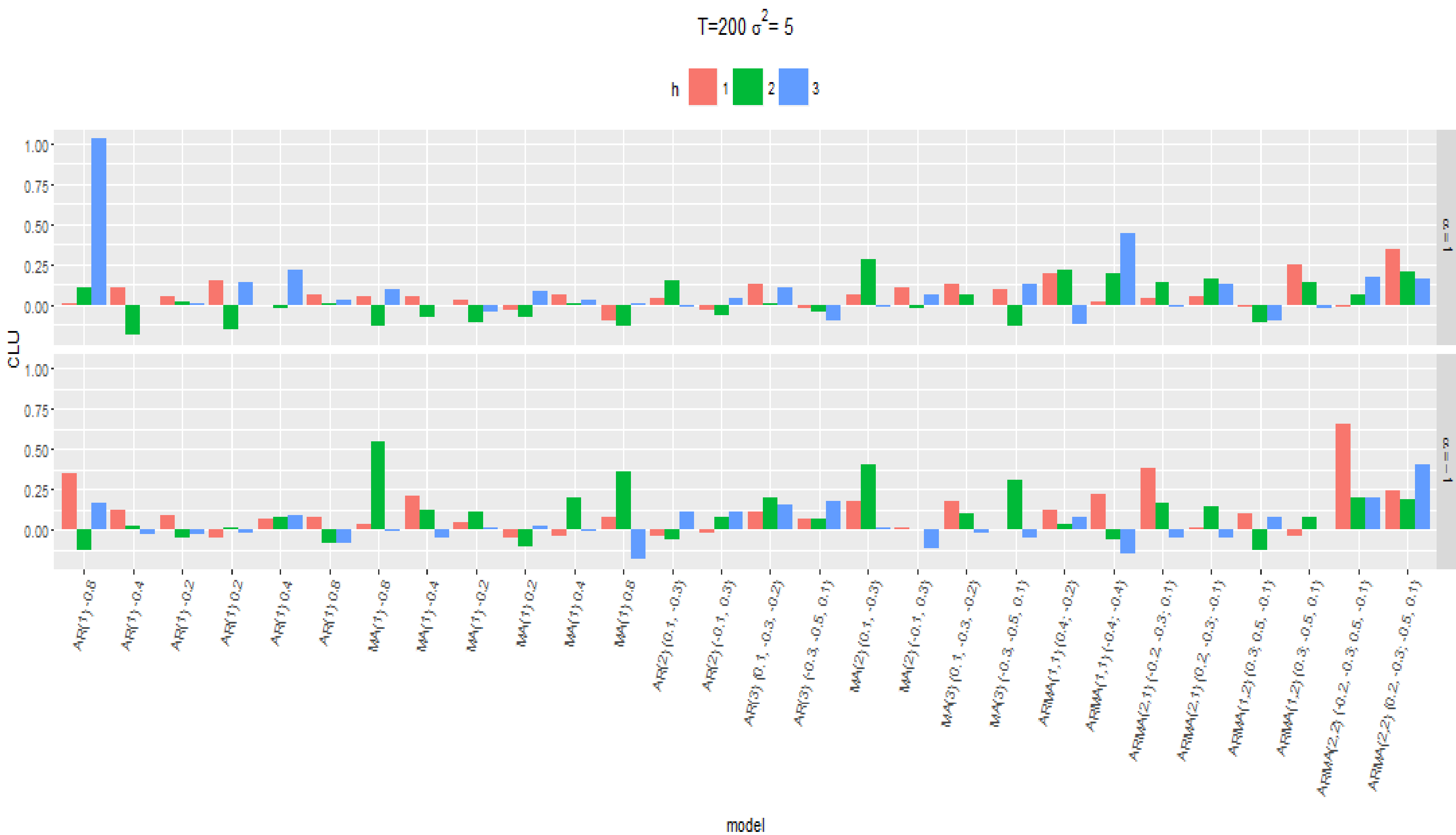

Figure 15. CLU by model at $T = 200$ and $\sigma^2 = 5$

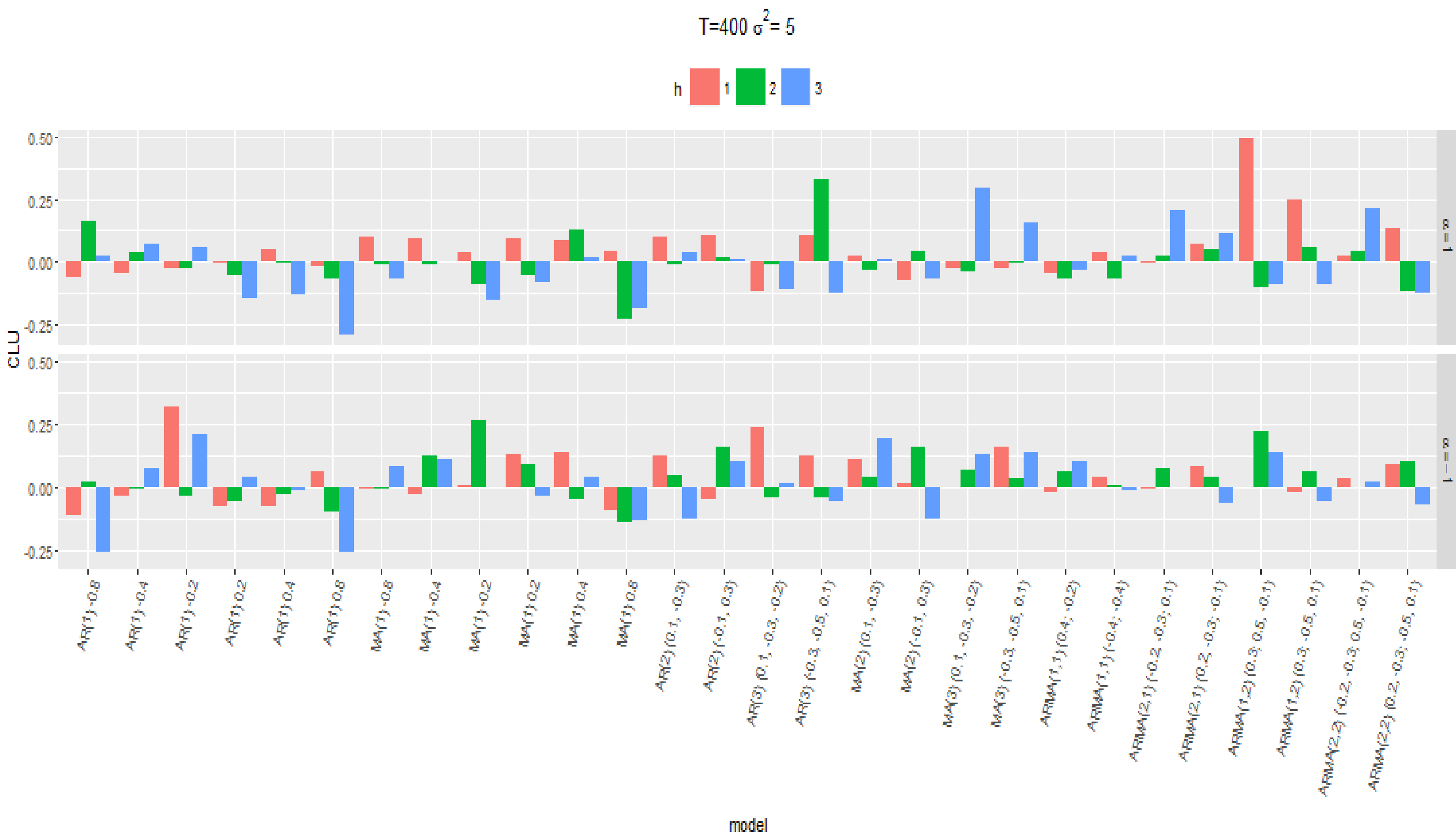

Figure 16. CLU by model at $T = 400$ and $\sigma^2 = 5$

Research Question 2

For research question 2, the author evaluated how the condition of linex unbiasedness differs when the variance, σ^2, of the series is less than, equal to and greater than 1. Figures 17 through 24, and Figures 53 through 72 show the comparison of the CLU values by series length at different variances, for all levels of h-step-ahead and shape parameter, when the model and parameter were fixed. For the purpose of illustration, this section only displayed all the figures of ARMA(p,q) models, which are Figures 17, 18, 19, 20, 21, 22, 23 and 24. All the figures of AR(p) and MA(q) models are displayed in Appendix B.

All these figures clearly showed that the lines were around zero when the variance was equal to or less than 1, but the lines fluctuated or had an L shape pattern when the variance was greater than 1. From the graphs, the difference of CLU values between $\sigma^2 = 0.5$ series and $\sigma^2 = 1$ series did not seem to be dramatic. However, for the $\sigma^2 = 5$ series, the lines either fluctuated and then turned into a more steady pattern after $T = 200$, or had a stable L shape pattern that first decreased and then approached zero as T increased to 200 and 400. Also, for the $\sigma^2 = 5$ series, the CLU values sometimes fluctuated in the same pattern in different forecast steps, see Figures 59, 60, 63, 69 and 71.

The overall CLU values of $\sigma^2 = 0.5$ series were lower compared to $\sigma^2 = 1$ series, but the CLU values of $\sigma^2 = 1$ series were occasionally lower than $\sigma^2 = 0.5$ series, especially when the model had order one. This can be seen when, after $T = 25$, AR(1), $\phi_1 = -0.2$ when $h = 3$ and $\alpha = 1$, see Figure 55. After $T = 50$,

AR(3), $(\phi_1, \phi_2, \phi_3) = (-0.3, -0.5, 0.1)$ when $h = 2$ and $\alpha = 1$, and ARMA(1,1), $(\phi_1; \theta_1) = (0.4; -0.2)$ when $h = 2$ and $\alpha = -1$. See Figures 61 and 17.

It also can be seen when, after $T = 100$, AR(1), $\phi_1 = -0.8$ and -0.2, and MA(1), $\theta_1 = 0.2$ and 0.4 when $h = 1$ and $\alpha = 1$, MA(1), $\theta_1 = -0.8$ and MA(2), $(\theta_1, \theta_2) = (0.1, -0.3)$ when $h = 1$ and $\alpha = -1$, AR(3), $(\phi_1, \phi_2, \phi_3) = (-0.3, -0.5, 0.1)$ and MA(1), $\theta_1 = 0.4$ when $h = 2$ and $\alpha = 1$, AR(3), $(\phi_1, \phi_2, \phi_3) = (0.1, -0.3, -0.2)$ and MA(2), (θ_1, θ_2)= $(0.1, -0.3)$ when $h = 2$ and $\alpha = -1$, and AR(1), $\phi_1 = -0.8$ and -0.2, when $h = 3$ and $\alpha = 1$. See Figures 53, 63, 66, 67, 61, 62 and 69.

The $\sigma^2 = 1$ series had lower CLU values than $\sigma^2 = 0.5$ series also appeared when, after $T = 200$, AR(1), $\phi_1 = -0.2$ when $h = 3$ and $\alpha = -1$ and AR(1), $\phi_1 = 0.4$, when $h = 1$ and $\alpha = 1$. See Figures 55 and 57.

Although the $\sigma^2 = 5$ series had greater variance, it sometimes produced CLU values that were lower than the case of $\sigma^2 = 0.5$ and 1 when T increased. This was observed when, after $T = 50$, MA(1), $\theta_1 = -0.8$ and 0.2 when $h = 2$ and $\alpha = 1$, AR(1), $\phi_1 = -0.2$ when $h = 2$ and $\alpha = -1$, and MA(2), $(\theta_1, \theta_2) = (-0.1, 0.3)$ when $h = 3$ and $\alpha = -1$. See Figures 55, 63 and 70.

This was also observed when, after $T = 100$, MA(1), $\theta_1 = 0.4$ when $h = 1$ and $\alpha = -1$, MA(1), $\theta_1 = -0.2$ and 0.8 when $h = 2$ and $\alpha = 1$, MA(1), $\theta_1 = -0.2$, AR(3), $(\phi_1, \phi_2, \phi_3) = (-0.3, -0.5, 0.1)$, ARMA(2,1), $(\phi_1, \phi_2; \theta_1) = (-0.2, -0.3; 0.1)$ and ARMA(1,2), $(\phi_1; \theta_1, \theta_2) = (0.3; -0.5, 0.1)$ when $h = 3$ and $\alpha = 1$, and MA(1), $\theta_1 = 0.8$ and ARMA(1,2), $(\phi_1; \theta_1, \theta_2) = (0.3; -0.5, 0.1)$ when $h = 3$ and $\alpha = -1$. See Figures 62, 65, 67, 68, 19 and 22.

As after $T = 200$, the CLU values of $\sigma^2 = 5$ series were lower than $\sigma^2 = 0.5$ and 1 series occurred at ARMA(2,1), $(\phi_1, \phi_2; \theta_1) = (-0.2, -0.3; 0.1)$ and $(0.2, -0.3; -0.1)$, and ARMA(1,2), $(\phi_1; \theta_1, \theta_2) = (0.3; -0.5, 0.1)$ when $h = 3$ and $\alpha = -1$. See Figures 19, 20 and 22.

Despite the fact that the series with bigger variances sometimes had the CLU values lower than the series with smaller variances, it did not represent the distance between CLU values and zero. Especially when the series had variance 5, the CLU values can be lower (negative values) and further from zero.

For the MA(2), when $\theta_1 = -0.2$ and $h = 2$, all variances had the CLU values close to zero and the lines were near each other. All the other graphs either showed an L shape patter or did not have a specific consistent pattern. Even the line graphs clearly visualized the CLU values, one needs to be aware of the scale on y axes to interpret the results. All figures displayed for this research question did not shown a significant difference between positive and negative α.

Approximately, for $\sigma^2 = 5$ series, the CLU values were higher than $\sigma^2 = 0.5$ series and $\sigma^2 = 1$ series when $T = 25, 50$ and 100, and the values became lower after $T = 200$, and approached zero at $T = 400$. The CLU values of $\sigma^2 = 1$ series were also generally higher than $\sigma^2 = 0.5$ series, but the difference between the two were not as dramatic as the $\sigma^2 = 5$ series. But as T increased, the CLU values obtained from all variance levels became small and close to zero, and all patterns started to die down and the lines became stable.

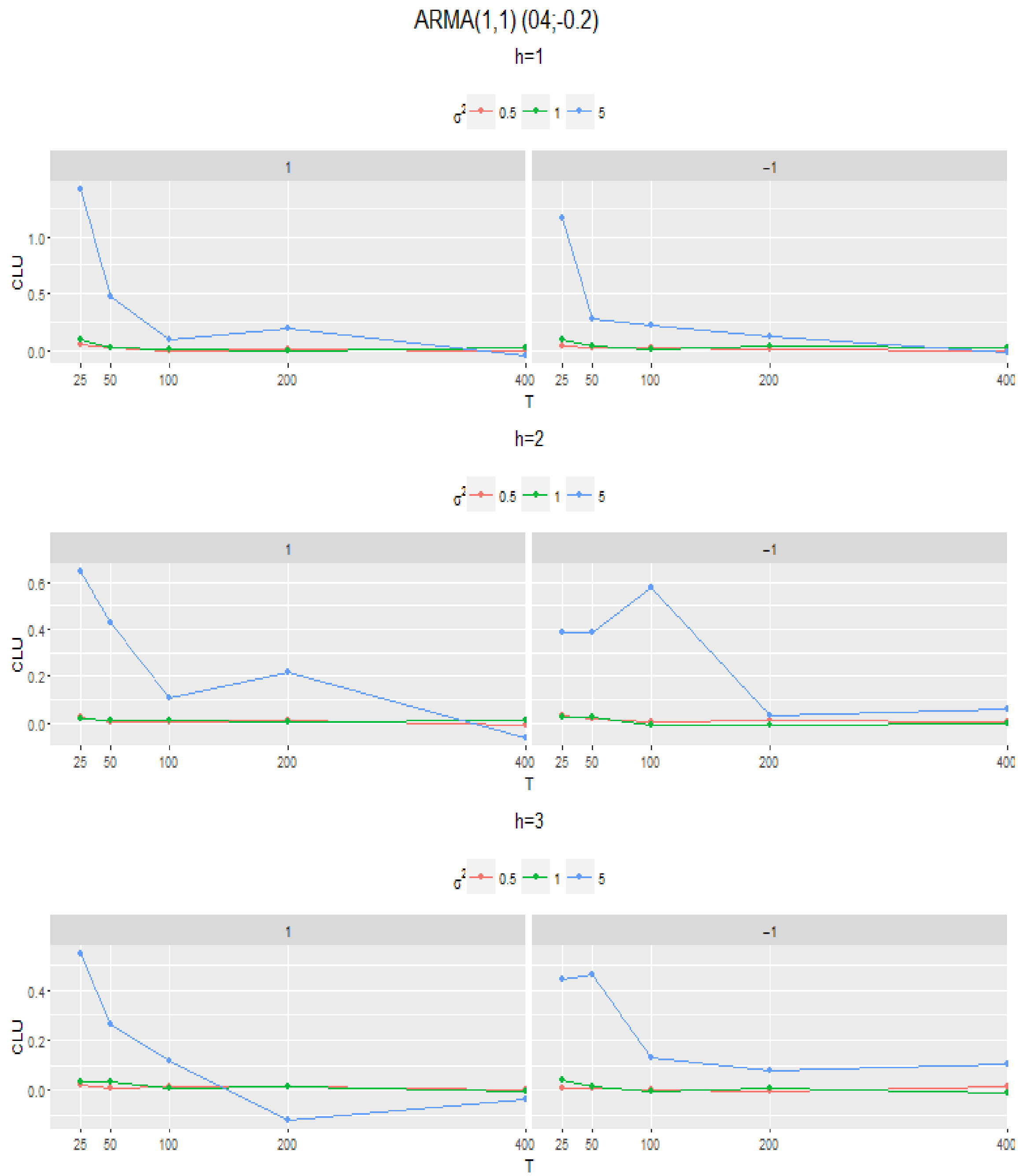

Figure 17. CLU of σ^2 by T for ARMA(1,1), $(\phi_1; \theta_1) = (0.4; -0.2)$

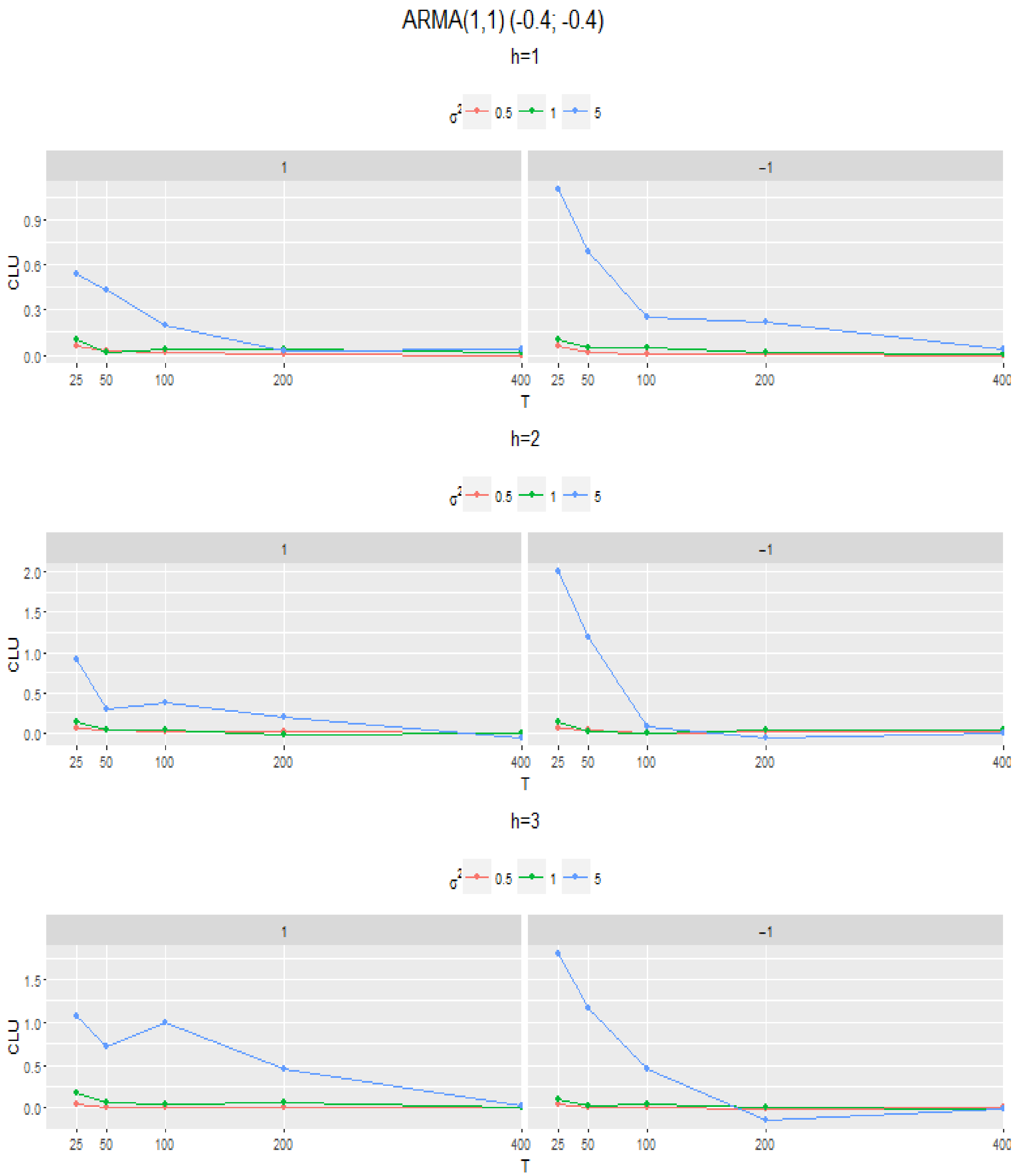

Figure 18. CLU of σ^2 by T for ARMA(1,1), $(\phi_1; \theta_1) = (-0.4; -0.4)$

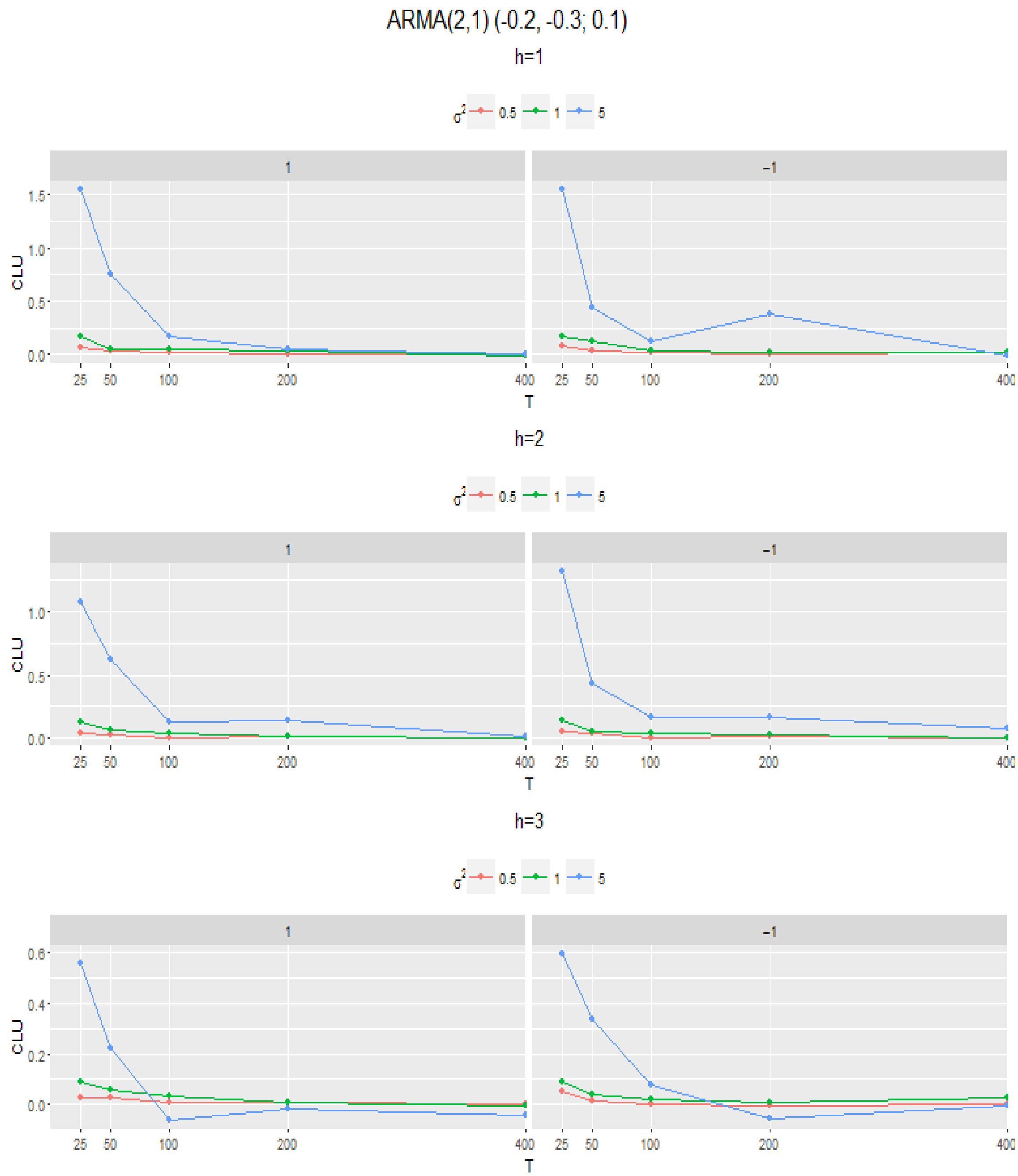

Figure 19. CLU of σ^2 by T for ARMA(2,1), $(\phi_1, \phi_2; \theta_1) = (-0.2, -0.3; 0.1)$

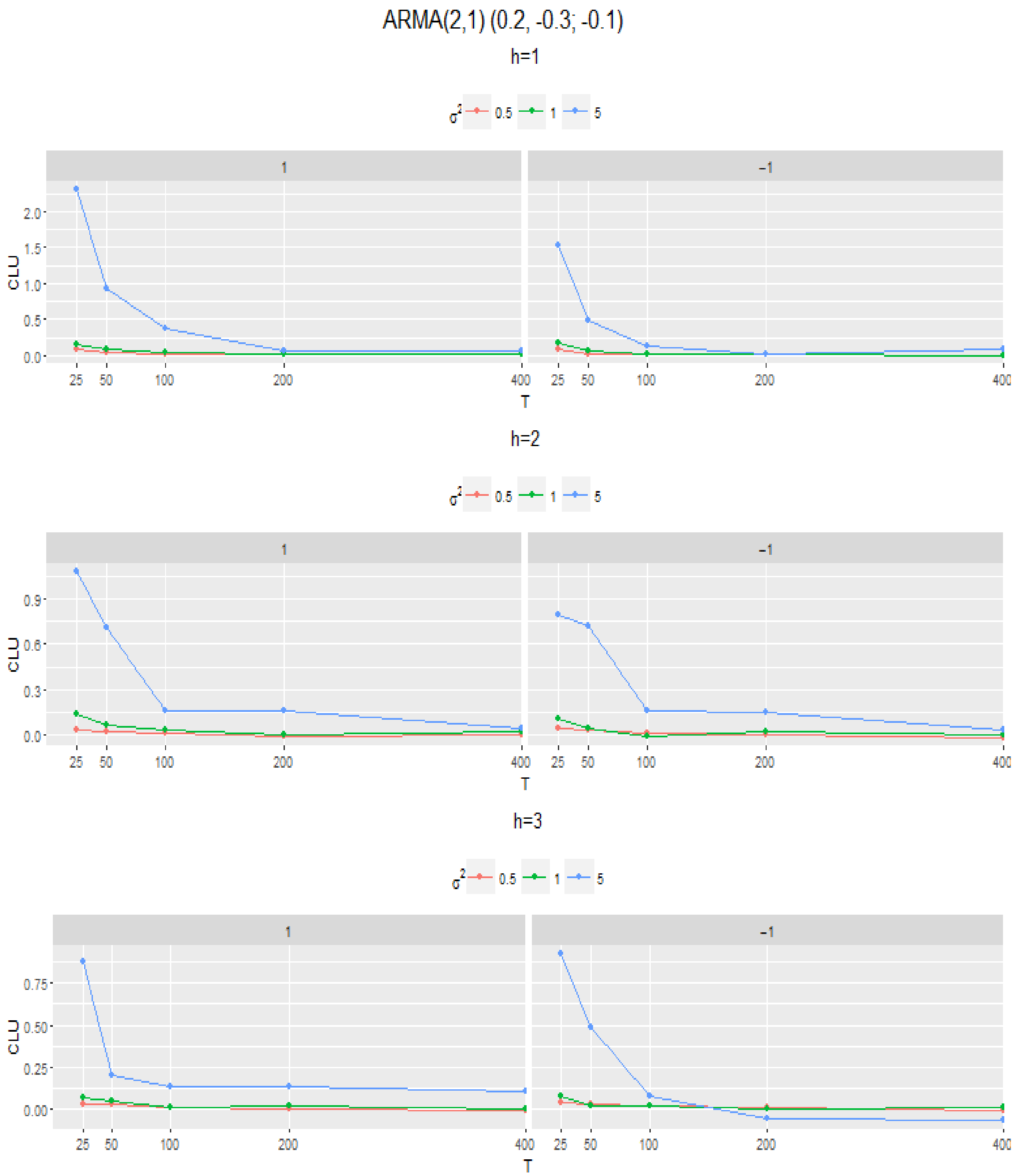

Figure 20. CLU of σ^2 by T for ARMA(2,1), $(\phi_1, \phi_2; \theta_1) = (0.2, -0.3; -0.1)$

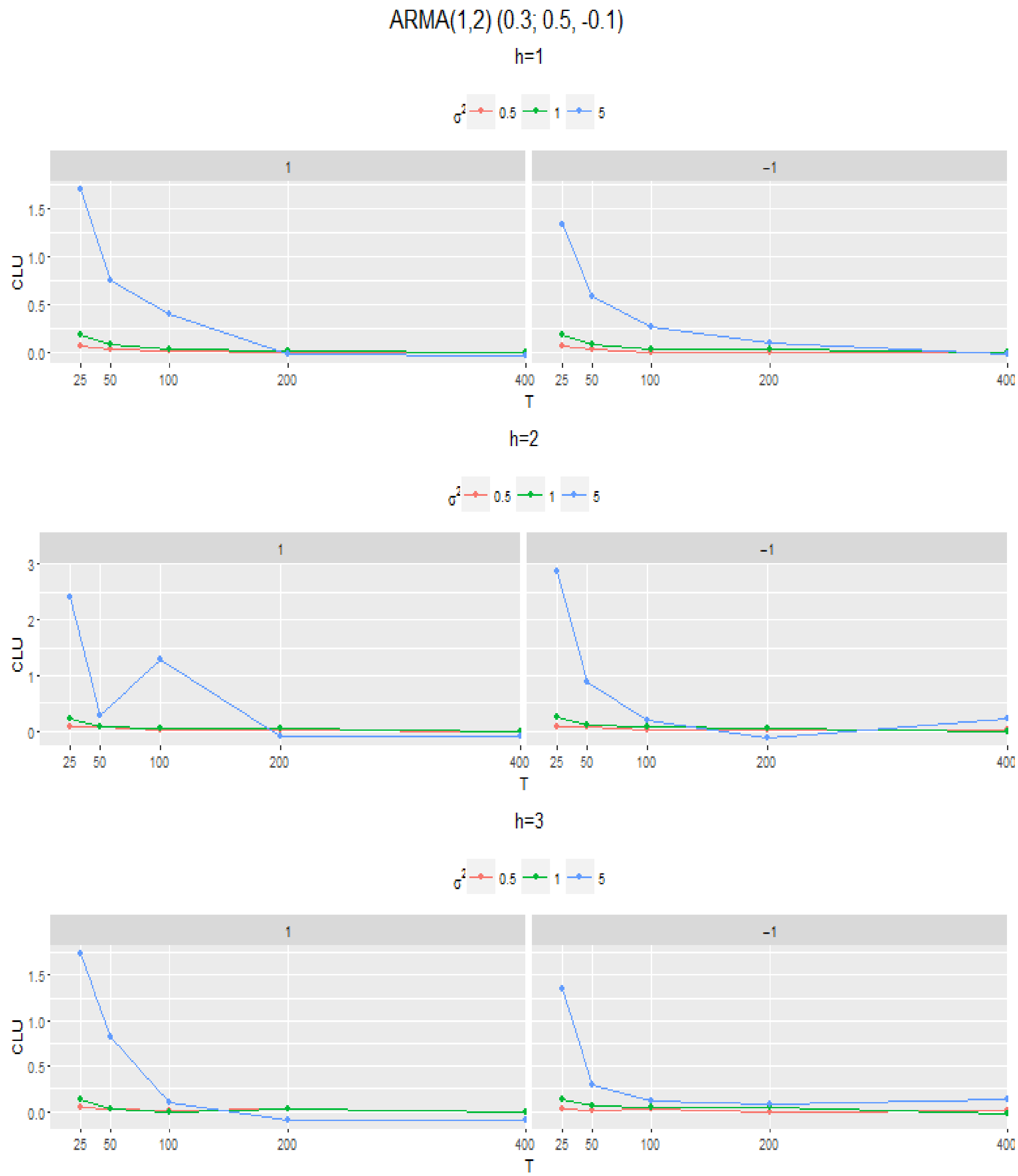

Figure 21. CLU of σ^2 by T for ARMA(1,2), $(\phi_1; \theta_1, \theta_2) = (0.3; 0.5, -0.1)$

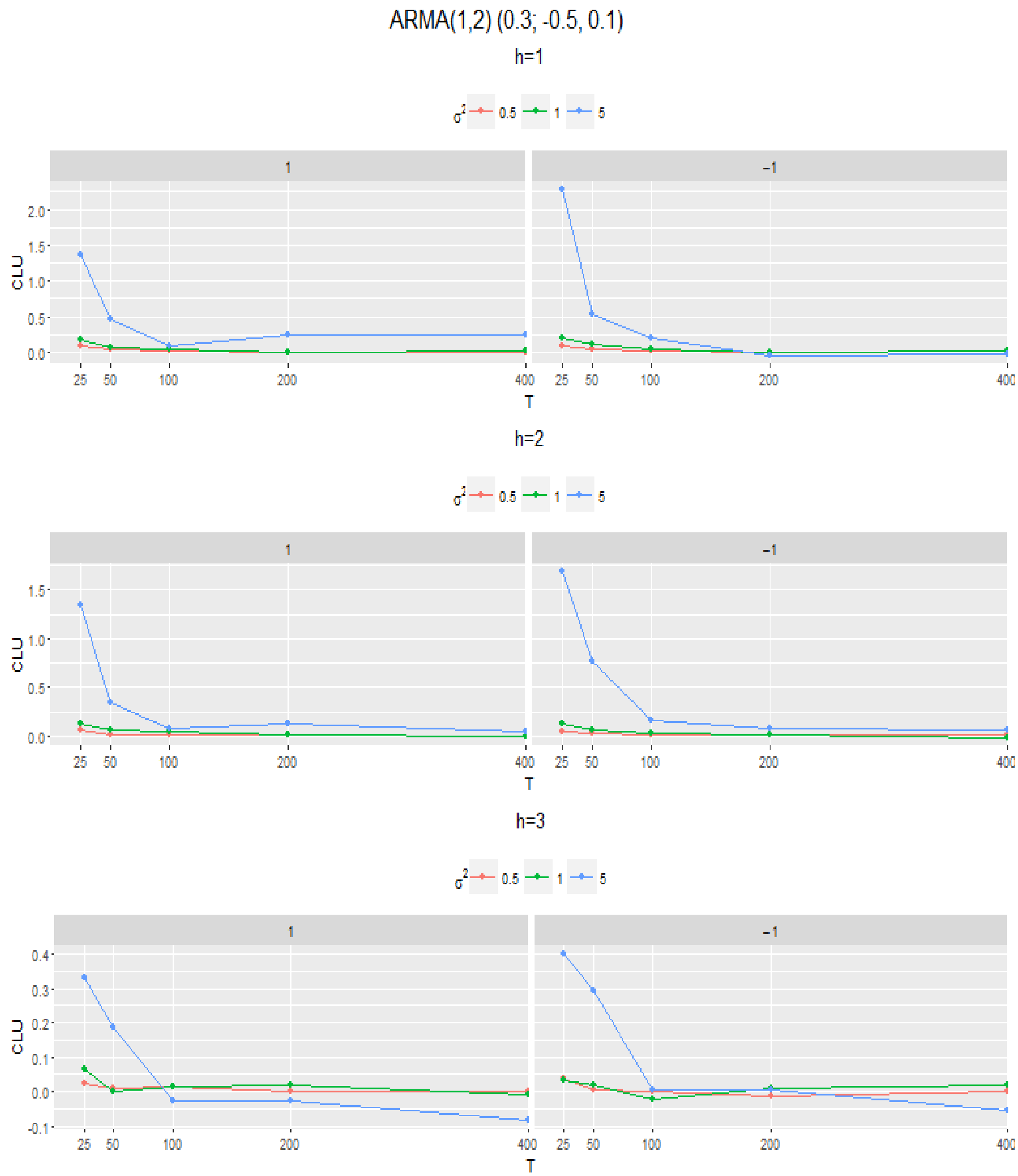

Figure 22. CLU of σ^2 by T for ARMA(1,2), $(\phi_1; \theta_1, \theta_2) = (0.3; -0.5, 0.1)$

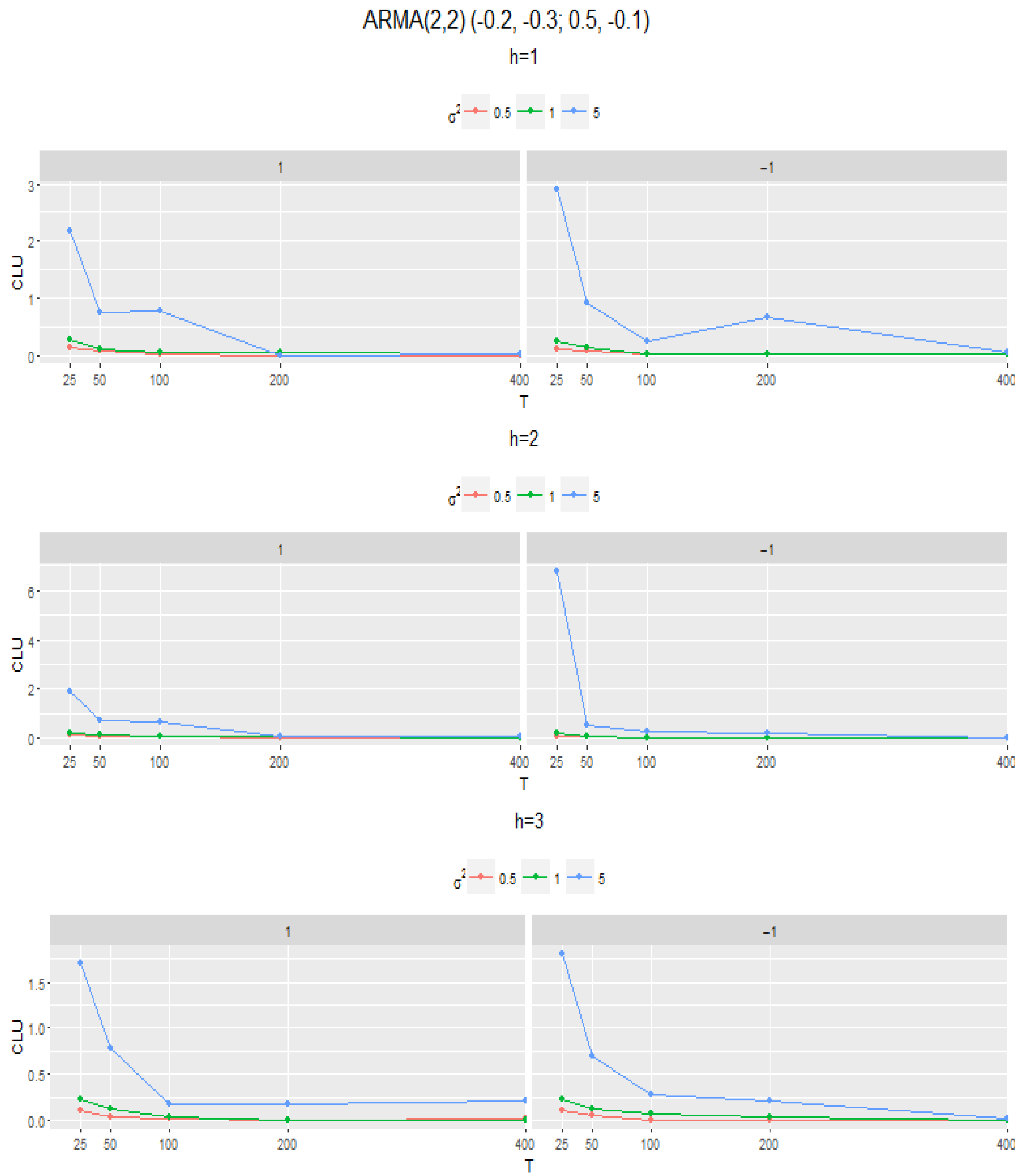

Figure 23. CLU of σ^2 by T for ARMA(2,2), $(\phi_1, \phi_2; \theta_1, \theta_2) = (-0.2, -0.3; 0.5, -0.1)$

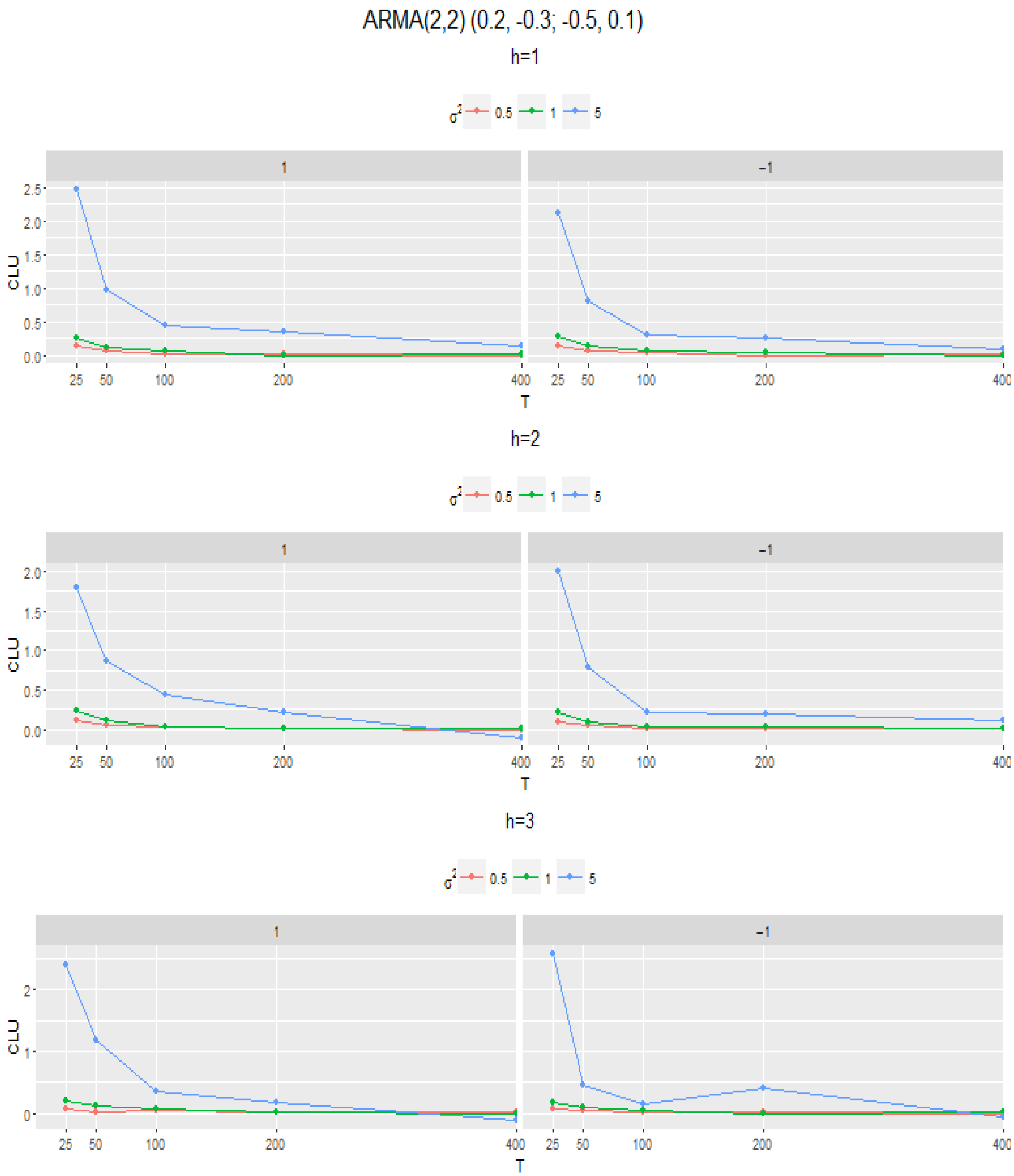

Figure 24. CLU of σ^2 by T for ARMA(2,2), $(\phi_1, \phi_2; \theta_1, \theta_2) = (0.2, -0.3; -0.5, 0.1)$

Research Question 3

For research question 3, the author explored how the condition of linex unbiasedness differs when the forecast step increases. Figures 25 through 34, and Figures 73 through 90 show the comparison of the CLU values by series length at different h-step-ahead, for all levels of variances and shape parameter, when the model and parameter were fixed. For the purpose of demonstration, this section only displayed all the figures of AR(p) models, which are Figures 25 through 34. All the figures of MA(q) and ARMA(p,q) models are displayed in Appendix C.

For the AR(1), $\phi_1 = -0.8$ and 0.8, $h = 3$ seemed to have the CLU values further from zero than $h = 2$ and $h = 1$ when T was small, such as 25, 50 and 100, for any σ^2. For the MA(q), most of the models seemed to have higher CLU values when $h = 1$, compared to when $h = 2$ and $h = 3$, while $T = 25$ and 50, for any σ^2. The CLU values also seemed to be higher when $h = 1$ for some ARMA(p,q), while T was small, for all σ^2. But the patterns were not consistent as T increased. Otherwise, there was no universal pattern to describe if the CLU values were always further from zero or closer to zero at any level of h. No α effect was observed.

An interesting pattern was observed. For fixed h, as T increased, similar to $\sigma^2 = 5$ case in the previous section, the CLU values tended to decline and lines had a less curvy L shape pattern. This pattern was clear for the ARMA(p,q), and higher order AR(p) and MA(q) models. For the AR(1) and MA(1), such pattern was less certain and the lines fluctuated more. Although lines did not clearly have an L shape for AR(1) and MA(1), the scale on y axes indicated the range of the

fluctuation was not very wide, for example, AR(1), $\phi_1 = 0.2$, the y axes had a range of 0.03 for $\sigma^2 = 0.5$, 0.06 for $\sigma^2 = 1$, and 0.5 for $\sigma^2 = 5$. Besides, from the figures displayed for this research question, for fixed model, it was observable that the lines for different h-step-ahead can have the same pattern. This can be seen in almost every figure, see the figures of this section for examples.

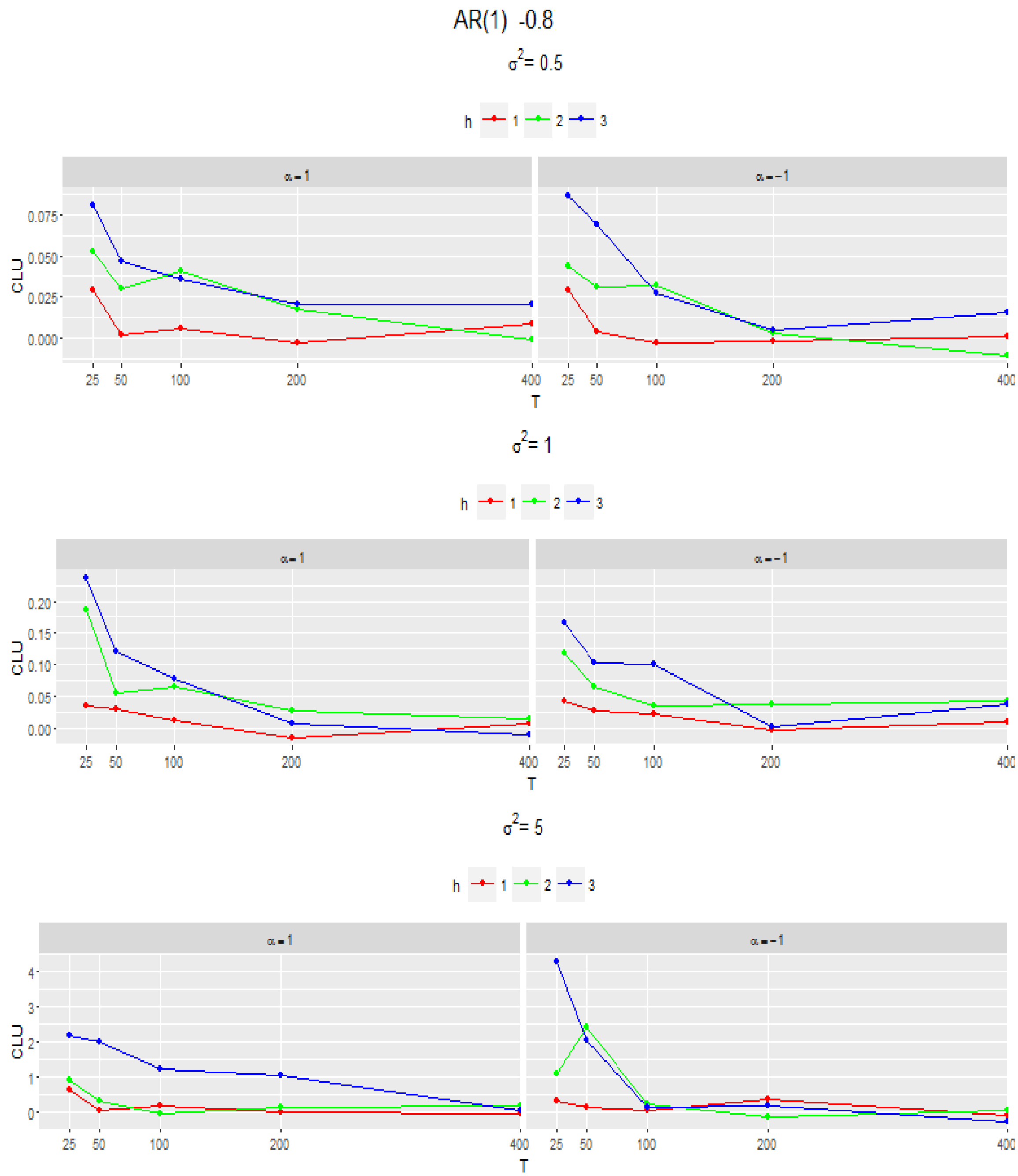

Figure 25. CLU of h by T for AR(1), $\phi_1 = -0.8$

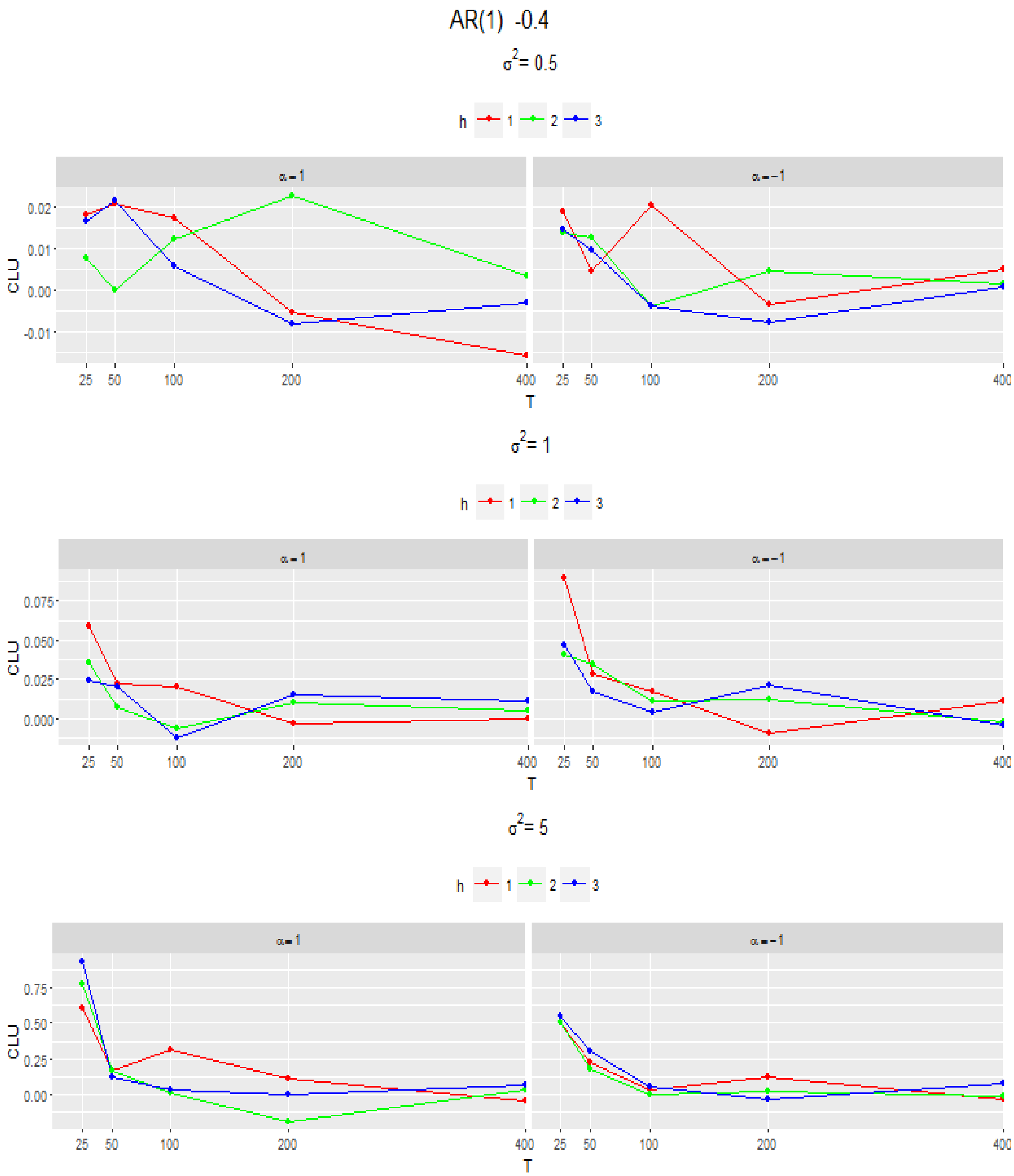

Figure 26. CLU of h by T for AR(1), $\phi_1 = -0.4$

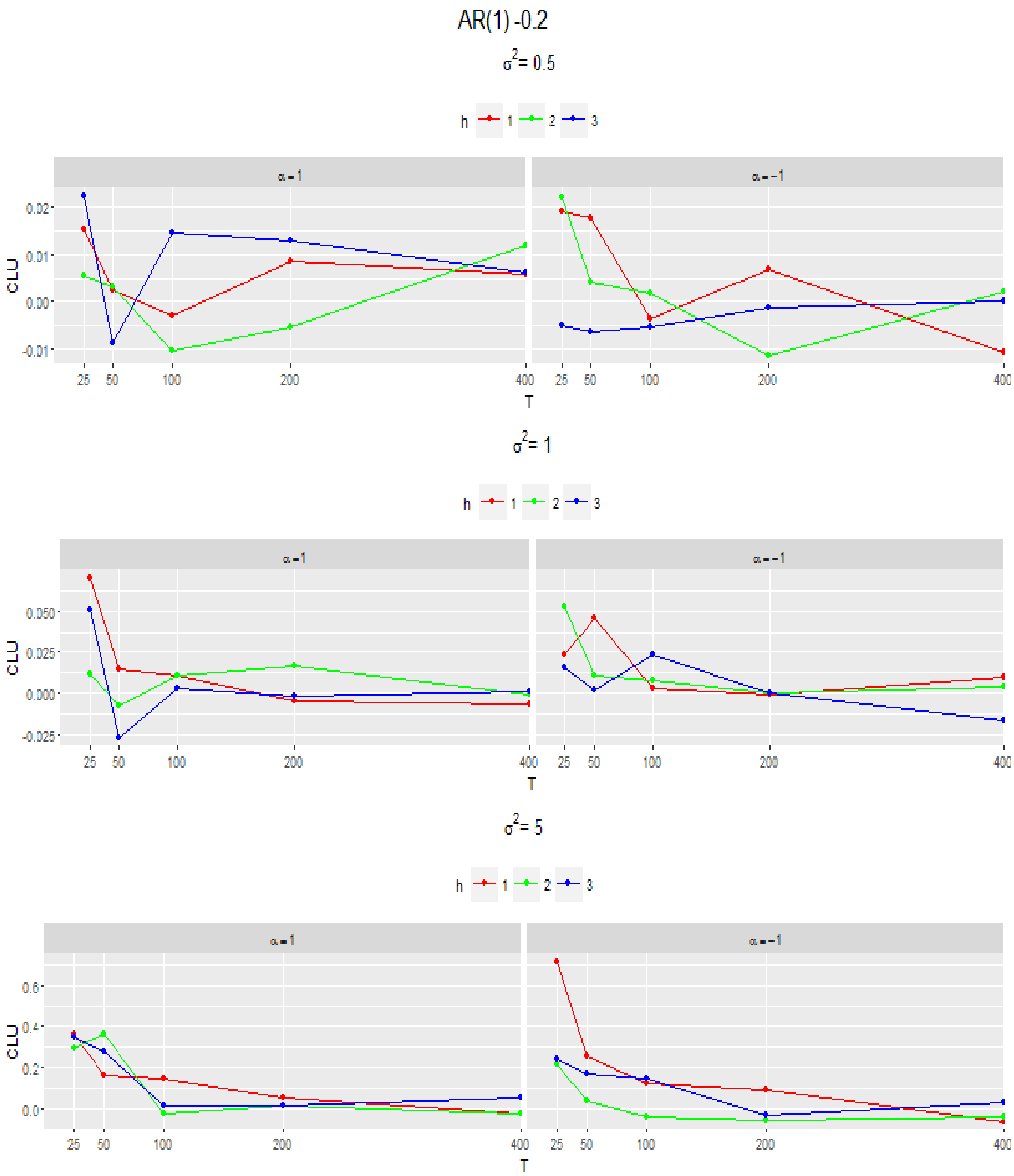

Figure 27. CLU of h by T for AR(1), $\phi_1 = -0.2$

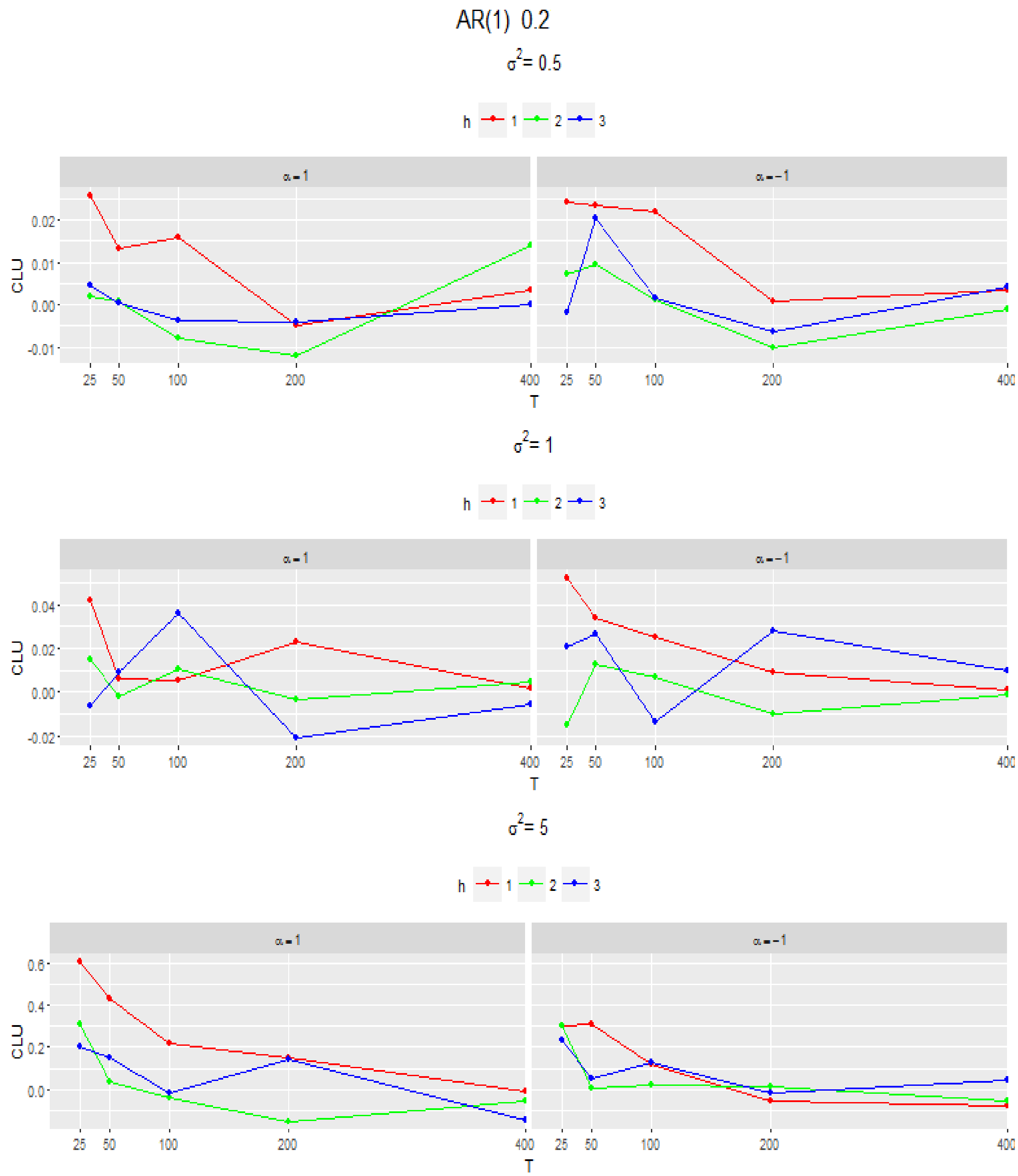

Figure 28. CLU of h by T for AR(1), $\phi_1 = 0.2$

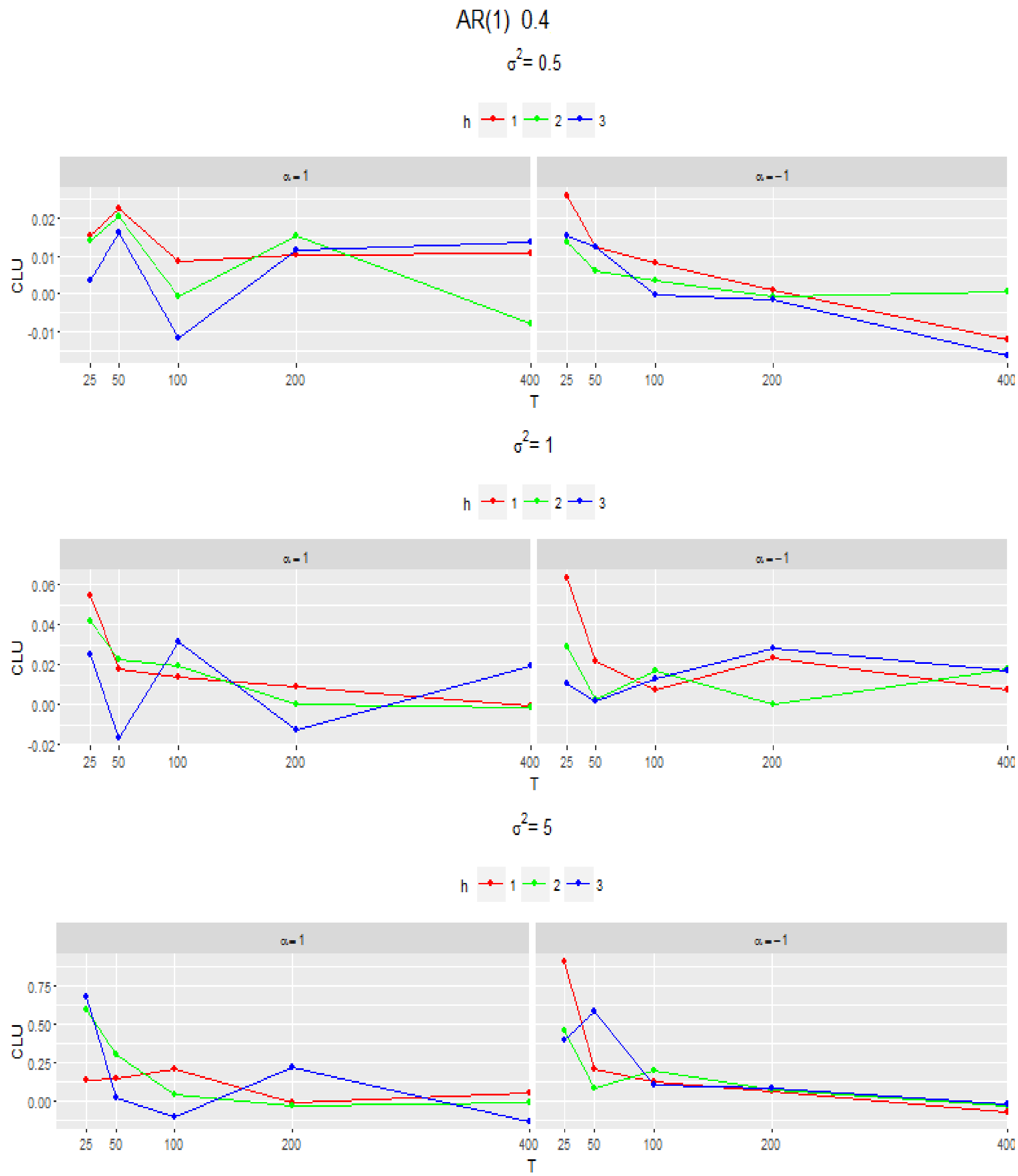

Figure 29. CLU of h by T for AR(1), $\phi_1 = 0.4$

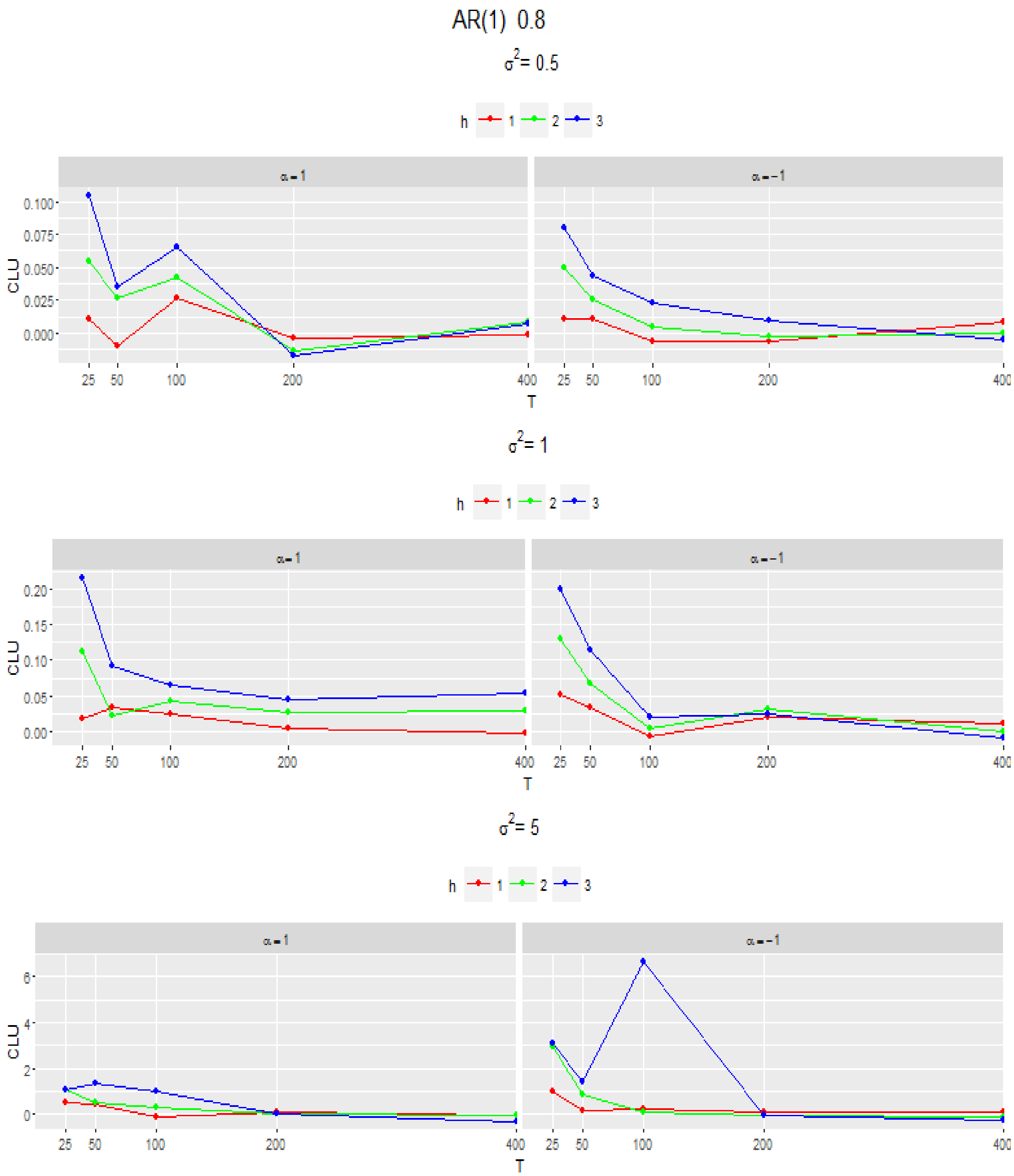

Figure 30. CLU of h by T for AR(1), $\phi_1 = 0.$)

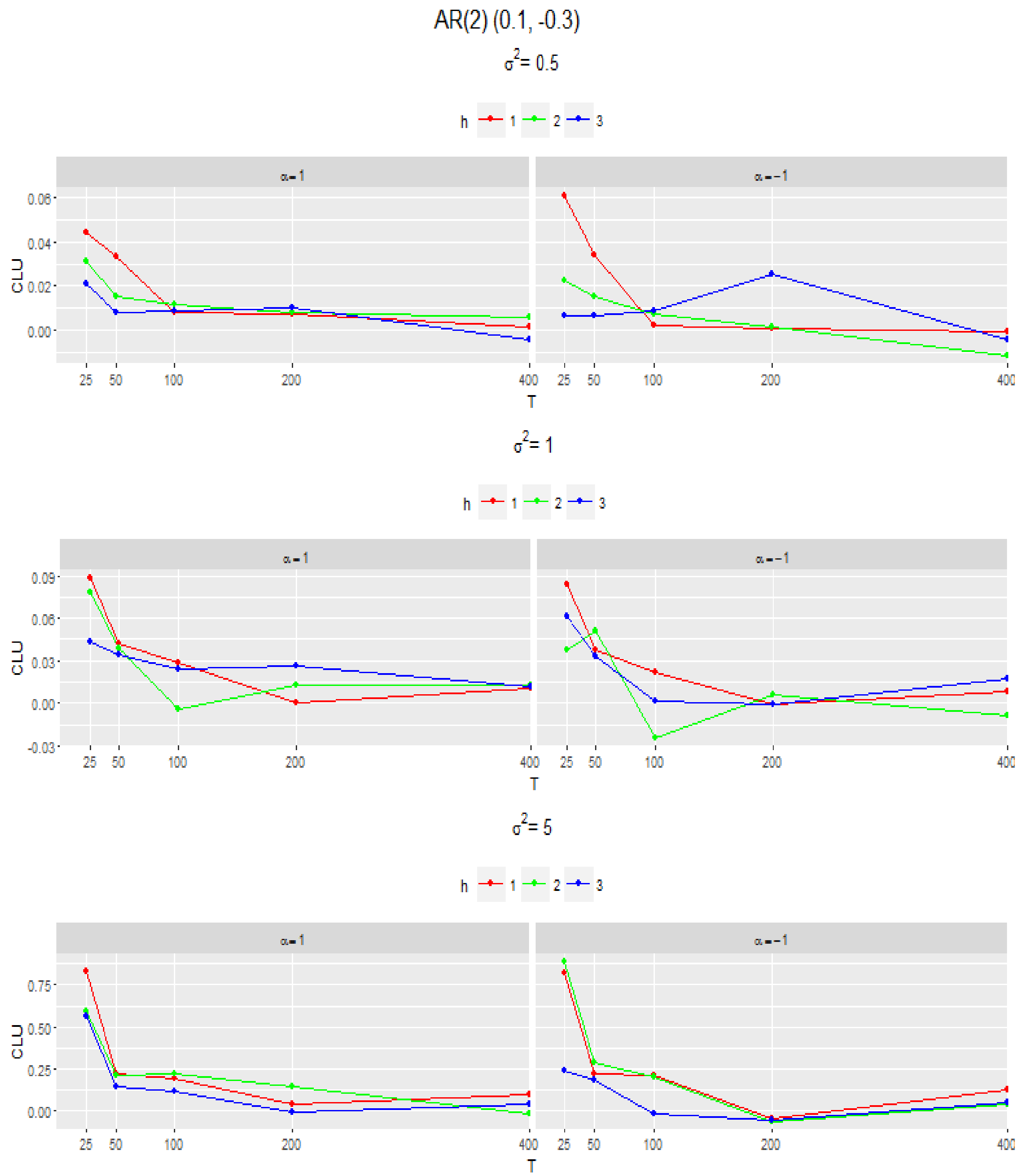

Figure 31. CLU of h by T for AR(2), $(\phi_1, \phi_2) = (0.1, -0.3)$

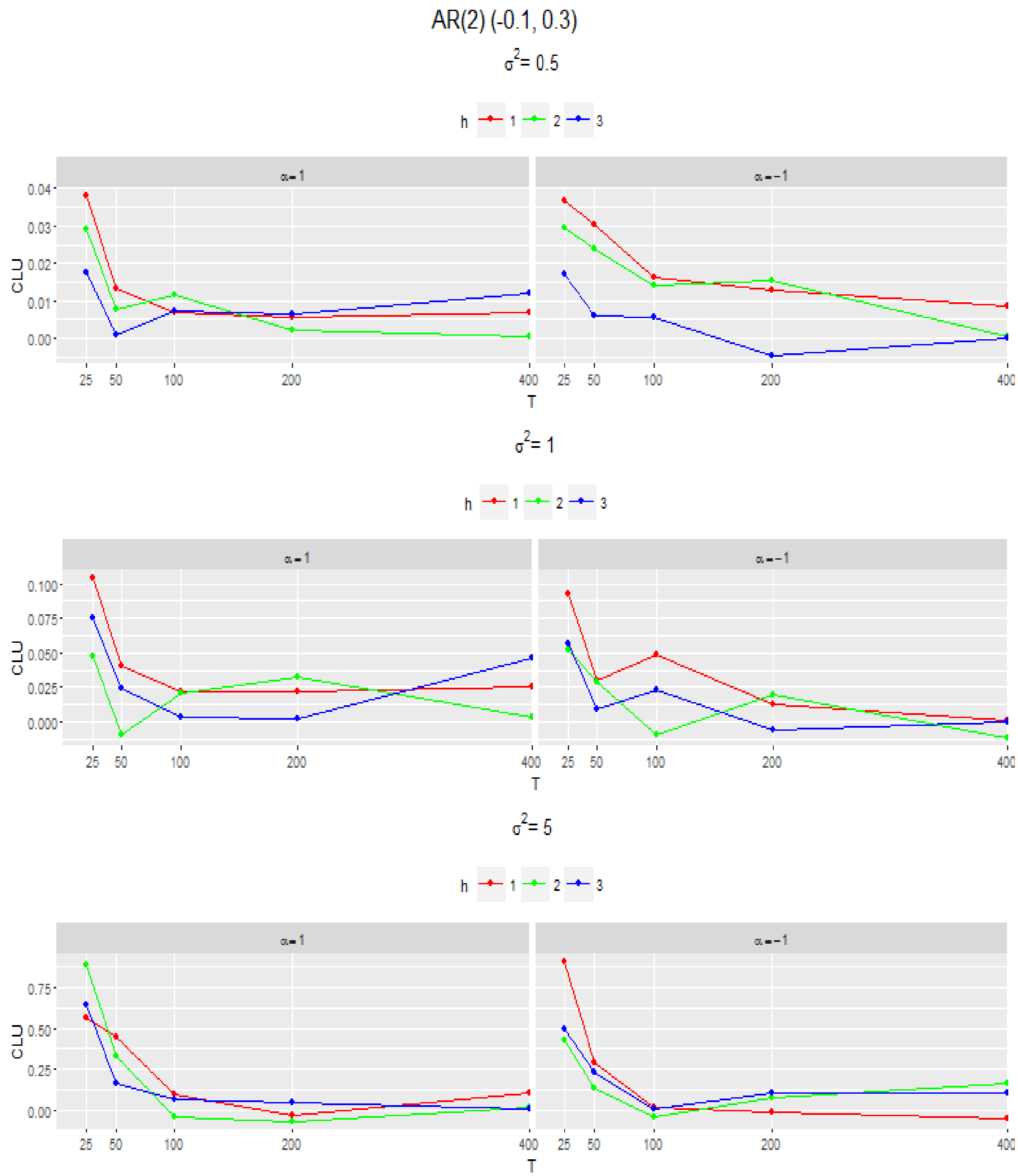

Figure 32. CLU of h by T for AR(2), $(\phi_1, \phi_2) = (-0.1, 0.3)$

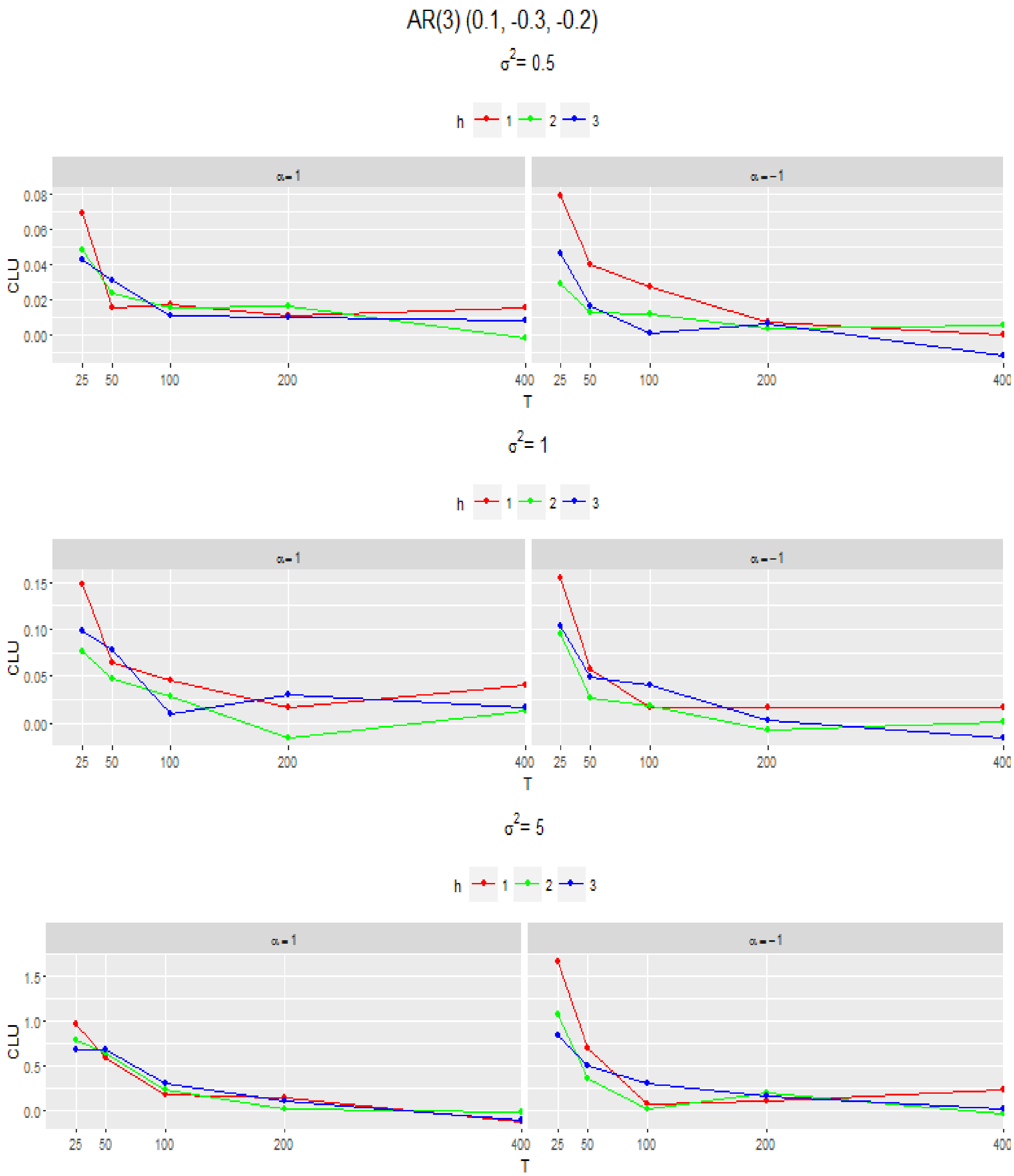

Figure 33. CLU of h by T for AR(3), $(\phi_1, \phi_2, \phi_3) = (0.1, -0.3, -0.2)$

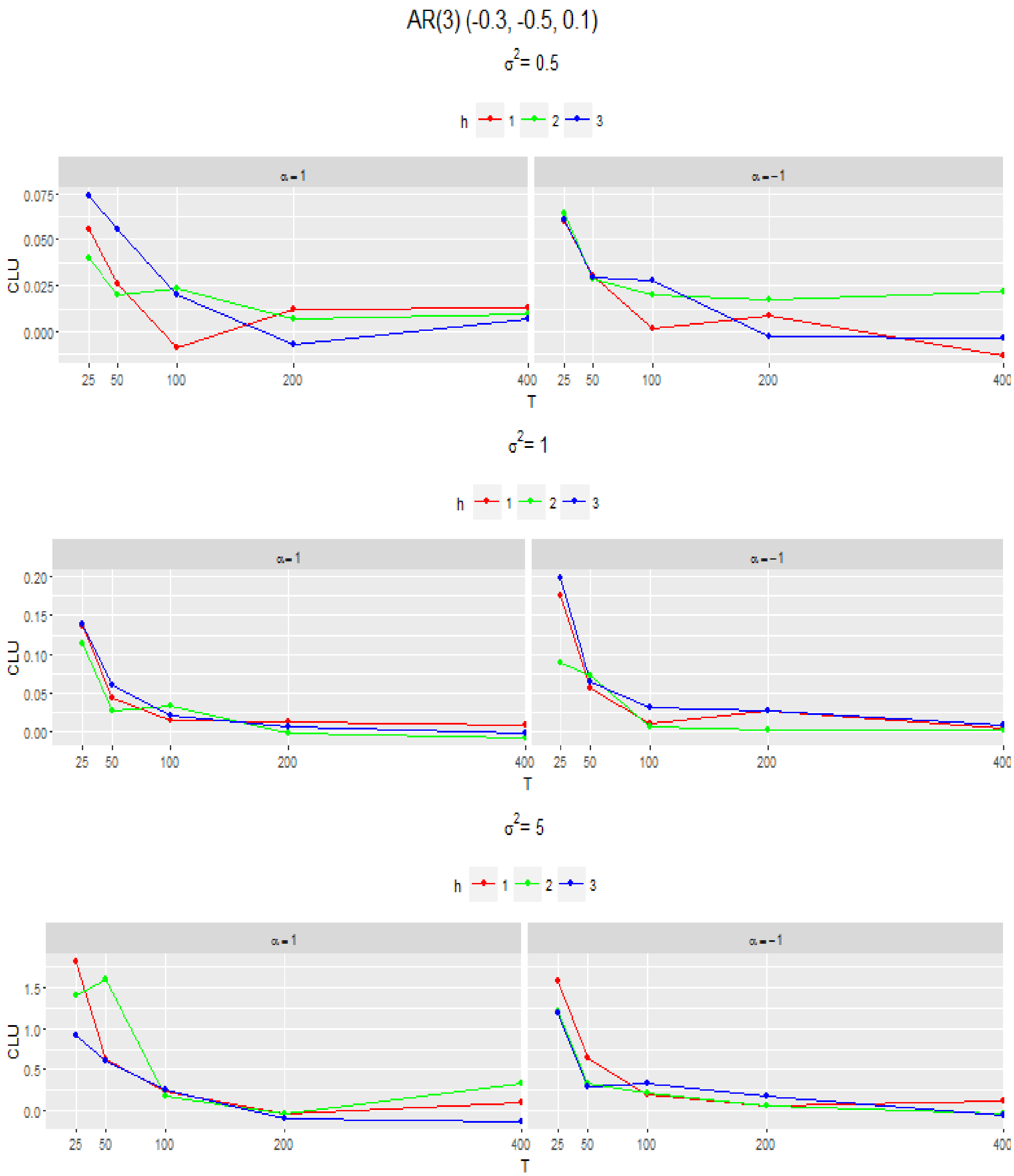

Figure 34. CLU of h by T for AR(3), $(\phi_1, \phi_2, \phi_3) = (-0.3, -0.5, 0.1)$

Research Question 4

For research question 4, the author assessed how the condition of linex unbiasedness differs when the length of series increases. Figures 35 through 43 show a pattern that for each model, at fixed variance, as series length increased, the CLU values decreased and approached zero, for all levels of h-step-ahead. The pattern was consistent for both α levels. As stated in the previous sections, all patterns were observed when series length was small, and patterns became less apparent as the series length increased. Even though the AR(p) and MA(q) models had the CLU values closer to zero compared to ARMA (p,q) models, as series length increased from 25, 50 and 100 to 200 and 400, all models had the CLU values approaching zero.

This phenomenon happened at all levels of variance, shape parameter, and forecast step. Recall, in research question 2, when the variance was big, the CLU values were further from zero, however, as T increases, the CLU values became closer to zero, regardless of the complexity of the time series models and the shape parameters. In research question 3, for all h levels, the CLU values decreased and approached zero as T increased.

Also, in research questions 2 and 3, an L shape pattern was mentioned. Although most of the elbow of an L shape seemed to occur at series length 100, for a more conservative analysis, the pattern seemed to be stable after series length 200. When series length was small, such as 25, 50 and 100, most of the CLU had positive

values, as series length increased to 200 and 400, the CLU values decreased and had a more even portion of positive and negative numbers. No shape parameter effect was found in the figures for this research question.

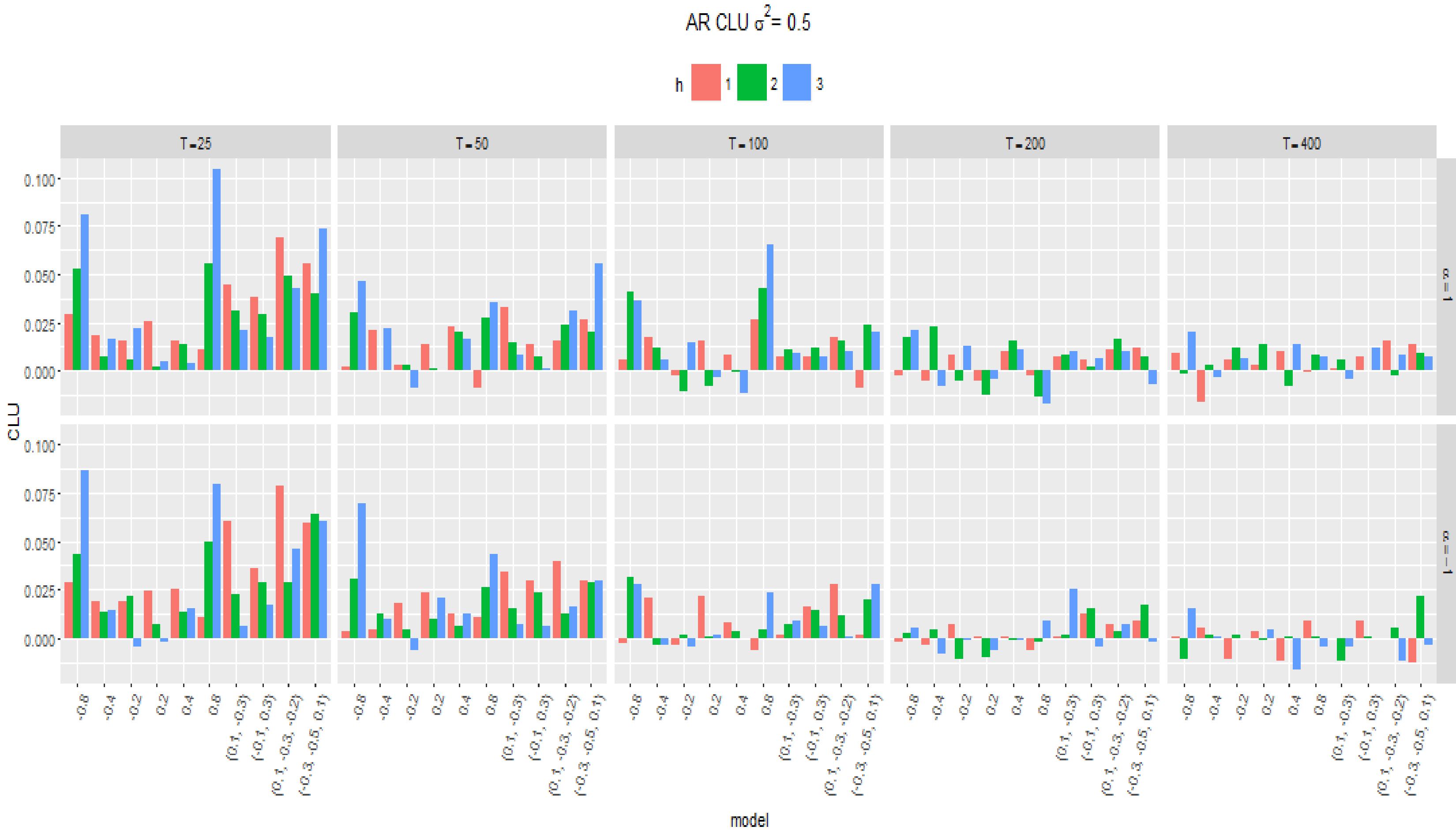

Figure 35. CLU of AR model at $\sigma^2 = 0.5$

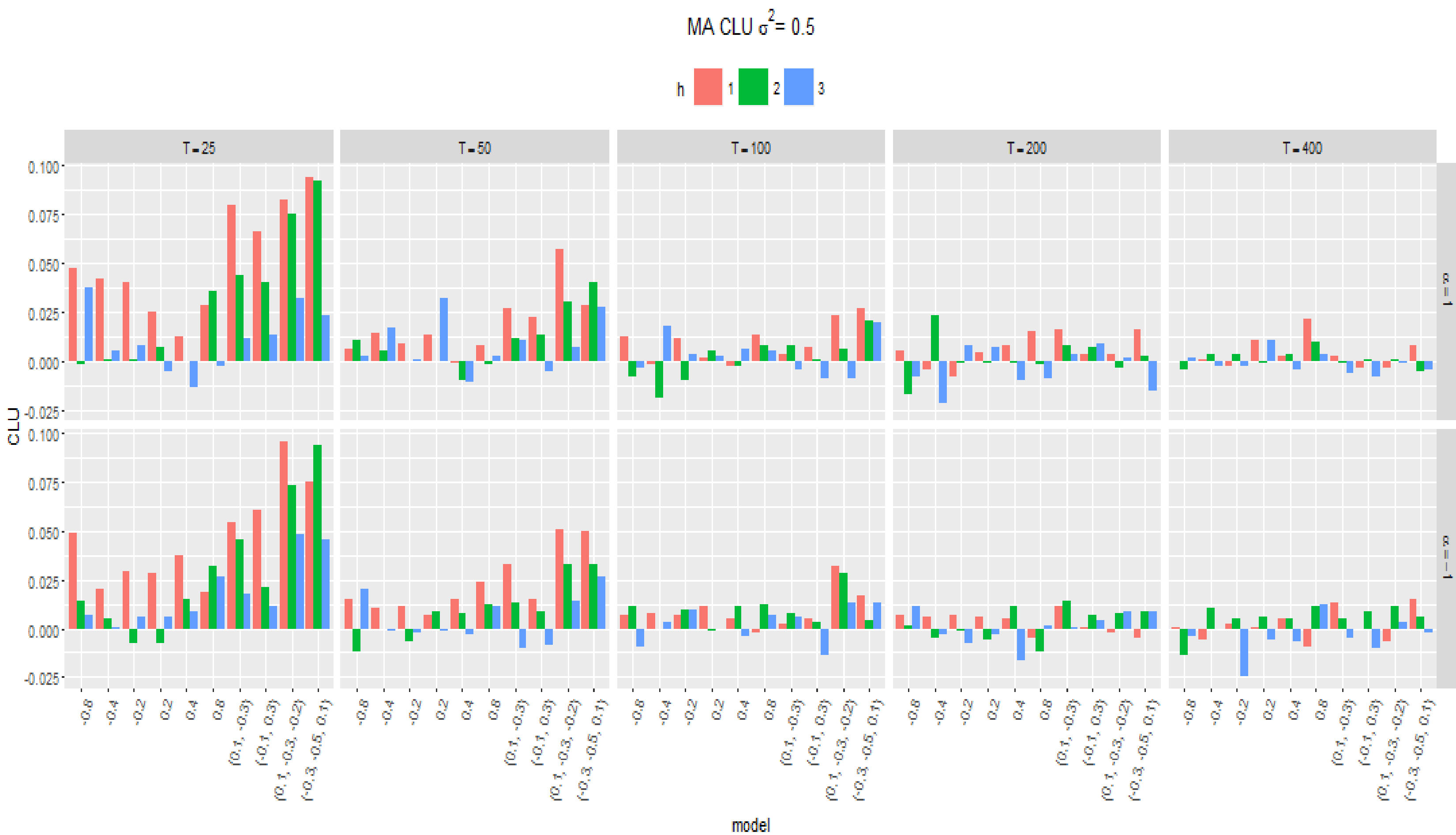

Figure 36. CLU of MA model at $\sigma^2 = 0.5$

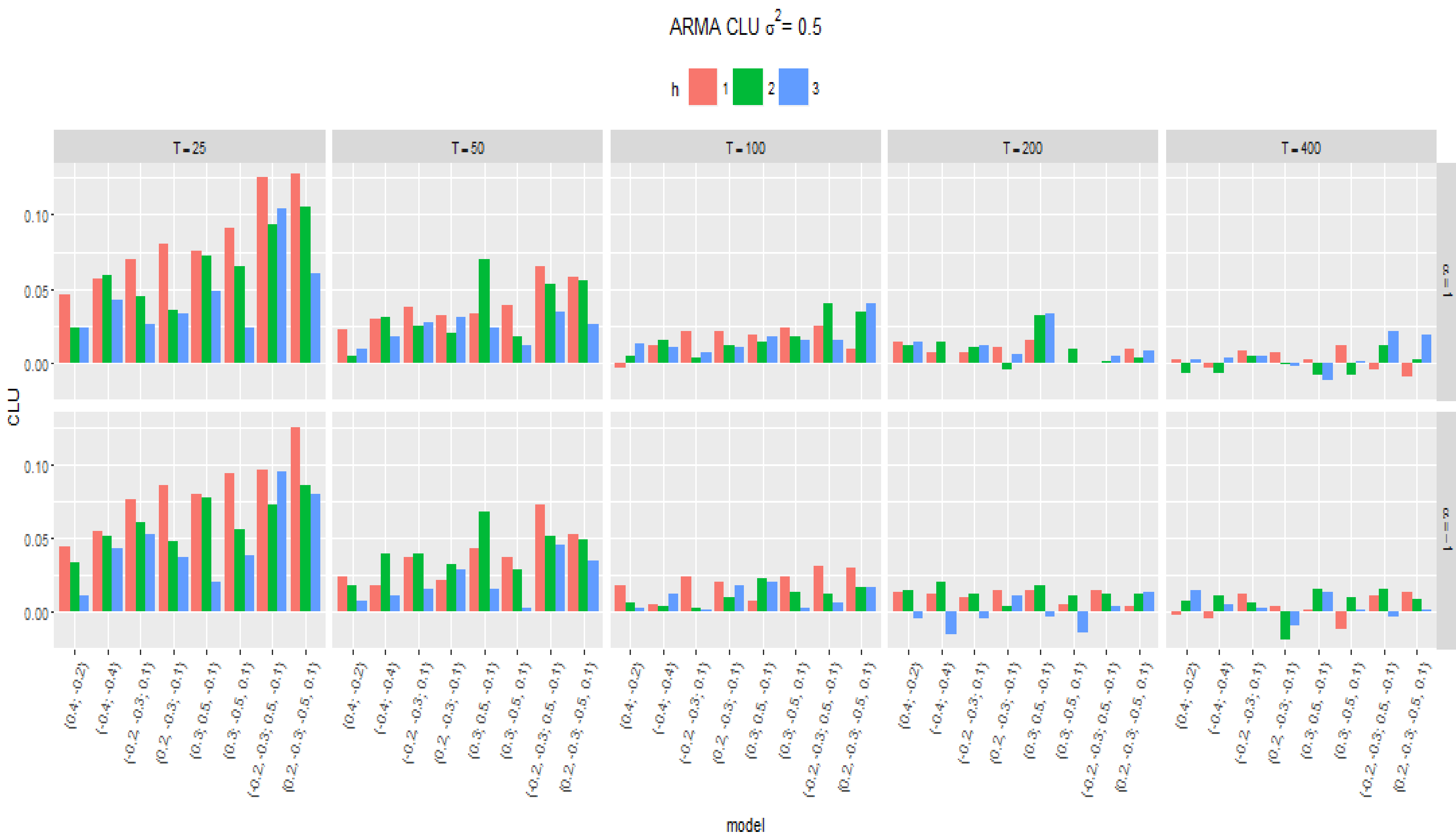

Figure 37. CLU of ARMA model at $\sigma^2 = 0.5$

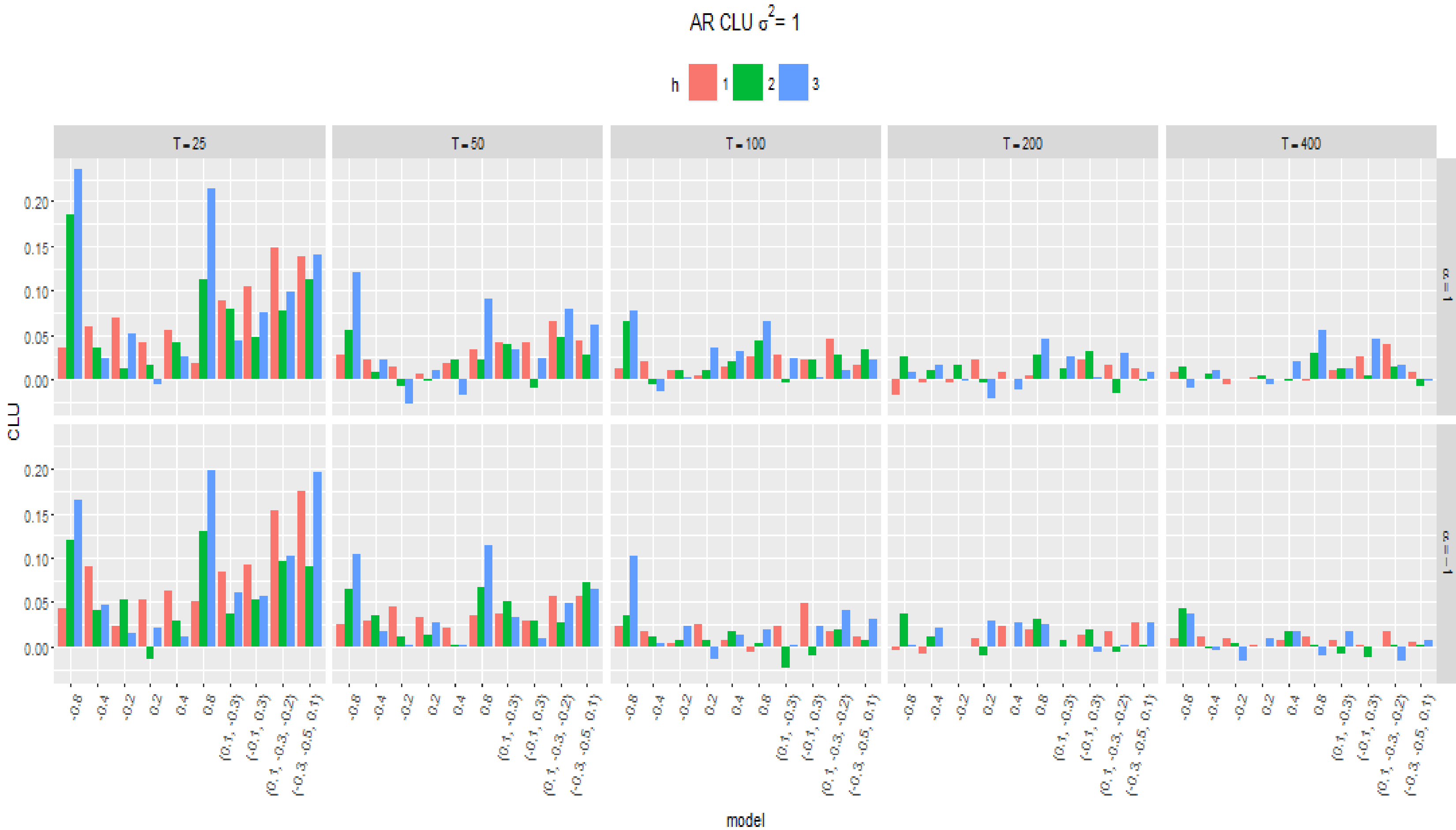

Figure 38. CLU of AR model at $\sigma^2 = 1$

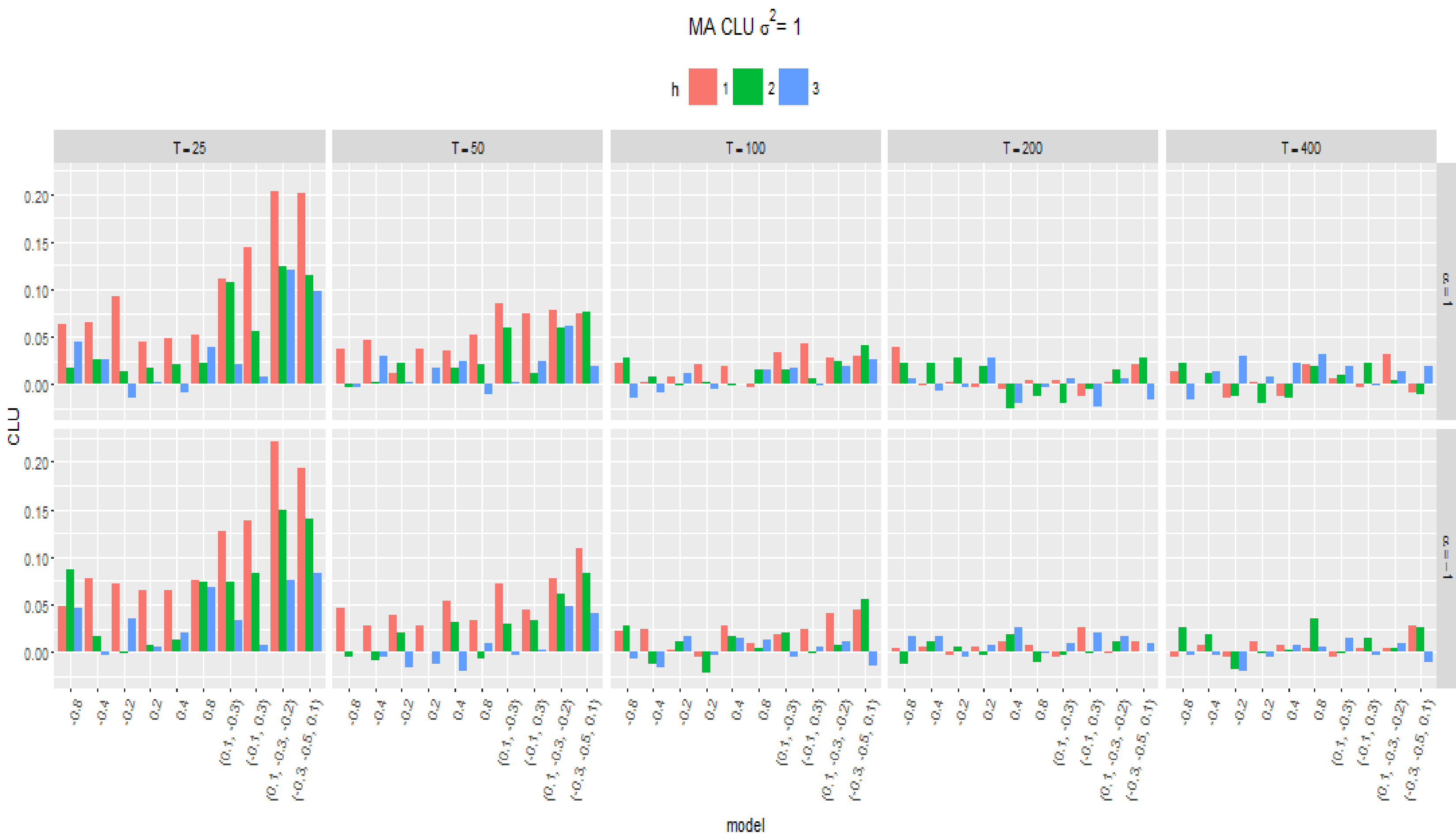

Figure 39. CLU of MA model at $\sigma^2 = 1$

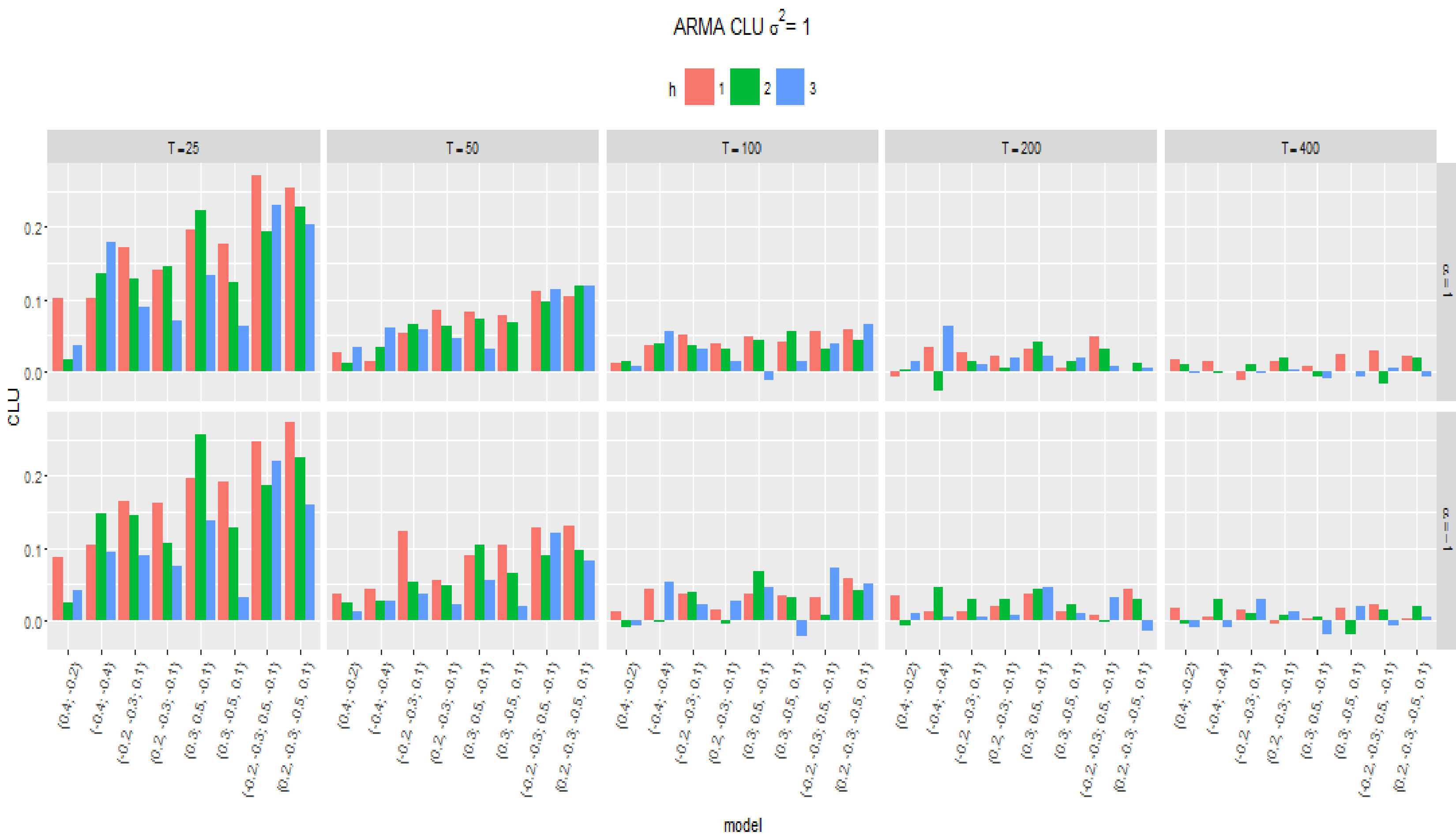

Figure 40. CLU of ARMA model at $\sigma^2 = 1$

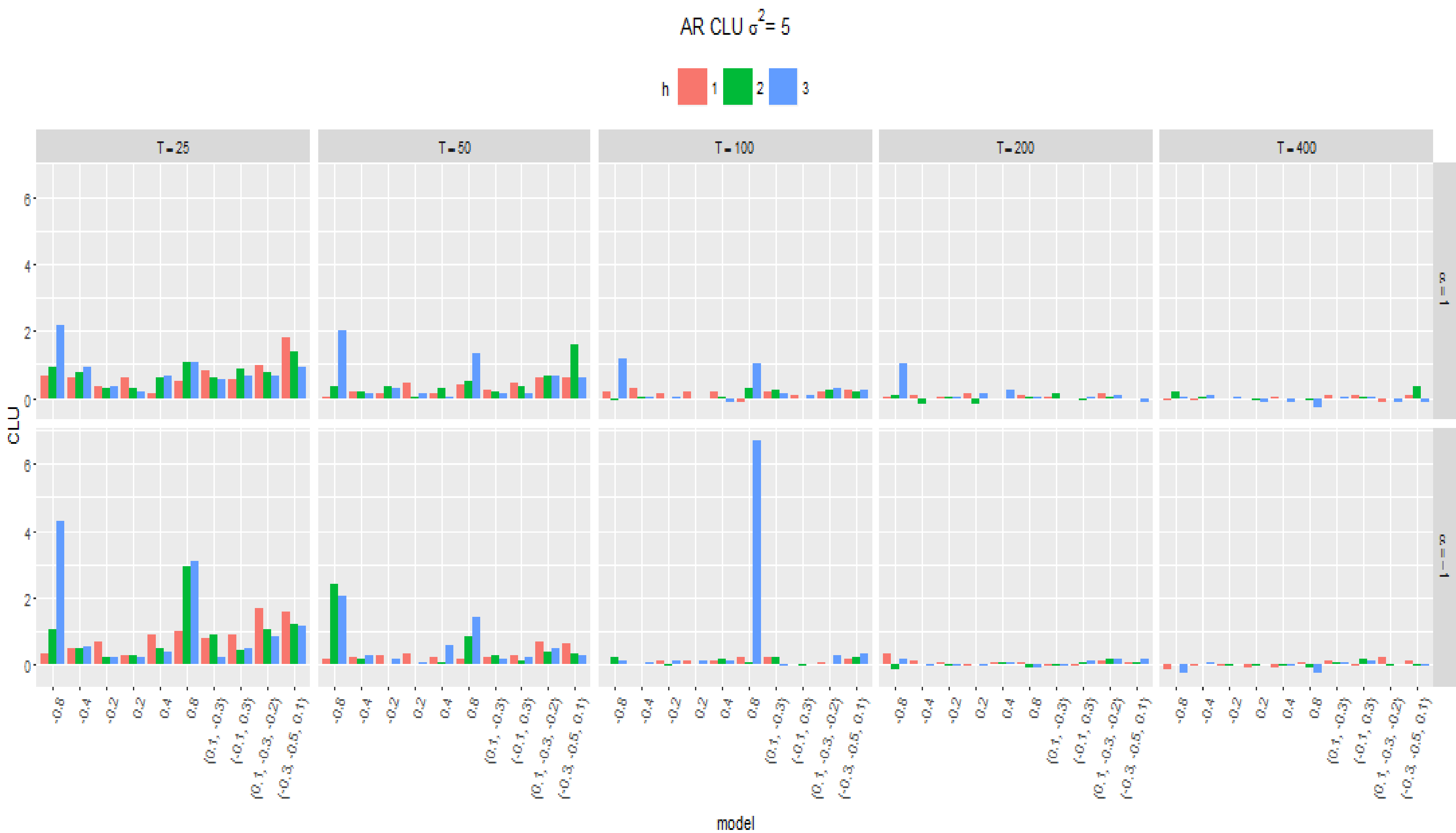

Figure 41. CLU of AR model at $\sigma^2 = 5$

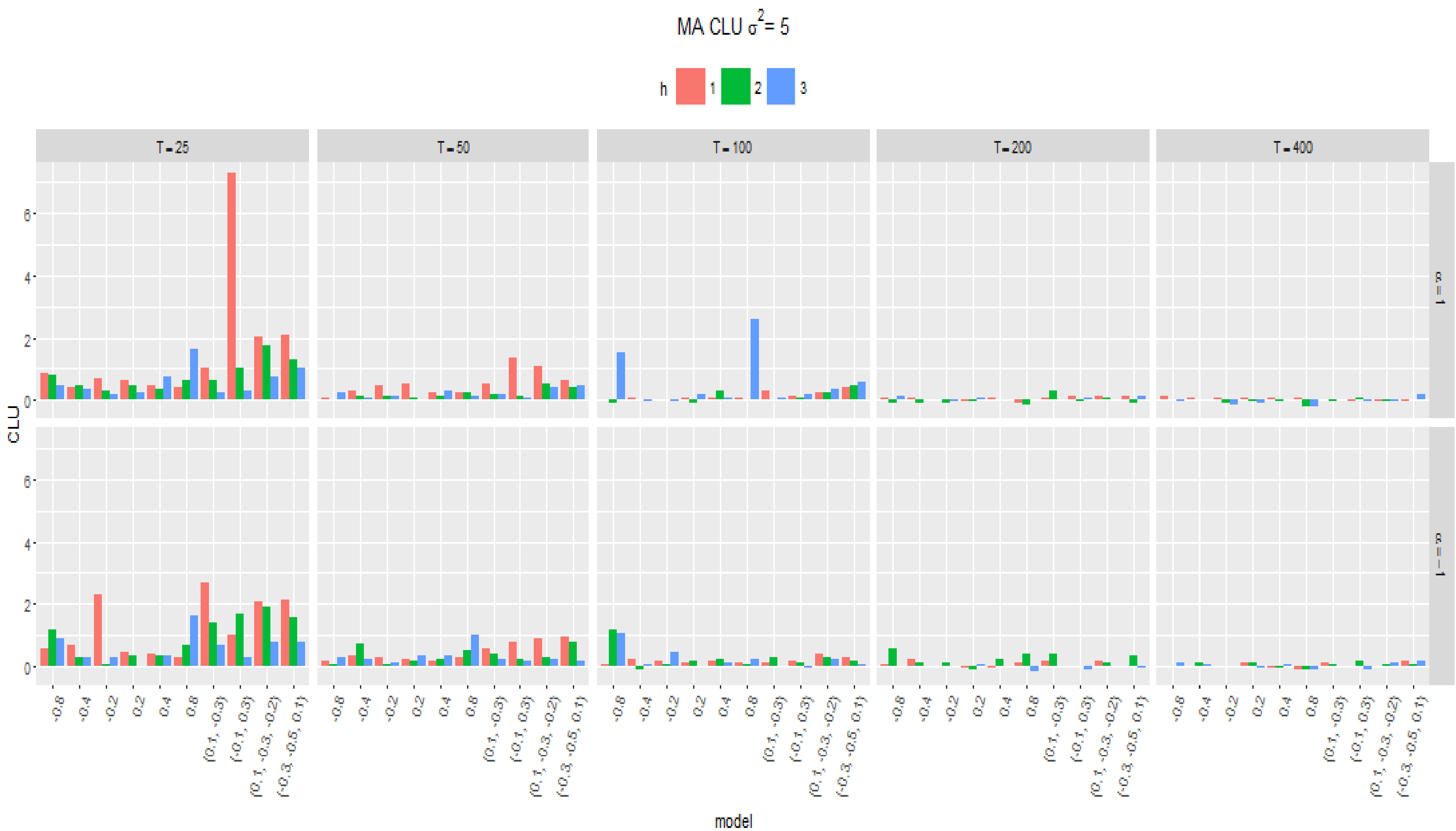

Figure 42. CLU of MA model at $\sigma^2 = 5$

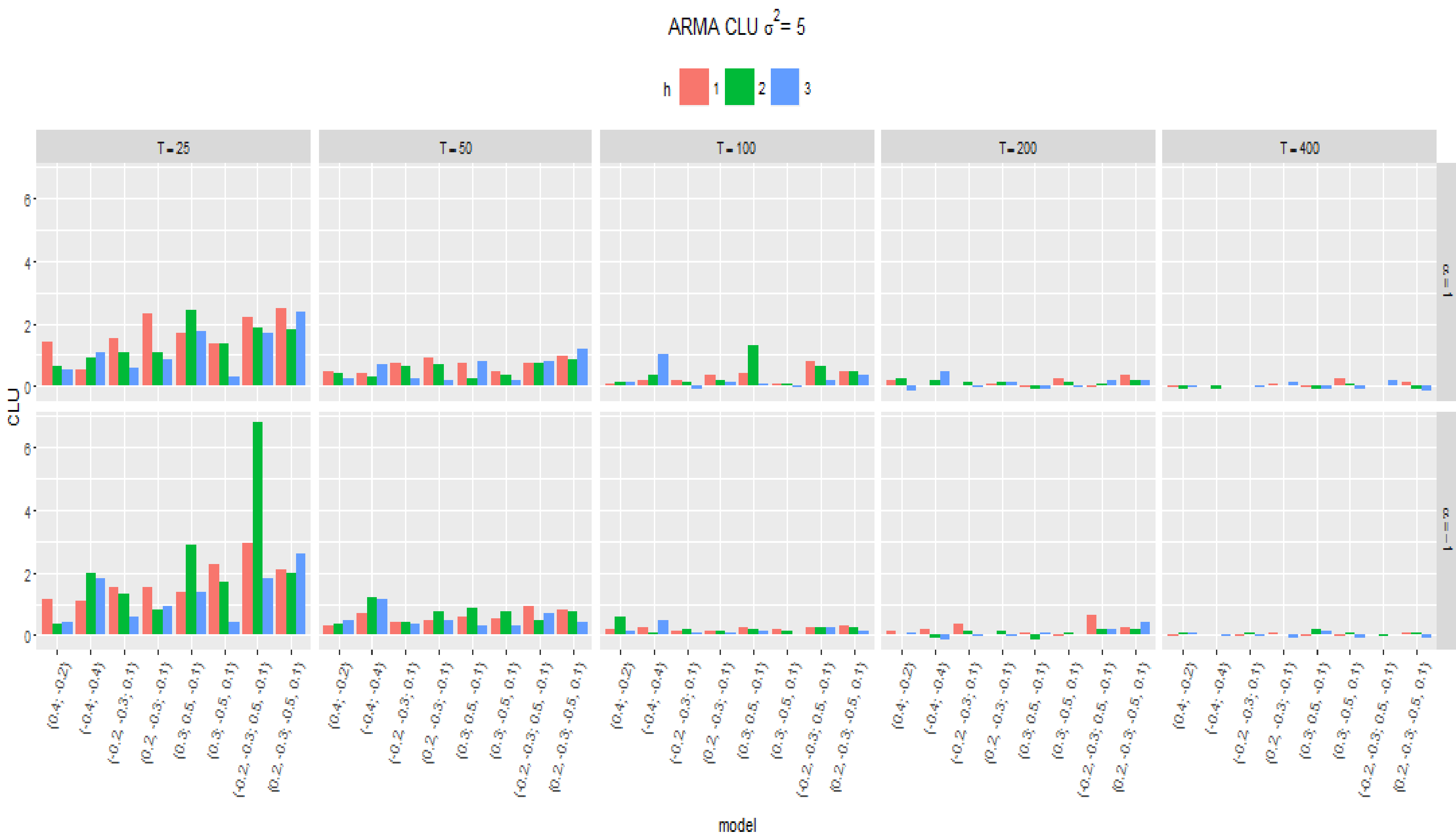

Figure 43. CLU of ARMA model at $\sigma^2 = 5$

Research Question 5

For research question 5, the author compared the condition of linex unbiasedness when the linex loss function shape parameter is positive 1 or negative 1. Based on all the tables and figures shown in the previous sections, no significant difference was observed when $\alpha = 1$ versus $\alpha = -1$. As mentioned previously, $\alpha = 1$ and -1 reflects the direction of asymmetry for an underprediction and overprediction. According to the results of this research discussed in the previous sections, the condition of linex unbiasedness was not affected by the sign of alpha. The positive and negative sign of α had equal performance in the simulation.

Research Question 6

For research question 6, the author analyzed the extra risk associated with the estimated linex unbiased predictors. Table 3 displays the theoretical risk of $\hat{Z}_{T+h}$, at all conditions. The theoretical risk of $\hat{Z}_{T+h}$ showed that the risk increased as the h-step-ahead increased, and also as the variance increased. The empirical risk of $\hat{\hat{Z}}_{T+h}$ can be found in Tables 4 through 12. The behavior of the empirical risk of $\hat{\hat{Z}}_{T+h}$ agreed with the theoretical risk of $\hat{Z}_{T+h}$ at almost any size of series length; when $\sigma^2 = 5$, the risk of $\hat{\hat{Z}}_{T+h}$ performed more closely to the risk of $\hat{Z}_{T+h}$ as the series length was larger.

In order to provide a thorough answer for this research question, relative efficiency was calculated and compared. Efficiency measures the optimality of risk of $\hat{\hat{Z}}_{T+h}$, and relative efficiency of the risk of $\hat{\hat{Z}}_{T+h}$ and the risk of $\hat{Z}_{T+h}$ is the ratio of their risk. Tables 28 through 36, which can be found in Appendix D, display the

numerical values of relative efficiency. Since the ratio equal to 1 is desired and 99.68% of the ratios obtained fell between 0.95 and 1.6, it is fair to say the risk of $\hat{\hat{Z}}_{T+h}$ was efficient for the risk of $\hat{Z}_{T+h}$.

The relative efficiency of $\hat{\hat{Z}}_{T+h}$ and $\hat{Z}_{T+h}$ was also visualized in Figures 44 through 52. For AR(p), MA(q) and ARMA(p,q) when $T = 25$ and 50, the ratio increased as the p and q increased; the ratio also increased as the variance increased. However, as the series length increased, all the uneven patterns even out. For the MA(q) and some AR(p) and ARMA(p,q) models, the ratio seemed to be slightly bigger when $h = 1$ than when $h = 2$ and $h = 3$, when $T = 25$, and $\sigma^2 = 0.5$ and 1, but as series length increased, the pattern disappeared.

Table 3

Risk of $\hat{Z}_{T+h}$ at All Conditions

Models	Parameters	$\sigma^2 = 0.5$			$\sigma^2 = 1$			$\sigma^2 = 5$		
		h=1	h=2	h=3	h=1	h=2	h=3	h=1	h=2	h=3
AR(1)	(-0.8)	0.250	0.410	0.512	0.500	0.820	1.025	2.500	4.100	5.124
	(-0.4)	0.250	0.290	0.296	0.500	0.580	0.593	2.500	2.900	2.964
	(-0.2)	0.250	0.260	0.260	0.500	0.520	0.521	2.500	2.600	2.604
	(0.2)	0.250	0.260	0.260	0.500	0.520	0.521	2.500	2.600	2.604
	(0.4)	0.250	0.290	0.296	0.500	0.580	0.593	2.500	2.900	2.964
	(0.8)	0.250	0.410	0.512	0.500	0.820	1.025	2.500	4.100	5.124
MA(1)	(-0.8)	0.250	0.253	0.274	0.500	0.505	0.547	2.500	2.525	2.735
	(-0.4)	0.250	0.253	0.277	0.500	0.505	0.553	2.500	2.525	2.765
	(-0.2)	0.250	0.253	0.274	0.500	0.505	0.547	2.500	2.525	2.735
	(0.2)	0.250	0.273	0.315	0.500	0.545	0.629	2.500	2.725	3.145
	(0.4)	0.250	0.410	0.410	0.500	0.820	0.820	2.500	4.100	4.100
	(0.8)	0.250	0.290	0.290	0.500	0.580	0.580	2.500	2.900	2.900
AR(2)	(0.1, -0.3)	0.250	0.260	0.260	0.500	0.520	0.520	2.500	2.600	2.600
	(-0.1, 0.3)	0.250	0.260	0.260	0.500	0.520	0.520	2.500	2.600	2.600
AR(3)	(0.1, -0.3, -0.2)	0.250	0.290	0.290	0.500	0.580	0.580	2.500	2.900	2.900
	(-0.3, -0.5, 0.1)	0.250	0.410	0.410	0.500	0.820	0.820	2.500	4.100	4.100
MA(2)	(0.1, -0.3)	0.250	0.253	0.275	0.500	0.505	0.550	2.500	2.525	2.750
	(-0.1, 0.3)	0.250	0.253	0.275	0.500	0.505	0.550	2.500	2.525	2.750
MA(3)	(0.1, -0.3, -0.2)	0.250	0.253	0.275	0.500	0.505	0.550	2.500	2.525	2.750
	(-0.3, -0.5, 0.1)	0.250	0.273	0.335	0.500	0.545	0.670	2.500	2.725	3.350
ARMA(1,1)	(0.4; -0.2)	0.250	0.260	0.262	0.500	0.520	0.523	2.500	2.600	2.616
	(-0.4; -0.4)	0.250	0.410	0.416	0.500	0.820	0.833	2.500	4.100	4.164
ARMA(2,1)	(-0.2, -0.3; 0.1)	0.250	0.253	0.278	0.500	0.505	0.556	2.500	2.525	2.781
	(0.2, -0.3; -0.1)	0.250	0.253	0.278	0.500	0.505	0.556	2.500	2.525	2.781
ARMA(1,2)	(0.3; 0.5, -0.1)	0.250	0.410	0.411	0.500	0.820	0.821	2.500	4.100	4.106
	(0.3; -0.5, 0.1)	0.250	0.260	0.261	0.500	0.520	0.521	2.500	2.600	2.606
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	0.250	0.273	0.335	0.500	0.545	0.670	2.500	2.725	3.350
	(0.2, -0.3; -0.5, 0.1)	0.250	0.273	0.295	0.500	0.545	0.590	2.500	2.725	2.950

Table 4

Risk of $\hat{\hat{Z}}_{T+h}$ *at* $\sigma^2 = 0.5$ *and* $h = 1$

Models	Parameters	$\alpha = 1$					$\alpha = 1$				
		T=25	T=50	T=100	T=200	T=400	T=25	T=50	T=100	T=200	T=400
AR(1)	(-0.8)	0.268	0.255	0.254	0.256	0.255	0.261	0.252	0.251	0.248	0.245
	(-0.4)	0.267	0.261	0.258	0.255	0.244	0.261	0.256	0.261	0.247	0.253
	(-0.2)	0.261	0.260	0.253	0.256	0.246	0.261	0.259	0.251	0.258	0.248
	(0.2)	0.263	0.264	0.261	0.241	0.244	0.262	0.264	0.250	0.254	0.246
	(0.4)	0.259	0.265	0.255	0.254	0.255	0.269	0.253	0.259	0.254	0.248
	(0.8)	0.256	0.250	0.258	0.257	0.254	0.261	0.250	0.249	0.245	0.259
MA(1)	(-0.8)	0.265	0.261	0.254	0.253	0.251	0.273	0.254	0.254	0.255	0.246
	(-0.4)	0.269	0.256	0.252	0.247	0.253	0.266	0.258	0.250	0.248	0.246
	(-0.2)	0.263	0.258	0.254	0.251	0.248	0.262	0.253	0.258	0.255	0.251
	(0.2)	0.262	0.260	0.254	0.256	0.258	0.268	0.258	0.255	0.252	0.246
	(0.4)	0.263	0.255	0.251	0.248	0.251	0.268	0.254	0.258	0.247	0.258
	(0.8)	0.265	0.257	0.256	0.253	0.257	0.263	0.266	0.250	0.246	0.250
AR(2)	(0.1, -0.3)	0.273	0.265	0.251	0.257	0.250	0.281	0.261	0.255	0.253	0.249
	(-0.1, 0.3)	0.274	0.260	0.251	0.250	0.252	0.273	0.262	0.256	0.252	0.257
AR(3)	(0.1, -0.3, -0.2)	0.300	0.259	0.254	0.255	0.254	0.294	0.279	0.271	0.244	0.253
	(-0.3, -0.5, 0.1)	0.280	0.268	0.249	0.249	0.259	0.293	0.265	0.259	0.247	0.244
MA(2)	(0.1, -0.3)	0.288	0.273	0.250	0.255	0.246	0.285	0.272	0.251	0.255	0.255
	(-0.1, 0.3)	0.289	0.263	0.259	0.256	0.245	0.291	0.260	0.259	0.250	0.253
MA(3)	(0.1, -0.3, -0.2)	0.308	0.281	0.260	0.255	0.247	0.307	0.272	0.266	0.257	0.251
	(-0.3, -0.5, 0.1)	0.304	0.273	0.266	0.256	0.253	0.299	0.277	0.260	0.251	0.253
ARMA(1,1)	(0.4; -0.2)	0.278	0.266	0.250	0.255	0.254	0.281	0.262	0.258	0.250	0.254
	(-0.4; -0.4)	0.276	0.269	0.257	0.253	0.251	0.279	0.259	0.255	0.258	0.253
ARMA(2,1)	(-0.2, -0.3; 0.1)	0.289	0.264	0.253	0.253	0.255	0.291	0.273	0.262	0.255	0.257
	(0.2, -0.3; -0.1)	0.291	0.263	0.260	0.254	0.257	0.296	0.259	0.261	0.253	0.250
ARMA(1,2)	(0.3; 0.5, -0.1)	0.292	0.264	0.261	0.250	0.251	0.294	0.275	0.260	0.252	0.249
	(0.3; -0.5, 0.1)	0.304	0.274	0.261	0.256	0.250	0.311	0.268	0.251	0.251	0.252
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	0.321	0.281	0.268	0.256	0.253	0.312	0.282	0.266	0.261	0.251
	(0.2, -0.3; -0.5, 0.1)	0.325	0.273	0.265	0.255	0.250	0.324	0.278	0.260	0.254	0.255

Table 5

Risk of $\hat{\hat{Z}}_{T+h}$ at $\sigma^2 = 0.5$ and $h = 2$

Models	Parameters	$\alpha = 1$					$\alpha = 1$				
		T=25	T=50	T=100	T=200	T=400	T=25	T=50	T=100	T=200	T=400
AR(1)	(-0.8)	0.438	0.417	0.432	0.422	0.414	0.437	0.424	0.431	0.410	0.408
	(-0.4)	0.293	0.291	0.295	0.291	0.295	0.302	0.297	0.297	0.292	0.291
	(-0.2)	0.268	0.259	0.262	0.256	0.264	0.274	0.265	0.259	0.257	0.258
	(0.2)	0.264	0.262	0.258	0.260	0.260	0.267	0.266	0.266	0.260	0.263
	(0.4)	0.294	0.303	0.292	0.290	0.287	0.303	0.290	0.298	0.285	0.292
	(0.8)	0.441	0.423	0.422	0.413	0.424	0.445	0.420	0.406	0.406	0.419
MA(1)	(-0.8)	0.408	0.409	0.415	0.397	0.412	0.421	0.408	0.413	0.418	0.398
	(-0.4)	0.290	0.293	0.286	0.300	0.284	0.291	0.289	0.291	0.286	0.297
	(-0.2)	0.258	0.260	0.260	0.258	0.261	0.260	0.260	0.262	0.261	0.258
	(0.2)	0.262	0.266	0.264	0.268	0.256	0.262	0.264	0.256	0.257	0.266
	(0.4)	0.295	0.290	0.290	0.288	0.295	0.296	0.296	0.299	0.299	0.294
	(0.8)	0.425	0.420	0.415	0.400	0.408	0.430	0.418	0.411	0.407	0.419
AR(2)	(0.1, -0.3)	0.271	0.262	0.258	0.257	0.259	0.266	0.269	0.255	0.246	0.254
	(-0.1, 0.3)	0.268	0.257	0.257	0.251	0.249	0.273	0.265	0.254	0.260	0.251
AR(3)	(0.1, -0.3, -0.2)	0.283	0.263	0.255	0.260	0.251	0.276	0.262	0.260	0.257	0.261
	(-0.3, -0.5, 0.1)	0.298	0.285	0.284	0.286	0.270	0.307	0.291	0.283	0.280	0.278
MA(2)	(0.1, -0.3)	0.278	0.268	0.253	0.247	0.253	0.279	0.261	0.255	0.254	0.251
	(-0.1, 0.3)	0.277	0.258	0.261	0.253	0.255	0.275	0.261	0.257	0.252	0.259
MA(3)	(0.1, -0.3, -0.2)	0.302	0.271	0.259	0.259	0.252	0.293	0.270	0.268	0.256	0.256
	(-0.3, -0.5, 0.1)	0.321	0.287	0.284	0.277	0.267	0.324	0.292	0.277	0.279	0.274
ARMA(1,1)	(0.4; -0.2)	0.274	0.262	0.269	0.263	0.259	0.279	0.269	0.265	0.262	0.255
	(-0.4; -0.4)	0.442	0.422	0.414	0.414	0.404	0.437	0.420	0.415	0.424	0.420
ARMA(2,1)	(-0.2, -0.3; 0.1)	0.281	0.269	0.253	0.260	0.258	0.282	0.280	0.252	0.252	0.255
	(0.2, -0.3; -0.1)	0.275	0.260	0.266	0.257	0.256	0.280	0.266	0.259	0.251	0.246
ARMA(1,2)	(0.3; 0.5, -0.1)	0.452	0.449	0.420	0.423	0.401	0.465	0.453	0.424	0.427	0.417
	(0.3; -0.5, 0.1)	0.296	0.270	0.262	0.270	0.254	0.294	0.276	0.267	0.261	0.263
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	0.316	0.296	0.289	0.281	0.276	0.314	0.303	0.283	0.274	0.281
	(0.2, -0.3; -0.5, 0.1)	0.330	0.303	0.295	0.280	0.276	0.319	0.299	0.278	0.281	0.276

Table 6

Risk of $\hat{\hat{Z}}_{T+h}$ *at* $\sigma^2 = 0.5$ *and* $h = 3$

Models	Parameters	$\alpha = 1$					$\alpha = 1$				
		T=25	T=50	T=100	T=200	T=400	T=25	T=50	T=100	T=200	T=400
AR(1)	(-0.8)	0.553	0.539	0.536	0.530	0.525	0.550	0.557	0.525	0.515	0.512
	(-0.4)	0.295	0.299	0.295	0.300	0.293	0.303	0.304	0.300	0.302	0.297
	(-0.2)	0.270	0.256	0.265	0.261	0.259	0.261	0.256	0.258	0.255	0.257
	(0.2)	0.264	0.260	0.257	0.257	0.259	0.262	0.265	0.254	0.254	0.256
	(0.4)	0.308	0.304	0.291	0.295	0.302	0.308	0.300	0.290	0.290	0.287
	(0.8)	0.570	0.525	0.535	0.506	0.529	0.563	0.543	0.518	0.514	0.526
MA(1)	(-0.8)	0.435	0.405	0.398	0.399	0.411	0.418	0.415	0.419	0.413	0.403
	(-0.4)	0.295	0.299	0.294	0.289	0.292	0.293	0.290	0.287	0.284	0.291
	(-0.2)	0.260	0.262	0.262	0.270	0.265	0.268	0.261	0.271	0.259	0.253
	(0.2)	0.256	0.270	0.266	0.263	0.263	0.260	0.258	0.262	0.254	0.264
	(0.4)	0.287	0.283	0.291	0.290	0.294	0.294	0.289	0.289	0.284	0.289
	(0.8)	0.410	0.425	0.406	0.399	0.405	0.419	0.422	0.407	0.407	0.409
AR(2)	(0.1, -0.3)	0.290	0.274	0.278	0.281	0.272	0.280	0.277	0.275	0.281	0.265
	(-0.1, 0.3)	0.293	0.283	0.275	0.276	0.283	0.290	0.274	0.280	0.281	0.274
AR(3)	(0.1, -0.3, -0.2)	0.295	0.285	0.285	0.282	0.284	0.305	0.280	0.280	0.278	0.278
	(-0.3, -0.5, 0.1)	0.355	0.343	0.322	0.315	0.316	0.340	0.334	0.322	0.320	0.314
MA(2)	(0.1, -0.3)	0.286	0.280	0.271	0.279	0.272	0.294	0.278	0.276	0.273	0.273
	(-0.1, 0.3)	0.284	0.281	0.270	0.278	0.269	0.282	0.276	0.273	0.274	0.272
MA(3)	(0.1, -0.3, -0.2)	0.296	0.293	0.281	0.273	0.279	0.297	0.289	0.276	0.282	0.269
	(-0.3, -0.5, 0.1)	0.356	0.342	0.341	0.343	0.337	0.352	0.346	0.344	0.339	0.348
ARMA(1,1)	(0.4; -0.2)	0.271	0.264	0.270	0.268	0.252	0.274	0.261	0.259	0.262	0.272
	(-0.4; -0.4)	0.464	0.445	0.439	0.438	0.440	0.461	0.451	0.441	0.421	0.428
ARMA(2,1)	(-0.2, -0.3; 0.1)	0.291	0.285	0.276	0.276	0.279	0.298	0.283	0.273	0.267	0.276
	(0.2, -0.3; -0.1)	0.289	0.287	0.277	0.271	0.266	0.295	0.290	0.274	0.275	0.270
ARMA(1,2)	(0.3; 0.5, -0.1)	0.451	0.432	0.428	0.424	0.408	0.438	0.428	0.423	0.425	0.427
	(0.3; -0.5, 0.1)	0.279	0.263	0.266	0.259	0.263	0.271	0.268	0.265	0.261	0.263
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	0.386	0.350	0.327	0.328	0.325	0.372	0.362	0.330	0.331	0.321
	(0.2, -0.3; -0.5, 0.1)	0.329	0.306	0.311	0.293	0.296	0.330	0.310	0.306	0.301	0.292

Table 7

Risk of $\hat{\hat{Z}}_{T+h}$ *at* $\sigma^2 = 1$ *and* $h = 1$

Models	Parameters	$\alpha = 1$					$\alpha = 1$				
		T=25	T=50	T=100	T=200	T=400	T=25	T=50	T=100	T=200	T=400
AR(1)	(-0.8)	0.527	0.520	0.505	0.490	0.500	0.535	0.523	0.515	0.492	0.502
	(-0.4)	0.537	0.507	0.521	0.496	0.505	0.553	0.512	0.521	0.497	0.505
	(-0.2)	0.543	0.511	0.510	0.504	0.488	0.512	0.518	0.496	0.509	0.499
	(0.2)	0.524	0.504	0.510	0.512	0.502	0.530	0.521	0.507	0.512	0.505
	(0.4)	0.538	0.513	0.517	0.512	0.495	0.549	0.513	0.517	0.507	0.498
	(0.8)	0.517	0.533	0.514	0.498	0.499	0.535	0.521	0.491	0.506	0.509
MA(1)	(-0.8)	0.537	0.522	0.511	0.531	0.518	0.535	0.513	0.517	0.494	0.504
	(-0.4)	0.543	0.532	0.491	0.505	0.506	0.552	0.501	0.529	0.500	0.508
	(-0.2)	0.557	0.518	0.503	0.511	0.484	0.547	0.523	0.499	0.498	0.495
	(0.2)	0.539	0.524	0.521	0.502	0.503	0.533	0.511	0.494	0.493	0.503
	(0.4)	0.519	0.518	0.512	0.501	0.498	0.537	0.528	0.510	0.508	0.508
	(0.8)	0.541	0.535	0.500	0.509	0.511	0.544	0.514	0.511	0.509	0.502
AR(2)	(0.1, -0.3)	0.557	0.523	0.512	0.498	0.511	0.552	0.523	0.503	0.506	0.501
	(-0.1, 0.3)	0.579	0.516	0.509	0.509	0.510	0.548	0.517	0.519	0.506	0.509
AR(3)	(0.1, -0.3, -0.2)	0.604	0.551	0.531	0.512	0.520	0.585	0.539	0.502	0.511	0.508
	(-0.3, -0.5, 0.1)	0.588	0.532	0.515	0.512	0.502	0.603	0.529	0.513	0.518	0.500
MA(2)	(0.1, -0.3)	0.586	0.550	0.521	0.503	0.504	0.581	0.554	0.513	0.495	0.503
	(-0.1, 0.3)	0.585	0.544	0.516	0.502	0.494	0.584	0.538	0.502	0.509	0.505
MA(3)	(0.1, -0.3, -0.2)	0.615	0.548	0.522	0.513	0.505	0.644	0.558	0.516	0.504	0.497
	(-0.3, -0.5, 0.1)	0.638	0.545	0.519	0.515	0.497	0.608	0.569	0.512	0.509	0.520
ARMA(1,1)	(0.4; -0.2)	0.557	0.521	0.512	0.497	0.507	0.555	0.530	0.503	0.515	0.506
	(-0.4; -0.4)	0.559	0.514	0.523	0.515	0.503	0.565	0.519	0.512	0.515	0.506
ARMA(2,1)	(-0.2, -0.3; 0.1)	0.599	0.526	0.521	0.510	0.477	0.611	0.565	0.528	0.512	0.517
	(0.2, -0.3; -0.1)	0.577	0.549	0.527	0.513	0.515	0.601	0.547	0.503	0.518	0.493
ARMA(1,2)	(0.3; 0.5, -0.1)	0.603	0.555	0.534	0.523	0.512	0.613	0.550	0.519	0.513	0.514
	(0.3; -0.5, 0.1)	0.611	0.530	0.522	0.497	0.509	0.621	0.563	0.521	0.503	0.506
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	0.660	0.578	0.534	0.521	0.510	0.651	0.583	0.528	0.513	0.508
	(0.2, -0.3; -0.5, 0.1)	0.667	0.564	0.527	0.508	0.525	0.662	0.594	0.551	0.537	0.498

Table 8

Risk of $\hat{\hat{Z}}_{T+h}$ at $\sigma^2 = 1$ and $h = 2$

Models	Parameters	$\alpha = 1$					$\alpha = 1$				
		T=25	T=50	T=100	T=200	T=400	T=25	T=50	T=100	T=200	T=400
AR(1)	(-0.8)	0.943	0.843	0.847	0.834	0.826	0.876	0.847	0.833	0.848	0.839
	(-0.4)	0.589	0.585	0.581	0.584	0.578	0.599	0.602	0.586	0.598	0.575
	(-0.2)	0.546	0.522	0.533	0.525	0.528	0.552	0.520	0.538	0.523	0.523
	(0.2)	0.540	0.508	0.515	0.514	0.536	0.521	0.531	0.515	0.524	0.525
	(0.4)	0.594	0.609	0.591	0.581	0.569	0.595	0.587	0.589	0.582	0.594
	(0.8)	0.878	0.847	0.854	0.832	0.841	0.907	0.843	0.815	0.848	0.816
MA(1)	(-0.8)	0.828	0.814	0.847	0.844	0.828	0.877	0.810	0.845	0.812	0.827
	(-0.4)	0.610	0.583	0.585	0.590	0.588	0.611	0.582	0.571	0.589	0.581
	(-0.2)	0.529	0.527	0.516	0.539	0.532	0.523	0.523	0.534	0.530	0.512
	(0.2)	0.522	0.531	0.528	0.521	0.514	0.529	0.532	0.509	0.510	0.524
	(0.4)	0.595	0.598	0.573	0.575	0.579	0.582	0.588	0.587	0.582	0.580
	(0.8)	0.818	0.839	0.841	0.807	0.815	0.874	0.815	0.827	0.810	0.842
AR(2)	(0.1, -0.3)	0.553	0.519	0.506	0.501	0.517	0.537	0.536	0.506	0.515	0.499
	(-0.1, 0.3)	0.537	0.500	0.507	0.516	0.501	0.532	0.517	0.503	0.517	0.503
AR(3)	(0.1, -0.3, -0.2)	0.551	0.528	0.525	0.511	0.508	0.552	0.532	0.515	0.505	0.502
	(-0.3, -0.5, 0.1)	0.608	0.554	0.559	0.548	0.531	0.606	0.567	0.549	0.550	0.539
MA(2)	(0.1, -0.3)	0.566	0.533	0.503	0.504	0.509	0.559	0.520	0.521	0.505	0.507
	(-0.1, 0.3)	0.547	0.512	0.510	0.509	0.521	0.553	0.539	0.502	0.503	0.516
MA(3)	(0.1, -0.3, -0.2)	0.573	0.546	0.535	0.521	0.508	0.603	0.550	0.515	0.514	0.498
	(-0.3, -0.5, 0.1)	0.611	0.592	0.569	0.560	0.548	0.619	0.574	0.571	0.545	0.557
ARMA(1,1)	(0.4; -0.2)	0.530	0.532	0.532	0.527	0.531	0.540	0.554	0.519	0.521	0.514
	(-0.4; -0.4)	0.907	0.833	0.846	0.800	0.812	0.910	0.843	0.832	0.848	0.854
ARMA(2,1)	(-0.2, -0.3; 0.1)	0.583	0.525	0.527	0.517	0.524	0.586	0.546	0.527	0.518	0.503
	(0.2, -0.3; -0.1)	0.583	0.551	0.513	0.518	0.510	0.564	0.530	0.517	0.516	0.514
ARMA(1,2)	(0.3; 0.5, -0.1)	0.940	0.873	0.856	0.864	0.818	1.009	0.877	0.868	0.847	0.829
	(0.3; -0.5, 0.1)	0.603	0.562	0.551	0.523	0.529	0.603	0.552	0.536	0.542	0.515
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	0.659	0.610	0.559	0.553	0.527	0.657	0.588	0.565	0.550	0.545
	(0.2, -0.3; -0.5, 0.1)	0.675	0.613	0.572	0.547	0.550	0.670	0.598	0.572	0.549	0.559

Table 9

Risk of $\hat{\hat{Z}}_{T+h}$ *at* $\sigma^2 = 1$ *and* $h = 3$

Models	Parameters	$\alpha = 1$					$\alpha = 1$				
		T=25	T=50	T=100	T=200	T=400	T=25	T=50	T=100	T=200	T=400
AR(1)	(-0.8)	1.183	1.099	1.088	1.036	1.022	1.099	1.081	1.085	1.012	1.073
	(-0.4)	0.595	0.596	0.584	0.607	0.597	0.627	0.601	0.602	0.602	0.595
	(-0.2)	0.551	0.507	0.535	0.539	0.528	0.516	0.525	0.533	0.519	0.514
	(0.2)	0.514	0.522	0.538	0.507	0.511	0.548	0.539	0.515	0.539	0.522
	(0.4)	0.608	0.593	0.614	0.597	0.583	0.613	0.598	0.601	0.615	0.612
	(0.8)	1.140	1.101	1.048	1.042	1.063	1.151	1.074	1.014	1.066	1.013
MA(1)	(-0.8)	0.849	0.829	0.810	0.824	0.820	0.847	0.834	0.828	0.831	0.834
	(-0.4)	0.599	0.602	0.574	0.584	0.581	0.571	0.579	0.577	0.592	0.587
	(-0.2)	0.524	0.521	0.528	0.519	0.539	0.545	0.514	0.533	0.520	0.507
	(0.2)	0.527	0.536	0.516	0.546	0.524	0.525	0.528	0.522	0.520	0.520
	(0.4)	0.582	0.610	0.580	0.576	0.588	0.602	0.570	0.594	0.588	0.574
	(0.8)	0.829	0.804	0.826	0.817	0.856	0.874	0.833	0.831	0.830	0.817
AR(2)	(0.1, -0.3)	0.580	0.565	0.567	0.555	0.558	0.584	0.569	0.544	0.547	0.560
	(-0.1, 0.3)	0.615	0.571	0.569	0.558	0.580	0.583	0.556	0.554	0.532	0.556
AR(3)	(0.1, -0.3, -0.2)	0.599	0.591	0.561	0.560	0.560	0.598	0.577	0.570	0.550	0.557
	(-0.3, -0.5, 0.1)	0.712	0.660	0.644	0.614	0.635	0.749	0.677	0.642	0.641	0.634
MA(2)	(0.1, -0.3)	0.563	0.566	0.564	0.547	0.561	0.587	0.563	0.542	0.541	0.560
	(-0.1, 0.3)	0.571	0.569	0.545	0.549	0.551	0.574	0.557	0.542	0.566	0.553
MA(3)	(0.1, -0.3, -0.2)	0.629	0.586	0.555	0.545	0.568	0.612	0.581	0.563	0.556	0.556
	(-0.3, -0.5, 0.1)	0.731	0.676	0.676	0.659	0.682	0.713	0.714	0.674	0.671	0.654
ARMA(1,1)	(0.4; -0.2)	0.555	0.546	0.528	0.528	0.532	0.543	0.538	0.521	0.536	0.516
	(-0.4; -0.4)	1.001	0.904	0.913	0.908	0.897	0.928	0.880	0.895	0.880	0.871
ARMA(2,1)	(-0.2, -0.3; 0.1)	0.591	0.568	0.556	0.544	0.540	0.599	0.578	0.558	0.558	0.562
	(0.2, -0.3; -0.1)	0.596	0.564	0.563	0.560	0.547	0.600	0.551	0.569	0.544	0.560
ARMA(1,2)	(0.3; 0.5, -0.1)	0.918	0.851	0.823	0.872	0.820	0.951	0.864	0.865	0.879	0.825
	(0.3; -0.5, 0.1)	0.569	0.526	0.536	0.531	0.517	0.561	0.524	0.518	0.521	0.532
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	0.817	0.733	0.684	0.673	0.657	0.797	0.730	0.698	0.648	0.650
	(0.2, -0.3; -0.5, 0.1)	0.703	0.647	0.619	0.582	0.584	0.677	0.629	0.598	0.573	0.583

Table 10

Risk of $\hat{\hat{Z}}_{T+h}$ at $\sigma^2 = 5$ and $h = 1$

Models	Parameters	$\alpha = 1$					$\alpha = 1$				
		T=25	T=50	T=100	T=200	T=400	T=25	T=50	T=100	T=200	T=400
AR(1)	(-0.8)	3.060	2.531	2.579	2.519	2.400	2.764	2.596	2.555	2.830	2.400
	(-0.4)	2.986	2.603	2.756	2.569	2.478	2.932	2.666	2.485	2.627	2.442
	(-0.2)	2.778	2.607	2.636	2.554	2.471	3.169	2.698	2.594	2.581	2.458
	(0.2)	3.013	2.884	2.674	2.665	2.495	2.733	2.759	2.590	2.422	2.388
	(0.4)	2.518	2.604	2.682	2.461	2.531	3.278	2.700	2.596	2.602	2.453
	(0.8)	2.942	2.876	2.390	2.585	2.504	3.411	2.561	2.683	2.561	2.526
MA(1)	(-0.8)	3.208	2.518	2.496	2.526	2.576	2.967	2.586	2.551	2.535	2.466
	(-0.4)	2.768	2.713	2.574	2.532	2.589	3.006	2.752	2.702	2.663	2.453
	(-0.2)	3.054	2.928	2.523	2.510	2.532	4.662	2.722	2.583	2.519	2.597
	(0.2)	3.003	2.969	2.517	2.414	2.577	2.866	2.721	2.590	2.430	2.637
	(0.4)	2.874	2.730	2.551	2.579	2.538	2.778	2.610	2.659	2.448	2.463
	(0.8)	2.803	2.648	2.561	2.429	2.532	2.684	2.716	2.548	2.573	2.417
AR(2)	(0.1, -0.3)	3.127	2.658	2.646	2.519	2.598	3.160	2.663	2.646	2.443	2.605
	(-0.1, 0.3)	2.897	2.903	2.545	2.450	2.580	3.185	2.685	2.504	2.429	2.412
AR(3)	(0.1, -0.3, -0.2)	3.157	2.955	2.630	2.584	2.387	3.840	2.991	2.476	2.572	2.685
	(-0.3, -0.5, 0.1)	4.016	3.013	2.660	2.463	2.570	3.789	3.036	2.615	2.521	2.611
MA(2)	(0.1, -0.3)	3.294	2.850	2.786	2.521	2.541	4.942	2.951	2.572	2.635	2.598
	(-0.1, 0.3)	9.538	3.706	2.609	2.586	2.435	3.291	3.182	2.656	2.496	2.526
MA(3)	(0.1, -0.3, -0.2)	4.154	3.436	2.679	2.566	2.484	4.206	3.226	2.780	2.632	2.474
	(-0.3, -0.5, 0.1)	4.131	2.959	2.822	2.530	2.483	4.250	3.274	2.707	2.472	2.613
ARMA(1,1)	(0.4; -0.2)	3.689	2.860	2.544	2.704	2.446	3.469	2.682	2.656	2.599	2.488
	(-0.4; -0.4)	2.769	2.794	2.631	2.510	2.521	3.375	3.064	2.679	2.691	2.531
ARMA(2,1)	(-0.2, -0.3; 0.1)	3.708	3.058	2.589	2.489	2.474	3.719	2.756	2.549	2.842	2.497
	(0.2, -0.3; -0.1)	4.423	3.276	2.751	2.498	2.558	3.665	2.834	2.538	2.482	2.544
ARMA(1,2)	(0.3; 0.5, -0.1)	3.869	3.089	2.798	2.451	2.485	3.502	2.953	2.683	2.539	2.464
	(0.3; -0.5, 0.1)	3.515	2.824	2.497	2.674	2.718	4.414	2.925	2.587	2.428	2.426
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	4.121	2.985	3.178	2.450	2.484	4.886	3.173	2.628	3.122	2.511
	(0.2, -0.3; -0.5, 0.1)	4.415	3.206	2.834	2.772	2.604	4.049	3.022	2.674	2.664	2.575

Table 11

Risk of $\hat{\hat{Z}}_{T+h}$ *at* $\sigma^2 = 5$ *and* $h = 2$

Models	Parameters	$\alpha = 1$					$\alpha = 1$				
		T=25	T=50	T=100	T=200	T=400	T=25	T=50	T=100	T=200	T=400
AR(1)	(-0.8)	4.684	4.272	3.989	4.146	4.271	4.880	6.330	4.214	3.892	4.101
	(-0.4)	3.564	3.025	2.862	2.723	2.932	3.272	3.033	2.915	2.934	2.892
	(-0.2)	2.886	2.976	2.550	2.570	2.578	2.818	2.628	2.561	2.559	2.548
	(0.2)	2.881	2.621	2.572	2.489	2.584	2.859	2.591	2.602	2.604	2.530
	(0.4)	3.388	3.179	2.914	2.884	2.891	3.219	2.956	3.049	2.977	2.888
	(0.8)	4.883	4.422	4.364	4.098	4.035	6.693	4.770	4.050	3.958	3.937
MA(1)	(-0.8)	4.822	4.005	4.012	3.982	4.118	5.137	4.175	5.276	4.609	4.128
	(-0.4)	3.344	3.057	2.927	2.876	2.859	3.228	3.613	2.765	2.938	2.935
	(-0.2)	2.860	2.738	2.613	2.508	2.487	2.658	2.655	2.619	2.652	2.589
	(0.2)	3.052	2.684	2.527	2.544	2.531	3.000	2.736	2.727	2.492	2.486
	(0.4)	3.284	3.030	3.129	2.941	2.886	3.230	3.119	3.091	3.071	2.839
	(0.8)	4.636	4.293	4.136	3.997	3.862	4.687	4.611	4.083	4.450	3.959
AR(2)	(0.1, -0.3)	2.991	2.688	2.744	2.682	2.491	3.289	2.779	2.698	2.474	2.564
	(-0.1, 0.3)	3.302	2.805	2.469	2.482	2.520	2.872	2.636	2.485	2.582	2.658
AR(3)	(0.1, -0.3, -0.2)	3.104	3.049	2.728	2.474	2.451	3.372	2.748	2.500	2.644	2.505
	(-0.3, -0.5, 0.1)	3.896	4.192	2.823	2.645	3.039	3.679	2.908	2.828	2.764	2.680
MA(2)	(0.1, -0.3)	3.025	2.612	2.539	2.770	2.486	3.747	2.873	2.788	2.907	2.520
	(-0.1, 0.3)	3.374	2.587	2.538	2.503	2.570	4.046	2.677	2.623	2.493	2.665
MA(3)	(0.1, -0.3, -0.2)	4.020	2.930	2.711	2.568	2.516	4.130	2.671	2.714	2.568	2.600
	(-0.3, -0.5, 0.1)	3.729	3.004	3.113	2.596	2.708	4.027	3.397	2.856	2.965	2.791
ARMA(1,1)	(0.4; -0.2)	3.159	2.973	2.648	2.816	2.553	2.881	2.901	3.141	2.599	2.673
	(-0.4; -0.4)	4.777	4.262	4.425	4.272	3.975	5.833	5.174	4.107	4.008	4.076
ARMA(2,1)	(-0.2, -0.3; 0.1)	3.392	3.009	2.607	2.611	2.528	3.629	2.833	2.653	2.680	2.620
	(0.2, -0.3; -0.1)	3.365	3.108	2.637	2.676	2.534	3.058	3.129	2.649	2.661	2.552
ARMA(1,2)	(0.3; 0.5, -0.1)	6.126	4.274	5.315	3.941	4.004	6.610	4.860	4.224	3.958	4.260
	(0.3; -0.5, 0.1)	3.684	2.853	2.626	2.708	2.634	4.022	3.269	2.681	2.655	2.649
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	4.192	3.265	3.265	2.745	2.720	9.126	3.041	2.879	2.884	2.736
	(0.2, -0.3; -0.5, 0.1)	4.142	3.378	3.039	2.869	2.571	4.265	3.289	2.820	2.864	2.813

Table 12

Risk of $\hat{\hat{Z}}_{T+h}$ *at* $\sigma^2 = 5$ *and* $h = 3$

Models	Parameters	$\alpha = 1$					$\alpha = 1$				
		T=25	T=50	T=100	T=200	T=400	T=25	T=50	T=100	T=200	T=400
AR(1)	(-0.8)	6.827	6.912	6.127	6.074	5.052	8.996	6.926	5.223	5.218	4.855
	(-0.4)	3.829	3.064	2.991	2.969	3.019	3.483	3.261	2.997	2.919	3.054
	(-0.2)	2.949	2.835	2.650	2.600	2.646	2.858	2.762	2.757	2.607	2.627
	(0.2)	2.796	2.747	2.578	2.779	2.473	2.844	2.658	2.739	2.591	2.648
	(0.4)	3.591	2.945	2.891	3.192	2.795	3.295	3.496	3.045	3.093	2.965
	(0.8)	5.735	6.179	6.031	5.079	4.805	7.802	6.253	11.632	4.950	4.793
MA(1)	(-0.8)	4.480	4.363	5.575	4.207	4.002	4.864	4.365	5.159	4.131	4.166
	(-0.4)	3.189	3.062	2.890	2.902	2.915	3.182	3.136	2.961	2.859	2.968
	(-0.2)	2.809	2.693	2.604	2.564	2.456	2.928	2.722	3.020	2.586	2.843
	(0.2)	2.870	2.630	2.767	2.708	2.507	2.628	2.888	2.589	2.607	2.539
	(0.4)	3.625	3.195	2.952	2.913	2.910	3.236	3.173	2.968	2.846	2.907
	(0.8)	5.652	4.219	6.731	4.064	3.929	5.638	5.023	4.346	3.879	3.942
AR(2)	(0.1, -0.3)	3.153	2.839	2.833	2.679	2.795	2.868	2.847	2.750	2.668	2.776
	(-0.1, 0.3)	3.302	2.896	2.794	2.781	2.765	3.168	2.962	2.774	2.828	2.875
AR(3)	(0.1, -0.3, -0.2)	3.203	3.296	2.979	2.831	2.639	3.369	3.109	2.912	2.838	2.763
	(-0.3, -0.5, 0.1)	3.754	3.568	3.352	2.981	3.016	3.988	3.212	3.401	3.315	3.044
MA(2)	(0.1, -0.3)	2.990	2.927	2.846	2.757	2.765	3.418	2.949	2.742	2.719	2.695
	(-0.1, 0.3)	3.136	2.843	2.927	2.797	2.680	3.098	2.931	2.725	2.675	2.639
MA(3)	(0.1, -0.3, -0.2)	3.334	3.056	3.063	2.743	2.733	3.400	2.897	2.946	2.707	2.876
	(-0.3, -0.5, 0.1)	4.185	3.749	3.929	3.483	3.483	3.897	3.451	3.378	3.275	3.486
ARMA(1,1)	(0.4; -0.2)	3.074	2.832	2.718	2.499	2.602	2.967	2.982	2.733	2.706	2.709
	(-0.4; -0.4)	5.253	5.011	5.352	4.760	4.379	6.022	5.438	4.715	4.180	4.356
ARMA(2,1)	(-0.2, -0.3; 0.1)	3.138	2.876	2.630	2.700	2.630	3.223	2.976	2.749	2.640	2.695
	(0.2, -0.3; -0.1)	3.495	2.844	2.864	2.816	2.820	3.476	3.123	2.753	2.623	2.651
ARMA(1,2)	(0.3; 0.5, -0.1)	5.681	4.936	4.214	4.004	4.079	5.303	4.416	4.246	4.217	4.296
	(0.3; -0.5, 0.1)	2.914	2.772	2.554	2.569	2.551	2.968	2.869	2.610	2.624	2.577
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	4.589	3.852	3.362	3.366	3.470	4.723	3.813	3.445	3.405	3.258
	(0.2, -0.3; -0.5, 0.1)	4.901	3.857	3.119	3.028	2.734	5.114	3.108	2.982	3.254	2.814

Figure 44. Relative efficiency of AR model at h=1

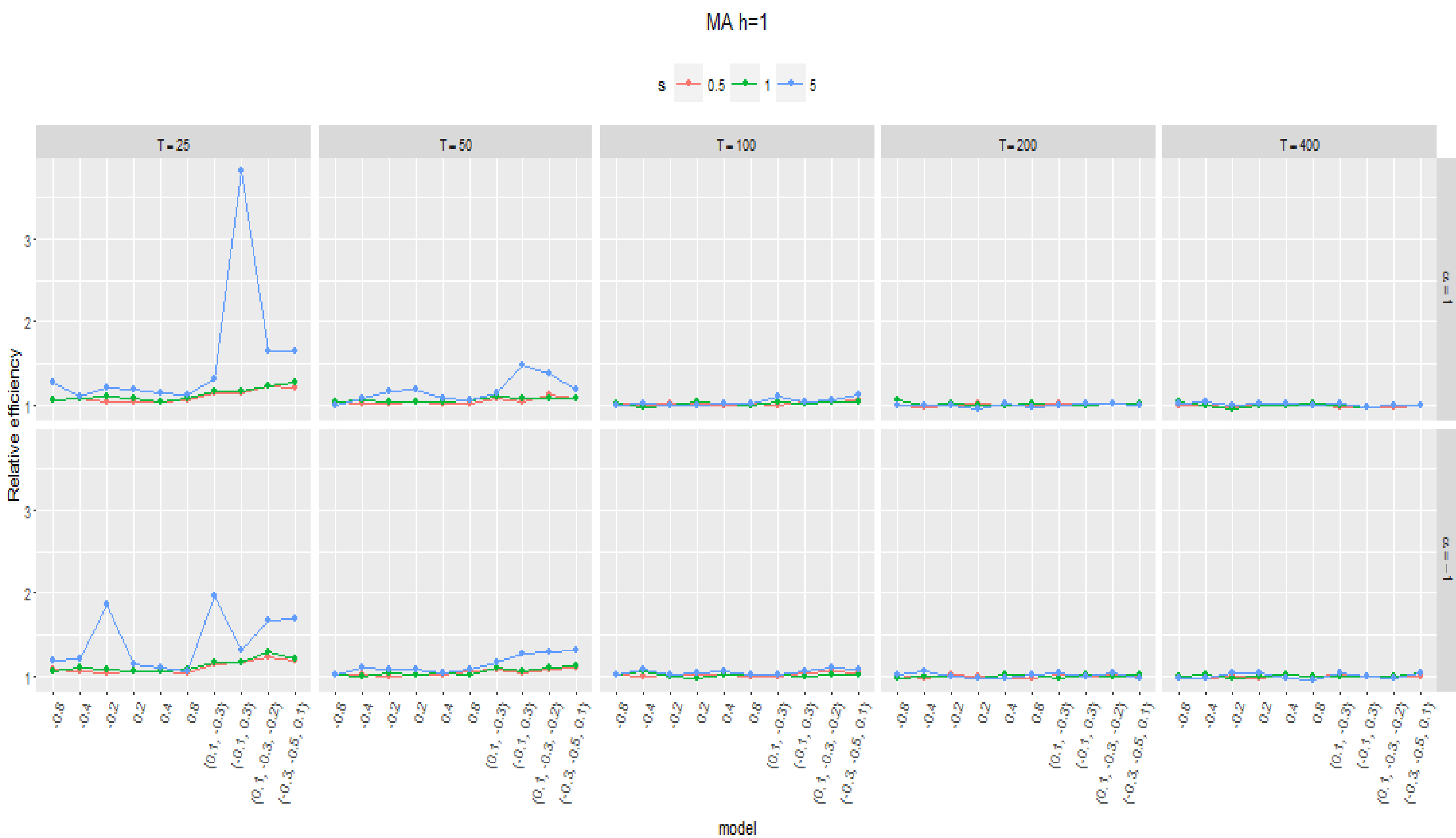

Figure 45. Relative efficiency of MA model at h=1

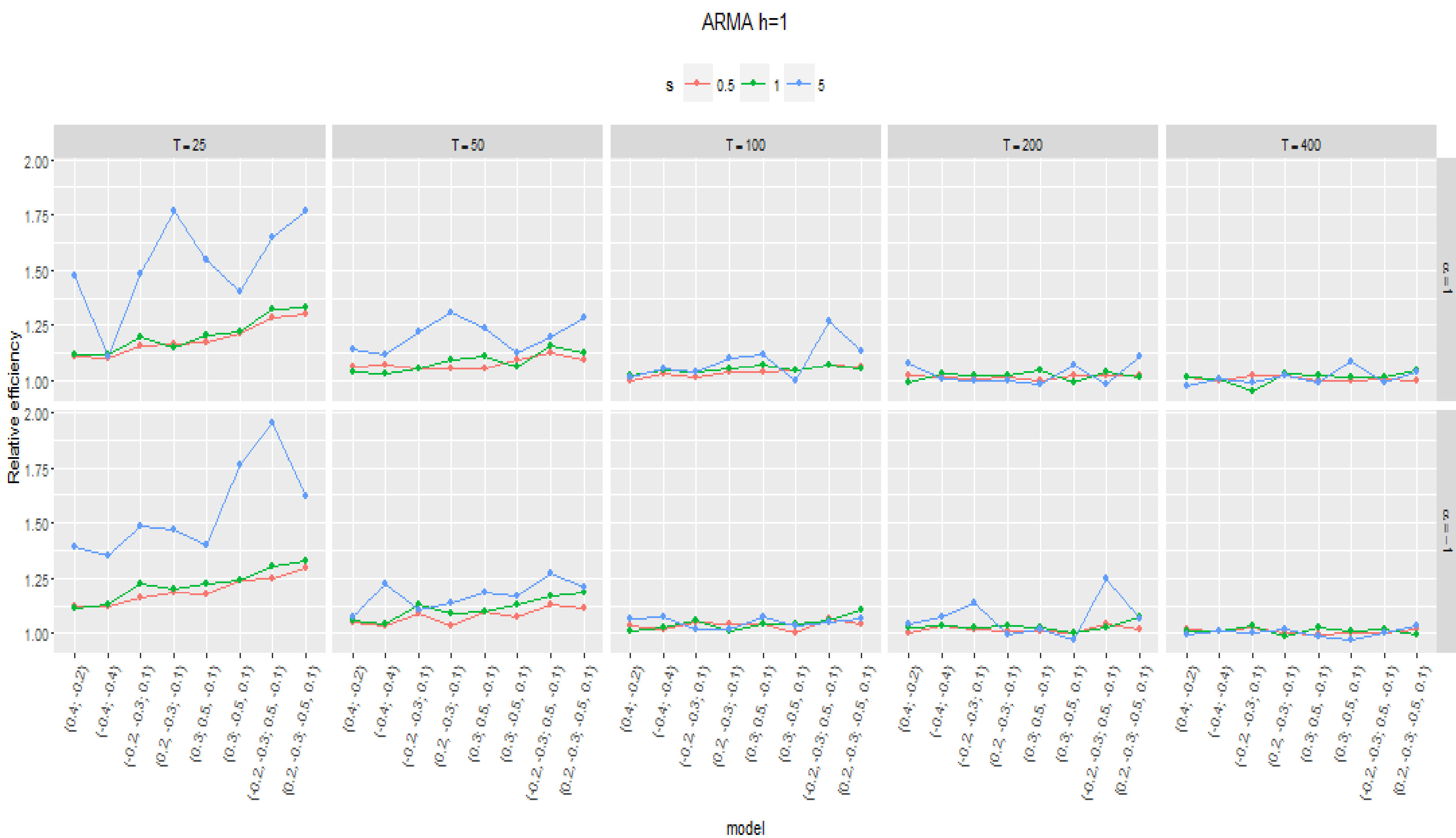

Figure 46. Relative efficiency of ARMA model at h=1

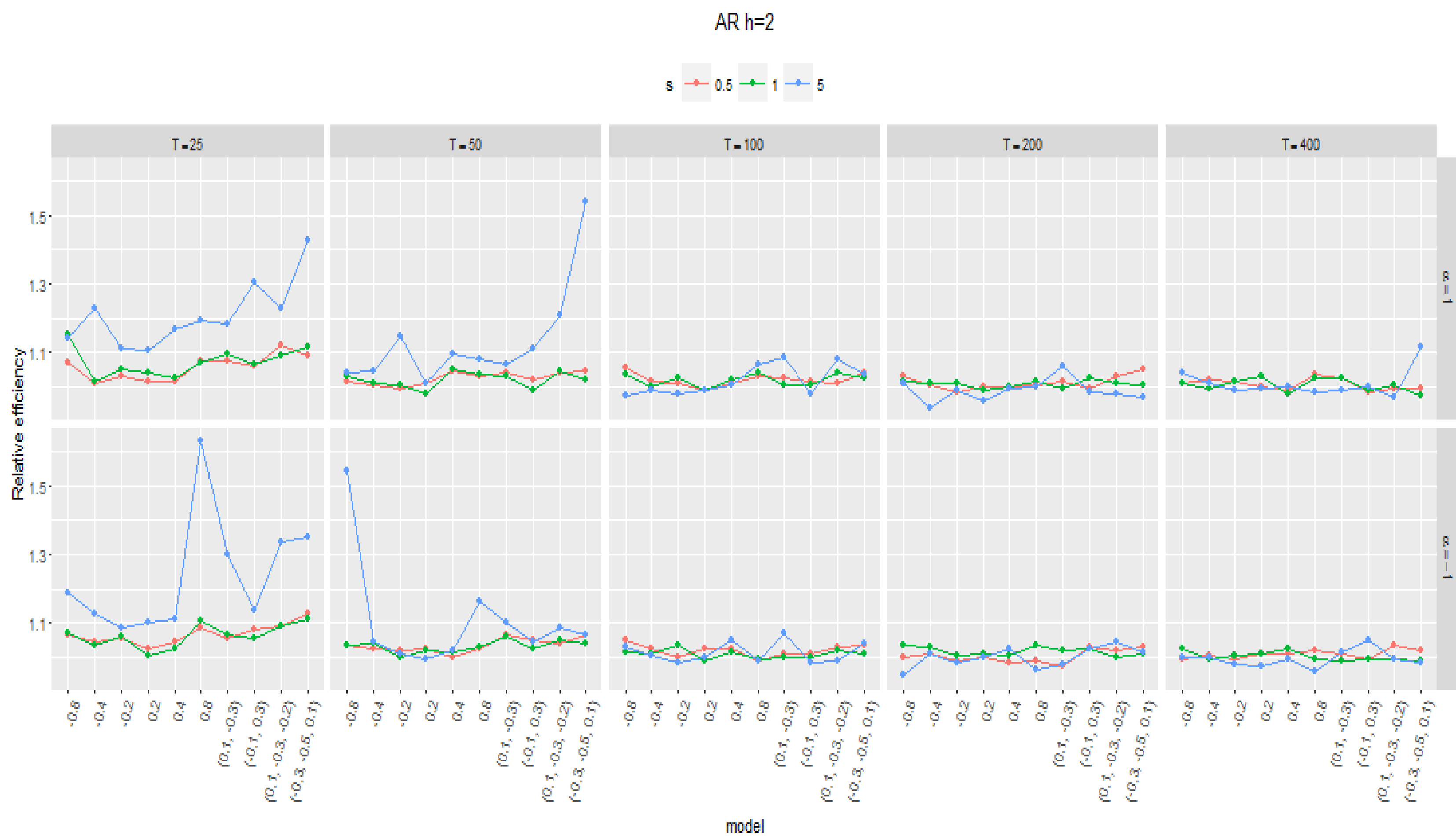

Figure 47. Relative efficiency of AR model at h=2

Figure 48. Relative efficiency of MA model at h=2

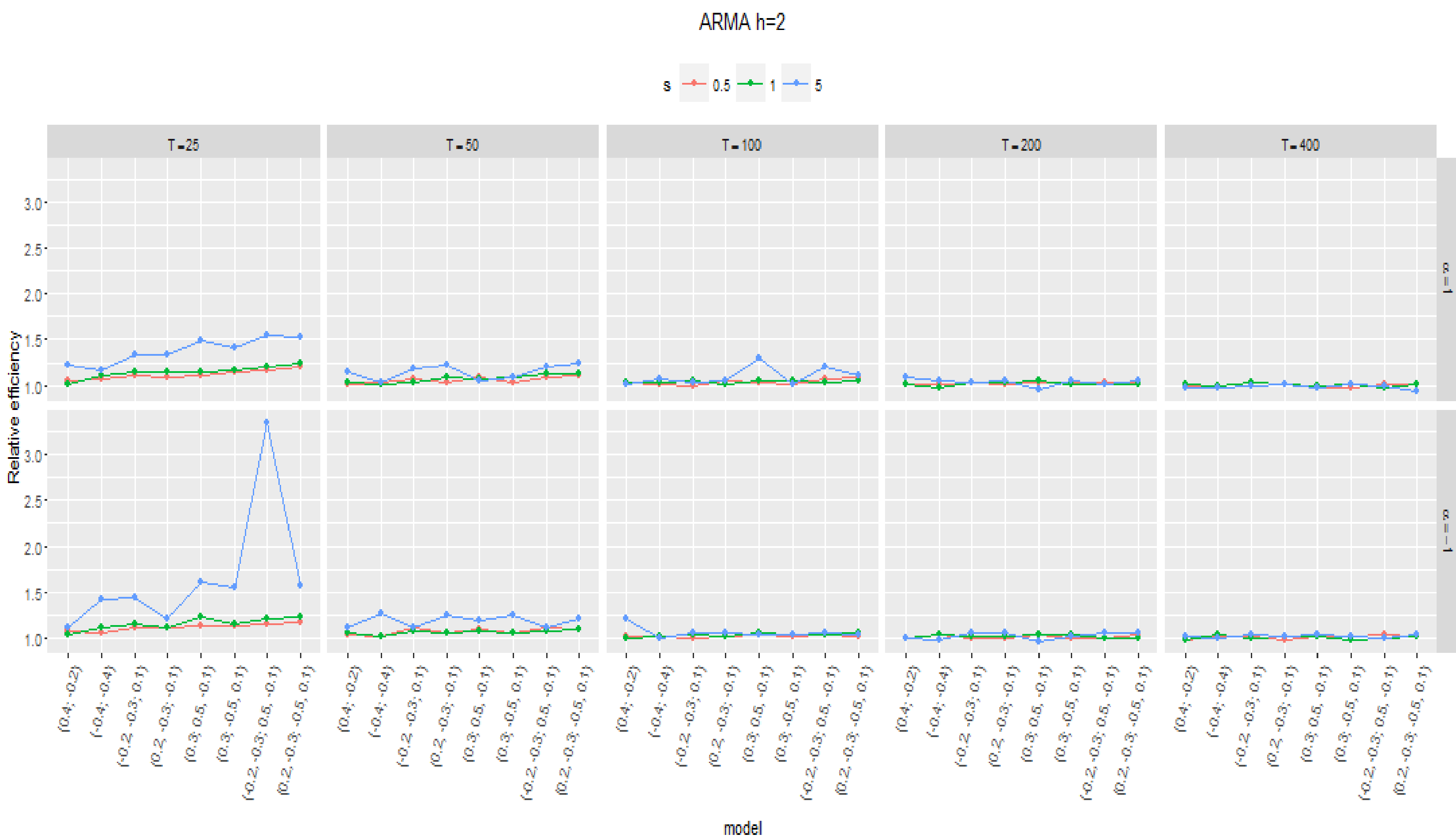

Figure 49. Relative efficiency of ARMA model at h=2

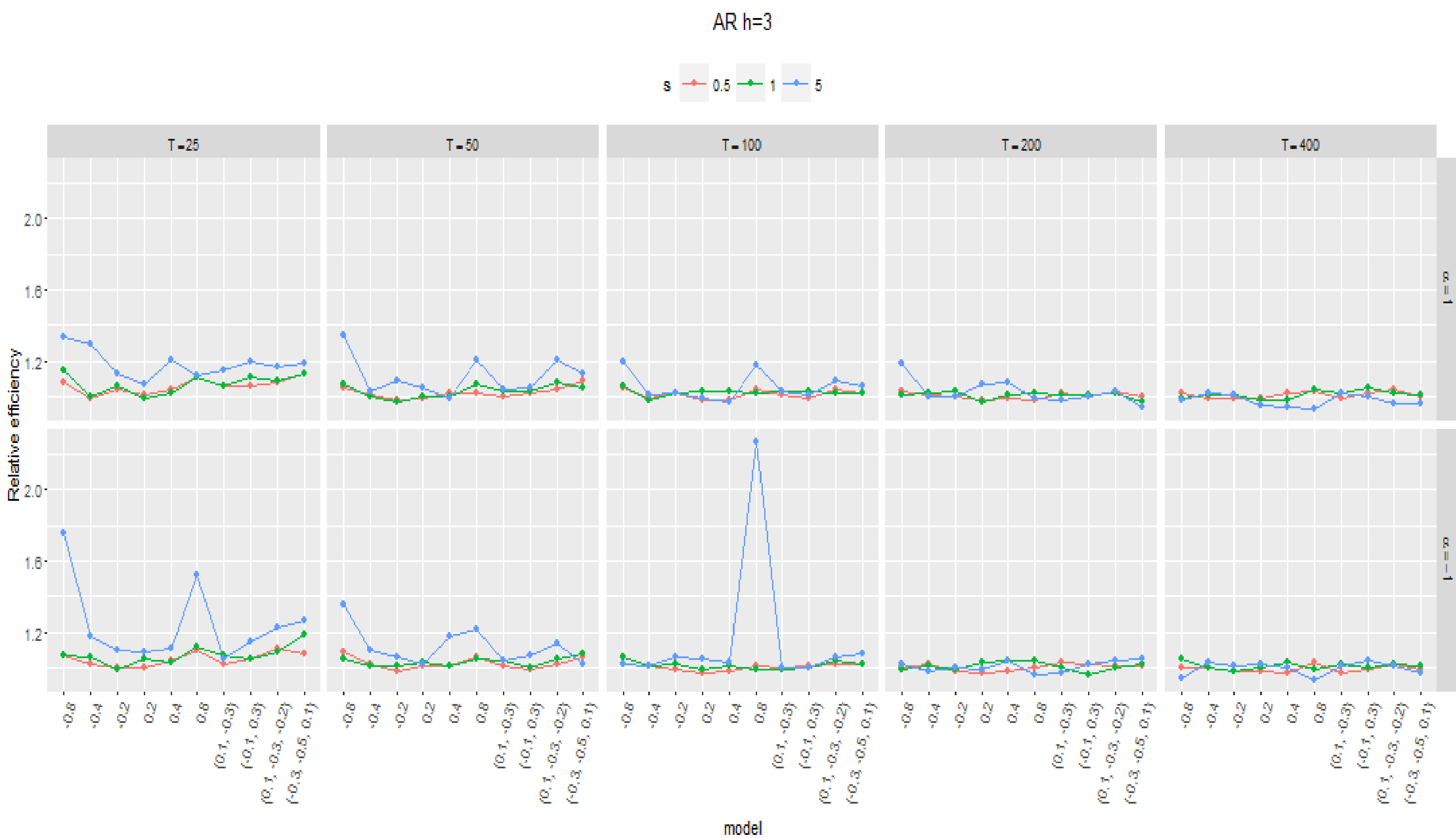

Figure 50. Relative efficiency of AR modelsat h=3

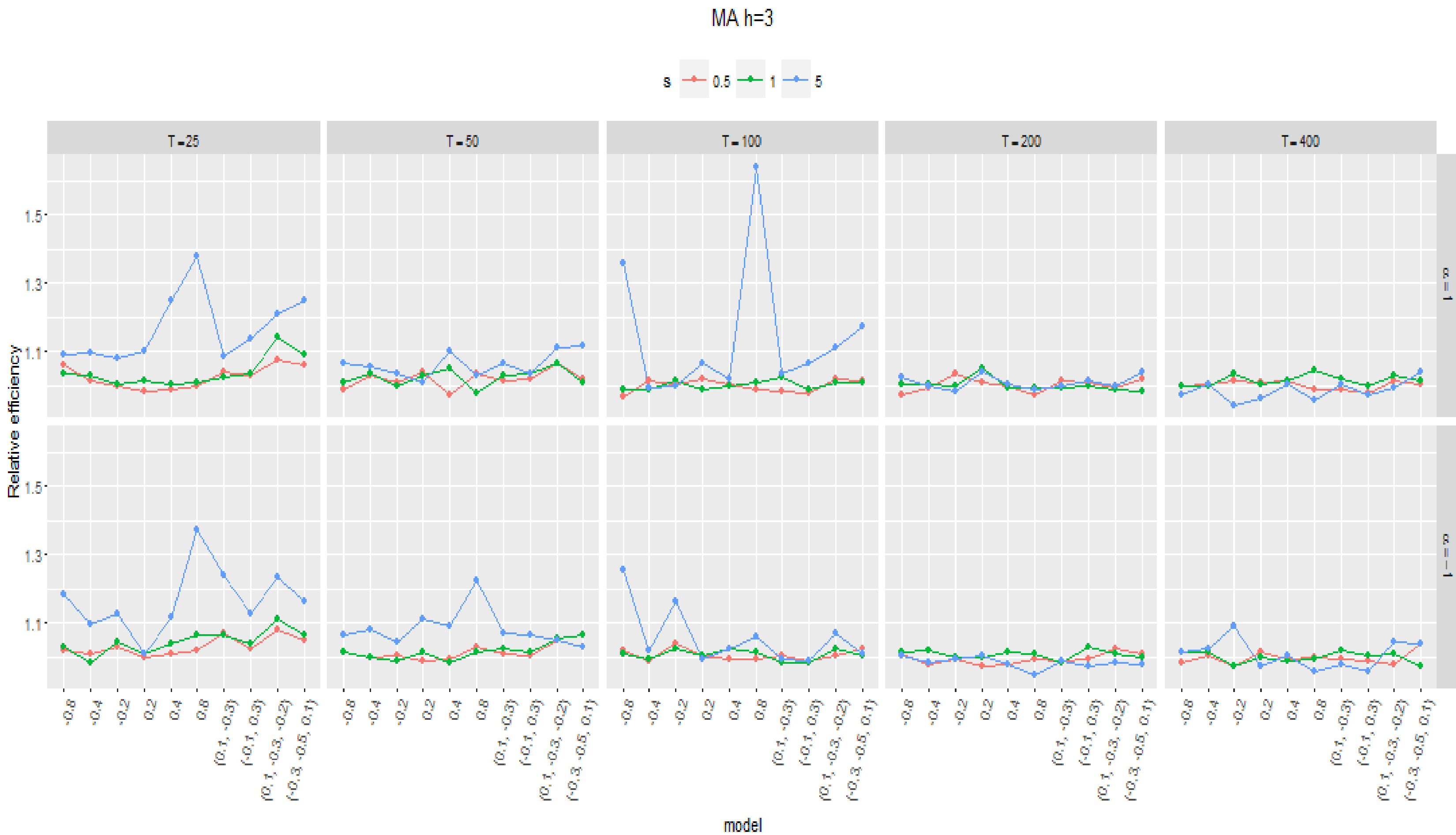

Figure 51. Relative efficiency of MA model at h=3

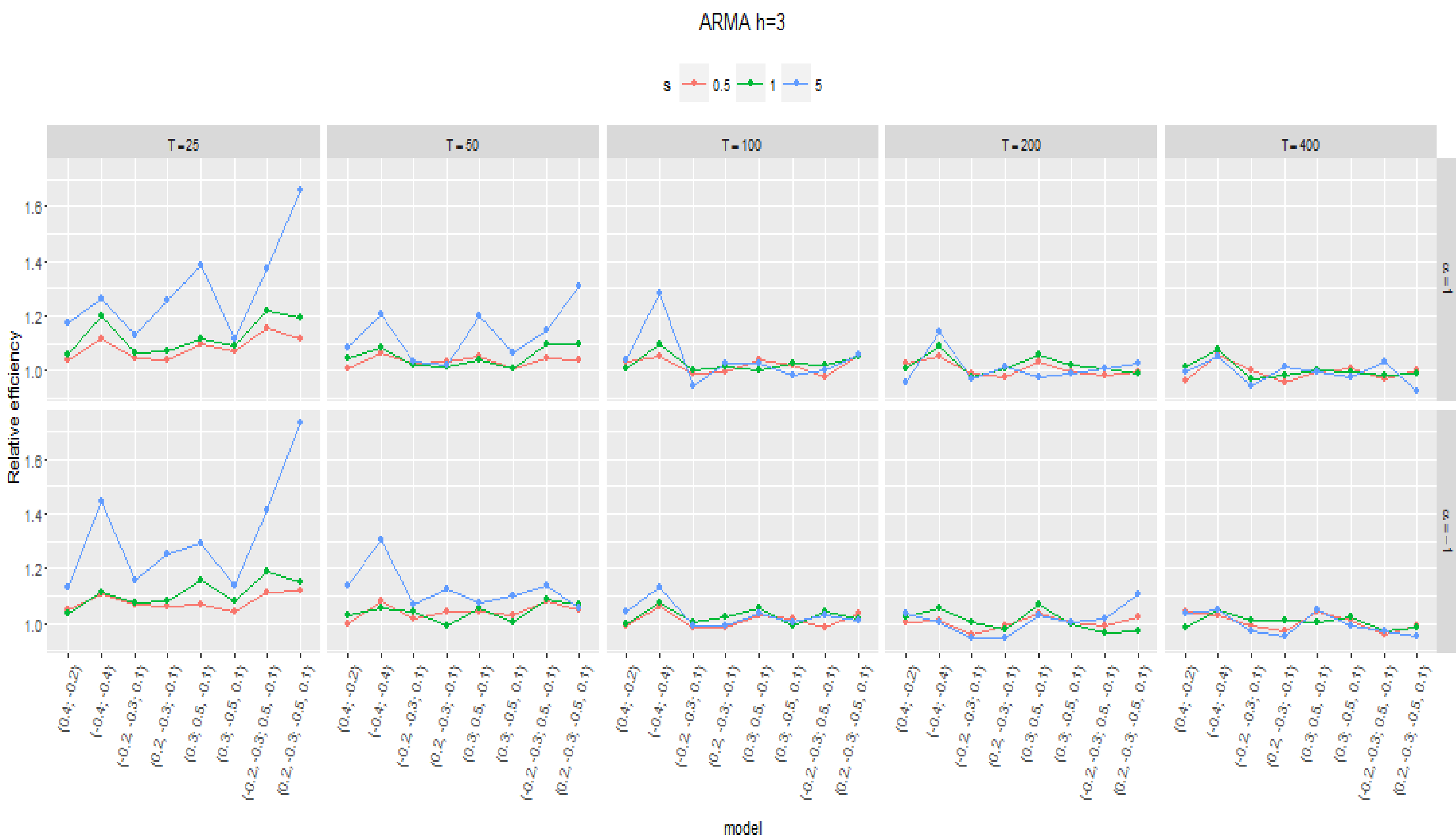

Figure 52. Relative efficiency of ARMA model at h=3

CHAPTER V

CONCLUSIONS

The linex loss is a useful function for asymmetric loss, and the linex unbiased prediction has been developed and applied to real word forecasting. This study demonstrated the linex unbiased prediction when the time series parameters are unknown and being estimated. Varying parameters, variance, series length, forecast steps and the loss function shape parameters allowed the examination to cover more possible outcomes and gain insight to the real world.

This chapter summarized and discussed the results in light of the research questions. The Summary of Findings were discussed first, followed by Limitations of the Study and Recommendations for future research.

Summary of Findings

The linex unbiased predictor, $\hat{Z}_{T+h}$ is known to be risk unbiased. The primary focus of this dissertation was to study how the estimated linex unbiased predictor, $\hat{\hat{Z}}_{T+h}$ behaves for the AR(p), MA(q) and ARMA(p,q) time series models. By computing values of the CLU, which for $\hat{Z}_{T+h}$ is zero, under different variances, series lengths, forecast steps and shape parameters, one is able to measure how much the CLU departs from zero when replacing $\hat{Z}_{T+h}$ with $\hat{\hat{Z}}_{T+h}$.

The conclusion for this study can be summarized as follows. No true zero CLU value was found in this study. In general, a simpler model produced CLU

values that were closer to zero compared to a complex model. For example, for the AR(1) models, the CLU values were closer to zero when ϕ_1 was small; for the MA(q) models, the CLU values were closer to zero when q=1; for the ARMA(p,q) models, the CLU values were closer to zero at ARMA(1,1). Except the nearly non-stationary AR(1) cases, which did not seem to have the CLU quantities like other stationary AR(1) did, and produced values that were as high as ARMA(p,q) models. The nearly non-stationary models and other more complex models required larger series length to approximate the linex unbiasedness.

Also, the AR(2), AR(3), MA(2), MA(3) and all ARMA(p,q) models each had two sets of parameters, and no significant difference was found between the parameter sets. The parameter sets were chosen based on the condition of having complex roots, real roots or both. The results of this study have showed that the condition of linex unbiasedness was not affected by whether time series models have real roots, complex roots, or both, when parameter values are small. As long as the series is stationary, having real roots or complex roots need not be of great concern in this study.

Moreover, the condition of linex unbiasedness was highly affected by the variance. From the figures presented in Chapter IV, the lines in graphs for $\sigma^2 = 5$ series fluctuated, whereas $\sigma^2 = 0.5$ and $\sigma^2 = 1$ series were more stable and consistently around zero. Greater variance required greater series length to approach the linex unbiasedness. The CLU values for $\sigma^2 = 5$ series had a pattern that was different from $\sigma^2 = 1$ and $\sigma^2 = 5$ series. Furthermore, for fixed variance, all three h-step-ahead showed that increasing series length decreased the CLU values.

For small T such as 25, $h = 1$ seemed to have higher CLU values for MA(q) models. Since the conditional mean of MA models became zero after h > q, this might affect the CLU values when the series length was small. Although the conditional prediction error variance increased as h increased, the CLU values for different h levels were essentially determined by the value of parameters, Z_{T+h}, observed series (i.e. $Z_T, Z_{T-1}, \ldots$), variance, α, and the series mean, which was zero in this study.

The sign of linex loss function shape parameter showed no effect on the condition of linex unbiasedness. This was a reasonable result since the linex loss function is designed for handling asymmetric loss, the unbiasedness should not depend on whether the loss is greater for over or under prediction.

Nevertheless, for any model and any condition, as series length increased, the CLU values approached zero. Despite the models, variances and forecast steps, all the unstable or unequal patterns started to became less apparent as series length increases. Having conditions like those from the current study, series length greater than 200 is recommended. Also, the empirical risk agreed with the theoretical risk, the risk increased as variance and forecast step increased. And as series length increased, the relative efficiency got small and close to one. This result corresponded to the result from the study of Patton and Timmermann (2007), who also showed that the risk increased as the forecast step increased, and the study of Fuller and Hasza (1980), who showed that the empirical risk was slightly larger than the theoretical approximation when series length was small.

Series length is essentially sample size. This study has showed that sample size had a strong effect on the condition of linex unbiasedness. Smaller sample size

usually introduces more bias and risk, and a prediction will most likely perform worse. Sufficiently large sample size can perform a more successful prediction and can better approximate the condition of unbiasedness. This research corresponds with the large sample theory, which indicated as the sample size increases, $\hat{\sigma^2} \to \sigma^2$, $\hat{\phi} \to \phi$, $\hat{\theta} \to \theta$ and $\hat{\hat{Z}} \to \hat{Z}$. When parameters of AR(p), MA(q) and ARMA(p,q) are unknown and being estimated, the prediction is asymptotically linex unbiased.

Limitations of the Study

This study was based on stationary univariate time series models in the time domain, so the results only apply to stationary or invertible AR, MA, ARMA models. Thus, the results should not be applied to non-stationary or non-invertible time series and multivariate time series models.

The time series were assumed to be Gaussian, and the loss function was chosen to be the linex loss. These assumptions should be considered based on the nature of the question before applying the results to other designs.

For the purpose of convenience and processing speed of the simulation, some parameters were arbitrarily chosen with values less than or equal to 0.5. All the results of time series models were limited with such a condition. Time series model like ARIMA(p,d,q) was not covered in this study. Thus, the results from ARMA models do not apply to ARIMA.

In this study, $\sigma^2 = 5$ was considered as large variance. However, it is possible to consider $\sigma^2 = 5$ as a small variance in real world applications; but this must be each researcher's own judgment. In addition, for the shape parameter, 1 and -1 were used, thus the magnitude of asymmetry can be considered as small. The CLU

values of this study should not be generalized to bigger magnitudes of asymmetry without further evidence. Finally, 3-step-ahead is usually considered as a short term prediction, the results of this study also should not be generalized to long term prediction without supplementary declaration.

Recommendations for Future Research

The recommendation for future research is extended from the limitations in this study. It is legitimate to hypothesize that using bigger numeric values for model parameters can produce more variety in the results. Considering a AR(3) model with 1 as one of the parameter values, this might have different results when comparing models. Differenced series like ARIMA(p,d,q) was not included in the current study. In practice, a series is not always stationary as desired; hence, the condition of linex unbiasedness for differenced series needs to be explored. The current study also suggests to extend the linex unbiasedness to the ARCH and GARCH models. Although the linex unbiased predictor have been applied to the ARCH and GARCH processes in many researches, the property of linex unbiasedness under these models has not yet been well established. Therefore, providing a source of such models to examine the suggested sample size in order to approximate the linex unbiased prediction will be valuable.

Increasing the magnitude of asymmetry is another suggestion. Since the degree of asymmetric loss varies in real applications, it will be practical to investigate how the linex unbiased prediction behaves as the magnitude of asymmetry changes. The time series used in this study were simulated from Gaussian distribution, however, real world series are not always normally

distributed. Thus, generating data from non-Gaussian distribution and investigating its linex unbiasedness condition will make linex unbiased predictor more useful in practical use. Last but not least, researchers have been using linex unbiased predictor in practical application, but the acceptable range of the CLU values has not been officially established, such as the closeness of the CLU values to zero. This is an interesting area for future research.

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APPENDIXA

TABLES OF THE CLU BY MODEL

Table 13

CLU by Model at $T = 25$ and $\sigma^2 = 0.5$

		T=25					
Models	Parameters	$\alpha = 1$			$\alpha = -1$		
		h=1	h=2	h=3	h=1	h=2	h=3
AR(1)	(-0.8)	0.029	0.053	0.081	0.029	0.044	0.087
	(-0.4)	0.018	0.008	0.017	0.019	0.014	0.015
	(-0.2)	0.015	0.005	0.022	0.019	0.022	-0.005
	(0.2)	0.026	0.002	0.004	0.024	0.007	-0.002
	(0.4)	0.015	0.014	0.004	0.026	0.014	0.015
	(0.8)	0.011	0.055	0.104	0.011	0.049	0.080
MA(1)	(-0.8)	0.047	-0.002	0.037	0.049	0.014	0.007
	(-0.4)	0.042	0.001	0.006	0.020	0.005	0.000*
	(-0.2)	0.040	0.001	0.008	0.029	-0.008	0.006
	(0.2)	0.025	0.007	-0.005	0.028	-0.007	0.006
	(0.4)	0.012	0.000*	-0.013	0.037	0.014	0.009
	(0.8)	0.029	0.036	-0.002	0.019	0.032	0.027
AR(2)	(0.1, -0.3)	0.044	0.031	0.021	0.061	0.022	0.006
	(-0.1, 0.3)	0.038	0.029	0.018	0.037	0.029	0.017
AR(3)	(0.1, -0.3, -0.2)	0.069	0.049	0.043	0.079	0.029	0.046
	(-0.3, -0.5, 0.1)	0.055	0.040	0.074	0.060	0.064	0.061
MA(2)	(0.1, -0.3)	0.080	0.044	0.011	0.054	0.046	0.017
	(-0.1, 0.3)	0.066	0.040	0.013	0.060	0.021	0.011
MA(3)	(0.1, -0.3, -0.2)	0.083	0.075	0.033	0.095	0.073	0.048
	(-0.3, -0.5, 0.1)	0.094	0.092	0.024	0.075	0.094	0.045
ARMA(1,1)	(0.4; -0.2)	0.046	0.023	0.024	0.044	0.033	0.010
	(-0.4; -0.4)	0.057	0.059	0.043	0.054	0.051	0.043
ARMA(2,1)	(-0.2, -0.3; 0.1)	0.069	0.045	0.026	0.075	0.060	0.052
	(0.2, -0.3; -0.1)	0.080	0.035	0.033	0.086	0.047	0.037
ARMA(1,2)	(0.3; 0.5, -0.1)	0.076	0.072	0.048	0.080	0.077	0.020
	(0.3; -0.5, 0.1)	0.091	0.065	0.024	0.094	0.056	0.037
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	0.126	0.093	0.104	0.096	0.072	0.095
	(0.2, -0.3; -0.5, 0.1)	0.128	0.106	0.060	0.125	0.085	0.080

Note. $0.000^* < \pm\ 0.0001$.

Table 14

CLU by Model at $T = 50$ and $\sigma^2 = 0.5$

Models	Parameters	T=50					
		$\alpha = 1$			$\alpha = -1$		
		h=1	h=2	h=3	h=1	h=2	h=3
AR(1)	(-0.8)	0.002	0.030	0.047	0.004	0.031	0.070
	(-0.4)	0.021	0.000*	0.022	0.005	0.013	0.010
	(-0.2)	0.003	0.003	-0.008	0.018	0.004	-0.006
	(0.2)	0.013	0.001	0.000*	0.024	0.010	0.021
	(0.4)	0.023	0.020	0.016	0.012	0.006	0.013
	(0.8)	-0.009	0.027	0.035	0.011	0.026	0.043
MA(1)	(-0.8)	0.006	0.011	0.003	0.015	-0.012	0.020
	(-0.4)	0.014	0.005	0.017	0.010	0.000*	-0.001
	(-0.2)	0.009	-0.001	0.001	0.011	-0.007	-0.002
	(0.2)	0.013	0.000*	0.032	0.007	0.009	-0.001
	(0.4)	-0.001	-0.010	-0.010	0.015	0.008	-0.003
	(0.8)	0.008	-0.002	0.002	0.024	0.013	0.012
AR(2)	(0.1, -0.3)	0.033	0.015	0.008	0.034	0.015	0.007
	(-0.1, 0.3)	0.013	0.008	0.001	0.030	0.024	0.006
AR(3)	(0.1, -0.3, -0.2)	0.016	0.023	0.031	0.040	0.013	0.017
	(-0.3, -0.5, 0.1)	0.026	0.020	0.055	0.030	0.029	0.030
MA(2)	(0.1, -0.3)	0.027	0.011	0.011	0.032	0.013	-0.010
	(-0.1, 0.3)	0.022	0.013	-0.006	0.015	0.008	-0.008
MA(3)	(0.1, -0.3, -0.2)	0.058	0.031	0.007	0.050	0.033	0.014
	(-0.3, -0.5, 0.1)	0.029	0.040	0.028	0.050	0.033	0.027
ARMA(1,1)	(0.4; -0.2)	0.023	0.005	0.009	0.024	0.018	0.007
	(-0.4; -0.4)	0.029	0.031	0.017	0.018	0.039	0.011
ARMA(2,1)	(-0.2, -0.3; 0.1)	0.037	0.025	0.026	0.037	0.039	0.015
	(0.2, -0.3; -0.1)	0.032	0.020	0.030	0.021	0.031	0.028
ARMA(1,2)	(0.3; 0.5, -0.1)	0.033	0.070	0.023	0.043	0.068	0.016
	(0.3; -0.5, 0.1)	0.038	0.017	0.012	0.037	0.028	0.003
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	0.065	0.054	0.034	0.072	0.051	0.045
	(0.2, -0.3; -0.5, 0.1)	0.058	0.056	0.026	0.053	0.049	0.034

Note. 0.000* < ± 0.0001.

Table 15

CLU by Model at $T = 100$ and $\sigma^2 = 0.5$

T=100							
Models	Parameters	$\alpha = 1$			$\alpha = -1$		
		h=1	h=2	h=3	h=1	h=2	h=3
AR(1)	(-0.8)	0.006	0.041	0.036	-0.003	0.032	0.028
	(-0.4)	0.017	0.012	0.006	0.020	-0.004	-0.004
	(-0.2)	-0.003	-0.010	0.015	-0.003	0.002	-0.005
	(0.2)	0.016	-0.008	-0.004	0.022	0.001	0.002
	(0.4)	0.009	-0.001	-0.011	0.008	0.004	0.000*
	(0.8)	0.026	0.043	0.065	-0.006	0.005	0.023
MA(1)	(-0.8)	0.013	-0.009	-0.003	0.006	0.011	-0.009
	(-0.4)	-0.002	-0.019	0.018	0.007	-0.001	0.003
	(-0.2)	0.011	-0.010	0.004	0.007	0.009	0.010
	(0.2)	0.002	0.006	0.003	0.011	-0.001	-0.001
	(0.4)	-0.002	-0.003	0.006	0.005	0.012	-0.004
	(0.8)	0.013	0.008	0.005	-0.002	0.012	0.007
AR(2)	(0.1, -0.3)	0.008	0.011	0.009	0.002	0.007	0.009
	(-0.1, 0.3)	0.007	0.012	0.007	0.016	0.014	0.006
AR(3)	(0.1, -0.3, -0.2)	0.017	0.016	0.011	0.028	0.012	0.001
	(-0.3, -0.5, 0.1)	-0.009	0.024	0.020	0.002	0.020	0.028
MA(2)	(0.1, -0.3)	0.003	0.007	-0.004	0.002	0.007	0.006
	(-0.1, 0.3)	0.007	0.001	-0.009	0.005	0.004	-0.014
MA(3)	(0.1, -0.3, -0.2)	0.023	0.006	-0.009	0.032	0.029	0.013
	(-0.3, -0.5, 0.1)	0.027	0.020	0.020	0.016	0.004	0.013
ARMA(1,1)	(0.4; -0.2)	-0.003	0.004	0.013	0.018	0.006	0.003
	(-0.4; -0.4)	0.012	0.015	0.011	0.005	0.003	0.011
ARMA(2,1)	(-0.2, -0.3; 0.1)	0.021	0.003	0.007	0.024	0.002	0.001
	(0.2, -0.3; -0.1)	0.021	0.011	0.011	0.019	0.009	0.018
ARMA(1,2)	(0.3; 0.5, -0.1)	0.019	0.014	0.017	0.007	0.022	0.020
	(0.3; -0.5, 0.1)	0.024	0.018	0.016	0.023	0.013	0.002
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	0.024	0.040	0.015	0.030	0.012	0.006
	(0.2, -0.3; -0.5, 0.1)	0.009	0.034	0.040	0.029	0.016	0.016

Note. $0.000^* < \pm\ 0.0001$.

Table 16

CLU by Model at $T = 200$ and $\sigma^2 = 0.5$

T=200								
Models	Parameters	$\alpha = 1$			$\alpha = -1$		
		h=1	h=2	h=3	h=1	h=2	h=3
AR(1)	(-0.8)	-0.003	0.018	0.021	-0.002	0.003	0.005
	(-0.4)	-0.006	0.023	-0.008	-0.004	0.005	-0.008
	(-0.2)	0.008	-0.005	0.013	0.007	-0.011	-0.001
	(0.2)	-0.005	-0.012	-0.004	0.001	-0.010	-0.006
	(0.4)	0.010	0.015	0.012	0.001	-0.001	-0.001
	(0.8)	-0.003	-0.013	-0.017	-0.006	-0.002	0.009
MA(1)	(-0.8)	0.005	-0.017	-0.008	0.007	0.001	0.011
	(-0.4)	-0.005	0.023	-0.021	0.006	-0.005	-0.003
	(-0.2)	-0.008	-0.001	0.008	0.006	-0.001	-0.008
	(0.2)	0.004	-0.001	0.007	0.006	-0.006	-0.003
	(0.4)	0.008	-0.001	-0.010	0.005	0.011	-0.016
	(0.8)	0.015	-0.002	-0.009	-0.005	-0.012	0.001
AR(2)	(0.1, -0.3)	0.007	0.008	0.010	0.001	0.001	0.025
	(-0.1, 0.3)	0.005	0.002	0.006	0.013	0.015	-0.005
AR(3)	(0.1, -0.3, -0.2)	0.011	0.017	0.010	0.008	0.003	0.007
	(-0.3, -0.5, 0.1)	0.012	0.007	-0.007	0.009	0.017	-0.002
MA(2)	(0.1, -0.3)	0.016	0.008	0.003	0.011	0.014	0.001
	(-0.1, 0.3)	0.004	0.007	0.009	0.001	0.007	0.004
MA(3)	(0.1, -0.3, -0.2)	0.004	-0.003	0.001	-0.002	0.008	0.008
	(-0.3, -0.5, 0.1)	0.016	0.002	-0.015	-0.005	0.009	0.008
ARMA(1,1)	(0.4; -0.2)	0.014	0.011	0.014	0.013	0.014	-0.005
	(-0.4; -0.4)	0.007	0.013	-0.001	0.011	0.020	-0.016
ARMA(2,1)	(-0.2, -0.3; 0.1)	0.007	0.011	0.011	0.009	0.011	-0.004
	(0.2, -0.3; -0.1)	0.011	-0.005	0.005	0.015	0.003	0.010
ARMA(1,2)	(0.3; 0.5, -0.1)	0.015	0.032	0.033	0.014	0.017	-0.004
	(0.3; -0.5, 0.1)	0.000*	0.009	-0.001	0.005	0.010	-0.015
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	0.000*	0.001	0.005	0.015	0.011	0.004
	(0.2, -0.3; -0.5, 0.1)	0.009	0.004	0.009	0.003	0.011	0.013

Note. $0.000^* < \pm\ 0.0001$.

Table 17

CLU by Model at $T = 400$ and $\sigma^2 = 0.5$

		T = 400					
Models	Parameters	$\alpha = 1$			$\alpha = -1$		
		h=1	h=2	h=3	h=1	h=2	h=3
AR(1)	(-0.8)	0.009	-0.001	0.021	0.001	-0.011	0.015
	(-0.4)	-0.016	0.003	-0.003	0.005	0.001	0.001
	(-0.2)	0.006	0.012	0.006	-0.011	0.002	0.000*
	(0.2)	0.003	0.014	0.000*	0.003	-0.001	0.004
	(0.4)	0.011	-0.008	0.014	-0.012	0.001	-0.016
	(0.8)	-0.001	0.008	0.008	0.009	0.000*	-0.005
MA(1)	(-0.8)	0.000*	-0.004	0.001	0.001	-0.014	-0.004
	(-0.4)	0.001	0.003	-0.003	-0.006	0.010	0.000*
	(-0.2)	-0.003	0.003	-0.003	0.002	0.005	-0.025
	(0.2)	0.011	-0.001	0.011	0.000*	0.005	-0.005
	(0.4)	0.002	0.003	-0.004	0.005	0.005	-0.007
	(0.8)	0.021	0.010	0.003	-0.010	0.012	0.012
AR(2)	(0.1, -0.3)	0.001	0.006	-0.004	0.000*	-0.012	-0.005
	(-0.1, 0.3)	0.007	0.000*	0.012	0.009	0.001	0.000*
AR(3)	(0.1, -0.3, -0.2)	0.015	-0.002	0.008	0.000*	0.006	-0.012
	(-0.3, -0.5, 0.1)	0.013	0.009	0.007	-0.013	0.022	-0.003
MA(2)	(0.1, -0.3)	0.003	-0.001	-0.006	0.013	0.005	-0.005
	(-0.1, 0.3)	-0.003	0.000*	-0.009	0.000*	0.009	-0.010
MA(3)	(0.1, -0.3, -0.2)	-0.003	0.001	-0.001	-0.006	0.011	0.003
	(-0.3, -0.5, 0.1)	0.008	-0.006	-0.004	0.015	0.006	-0.002
ARMA(1,1)	(0.4; -0.2)	0.002	-0.007	0.002	-0.003	0.007	0.015
	(-0.4; -0.4)	-0.004	-0.007	0.003	-0.005	0.011	0.005
ARMA(2,1)	(-0.2, -0.3; 0.1)	0.008	0.004	0.004	0.012	0.006	0.003
	(0.2, -0.3; -0.1)	0.006	-0.001	-0.002	0.004	-0.019	-0.010
ARMA(1,2)	(0.3; 0.5, -0.1)	0.003	-0.008	-0.012	0.001	0.015	0.013
	(0.3; -0.5, 0.1)	0.011	-0.008	0.001	-0.012	0.010	0.001
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	-0.004	0.011	0.021	0.010	0.016	-0.004
	(0.2, -0.3; -0.5, 0.1)	-0.009	0.002	0.018	0.013	0.008	0.001

Note. 0.000* < ± 0.0001.

Table 18

CLU by Model at $T = 25$ and $\sigma^2 = 1$

Models	Parameters	$\alpha = 1$			$\alpha = -1$		
		h=1	h=2	h=3	h=1	h=2	h=3
AR(1)	(-0.8)	0.035	0.186	0.236	0.042	0.119	0.166
	(-0.4)	0.059	0.036	0.024	0.089	0.041	0.047
	(-0.2)	0.070	0.012	0.051	0.023	0.052	0.015
	(0.2)	0.042	0.015	-0.006	0.052	-0.015	0.021
	(0.4)	0.055	0.041	0.025	0.063	0.029	0.011
	(0.8)	0.019	0.112	0.214	0.051	0.129	0.198
MA(1)	(-0.8)	0.062	0.016	0.046	0.048	0.086	0.046
	(-0.4)	0.064	0.026	0.026	0.078	0.016	-0.003
	(-0.2)	0.093	0.013	-0.013	0.073	-0.001	0.036
	(0.2)	0.046	0.017	0.003	0.065	0.007	0.006
	(0.4)	0.049	0.022	-0.008	0.065	0.014	0.021
	(0.8)	0.052	0.023	0.040	0.075	0.074	0.068
AR(2)	(0.1, -0.3)	0.088	0.078	0.044	0.084	0.037	0.061
	(-0.1, 0.3)	0.104	0.048	0.075	0.093	0.052	0.057
AR(3)	(0.1, -0.3, -0.2)	0.147	0.076	0.099	0.154	0.095	0.103
	(-0.3, -0.5, 0.1)	0.137	0.113	0.139	0.175	0.090	0.197
MA(2)	(0.1, -0.3)	0.111	0.108	0.020	0.127	0.074	0.034
	(-0.1, 0.3)	0.145	0.056	0.009	0.137	0.083	0.008
MA(3)	(0.1, -0.3, -0.2)	0.204	0.124	0.120	0.221	0.149	0.076
	(-0.3, -0.5, 0.1)	0.201	0.115	0.098	0.194	0.141	0.083
ARMA(1,1)	(0.4; -0.2)	0.101	0.017	0.036	0.087	0.025	0.042
	(-0.4; -0.4)	0.103	0.136	0.179	0.104	0.147	0.094
ARMA(2,1)	(-0.2, -0.3; 0.1)	0.172	0.128	0.090	0.166	0.146	0.089
	(0.2, -0.3; -0.1)	0.141	0.145	0.071	0.163	0.107	0.075
ARMA(1,2)	(0.3; 0.5, -0.1)	0.195	0.223	0.133	0.197	0.257	0.138
	(0.3; -0.5, 0.1)	0.176	0.124	0.064	0.191	0.129	0.032
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	0.272	0.194	0.230	0.247	0.186	0.220
	(0.2, -0.3; -0.5, 0.1)	0.255	0.228	0.202	0.274	0.225	0.161

Table 19

CLU by Model at $T = 50$ and $\sigma^2 = 1$

T = 50							
Models	Parameters	$\alpha = 1$			$\alpha = -1$		
		h=1	h=2	h=3	h=1	h=2	h=3
AR(1)	(-0.8)	0.028	0.056	0.121	0.026	0.065	0.103
	(-0.4)	0.022	0.007	0.021	0.029	0.034	0.017
	(-0.2)	0.015	-0.008	-0.028	0.046	0.011	0.002
	(0.2)	0.006	-0.002	0.009	0.034	0.013	0.026
	(0.4)	0.018	0.022	-0.017	0.022	0.002	0.002
	(0.8)	0.034	0.022	0.091	0.035	0.067	0.114
MA(1)	(-0.8)	0.038	-0.002	-0.004	0.046	-0.006	0.000*
	(-0.4)	0.048	0.002	0.030	0.028	-0.008	-0.005
	(-0.2)	0.011	0.024	0.002	0.040	0.020	-0.016
	(0.2)	0.037	0.001	0.017	0.027	0.000*	-0.013
	(0.4)	0.036	0.017	0.024	0.054	0.031	-0.020
	(0.8)	0.053	0.020	-0.011	0.033	-0.007	0.009
AR(2)	(0.1, -0.3)	0.042	0.039	0.034	0.038	0.051	0.033
	(-0.1, 0.3)	0.041	-0.009	0.024	0.030	0.029	0.009
AR(3)	(0.1, -0.3, -0.2)	0.065	0.048	0.079	0.057	0.027	0.050
	(-0.3, -0.5, 0.1)	0.044	0.028	0.061	0.056	0.072	0.064
MA(2)	(0.1, -0.3)	0.086	0.060	0.002	0.072	0.031	-0.004
	(-0.1, 0.3)	0.074	0.012	0.026	0.044	0.033	0.003
MA(3)	(0.1, -0.3, -0.2)	0.077	0.060	0.062	0.077	0.060	0.049
	(-0.3, -0.5, 0.1)	0.074	0.076	0.019	0.109	0.083	0.040
ARMA(1,1)	(0.4; -0.2)	0.026	0.011	0.035	0.036	0.024	0.013
	(-0.4; -0.4)	0.013	0.034	0.060	0.044	0.027	0.026
ARMA(2,1)	(-0.2, -0.3; 0.1)	0.052	0.066	0.058	0.124	0.053	0.036
	(0.2, -0.3; -0.1)	0.083	0.064	0.047	0.056	0.049	0.021
ARMA(1,2)	(0.3; 0.5, -0.1)	0.083	0.072	0.032	0.089	0.104	0.055
	(0.3; -0.5, 0.1)	0.076	0.067	0.001	0.104	0.066	0.020
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	0.111	0.096	0.114	0.128	0.091	0.120
	(0.2, -0.3; -0.5, 0.1)	0.104	0.119	0.118	0.131	0.096	0.082

Note. 0.000* < ± 0.0001.

Table 20

CLU by Model at $T = 100$ and $\sigma^2 = 1$

T =100							
Models	Parameters	$\alpha = 1$			$\alpha = -1$		
		h=1	h=2	h=3	h=1	h=2	h=3
AR(1)	(-0.8)	0.011	0.065	0.078	0.023	0.035	0.101
	(-0.4)	0.020	-0.006	-0.013	0.017	0.011	0.004
	(-0.2)	0.010	0.011	0.003	0.003	0.008	0.023
	(0.2)	0.005	0.010	0.036	0.025	0.007	-0.014
	(0.4)	0.013	0.019	0.032	0.007	0.017	0.013
	(0.8)	0.025	0.043	0.066	-0.007	0.004	0.019
MA(1)	(-0.8)	0.024	0.028	-0.014	0.022	0.027	-0.008
	(-0.4)	0.002	0.008	-0.009	0.024	-0.013	-0.015
	(-0.2)	0.009	0.000*	0.012	0.001	0.011	0.018
	(0.2)	0.022	0.002	-0.004	-0.006	-0.022	-0.004
	(0.4)	0.019	0.000*	0.001	0.028	0.017	0.015
	(0.8)	-0.003	0.016	0.016	0.009	0.004	0.013
AR(2)	(0.1, -0.3)	0.028	-0.004	0.024	0.022	-0.025	0.001
	(-0.1, 0.3)	0.022	0.021	0.003	0.048	-0.009	0.023
AR(3)	(0.1, -0.3, -0.2)	0.045	0.028	0.010	0.017	0.019	0.041
	(-0.3, -0.5, 0.1)	0.016	0.034	0.022	0.011	0.007	0.031
MA(2)	(0.1, -0.3)	0.034	0.016	0.017	0.018	0.021	-0.005
	(-0.1, 0.3)	0.043	0.007	-0.001	0.025	-0.001	0.006
MA(3)	(0.1, -0.3, -0.2)	0.029	0.025	0.019	0.040	0.008	0.010
	(-0.3, -0.5, 0.1)	0.031	0.042	0.026	0.045	0.055	-0.015
ARMA(1,1)	(0.4; -0.2)	0.011	0.013	0.006	0.011	-0.009	-0.007
	(-0.4; -0.4)	0.037	0.039	0.054	0.044	-0.003	0.052
ARMA(2,1)	(-0.2, -0.3; 0.1)	0.050	0.036	0.031	0.036	0.038	0.021
	(0.2, -0.3; -0.1)	0.039	0.030	0.014	0.013	-0.005	0.026
ARMA(1,2)	(0.3; 0.5, -0.1)	0.048	0.043	-0.012	0.037	0.067	0.045
	(0.3; -0.5, 0.1)	0.041	0.056	0.015	0.035	0.031	-0.021
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	0.055	0.031	0.038	0.031	0.008	0.073
	(0.2, -0.3; -0.5, 0.1)	0.058	0.043	0.065	0.058	0.040	0.050

Note. $0.000^* < \pm\ 0.0001$.

Table 21

CLU by Model at $T = 200$ and $\sigma^2 = 1$

		T=200					
Models	Parameters	$\alpha = 1$			$\alpha = -1$		
		h=1	h=2	h=3	h=1	h=2	h=3
AR(1)	(-0.8)	-0.017	0.026	0.007	-0.004	0.037	0.001
	(-0.4)	-0.003	0.010	0.015	-0.009	0.012	0.021
	(-0.2)	-0.004	0.017	-0.002	-0.001	0.000*	0.000*
	(0.2)	0.023	-0.003	-0.021	0.009	-0.010	0.028
	(0.4)	0.009	0.000*	-0.012	0.024	0.000*	0.028
	(0.8)	0.005	0.028	0.046	0.020	0.031	0.025
MA(1)	(-0.8)	0.040	0.023	0.006	0.004	-0.012	0.017
	(-0.4)	-0.001	0.023	-0.006	0.005	0.011	0.016
	(-0.2)	0.002	0.028	-0.003	-0.004	0.006	-0.005
	(0.2)	-0.003	0.020	0.028	0.005	-0.004	0.007
	(0.4)	-0.004	-0.025	-0.020	0.011	0.020	0.026
	(0.8)	0.005	-0.012	-0.003	0.007	-0.010	-0.001
AR(2)	(0.1, -0.3)	0.000	0.013	0.026	-0.001	0.007	0.000*
	(-0.1, 0.3)	0.022	0.033	0.002	0.013	0.019	-0.006
AR(3)	(0.1, -0.3, -0.2)	0.016	-0.016	0.030	0.017	-0.007	0.002
	(-0.3, -0.5, 0.1)	0.012	-0.001	0.008	0.028	0.002	0.027
MA(2)	(0.1, -0.3)	0.005	-0.019	0.007	-0.006	-0.004	0.010
	(-0.1, 0.3)	-0.013	-0.005	-0.022	0.026	-0.002	0.020
MA(3)	(0.1, -0.3, -0.2)	0.003	0.016	0.006	-0.001	0.010	0.018
	(-0.3, -0.5, 0.1)	0.022	0.029	-0.016	0.012	0.001	0.009
ARMA(1,1)	(0.4; -0.2)	-0.007	0.002	0.014	0.033	-0.007	0.011
	(-0.4; -0.4)	0.034	-0.027	0.063	0.013	0.047	0.005
ARMA(2,1)	(-0.2, -0.3; 0.1)	0.027	0.013	0.010	0.013	0.030	0.005
	(0.2, -0.3; -0.1)	0.022	0.003	0.018	0.019	0.028	0.007
ARMA(1,2)	(0.3; 0.5, -0.1)	0.032	0.042	0.021	0.037	0.044	0.045
	(0.3; -0.5, 0.1)	0.005	0.013	0.019	0.011	0.021	0.008
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	0.048	0.031	0.006	0.007	-0.003	0.031
	(0.2, -0.3; -0.5, 0.1)	0.001	0.013	0.005	0.044	0.030	-0.014

Note. 0.000* $< \pm$ 0.0001.

Table 22

CLU by Model at $T = 400$ and $\sigma^2 = 1$

		T = 400					
Models	Parameters	$\alpha = 1$			$\alpha = -1$		
		h=1	h=2	h=3	h=1	h=2	h=3
AR(1)	(-0.8)	0.007	0.015	-0.010	0.010	0.042	0.037
	(-0.4)	0.000*	0.005	0.011	0.011	-0.002	-0.004
	(-0.2)	-0.007	-0.001	0.001	0.010	0.004	-0.016
	(0.2)	0.001	0.005	-0.006	0.001	-0.001	0.010
	(0.4)	0.000*	-0.001	0.019	0.007	0.017	0.017
	(0.8)	-0.002	0.030	0.055	0.012	0.001	-0.010
MA(1)	(-0.8)	0.015	0.022	-0.016	-0.005	0.026	-0.004
	(-0.4)	0.000*	0.013	0.014	0.008	0.019	-0.003
	(-0.2)	-0.013	-0.013	0.030	-0.005	-0.019	-0.020
	(0.2)	0.003	-0.019	0.009	0.011	-0.002	-0.005
	(0.4)	-0.013	-0.014	0.024	0.007	0.003	0.007
	(0.8)	0.022	0.019	0.033	0.004	0.035	0.006
AR(2)	(0.1, -0.3)	0.011	0.012	0.012	0.008	-0.008	0.017
	(-0.1, 0.3)	0.025	0.004	0.046	0.001	-0.012	0.000*
AR(3)	(0.1, -0.3, -0.2)	0.040	0.014	0.017	0.017	0.002	-0.016
	(-0.3, -0.5, 0.1)	0.008	-0.008	-0.002	0.005	0.002	0.008
MA(2)	(0.1, -0.3)	0.006	0.010	0.019	-0.006	-0.002	0.015
	(-0.1, 0.3)	-0.003	0.022	0.000*	0.004	0.015	-0.004
MA(3)	(0.1, -0.3, -0.2)	0.032	0.004	0.014	0.004	0.004	0.009
	(-0.3, -0.5, 0.1)	-0.008	-0.010	0.020	0.027	0.026	-0.011
ARMA(1,1)	(0.4; -0.2)	0.018	0.010	-0.002	0.017	-0.004	-0.009
	(-0.4; -0.4)	0.013	-0.003	0.000*	0.005	0.030	-0.009
ARMA(2,1)	(-0.2, -0.3; 0.1)	-0.012	0.009	-0.003	0.013	0.008	0.029
	(0.2, -0.3; -0.1)	0.014	0.020	0.003	-0.006	0.008	0.011
ARMA(1,2)	(0.3; 0.5, -0.1)	0.007	-0.007	-0.011	0.002	0.006	-0.018
	(0.3; -0.5, 0.1)	0.023	0.000*	-0.007	0.016	-0.020	0.020
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	0.030	-0.016	0.004	0.022	0.015	-0.007
	(0.2, -0.3; -0.5, 0.1)	0.022	0.019	-0.007	0.001	0.019	0.004

Note. 0.000* < ± 0.0001.

Table 23

CLU by Model at $T = 25$ and $\sigma^2 = 5$

Models	Parameters	T = 25					
		$\alpha = 1$			$\alpha = -1$		
		h=1	h=2	h=3	h=1	h=2	h=3
AR(1)	(-0.8)	0.648	0.895	2.178	0.328	1.076	4.291
	(-0.4)	0.607	0.770	0.936	0.501	0.507	0.552
	(-0.2)	0.363	0.291	0.346	0.719	0.215	0.243
	(0.2)	0.609	0.310	0.201	0.302	0.301	0.235
	(0.4)	0.143	0.593	0.683	0.916	0.466	0.399
	(0.8)	0.516	1.088	1.078	1.032	2.934	3.109
MA(1)	(-0.8)	0.859	0.825	0.453	0.561	1.170	0.883
	(-0.4)	0.406	0.434	0.321	0.648	0.302	0.288
	(-0.2)	0.665	0.281	0.196	2.308	0.027	0.286
	(0.2)	0.649	0.475	0.262	0.454	0.337	-0.025
	(0.4)	0.437	0.365	0.725	0.376	0.329	0.354
	(0.8)	0.420	0.617	1.616	0.295	0.660	1.611
AR(2)	(0.1, -0.3)	0.828	0.590	0.560	0.820	0.891	0.240
	(-0.1, 0.3)	0.565	0.886	0.642	0.909	0.427	0.498
AR(3)	(0.1, -0.3, -0.2)	0.955	0.784	0.675	1.669	1.078	0.830
	(-0.3, -0.5, 0.1)	1.814	1.401	0.918	1.575	1.201	1.183
MA(2)	(0.1, -0.3)	1.040	0.624	0.212	2.681	1.384	0.668
	(-0.1, 0.3)	7.288	1.008	0.300	1.032	1.660	0.290
MA(3)	(0.1, -0.3, -0.2)	2.007	1.746	0.724	2.095	1.884	0.773
	(-0.3, -0.5, 0.1)	2.063	1.275	1.031	2.136	1.575	0.755
ARMA(1,1)	(0.4; -0.2)	1.416	0.647	0.543	1.163	0.389	0.443
	(-0.4; -0.4)	0.539	0.910	1.080	1.102	2.003	1.797
ARMA(2,1)	(-0.2, -0.3; 0.1)	1.554	1.071	0.555	1.551	1.312	0.594
	(0.2, -0.3; -0.1)	2.313	1.080	0.875	1.529	0.792	0.922
ARMA(1,2)	(0.3; 0.5, -0.1)	1.707	2.401	1.735	1.337	2.869	1.343
	(0.3; -0.5, 0.1)	1.361	1.335	0.329	2.282	1.685	0.400
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	2.167	1.867	1.708	2.908	6.769	1.807
	(0.2, -0.3; -0.5, 0.1)	2.468	1.804	2.390	2.101	1.997	2.573

Table 24

CLU by Model at $T = 50$ and $\sigma^2 = 5$

		T = 50					
Models	Parameters	$\alpha = 1$			$\alpha = -1$		
		h=1	h=2	h=3	h=1	h=2	h=3
AR(1)	(-0.8)	0.054	0.331	1.996	0.151	2.400	2.038
	(-0.4)	0.172	0.171	0.125	0.221	0.183	0.303
	(-0.2)	0.161	0.361	0.278	0.254	0.037	0.170
	(0.2)	0.430	0.040	0.153	0.312	0.009	0.048
	(0.4)	0.145	0.302	0.021	0.209	0.081	0.584
	(0.8)	0.422	0.479	1.339	0.153	0.860	1.421
MA(1)	(-0.8)	0.043	-0.011	0.249	0.158	0.045	0.289
	(-0.4)	0.276	0.150	0.093	0.307	0.702	0.210
	(-0.2)	0.468	0.110	0.117	0.268	0.056	0.100
	(0.2)	0.531	0.067	0.022	0.230	0.156	0.305
	(0.4)	0.260	0.141	0.283	0.150	0.224	0.306
	(0.8)	0.236	0.227	0.146	0.294	0.517	0.979
AR(2)	(0.1, -0.3)	0.219	0.209	0.148	0.225	0.291	0.185
	(-0.1, 0.3)	0.451	0.331	0.163	0.289	0.134	0.236
AR(3)	(0.1, -0.3, -0.2)	0.588	0.647	0.684	0.694	0.360	0.504
	(-0.3, -0.5, 0.1)	0.625	1.584	0.605	0.644	0.330	0.291
MA(2)	(0.1, -0.3)	0.519	0.169	0.203	0.541	0.406	0.194
	(-0.1, 0.3)	1.339	0.110	0.086	0.804	0.193	0.176
MA(3)	(0.1, -0.3, -0.2)	1.085	0.526	0.393	0.883	0.247	0.209
	(-0.3, -0.5, 0.1)	0.622	0.428	0.468	0.962	0.774	0.184
ARMA(1,1)	(0.4; -0.2)	0.477	0.431	0.262	0.278	0.385	0.461
	(-0.4; -0.4)	0.426	0.295	0.718	0.686	1.191	1.167
ARMA(2,1)	(-0.2, -0.3; 0.1)	0.760	0.617	0.225	0.438	0.427	0.337
	(0.2, -0.3; -0.1)	0.928	0.710	0.204	0.489	0.726	0.486
ARMA(1,2)	(0.3; 0.5, -0.1)	0.749	0.269	0.823	0.597	0.882	0.300
	(0.3; -0.5, 0.1)	0.461	0.337	0.185	0.539	0.765	0.293
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	0.739	0.740	0.775	0.914	0.487	0.694
	(0.2, -0.3; -0.5, 0.1)	0.965	0.873	1.168	0.793	0.776	0.444

Table 25

CLU by Model at $T = 100$ and $\sigma^2 = 5$

		T = 100					
Models	Parameters	$\alpha = 1$			$\alpha = -1$		
		h=1	h=2	h=3	h=1	h=2	h=3
AR(1)	(-0.8)	0.184	-0.065	1.202	0.042	0.244	0.137
	(-0.4)	0.314	0.009	0.030	0.036	0.004	0.055
	(-0.2)	0.150	-0.026	0.011	0.126	-0.042	0.145
	(0.2)	0.215	-0.037	-0.018	0.117	0.022	0.125
	(0.4)	0.214	0.039	-0.101	0.132	0.195	0.105
	(0.8)	-0.115	0.302	1.009	0.216	0.068	6.687
MA(1)	(-0.8)	0.028	-0.126	1.510	0.056	1.189	1.053
	(-0.4)	0.072	0.018	-0.072	0.213	-0.102	0.074
	(-0.2)	0.019	0.013	-0.028	0.145	0.052	0.463
	(0.2)	0.059	-0.102	0.193	0.109	0.148	-0.006
	(0.4)	0.079	0.284	0.063	0.173	0.196	0.086
	(0.8)	0.095	0.020	2.565	0.116	0.034	0.236
AR(2)	(0.1, -0.3)	0.193	0.218	0.119	0.208	0.206	-0.013
	(-0.1, 0.3)	0.098	-0.039	0.063	0.020	-0.040	0.007
AR(3)	(0.1, -0.3, -0.2)	0.171	0.239	0.309	0.064	0.025	0.297
	(-0.3, -0.5, 0.1)	0.241	0.176	0.252	0.192	0.226	0.326
MA(2)	(0.1, -0.3)	0.318	-0.002	0.068	0.094	0.275	0.007
	(-0.1, 0.3)	0.145	0.067	0.194	0.189	0.087	-0.082
MA(3)	(0.1, -0.3, -0.2)	0.233	0.234	0.341	0.376	0.261	0.236
	(-0.3, -0.5, 0.1)	0.409	0.452	0.569	0.258	0.186	0.061
ARMA(1,1)	(0.4; -0.2)	0.092	0.109	0.118	0.217	0.579	0.131
	(-0.4; -0.4)	0.196	0.367	1.006	0.248	0.069	0.453
ARMA(2,1)	(-0.2, -0.3; 0.1)	0.169	0.135	-0.063	0.125	0.168	0.078
	(0.2, -0.3; -0.1)	0.365	0.165	0.141	0.135	0.163	0.083
ARMA(1,2)	(0.3; 0.5, -0.1)	0.401	1.297	0.093	0.272	0.199	0.115
	(0.3; -0.5, 0.1)	0.092	0.080	-0.028	0.192	0.157	0.005
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	0.779	0.649	0.182	0.252	0.225	0.273
	(0.2, -0.3; -0.5, 0.1)	0.452	0.447	0.357	0.308	0.225	0.136

Table 26

CLU by Model at $T = 200$ and $\sigma^2 = 5$

		T = 200					
Models	Parameters	$\alpha = 1$			$\alpha = -1$		
		h=1	h=2	h=3	h=1	h=2	h=3
AR(1)	(-0.8)	0.013	0.111	1.028	0.356	-0.122	0.168
	(-0.4)	0.108	-0.184	-0.003	0.128	0.021	-0.033
	(-0.2)	0.052	0.017	0.015	0.095	-0.053	-0.030
	(0.2)	0.150	-0.150	0.144	-0.052	0.012	-0.015
	(0.4)	-0.006	-0.027	0.220	0.065	0.080	0.090
	(0.8)	0.061	0.014	0.034	0.076	-0.078	-0.078
MA(1)	(-0.8)	0.055	-0.127	0.099	0.041	0.548	-0.010
	(-0.4)	0.052	-0.077	0.002	0.205	0.119	0.015
	(-0.2)	0.030	-0.113	-0.041	0.014	0.111	0.014
	(0.2)	-0.037	-0.072	0.089	-0.054	-0.105	0.024
	(0.4)	0.060	0.008	0.033	-0.041	0.204	-0.005
	(0.8)	-0.095	-0.132	0.012	0.083	0.364	-0.185
AR(2)	(0.1, -0.3)	0.044	0.148	-0.010	-0.045	-0.059	-0.055
	(-0.1, 0.3)	-0.029	-0.066	0.044	-0.014	0.081	0.108
AR(3)	(0.1, -0.3, -0.2)	0.134	0.013	0.104	0.114	0.197	0.152
	(-0.3, -0.5, 0.1)	-0.024	-0.042	-0.098	0.063	0.066	0.176
MA(2)	(0.1, -0.3)	0.067	0.278	-0.009	0.180	0.406	0.017
	(-0.1, 0.3)	0.105	-0.026	0.061	0.016	0.007	-0.112
MA(3)	(0.1, -0.3, -0.2)	0.132	0.068	-0.003	0.174	0.099	-0.022
	(-0.3, -0.5, 0.1)	0.099	-0.125	0.132	0.000*	0.312	-0.050
ARMA(1,1)	(0.4; -0.2)	0.199	0.219	-0.121	0.128	0.032	0.081
	(-0.4; -0.4)	0.024	0.197	0.448	0.220	-0.058	-0.147
ARMA(2,1)	(-0.2, -0.3; 0.1)	0.044	0.136	-0.016	0.382	0.166	-0.052
	(0.2, -0.3; -0.1)	0.058	0.160	0.132	0.019	0.147	-0.053
ARMA(1,2)	(0.3; 0.5, -0.1)	-0.016	-0.106	-0.102	0.102	-0.128	0.079
	(0.3; -0.5, 0.1)	0.251	0.137	-0.026	-0.045	0.085	0.003
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	-0.009	0.064	0.177	0.652	0.199	0.201
	(0.2, -0.3; -0.5, 0.1)	0.349	0.207	0.165	0.244	0.190	0.410

Note. $0.000^* < \pm\ 0.0001$.

Table 27

CLU by Model at $T = 400$ and $\sigma^2 = 5$

		T=400					
Models	Parameters	$\alpha = 1$			$\alpha = -1$		
		h=1	h=2	h=3	h=1	h=2	h=3
AR(1)	(-0.8)	-0.061	0.164	0.025	-0.116	0.022	-0.256
	(-0.4)	-0.046	0.040	0.070	-0.037	-0.006	0.077
	(-0.2)	-0.025	-0.028	0.056	-0.060	-0.037	0.033
	(0.2)	-0.007	-0.053	-0.145	-0.079	-0.058	0.041
	(0.4)	0.054	-0.004	-0.129	-0.076	-0.032	-0.014
	(0.8)	-0.019	-0.066	-0.289	0.062	-0.097	-0.256
MA(1)	(-0.8)	0.101	-0.010	-0.069	-0.007	-0.007	0.084
	(-0.4)	0.090	-0.013	0.005	-0.030	0.121	0.054
	(-0.2)	0.041	-0.090	-0.149	0.002	0.014	0.001
	(0.2)	0.091	-0.054	-0.078	0.130	0.087	-0.039
	(0.4)	0.084	-0.028	0.014	-0.041	-0.053	0.039
	(0.8)	0.043	-0.224	-0.185	-0.090	-0.143	-0.131
AR(2)	(0.1, -0.3)	0.099	-0.012	0.038	0.123	0.045	0.054
	(-0.1, 0.3)	0.104	0.016	0.011	-0.053	0.159	0.105
AR(3)	(0.1, -0.3, -0.2)	-0.118	-0.014	-0.105	0.235	-0.044	0.014
	(-0.3, -0.5, 0.1)	0.104	0.333	-0.120	0.127	-0.041	-0.059
MA(2)	(0.1, -0.3)	0.024	-0.033	0.013	0.107	0.039	-0.028
	(-0.1, 0.3)	-0.072	0.047	-0.064	0.014	0.162	-0.129
MA(3)	(0.1, -0.3, -0.2)	-0.022	-0.038	-0.031	0.000*	0.071	0.128
	(-0.3, -0.5, 0.1)	-0.024	-0.005	0.156	0.157	0.035	0.136
ARMA(1,1)	(0.4; -0.2)	-0.047	-0.065	-0.033	-0.022	0.059	0.104
	(-0.4; -0.4)	0.040	-0.064	0.027	0.042	0.008	-0.012
ARMA(2,1)	(-0.2, -0.3; 0.1)	-0.003	0.021	-0.041	-0.010	0.075	-0.004
	(0.2, -0.3; -0.1)	0.069	0.048	0.111	0.080	0.039	-0.066
ARMA(1,2)	(0.3; 0.5, -0.1)	-0.026	-0.104	-0.088	-0.002	0.219	0.138
	(0.3; -0.5, 0.1)	0.244	0.056	-0.085	-0.022	0.059	-0.057
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	0.023	0.047	0.215	0.035	-0.005	0.021
	(0.2, -0.3; -0.5, 0.1)	0.135	-0.112	-0.122	0.088	0.105	-0.070

Note. $0.000^* < \pm\ 0.0001$.

APPENDIXB

CLU OF σ^2 BY T FOR AR(P) AND MA(Q)

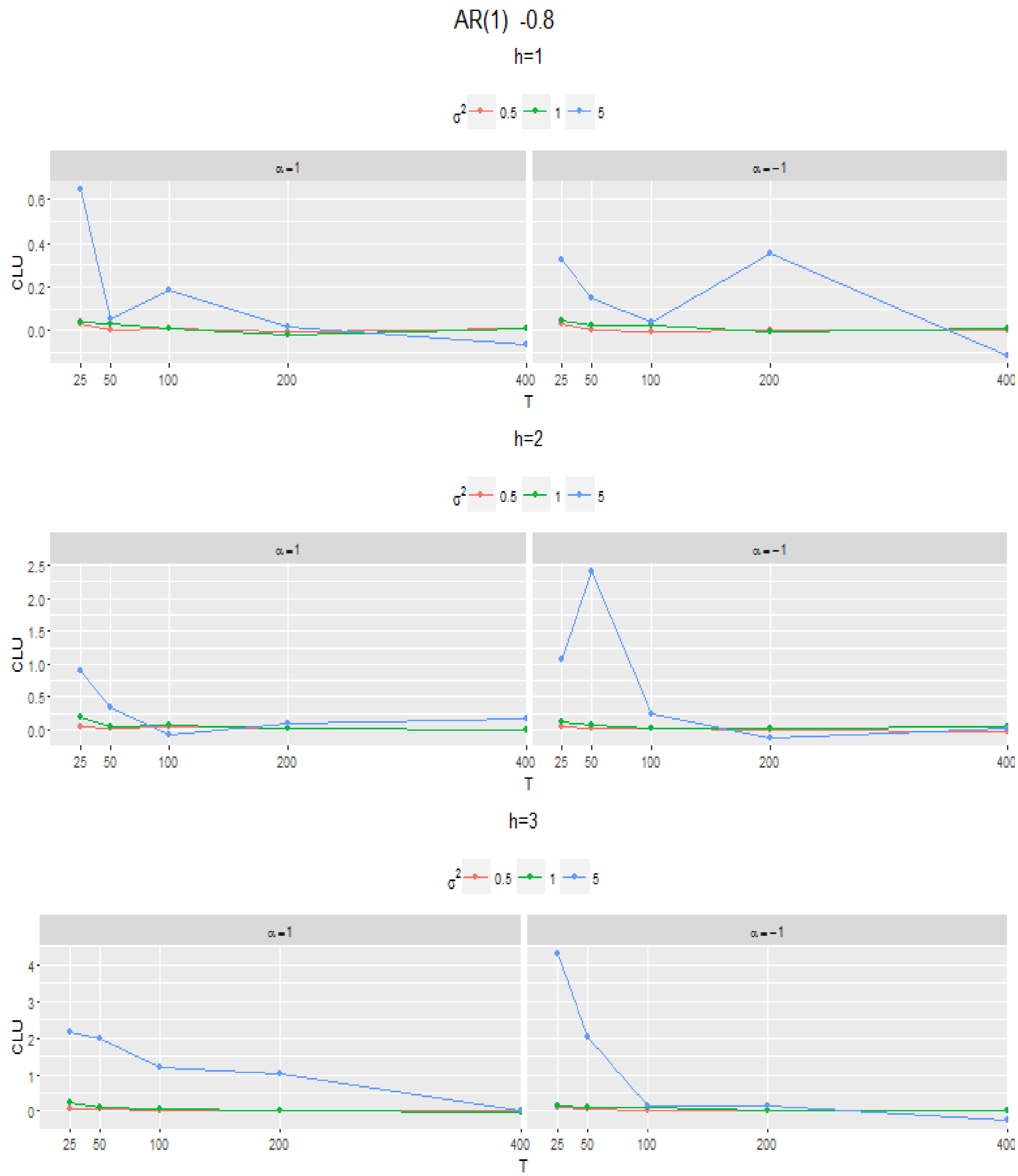

Figure 53. CLU of σ^2 by T for AR(1), $\phi_1 = -0.8$

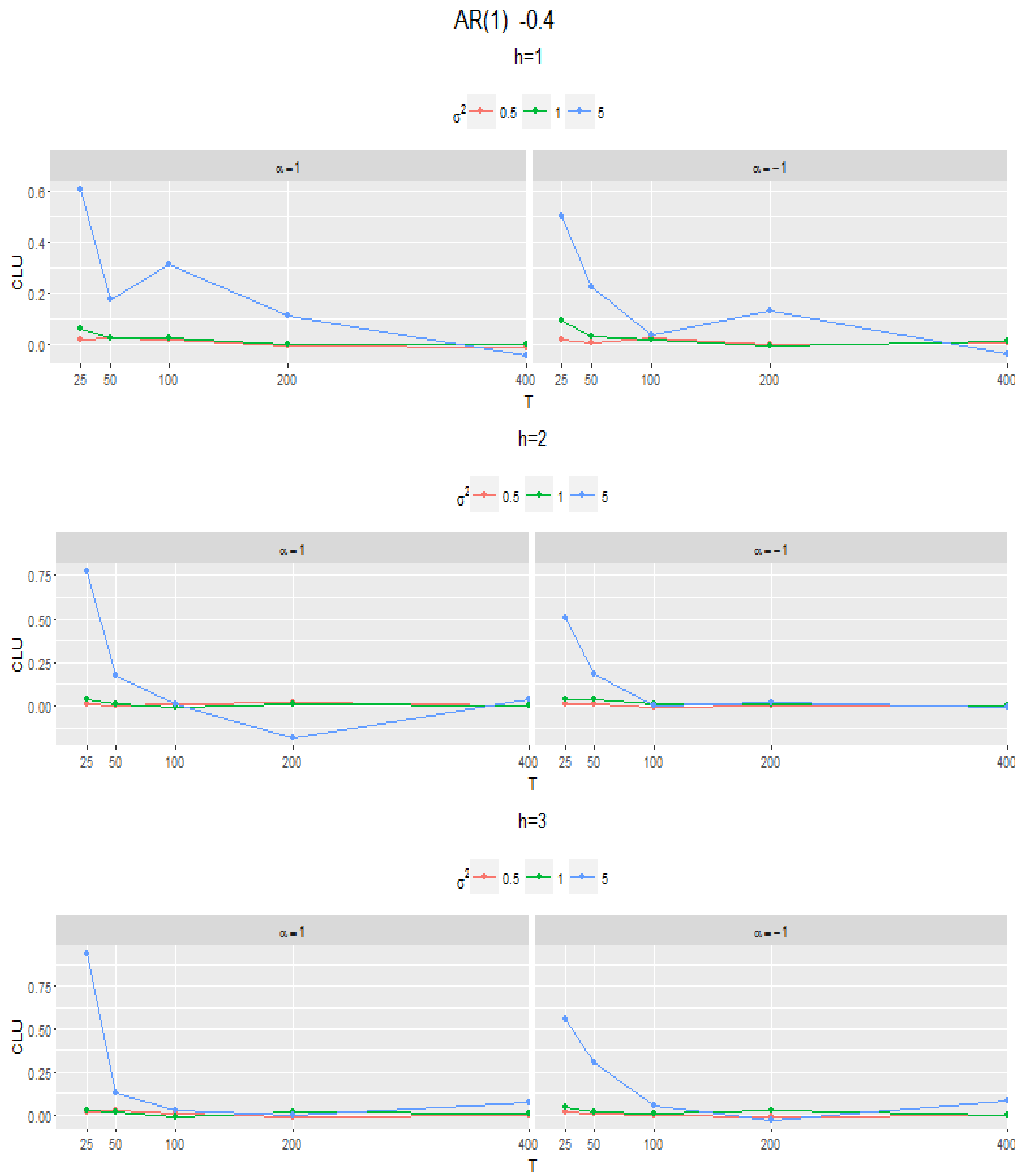

Figure 54. CLU of σ^2 by T for AR(1), $\phi_1 = -0.4$

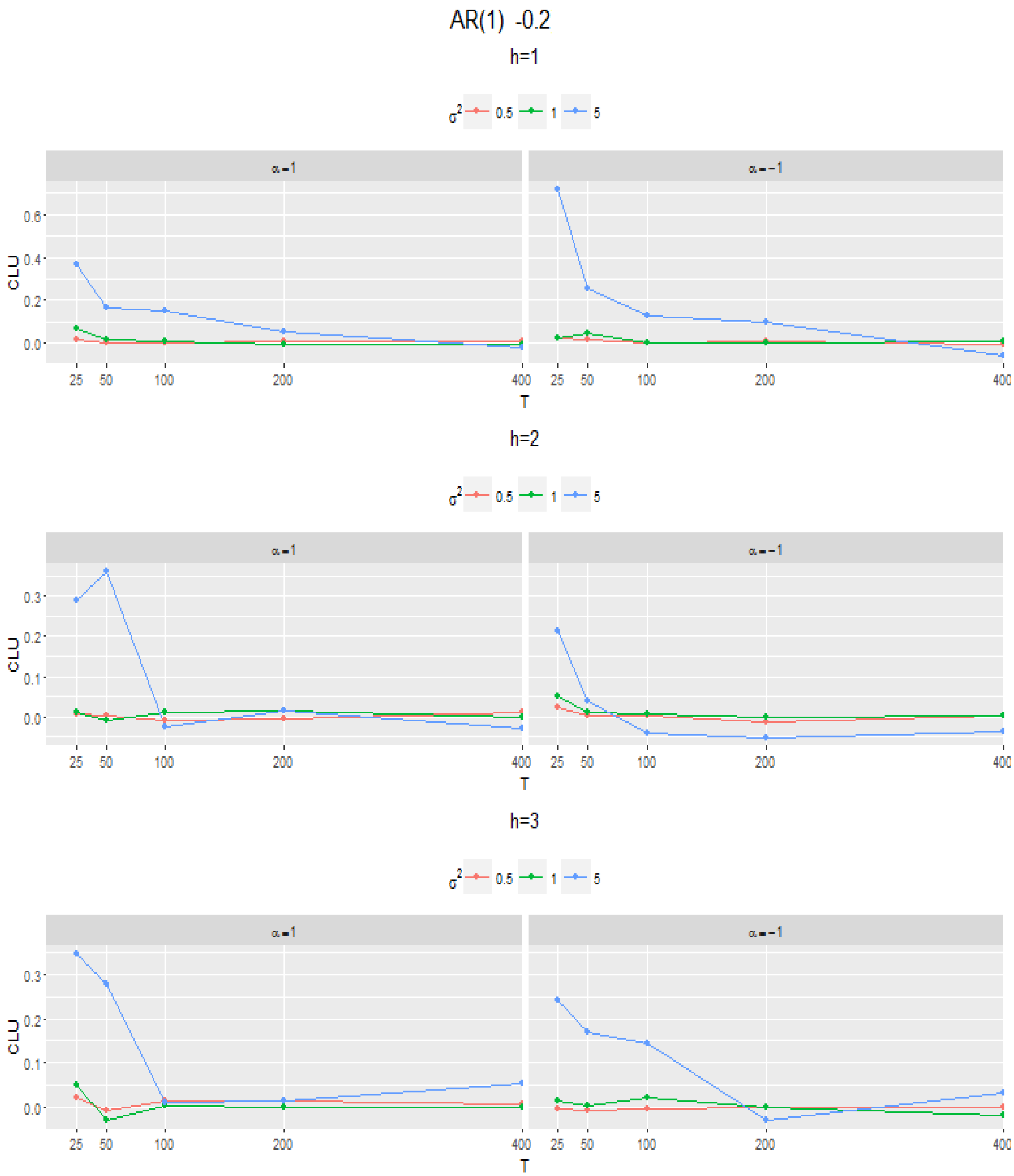

Figure 55. CLU of σ^2 by T for AR(1), $\phi_1 = -0.2$

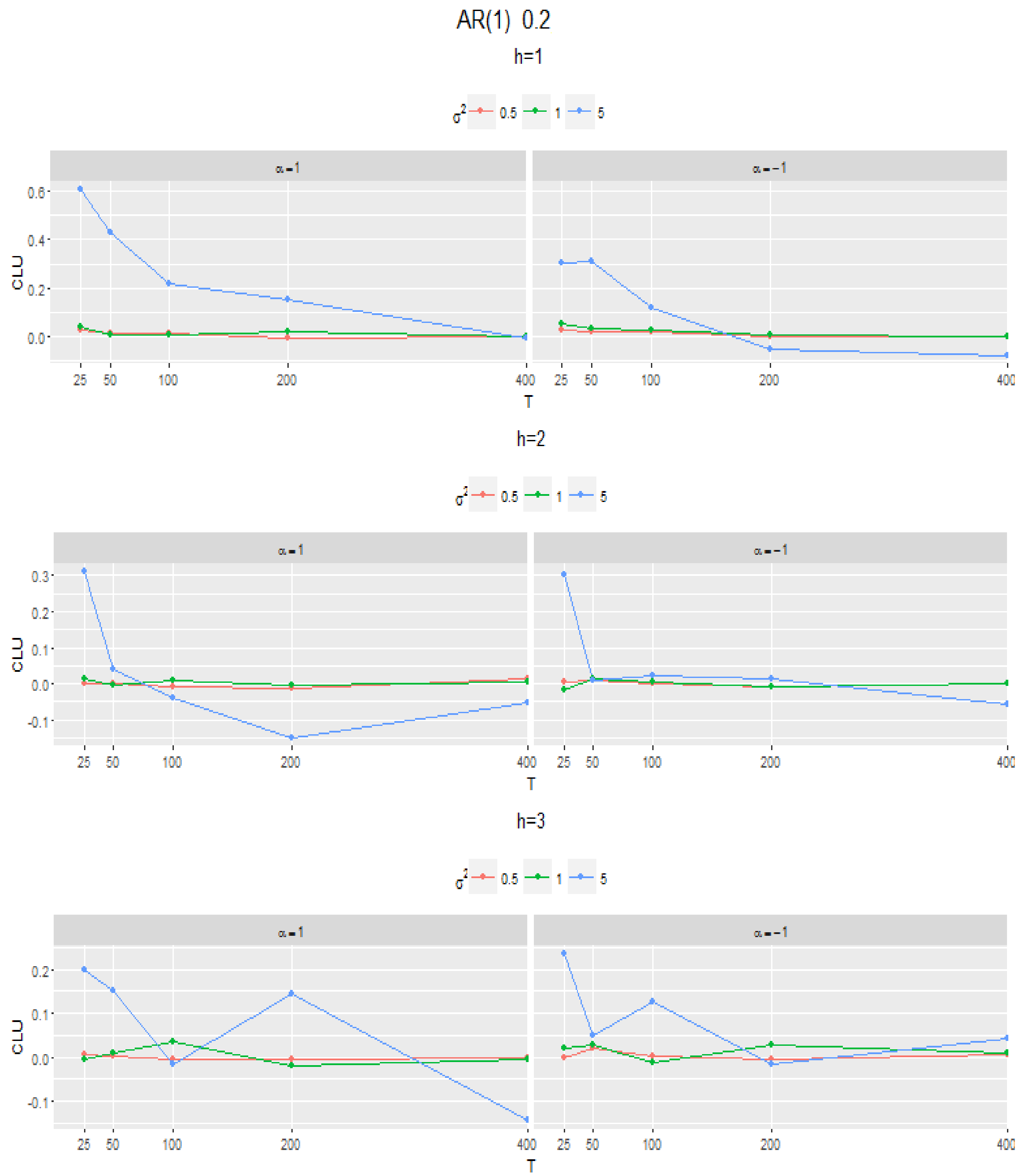

Figure 56. CLU of σ^2 by T for AR(1), $\phi_1 = 0.2$

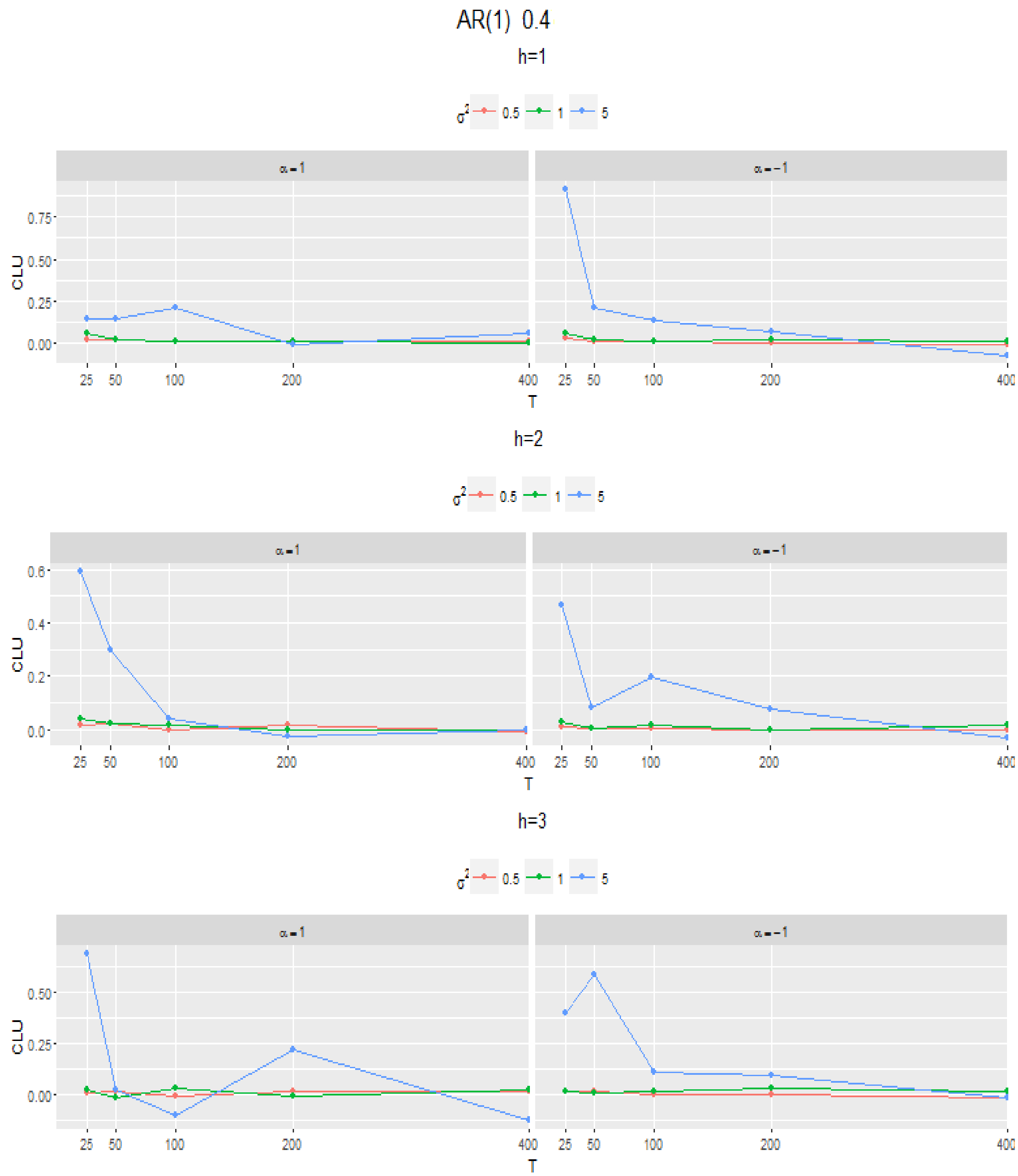

Figure 57. CLU of σ^2 by T for AR(1), $\phi_1 = 0.4$

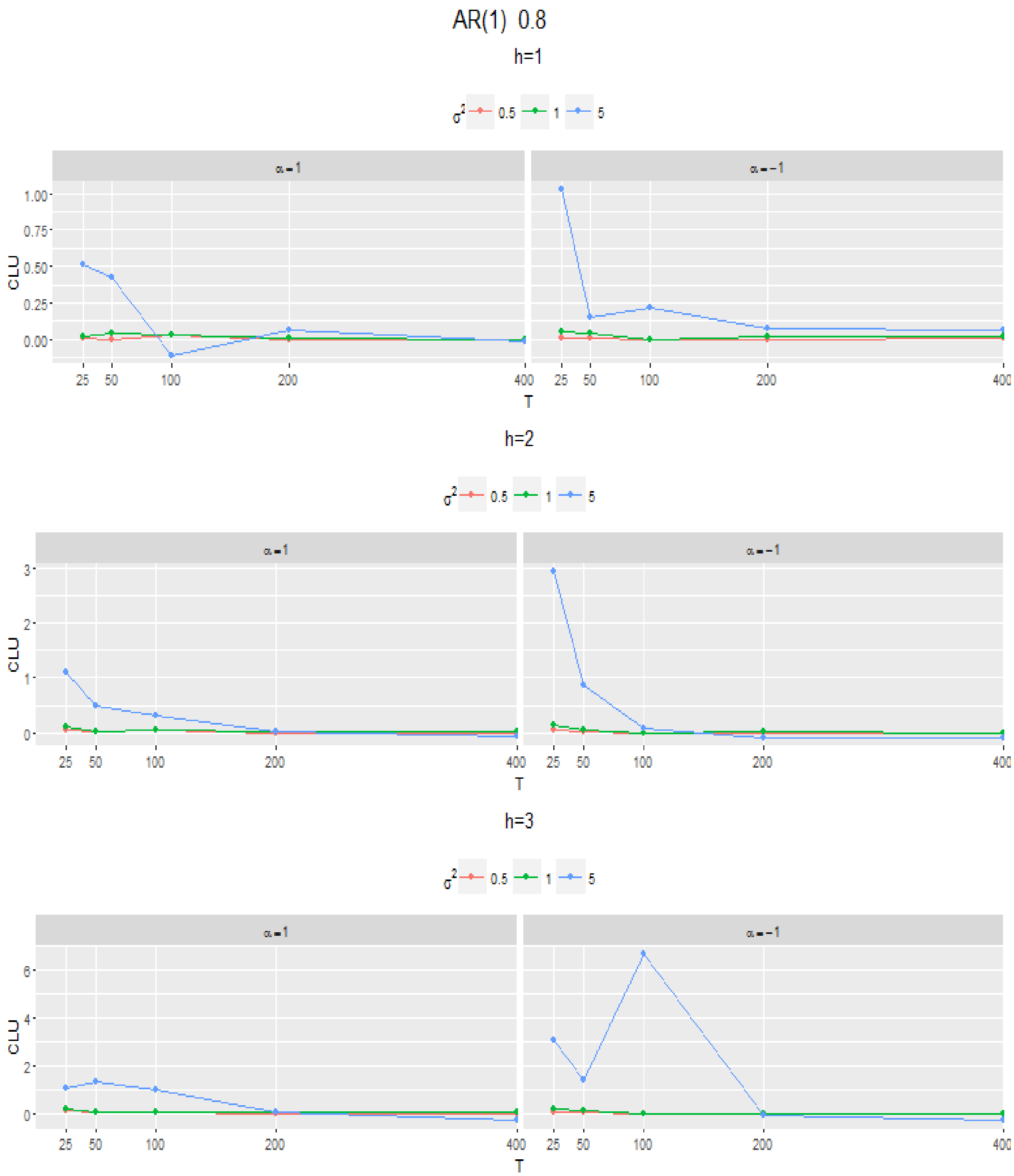

Figure 58. CLU of σ^2 by T for AR(1), $\phi_1 = 0.8$

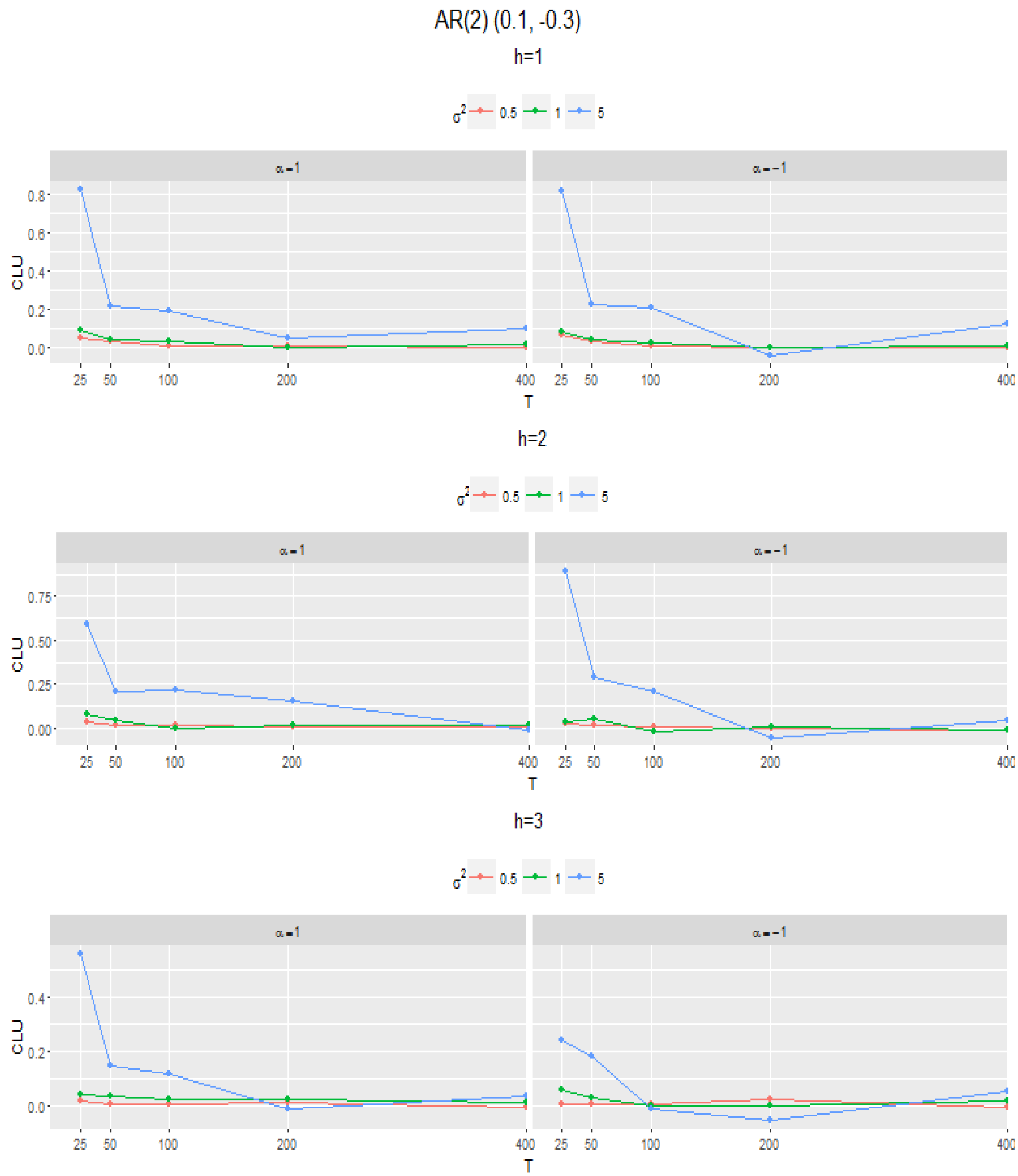

Figure 59. CLU of σ^2 by T for AR(2), $(\phi_1, \phi_2) = (0.1, -0.3)$

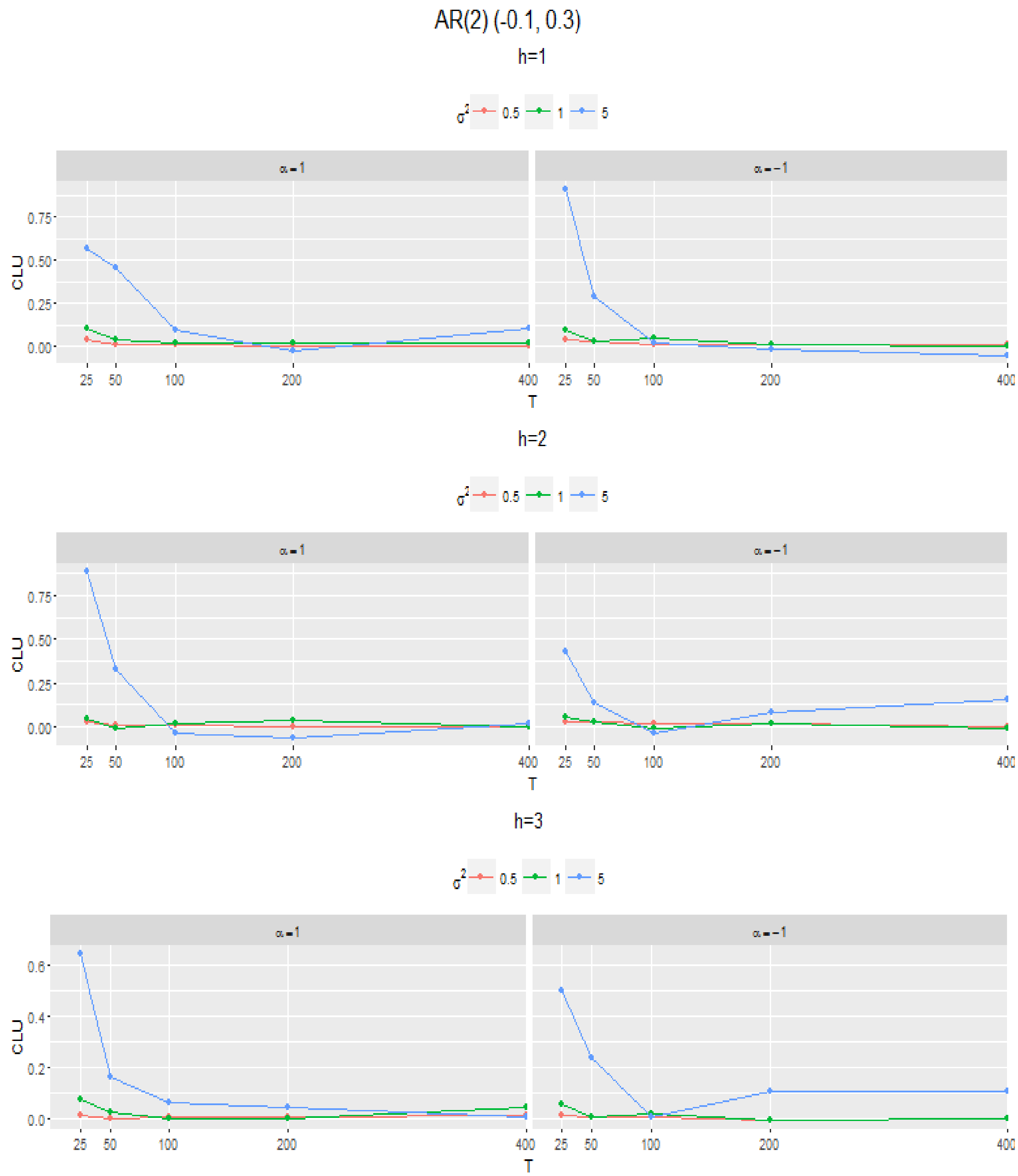

Figure 60. CLU of σ^2 by T for AR(2), $(\phi_1, \phi_2) = (-0.1, 0.3)$

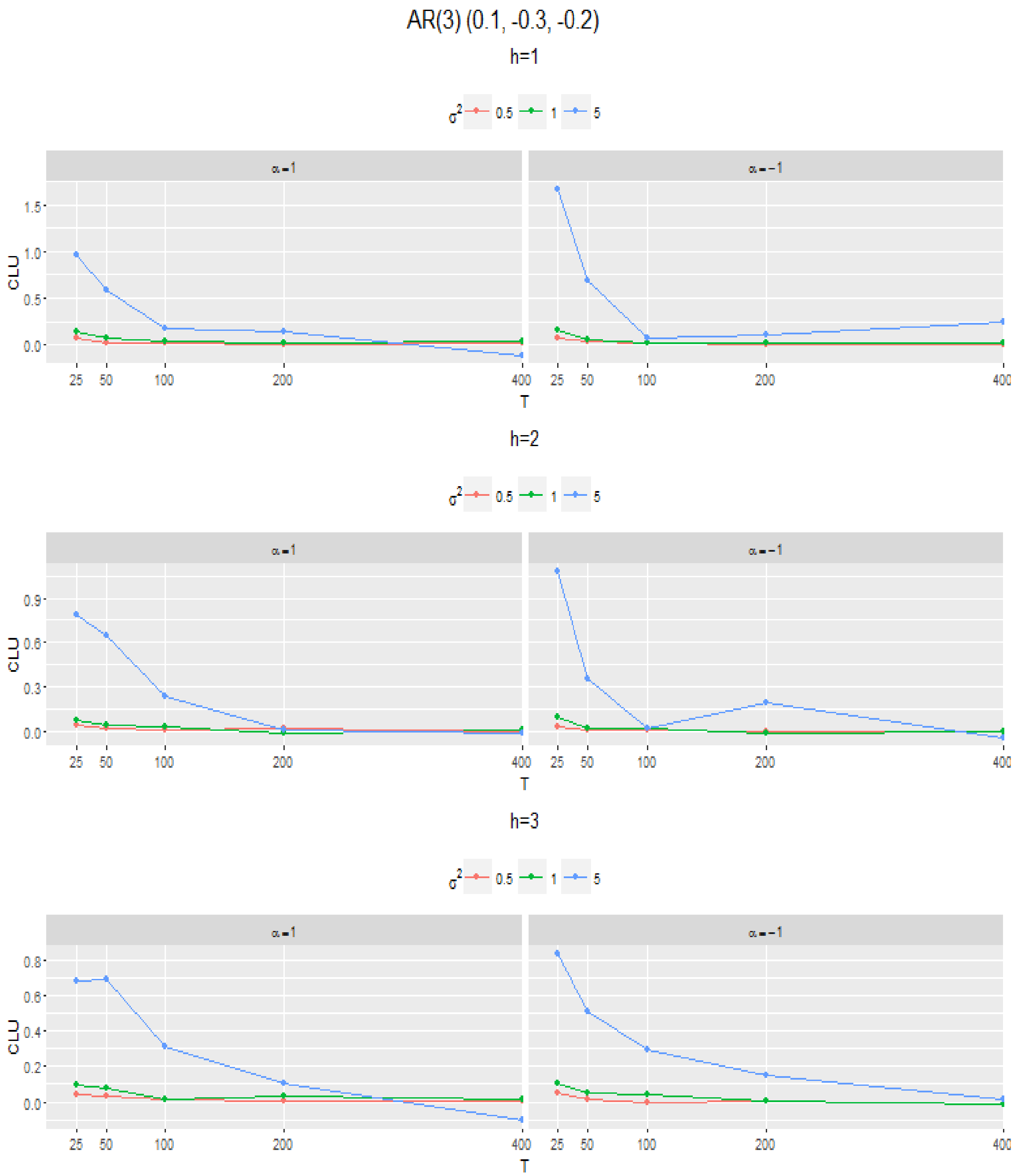

Figure 61. CLU of σ^2 by T for AR(3), $(\phi_1, \phi_2, \phi_3) = (0.1, -0.3, -0.2)$

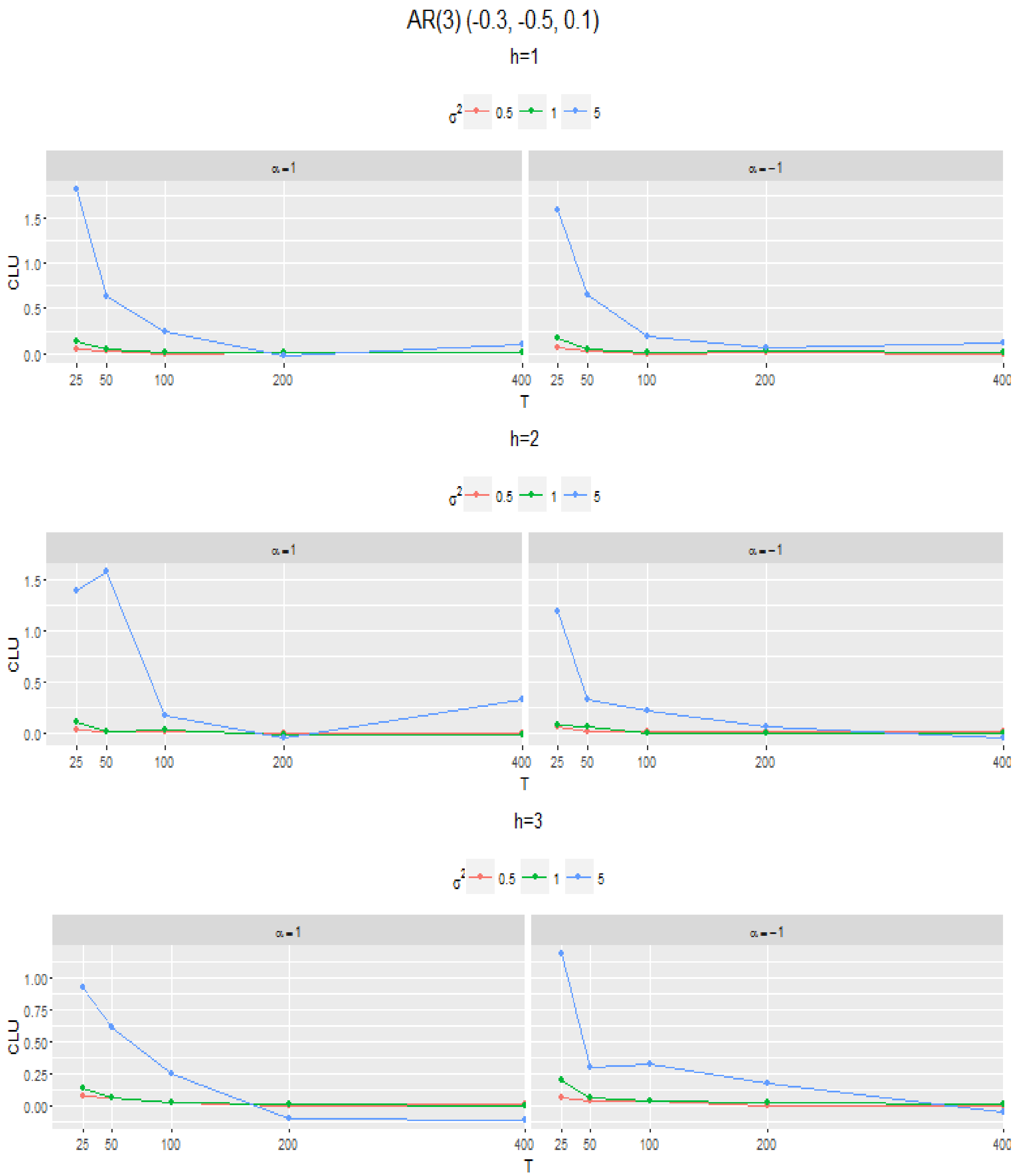

Figure 62. CLU of σ^2 by T for AR(3), $(\phi_1, \phi_2, \phi_3) = (-0.3, -0.5, 0.1)$

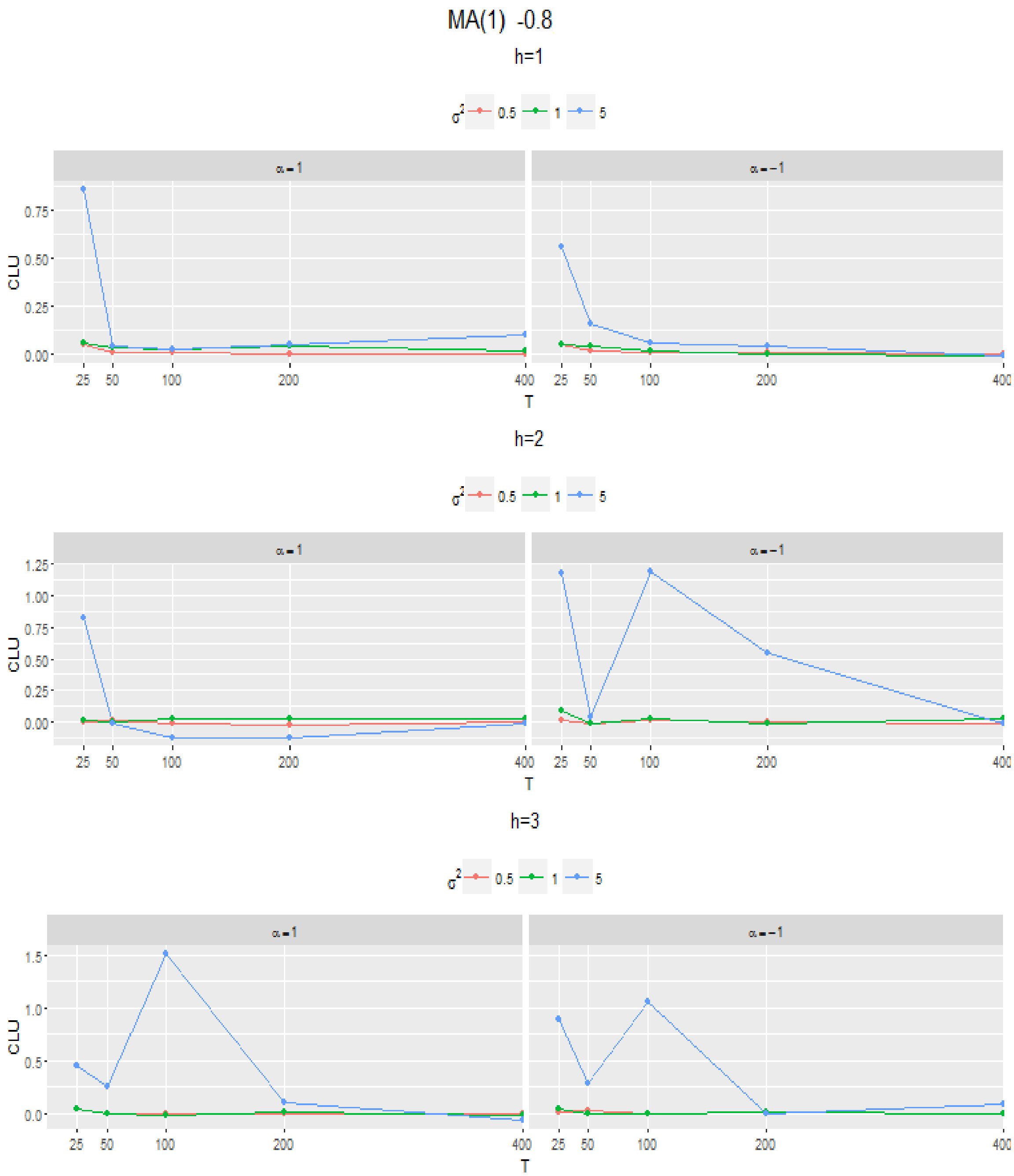

Figure 63. CLU of σ^2 by T for MA(1), $\theta_1 = -0.8$

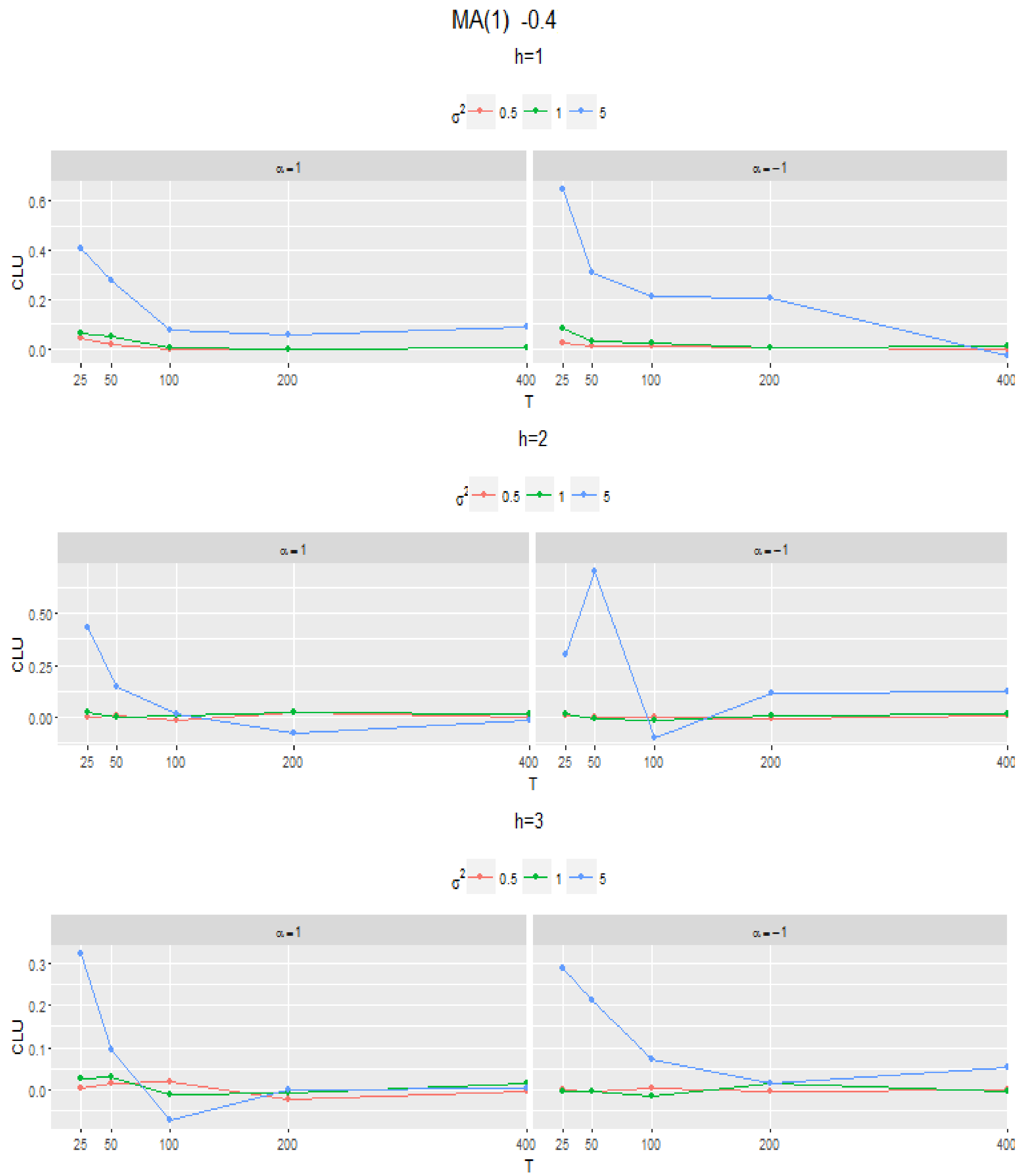

Figure 64. CLU of σ^2 by T for MA(1), $\theta_1 = -0.4$

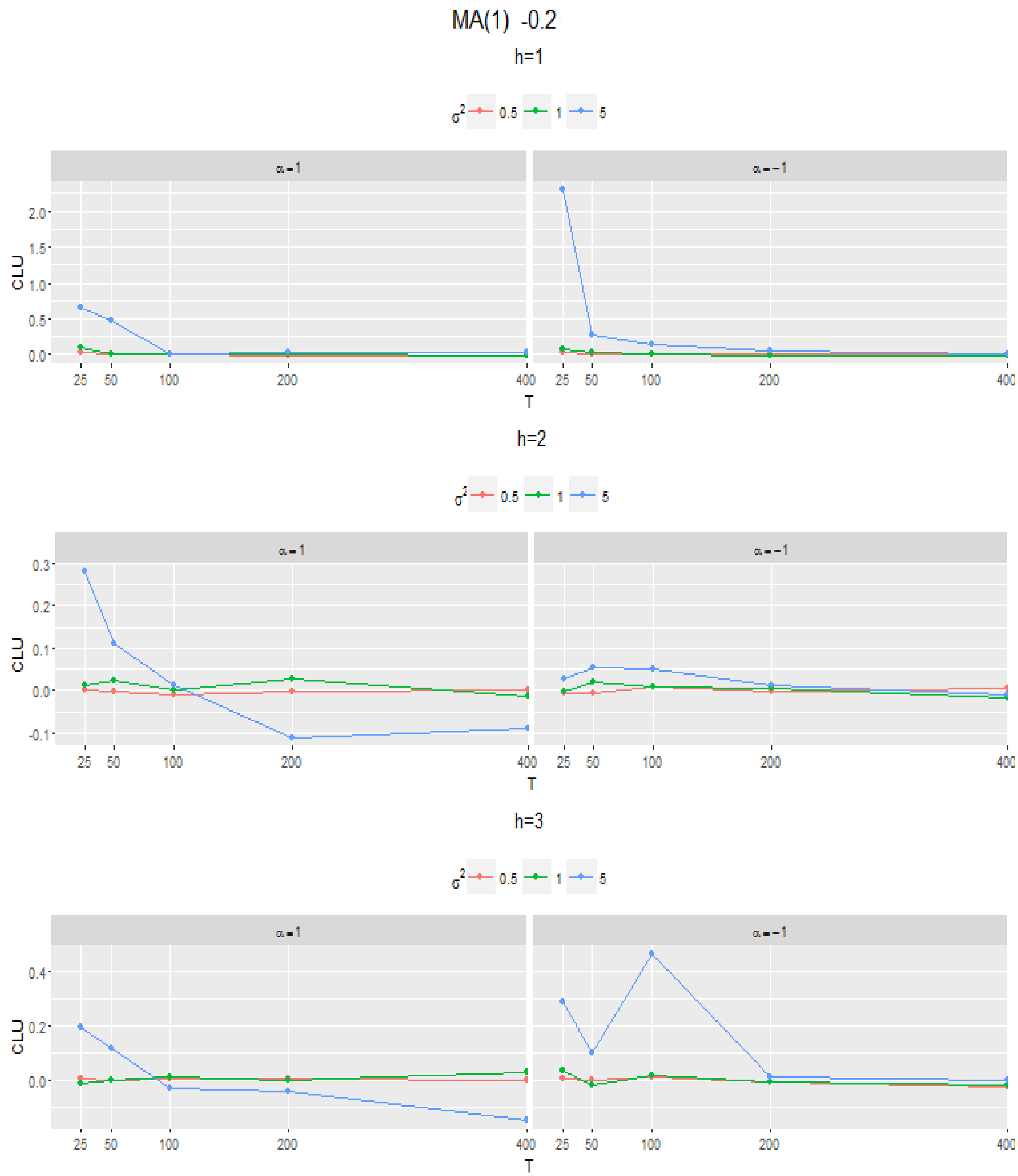

Figure 65. CLU of σ^2 by T for MA(1), $\theta_1 = -0.2$

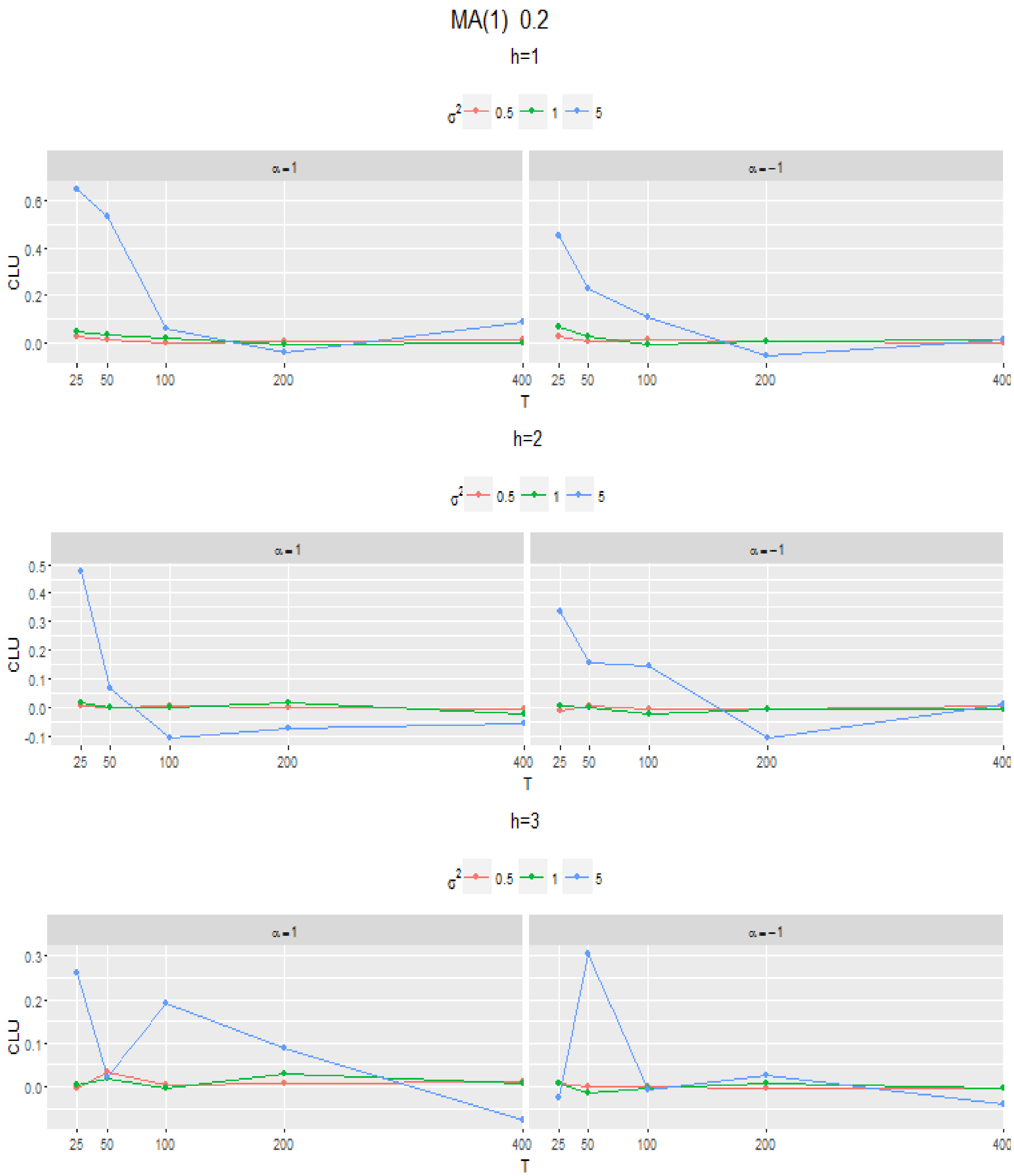

Figure 66. CLU of σ^2 by T for MA(1), $\theta_1 = 0.2$

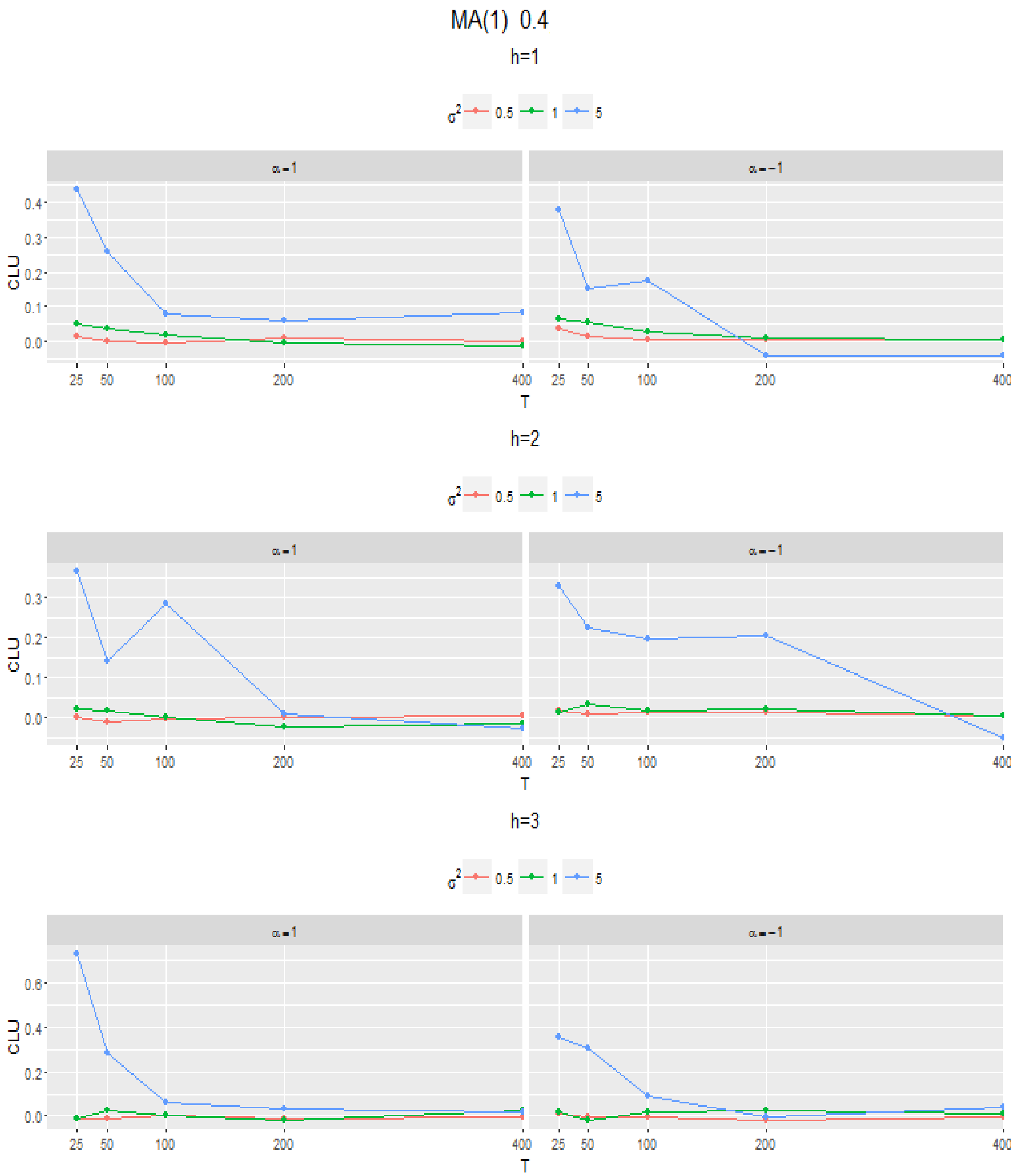

Figure 67. CLU of σ^2 by T for MA(1), $\theta_1 = 0.4$

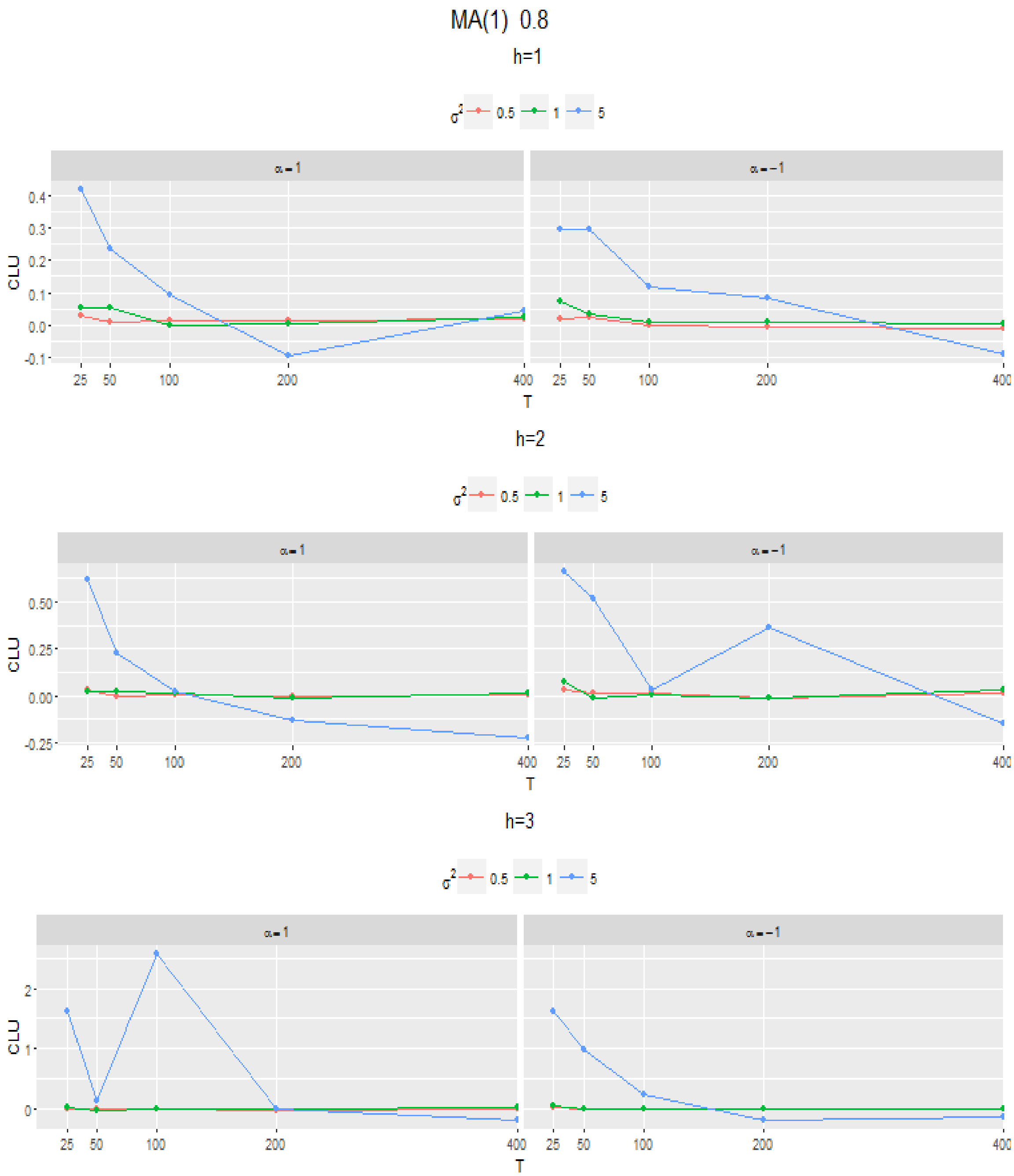

Figure 68. CLU of σ^2 by T for MA(1), $\theta_1 = 0.8$

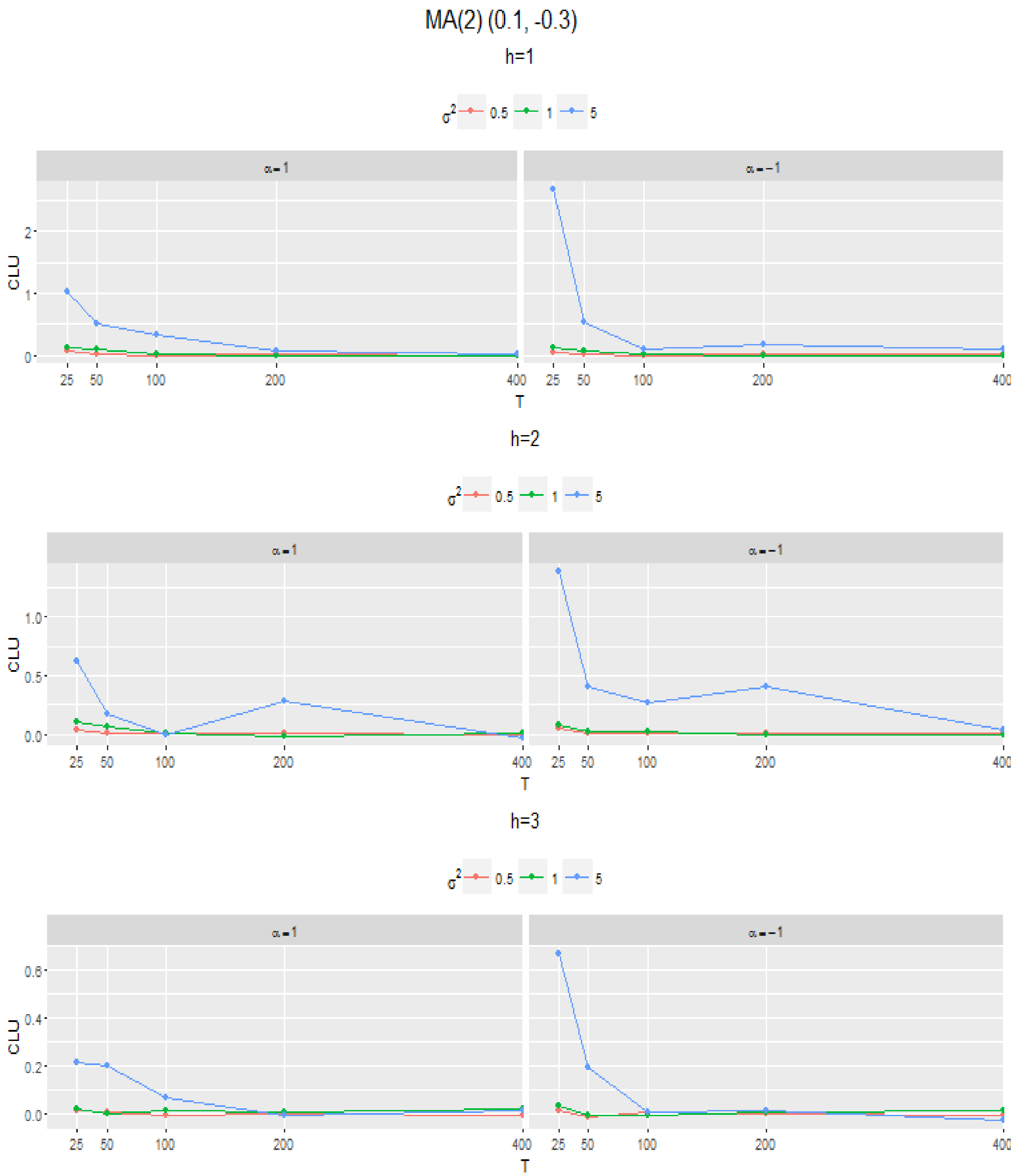

Figure 69. CLU of σ^2 by T for MA(2), $(\theta_1, \theta_2) = (0.1, -0.3)$

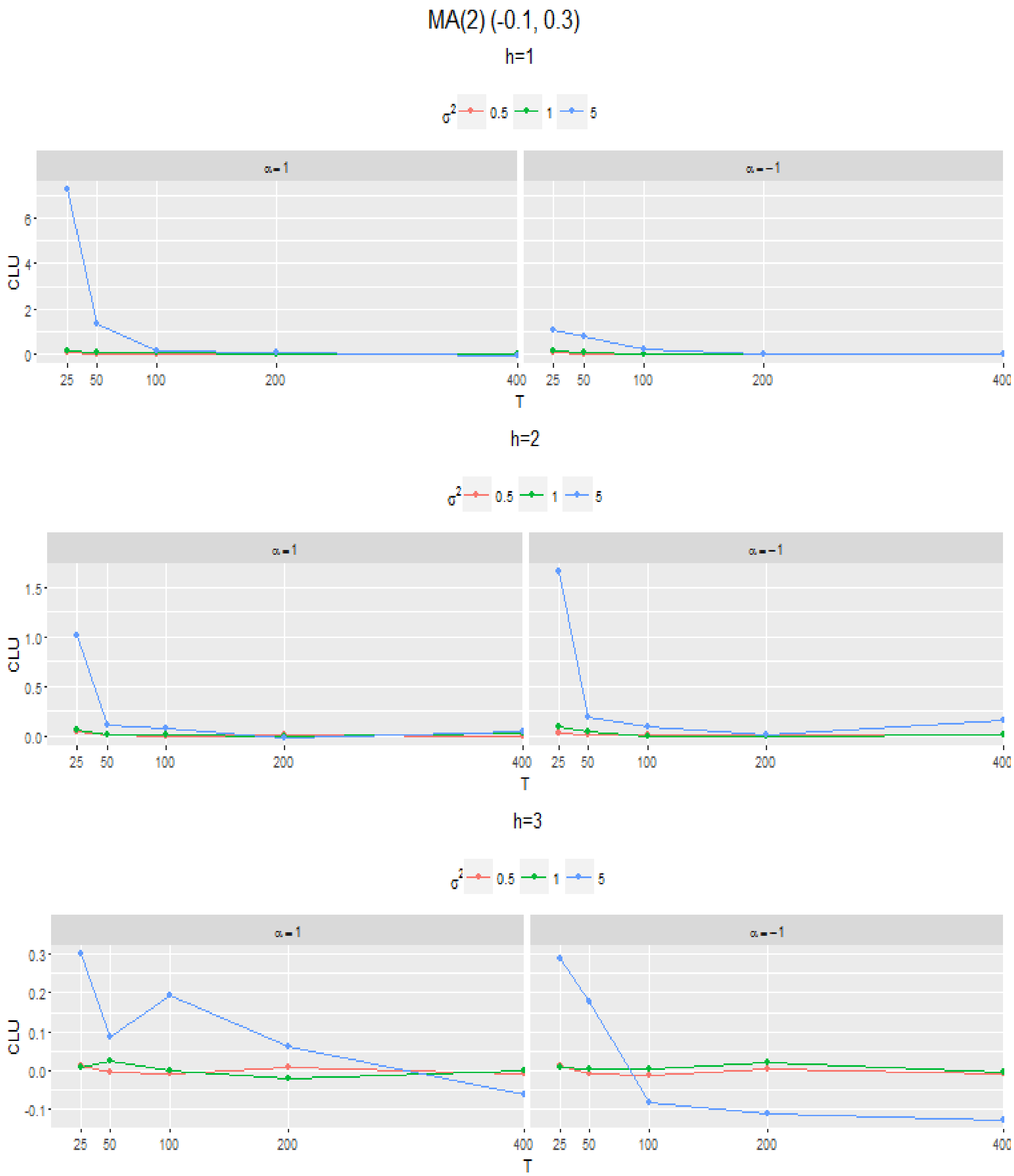

Figure 70. CLU of σ^2 by T for MA(2), $(\theta_1, \theta_2) = (-0.1, 0.3)$

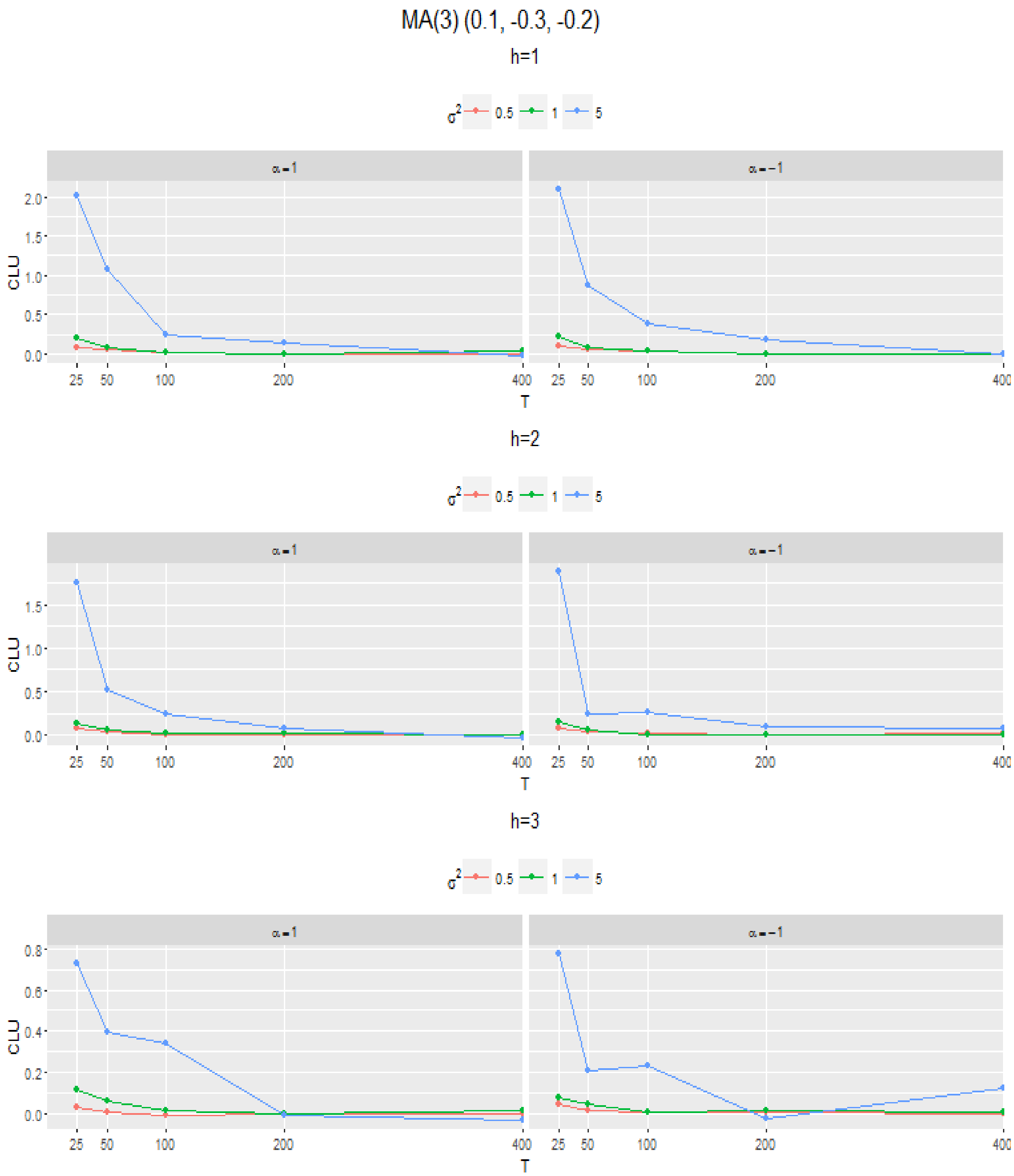

Figure 71. CLU of σ^2 by T for MA(3), $(\theta_1, \theta_2, \theta_3) = (0.1, -0.3, -0.2)$

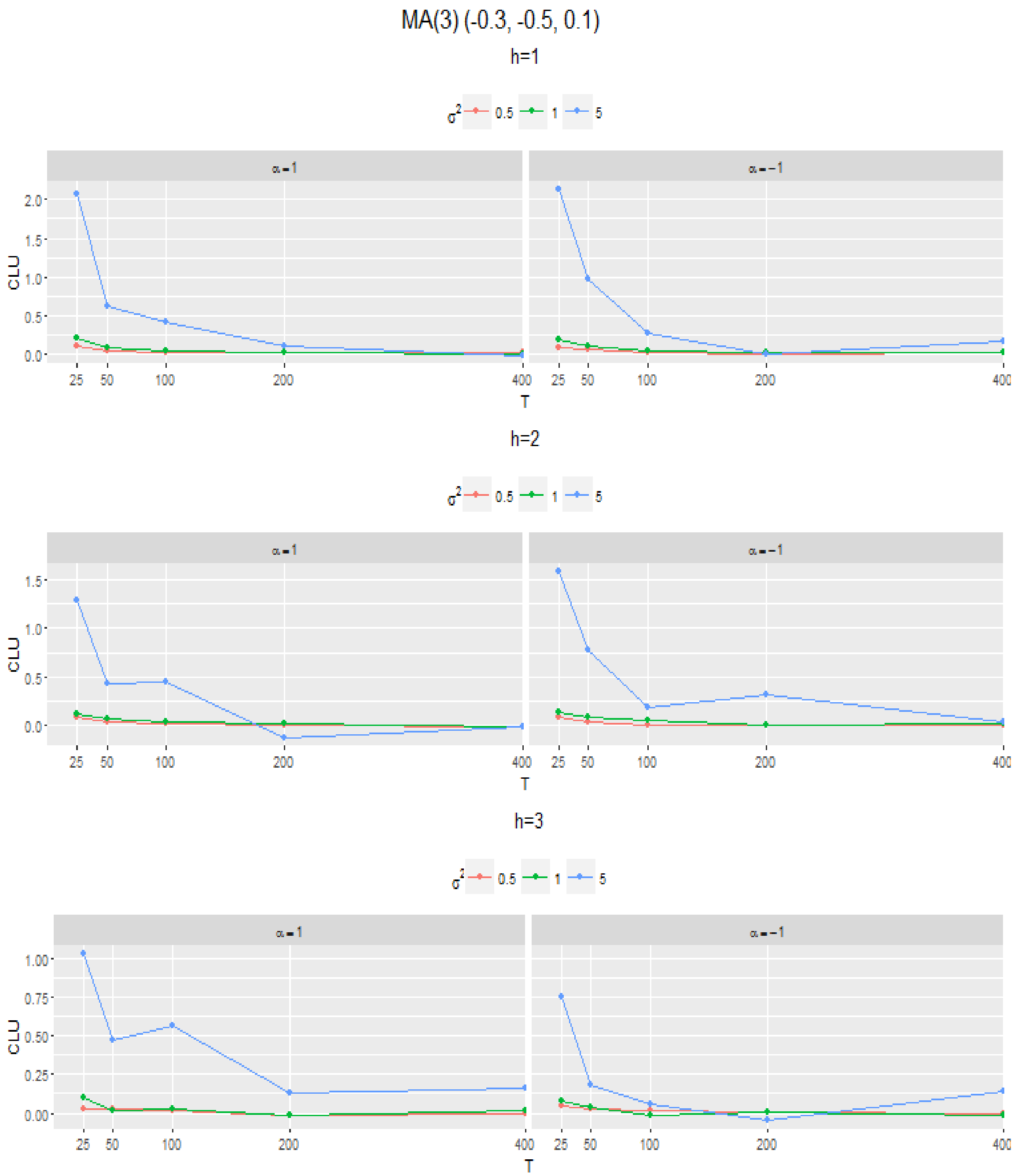

Figure 72. CLU of σ^2 by T for MA(3), $(\theta_1, \theta_2, \theta_3) = (0.1, -0.3, -0.2)$

APPENDIXC

CLU OF H BY T FOR MA(Q) AND ARMA(P,Q)

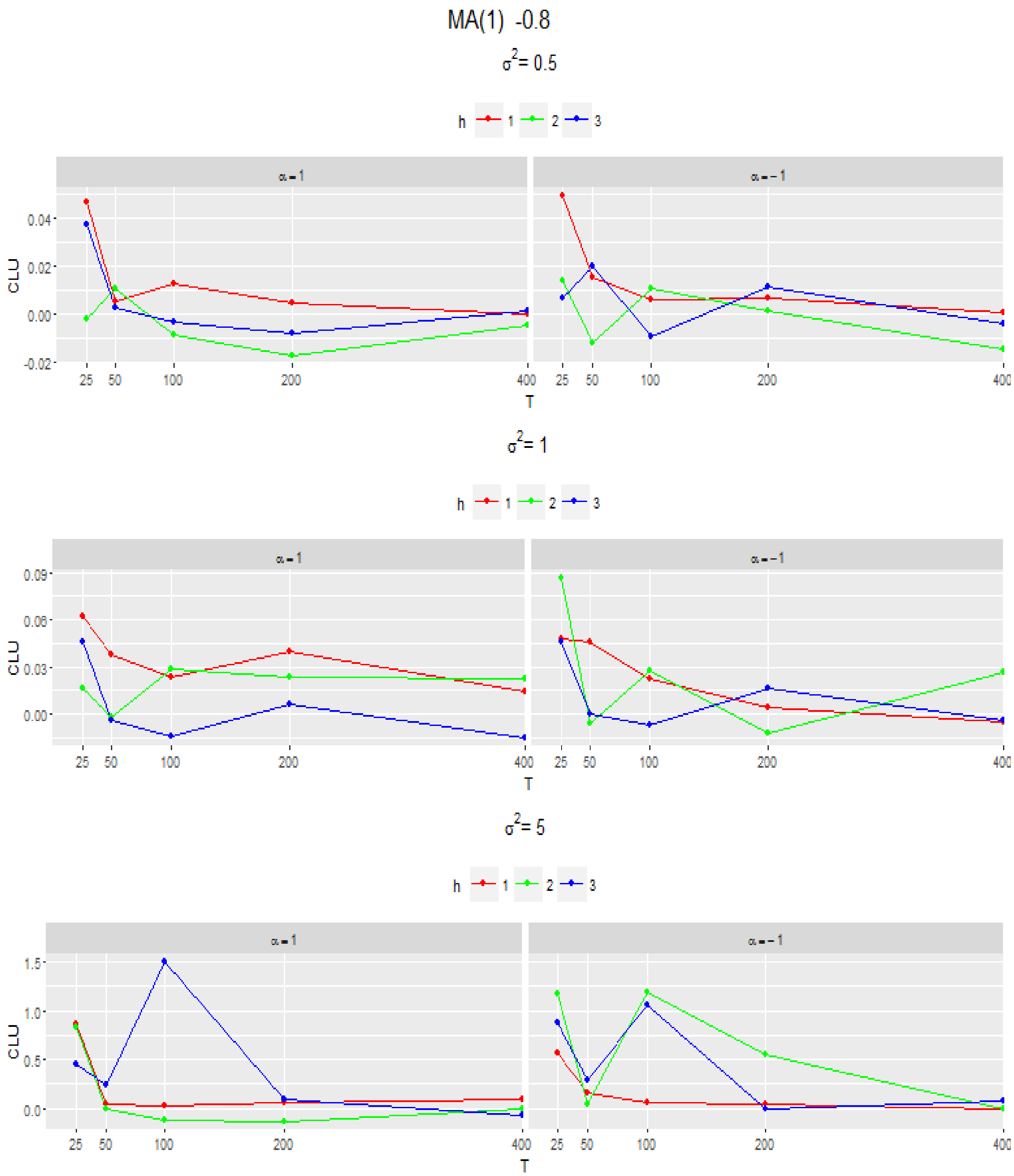

Figure 73. CLU of h by T for MA(1), $\theta_1 = -0.8$

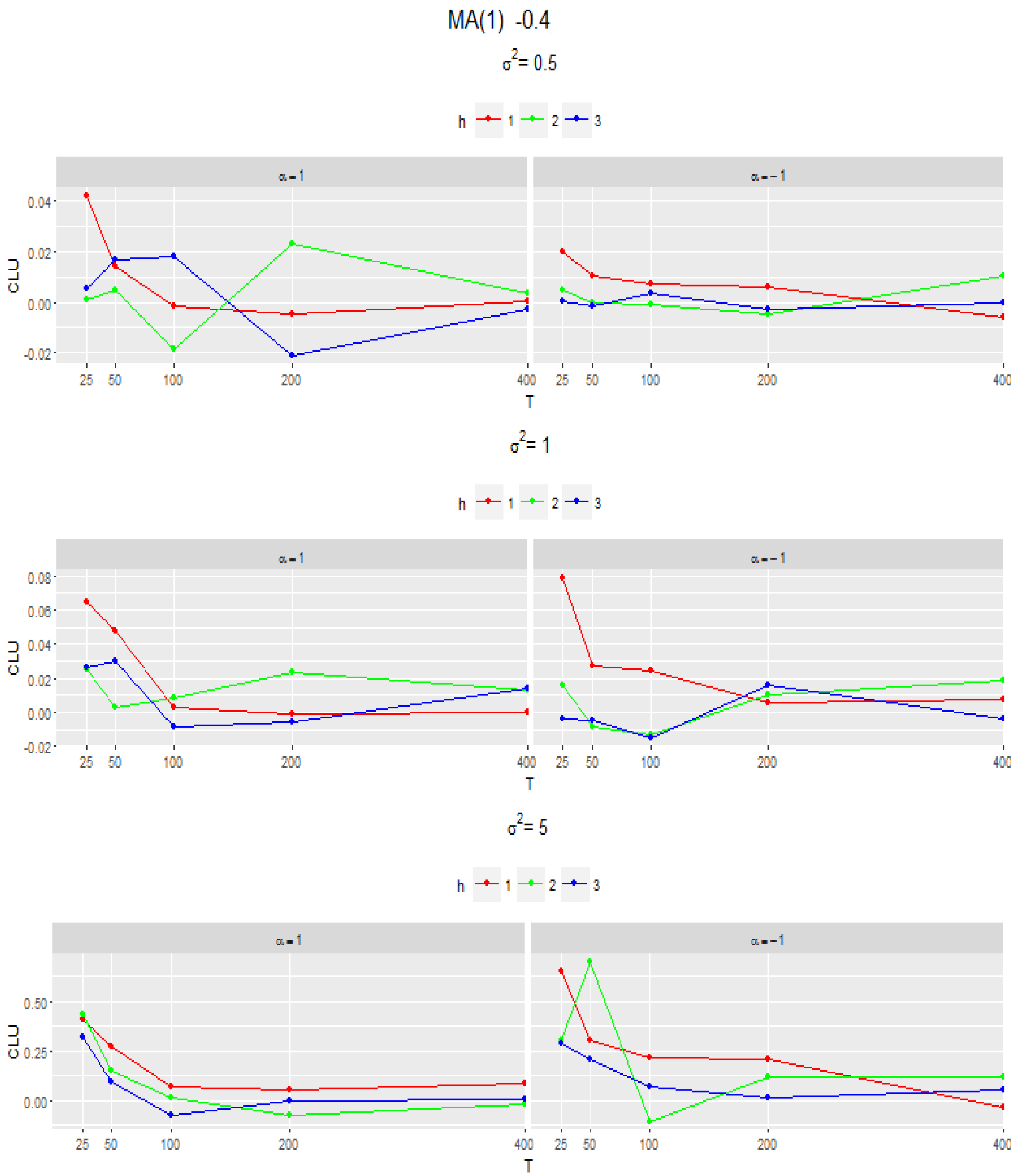

Figure 74. CLU of h by T for MA(1), $\theta_1 = -0.4$

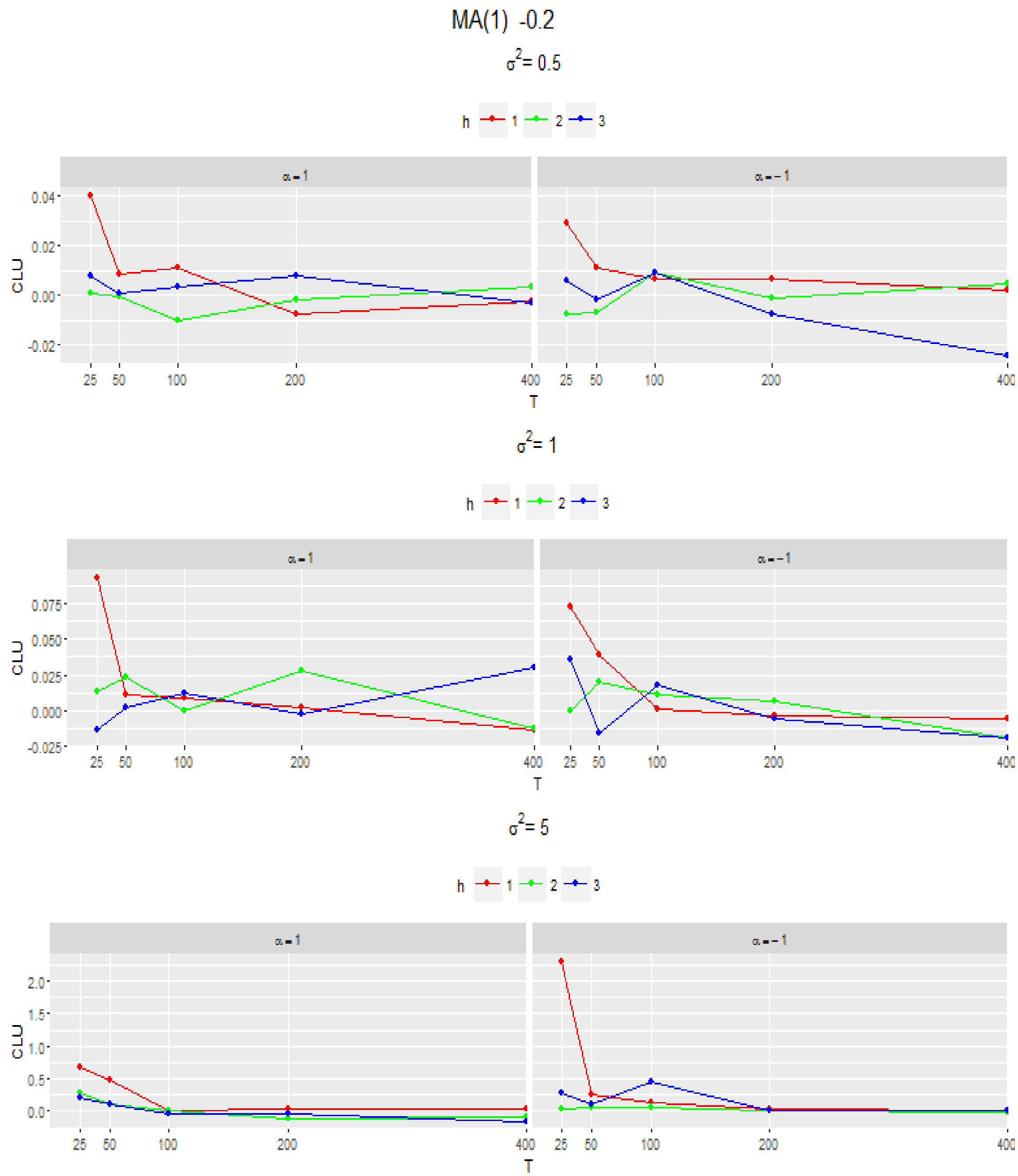

Figure 75. CLU of h by T for MA(1), $\theta_1 = -0.2$

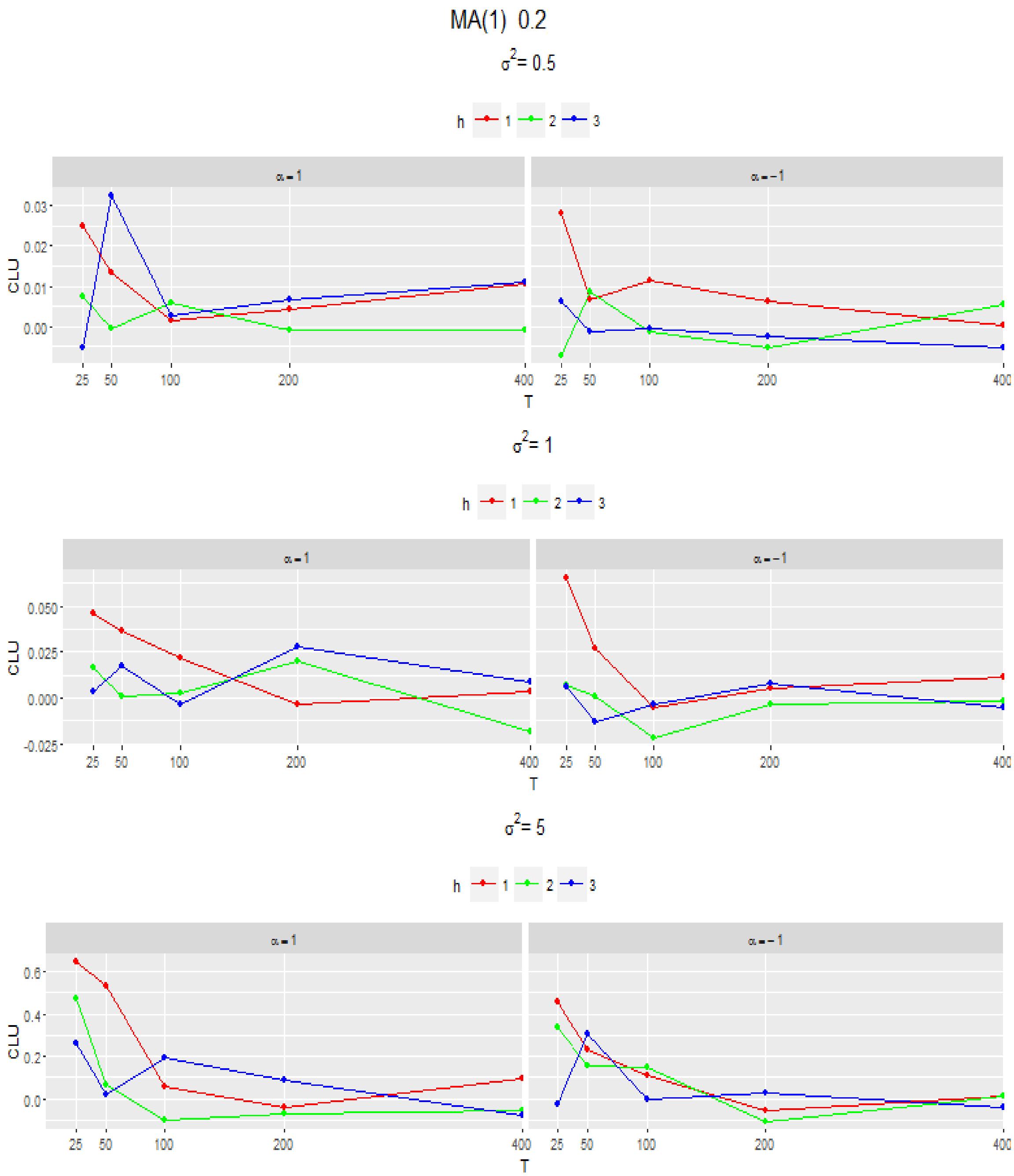

Figure 76. CLU of h by T for MA(1), $\theta_1 = 0.2$

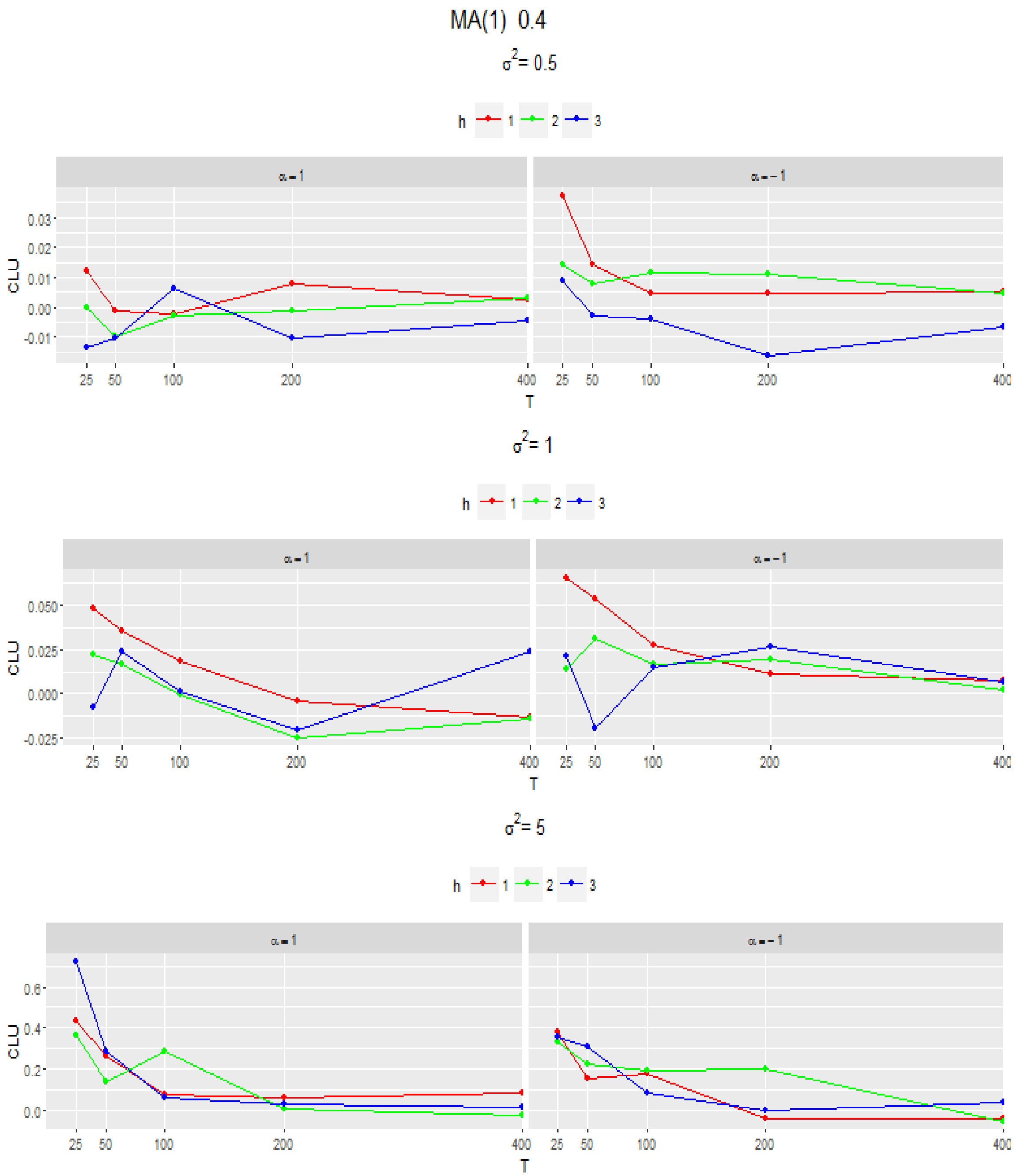

Figure 77. CLU of h by T for MA(1), $\theta_1 = 0.4$

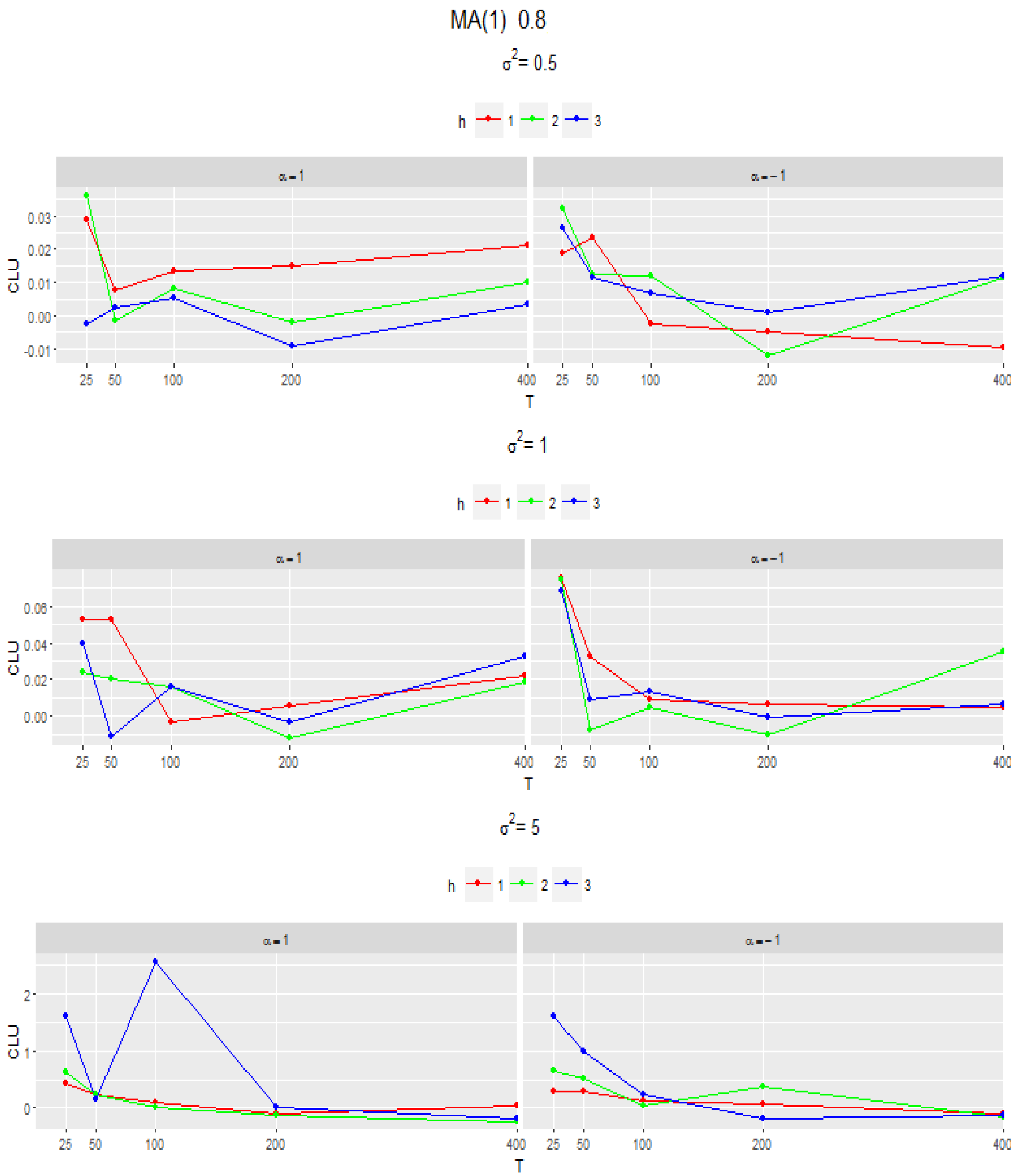

Figure 78. CLU of h by T for MA(1), $\theta_1 = 0.8$

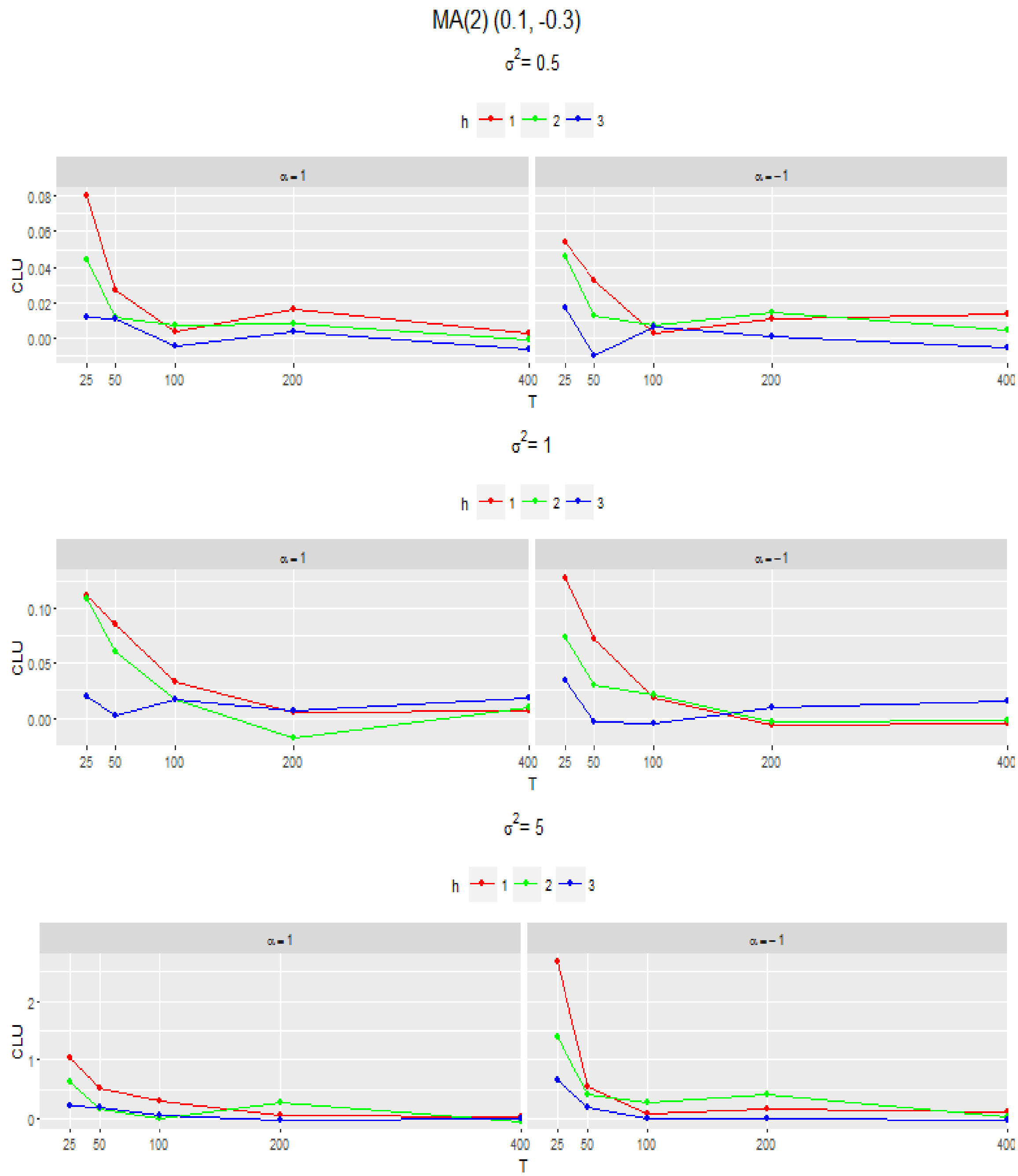

Figure 79. CLU of h by T for MA(2), $(\theta_1, \theta_2) = (0.1, -0.3)$

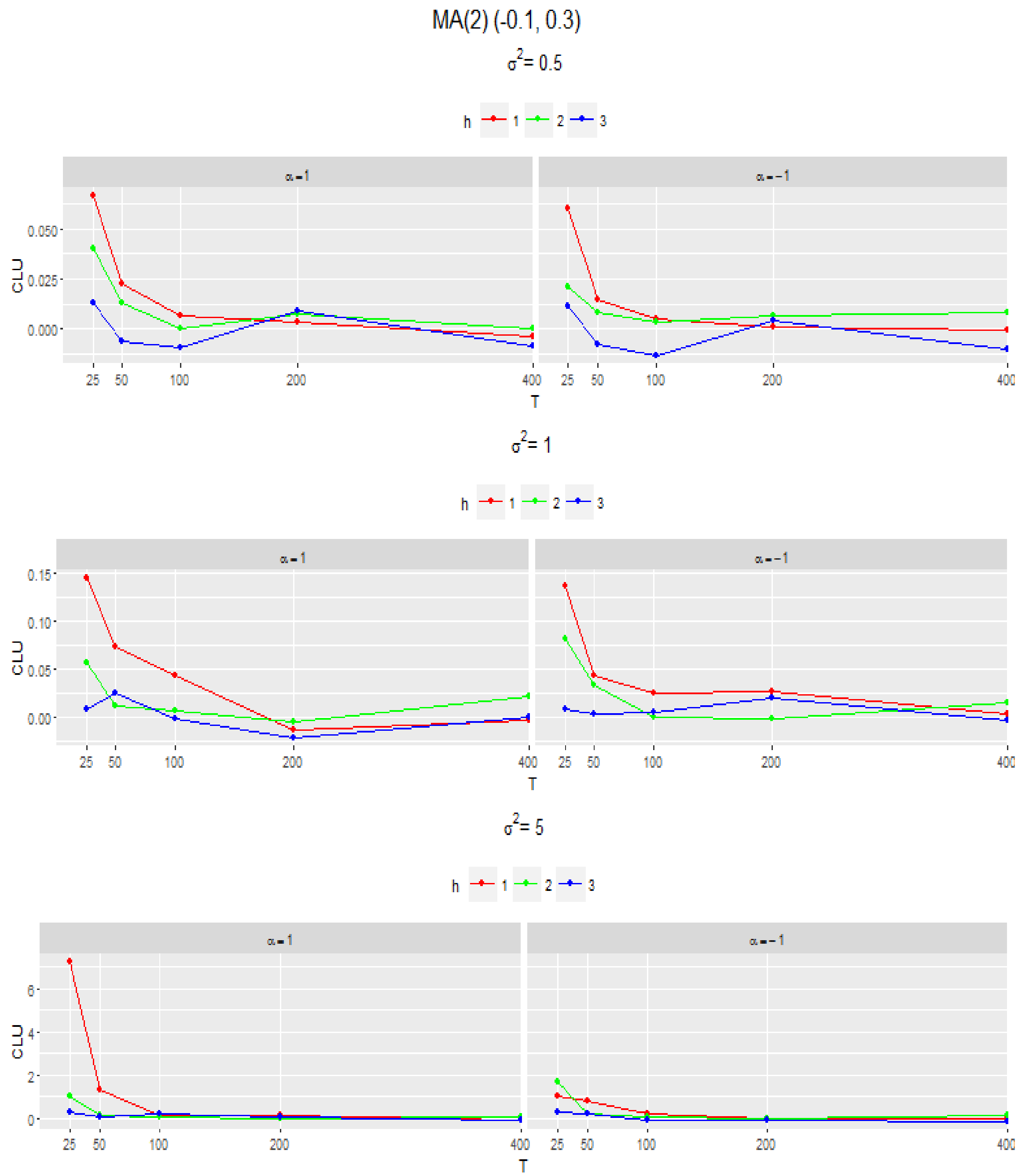

Figure 80. CLU of h by T for MA(2), $(\theta_1, \theta_2) = (-0.1, 0.3)$

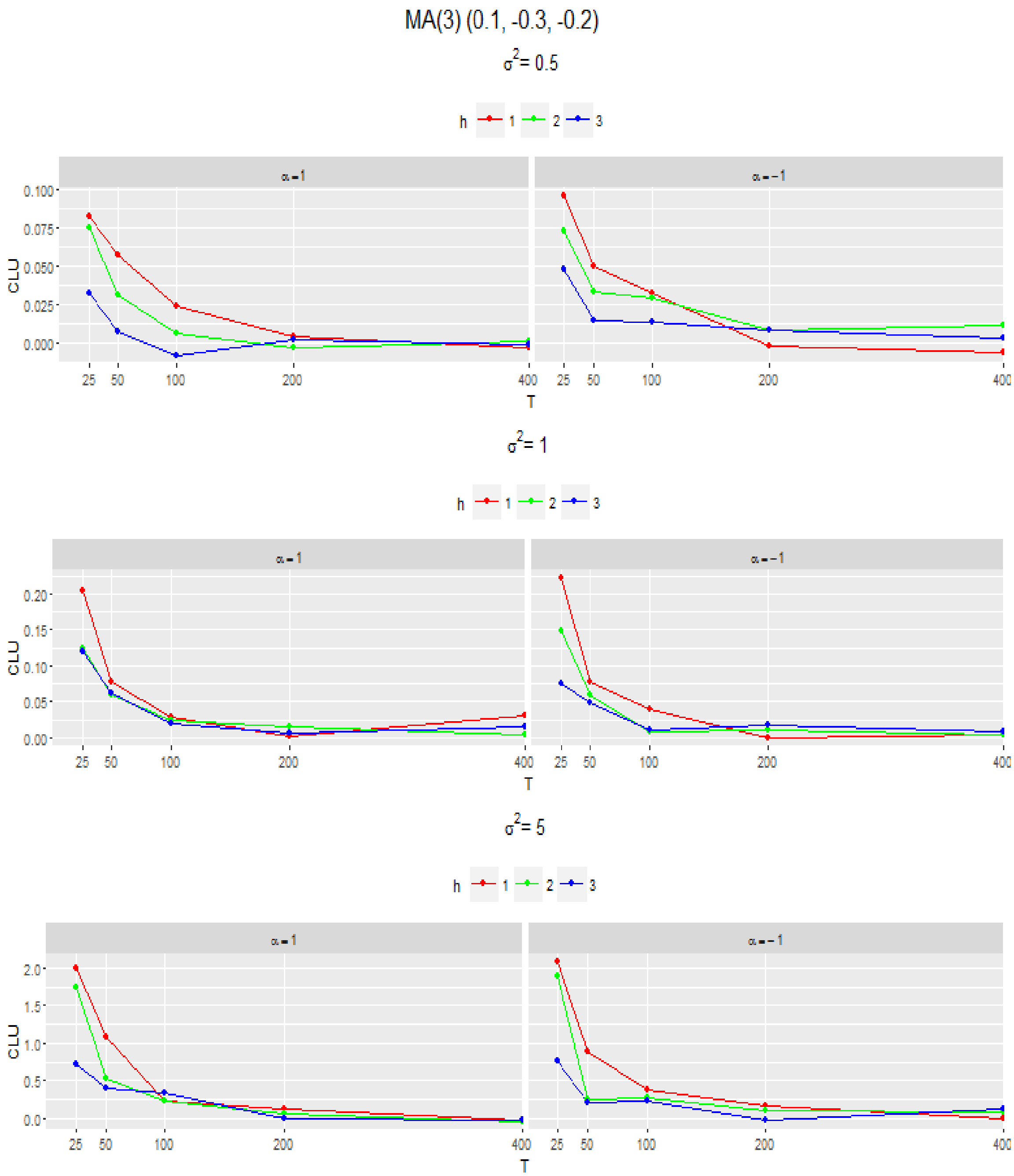

Figure 81. CLU of h by T for MA(3), $(\theta_1, \theta_2, \theta_3) = (0.1, -0.3, -0.2)$

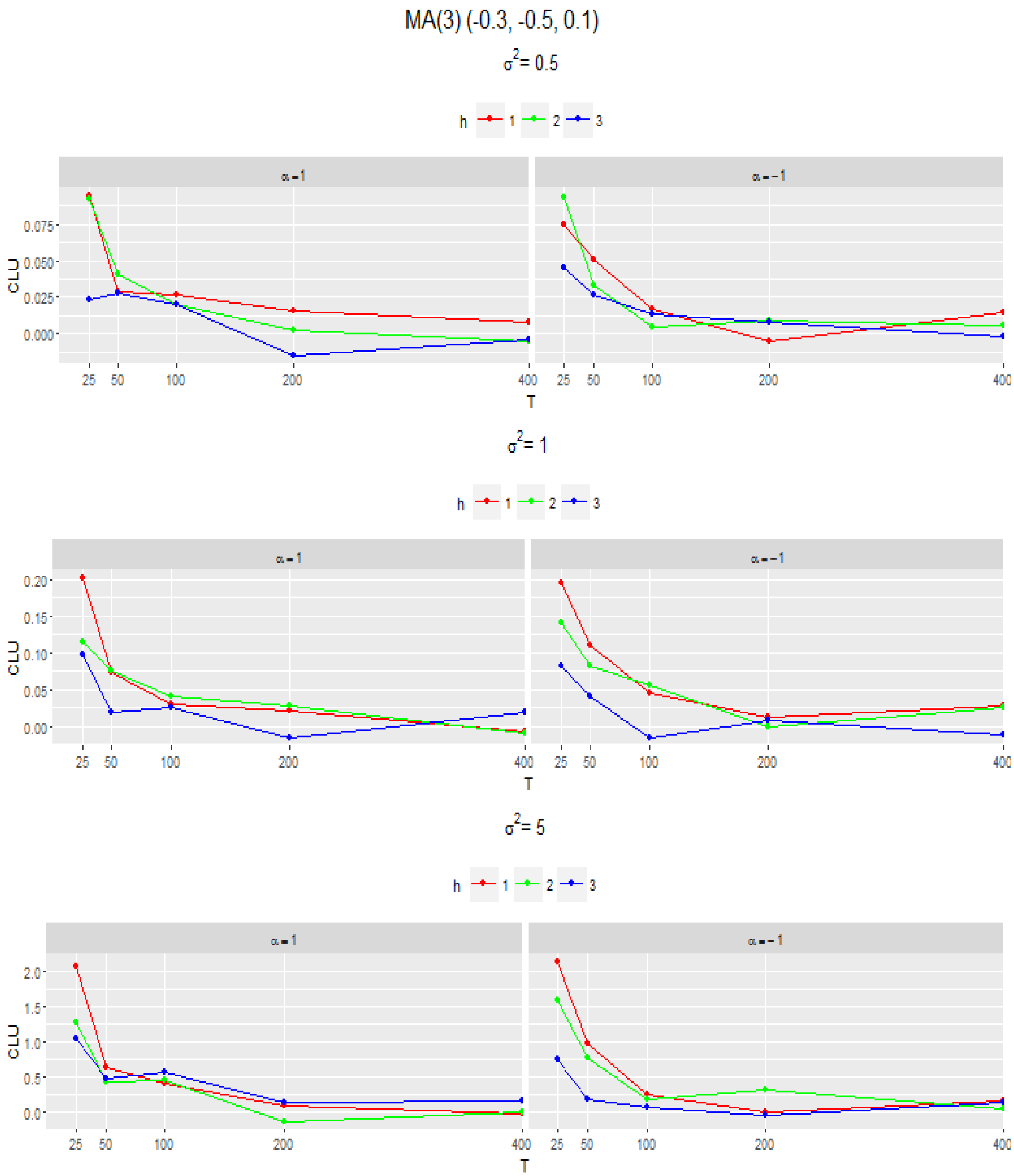

Figure 82. CLU of h by T for MA(3), $(\theta_1, \theta_2, \theta_3) = (0.1, -0.3, -0.2)$

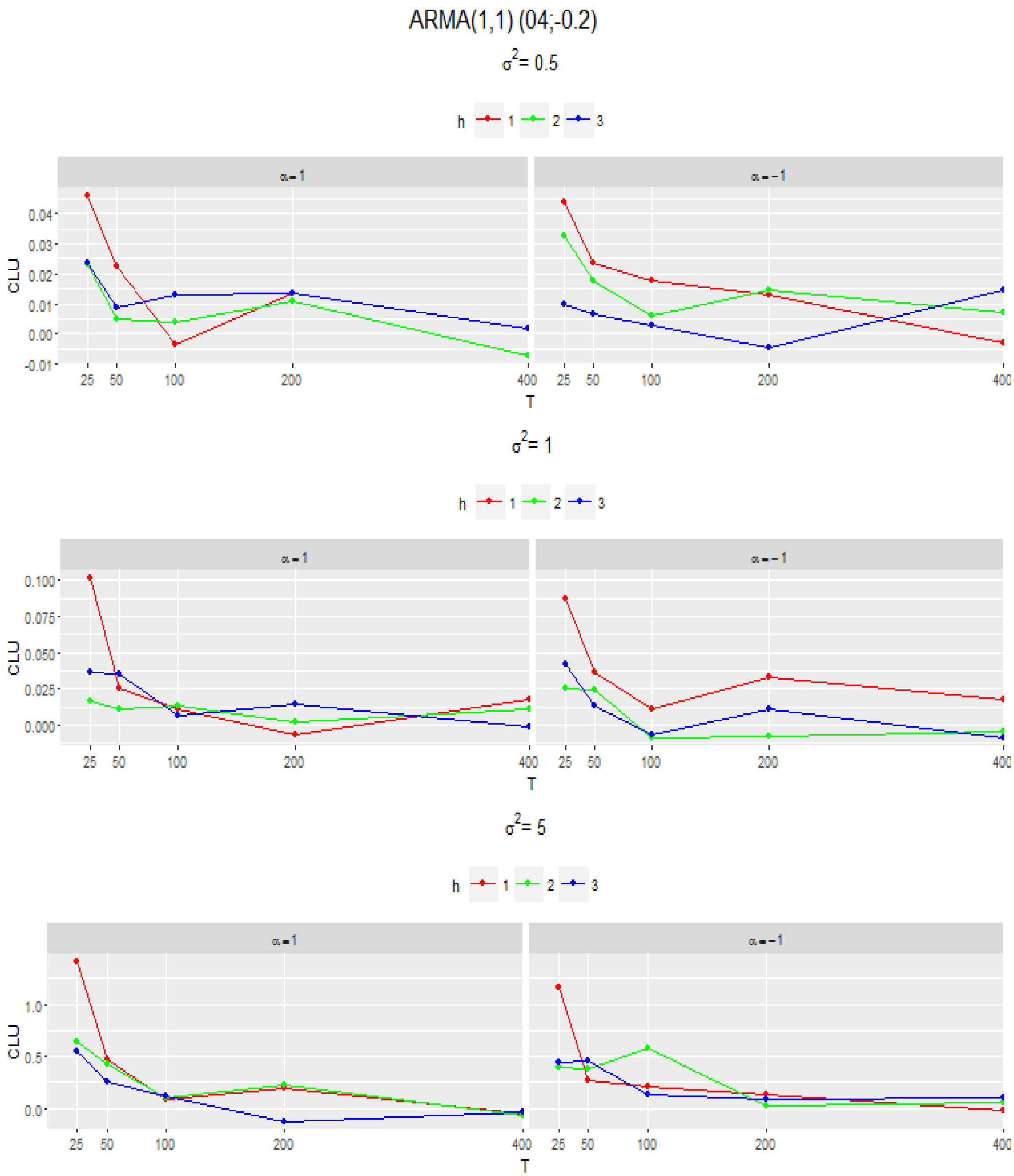

Figure 83. CLU of h by T for ARMA(1,1), $(\phi_1; \theta_1) = (0.4; -0.2)$

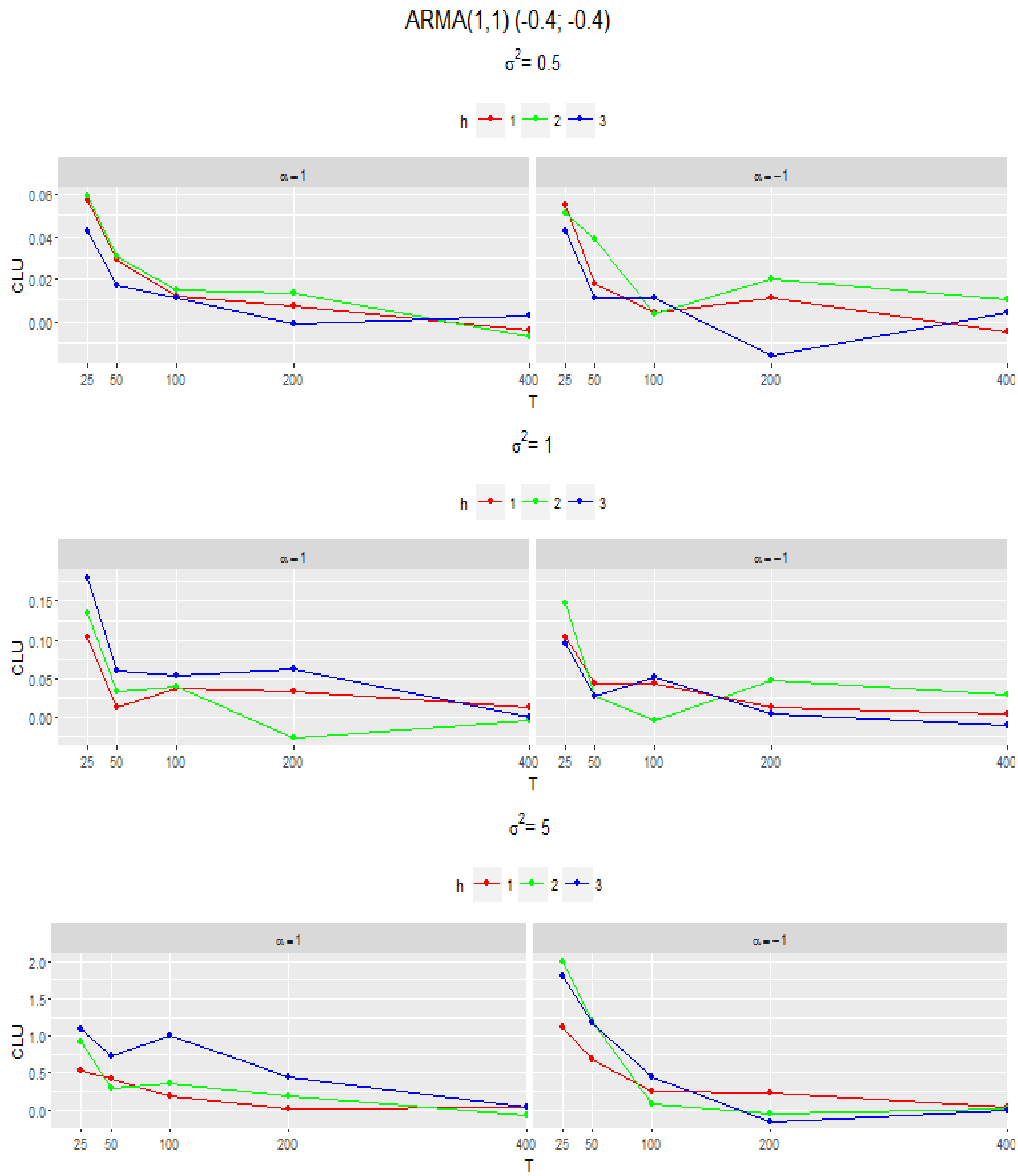

Figure 84. CLU of h by T for ARMA(1,1), $(\phi_1; \theta_1) = (-0.4; -0.4)$

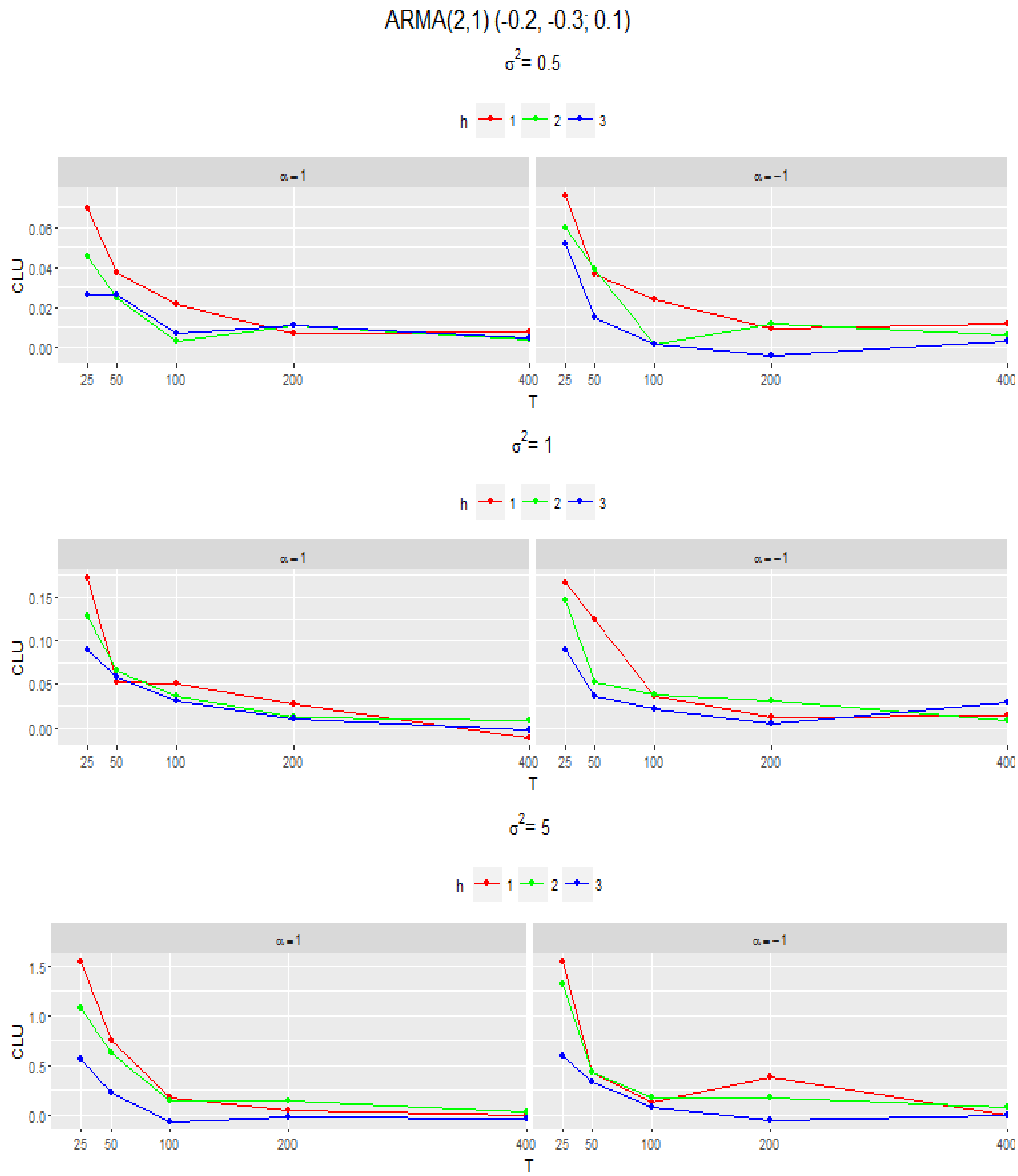

Figure 85. CLU of h by T for ARMA(2,1) $(\phi_1, \phi_2; \theta_1) = (-0.2, -0.3; 0.1)$

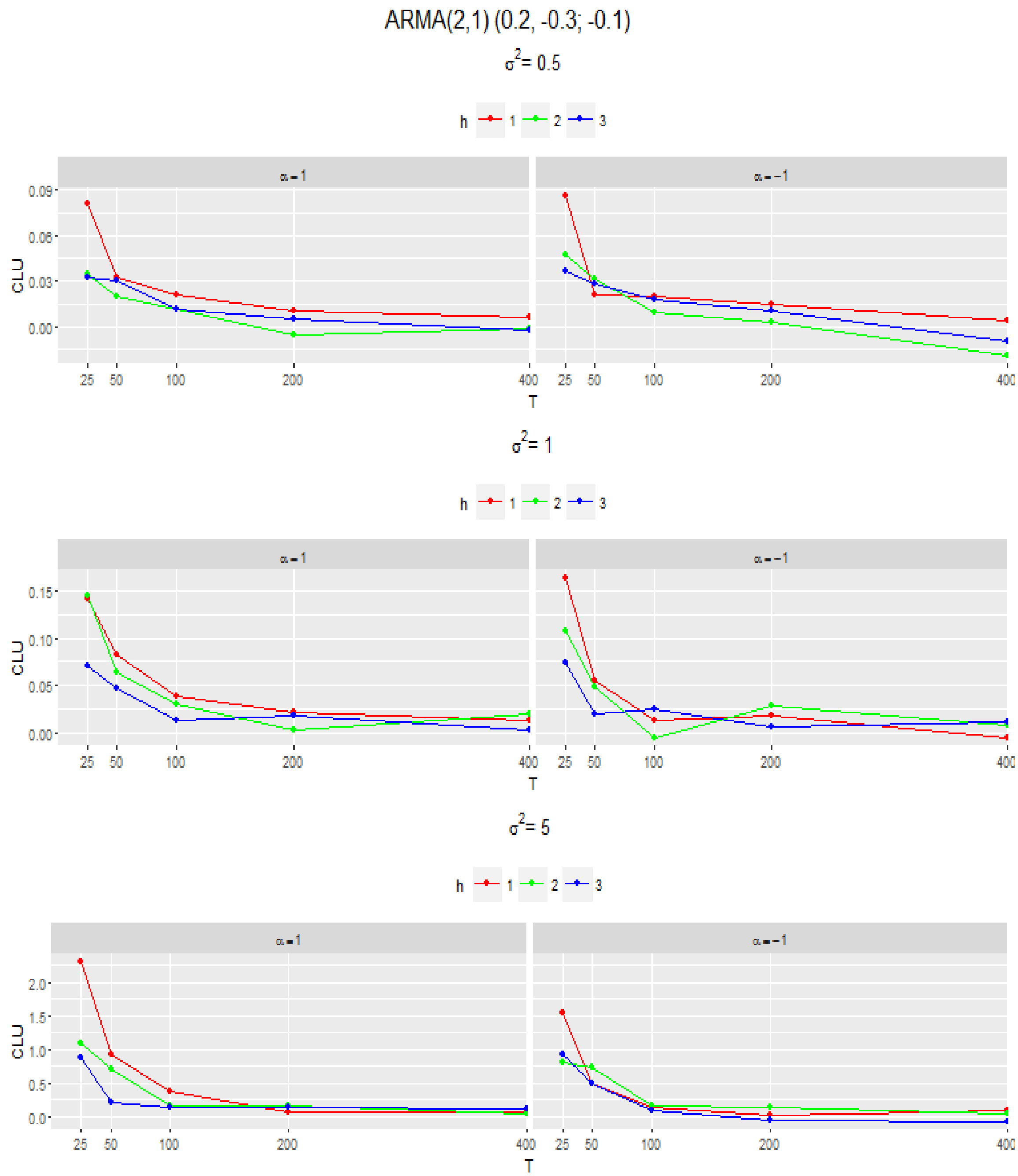

Figure 86. CLU of h by T for ARMA(2,1), $(\phi_1, \phi_2; \theta_1) = (0.2, -0.3; -0.1)$

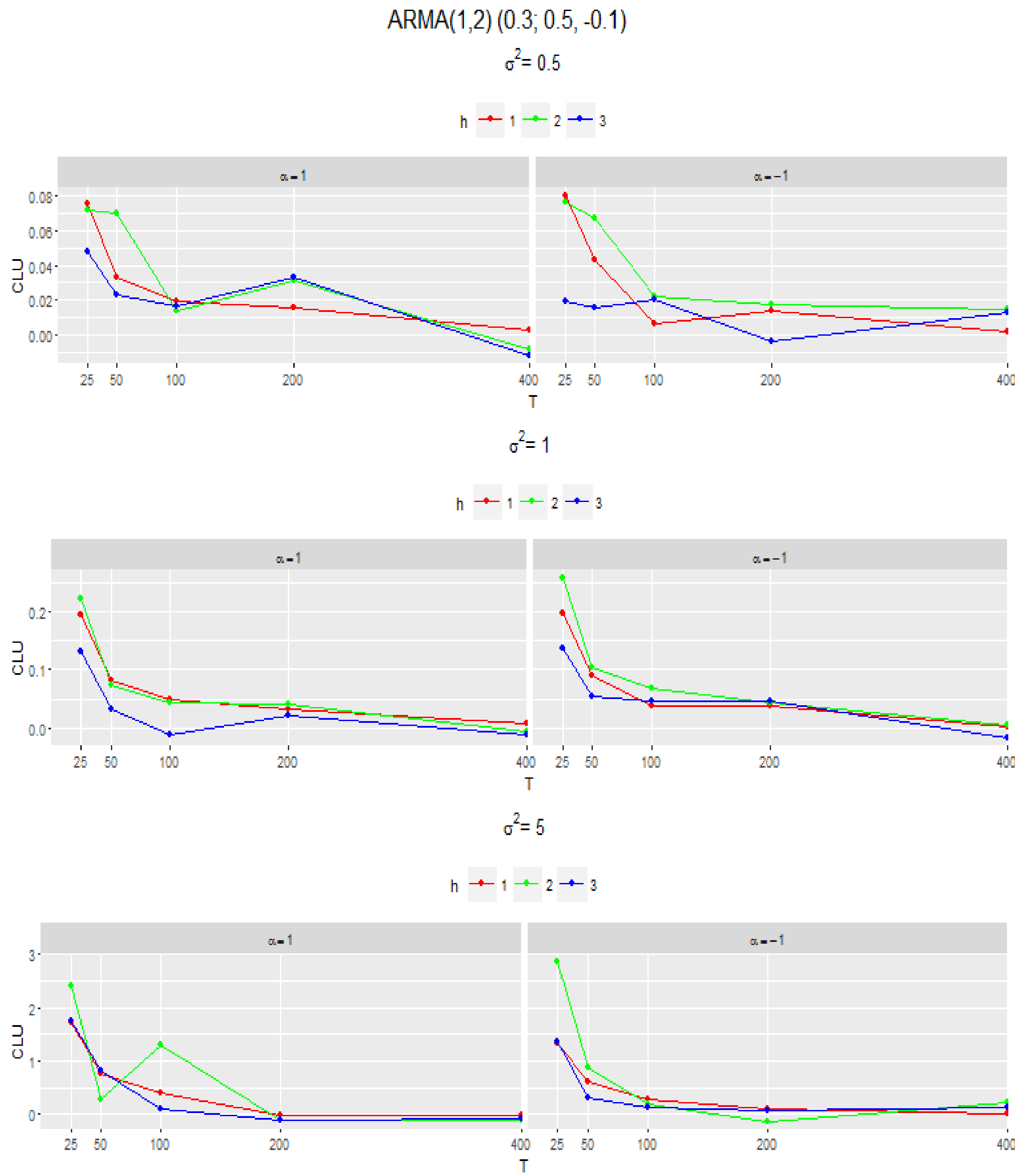

Figure 87. CLU of h by T for ARMA(1,2), $(\phi_1; \theta_1, \theta_2) = (0.3; 0.5, -0.1)$

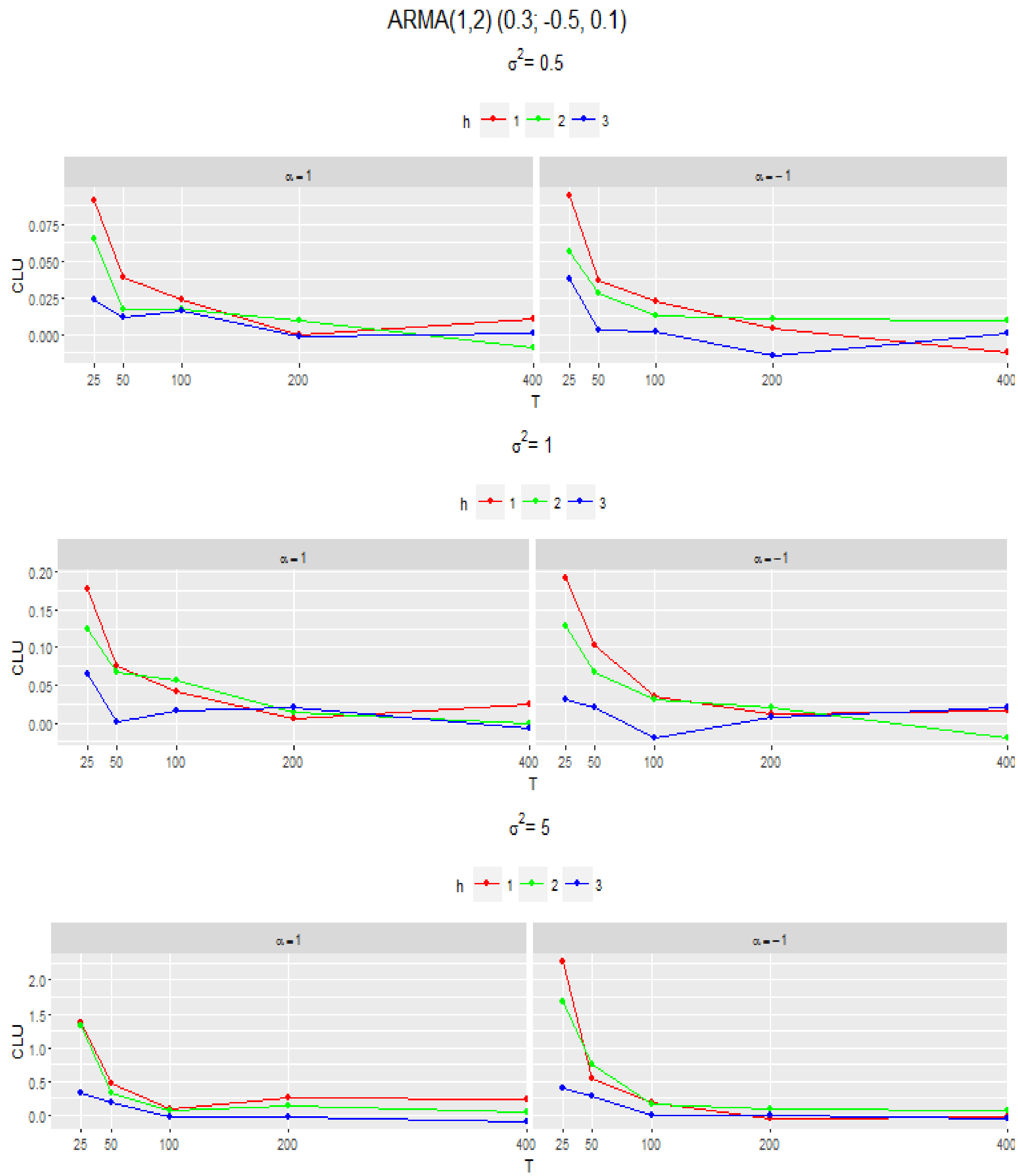

Figure 88. CLU of h by T for ARMA(1,2), $(\phi_1; \theta_1, \theta_2) = (0.3; -0.5, 0.1)$

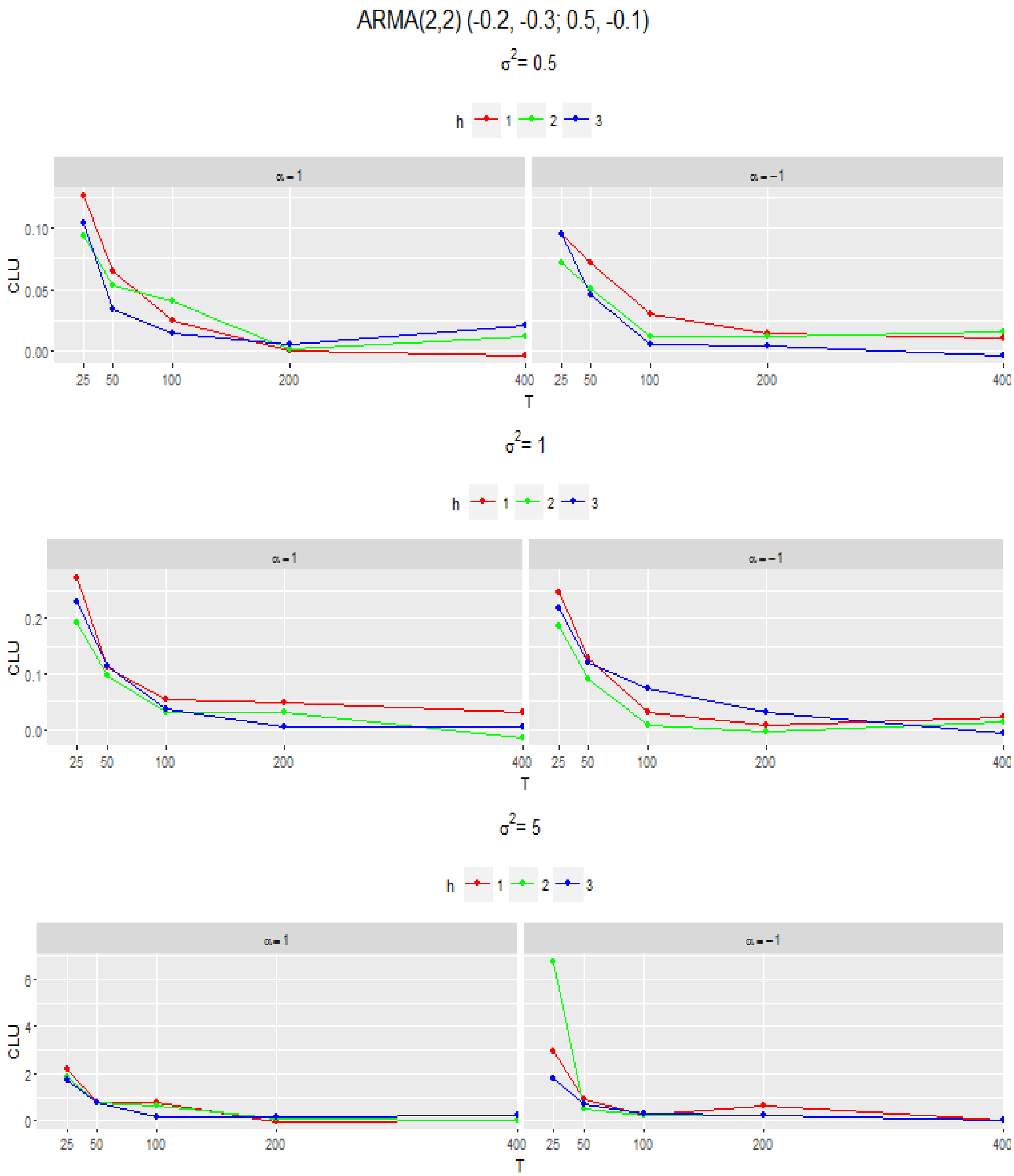

Figure 89. CLU of h by T for ARMA(2,2), $(\phi_1, \phi_2; \theta_1, \theta_2) = (-0.2, -0.3; 0.5, -0.1)$

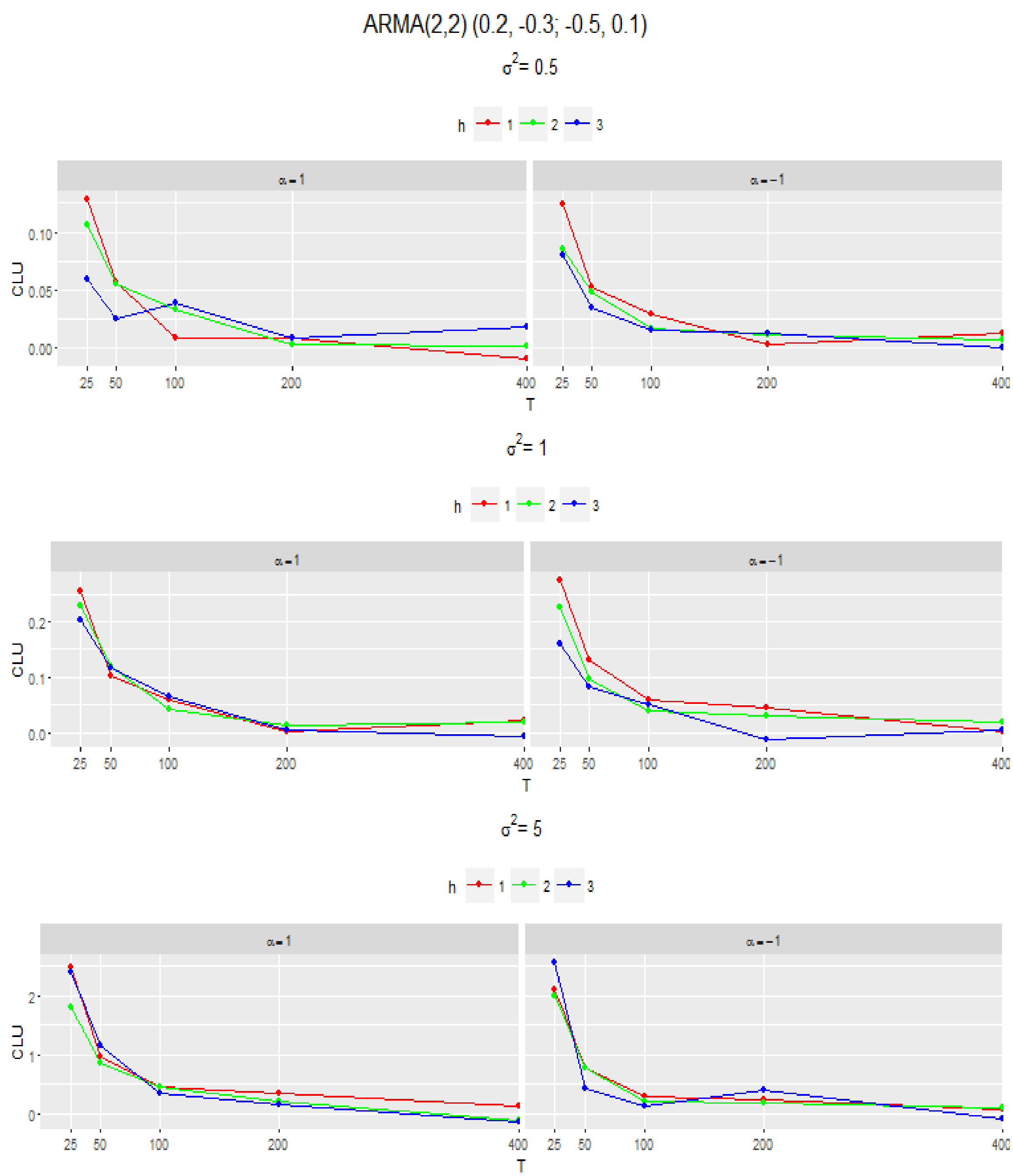

Figure 90. CLU of h by T for ARMA(2,2), $(\phi_1, \phi_2; \theta_1, \theta_2) = (0.2, -0.3; -0.5, 0.1)$

APPENDIXD

RELATIVE EFFICIENCY OF $\hat{\hat{Z}}_{T+H}$ AND $\hat{Z}_{T+H}$

Table 28

Relative Efficiency at $\sigma^2 = 0.5$ and $h = 1$

Models	Parameters	$\alpha = 1$					$\alpha = -1$				
		T=25	T=50	T=100	T=200	T=400	T=25	T=50	T=100	T=200	T=400
AR(1)	(-0.8)	1.072	1.022	1.017	1.022	1.020	1.045	1.009	1.006	0.992	0.981
	(-0.4)	1.070	1.044	1.032	1.022	0.975	1.043	1.023	1.043	0.987	1.010
	(-0.2)	1.043	1.040	1.013	1.022	0.985	1.043	1.035	1.005	1.030	0.991
	(0.2)	1.053	1.055	1.044	0.964	0.975	1.048	1.057	1.001	1.016	0.983
	(0.4)	1.036	1.060	1.020	1.017	1.022	1.075	1.014	1.035	1.015	0.991
	(0.8)	1.025	0.998	1.033	1.029	1.015	1.045	1.001	0.995	0.980	1.038
MA(1)	(-0.8)	1.060	1.043	1.017	1.012	1.004	1.092	1.017	1.017	1.020	0.984
	(-0.4)	1.077	1.024	1.010	0.988	1.010	1.062	1.030	1.000	0.991	0.984
	(-0.2)	1.052	1.031	1.017	1.006	0.991	1.050	1.012	1.030	1.018	1.006
	(0.2)	1.047	1.039	1.018	1.024	1.032	1.072	1.034	1.021	1.008	0.984
	(0.4)	1.051	1.018	1.006	0.994	1.005	1.073	1.016	1.033	0.988	1.031
	(0.8)	1.059	1.026	1.024	1.012	1.028	1.052	1.063	0.998	0.985	0.998
AR(2)	(0.1, -0.3)	1.093	1.062	1.006	1.026	0.999	1.123	1.042	1.019	1.011	0.995
	(-0.1, 0.3)	1.097	1.042	1.003	0.999	1.010	1.091	1.048	1.023	1.007	1.027
AR(3)	(0.1, -0.3, -0.2)	1.201	1.035	1.017	1.022	1.018	1.175	1.115	1.083	0.976	1.013
	(-0.3, -0.5, 0.1)	1.122	1.073	0.996	0.998	1.035	1.171	1.062	1.036	0.987	0.976
MA(2)	(0.1, -0.3)	1.154	1.090	1.000	1.022	0.983	1.141	1.090	1.005	1.021	1.019
	(-0.1, 0.3)	1.154	1.052	1.037	1.025	0.980	1.164	1.039	1.037	1.002	1.012
MA(3)	(0.1, -0.3, -0.2)	1.234	1.126	1.040	1.019	0.987	1.230	1.086	1.064	1.027	1.004
	(-0.3, -0.5, 0.1)	1.214	1.092	1.063	1.025	1.013	1.198	1.109	1.042	1.004	1.010
ARMA(1,1)	(0.4; -0.2)	1.110	1.064	0.999	1.020	1.017	1.122	1.048	1.031	1.002	1.018
	(-0.4; -0.4)	1.103	1.074	1.028	1.014	1.002	1.118	1.038	1.020	1.031	1.011
ARMA(2,1)	(-0.2, -0.3; 0.1)	1.158	1.057	1.013	1.011	1.021	1.164	1.093	1.047	1.022	1.027
	(0.2, -0.3; -0.1)	1.166	1.054	1.039	1.015	1.027	1.183	1.037	1.042	1.013	1.001
ARMA(1,2)	(0.3; 0.5, -0.1)	1.170	1.054	1.043	0.999	1.002	1.178	1.099	1.040	1.007	0.995
	(0.3; -0.5, 0.1)	1.214	1.097	1.045	1.022	1.002	1.242	1.073	1.004	1.006	1.006
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	1.283	1.123	1.071	1.023	1.011	1.249	1.126	1.064	1.043	1.002
	(0.2, -0.3; -0.5, 0.1)	1.301	1.091	1.060	1.020	1.000	1.296	1.111	1.041	1.016	1.020

Table 29

Relative Efficiency at $\sigma^2 = 0.5$ and $h = 2$

Models	Parameters	$\alpha = 1$					$\alpha = -1$				
		T=25	T=50	T=100	T=200	T=400	T=25	T=50	T=100	T=200	T=400
AR(1)	(-0.8)	1.069	1.016	1.054	1.029	1.010	1.066	1.035	1.052	0.999	0.994
	(-0.4)	1.009	1.005	1.017	1.002	1.018	1.043	1.023	1.024	1.007	1.004
	(-0.2)	1.032	0.996	1.009	0.983	1.016	1.055	1.021	0.998	0.989	0.993
	(0.2)	1.016	1.009	0.991	1.001	1.000	1.026	1.022	1.023	1.001	1.011
	(0.4)	1.014	1.043	1.008	1.001	0.991	1.046	1.001	1.027	0.983	1.007
	(0.8)	1.076	1.031	1.028	1.006	1.035	1.084	1.023	0.991	0.991	1.022
MA(1)	(-0.8)	0.995	0.998	1.013	0.968	1.004	1.027	0.996	1.006	1.019	0.970
	(-0.4)	1.000	1.009	0.988	1.036	0.980	1.005	0.995	1.004	0.988	1.024
	(-0.2)	0.991	1.001	1.001	0.993	1.005	1.001	1.001	1.006	1.002	0.992
	(0.2)	1.009	1.021	1.013	1.030	0.984	1.009	1.014	0.986	0.990	1.023
	(0.4)	1.017	0.999	0.999	0.993	1.018	1.021	1.021	1.032	1.030	1.013
	(0.8)	1.037	1.025	1.013	0.975	0.995	1.048	1.019	1.003	0.992	1.023
AR(2)	(0.1, -0.3)	1.073	1.039	1.023	1.017	1.024	1.055	1.066	1.011	0.974	1.007
	(-0.1, 0.3)	1.060	1.019	1.017	0.995	0.985	1.080	1.048	1.007	1.031	0.993
AR(3)	(0.1, -0.3, -0.2)	1.120	1.040	1.011	1.031	0.994	1.092	1.039	1.028	1.017	1.035
	(-0.3, -0.5, 0.1)	1.093	1.045	1.042	1.051	0.992	1.128	1.067	1.037	1.027	1.021
MA(2)	(0.1, -0.3)	1.101	1.061	1.004	0.980	1.001	1.103	1.035	1.009	1.005	0.993
	(-0.1, 0.3)	1.097	1.022	1.036	1.002	1.010	1.087	1.033	1.018	0.999	1.025
MA(3)	(0.1, -0.3, -0.2)	1.196	1.074	1.025	1.024	0.999	1.160	1.071	1.062	1.012	1.013
	(-0.3, -0.5, 0.1)	1.180	1.055	1.042	1.016	0.978	1.189	1.071	1.016	1.024	1.004
ARMA(1,1)	(0.4; -0.2)	1.053	1.007	1.033	1.010	0.996	1.072	1.036	1.019	1.006	0.982
	(-0.4; -0.4)	1.077	1.029	1.010	1.009	0.985	1.065	1.025	1.012	1.035	1.026
ARMA(2,1)	(-0.2, -0.3; 0.1)	1.112	1.065	1.001	1.030	1.023	1.118	1.109	1.000	0.998	1.011
	(0.2, -0.3; -0.1)	1.089	1.028	1.053	1.019	1.015	1.109	1.055	1.027	0.993	0.975
ARMA(1,2)	(0.3; 0.5, -0.1)	1.102	1.095	1.025	1.032	0.978	1.135	1.105	1.033	1.042	1.017
	(0.3; -0.5, 0.1)	1.137	1.039	1.006	1.037	0.978	1.132	1.061	1.028	1.003	1.013
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	1.160	1.087	1.062	1.031	1.014	1.153	1.110	1.040	1.005	1.030
	(0.2, -0.3; -0.5, 0.1)	1.212	1.111	1.083	1.028	1.014	1.172	1.098	1.018	1.030	1.013

Table 30

Relative Efficiency at $\sigma^2 = 0.5$ and $h = 3$

Models	Parameters	$\alpha = 1$					$\alpha = -1$				
		T=25	T=50	T=100	T=200	T=400	T=25	T=50	T=100	T=200	T=400
AR(1)	(-0.8)	1.080	1.053	1.046	1.035	1.024	1.074	1.087	1.025	1.004	0.999
	(-0.4)	0.995	1.008	0.995	1.011	0.989	1.023	1.027	1.011	1.018	1.003
	(-0.2)	1.037	0.982	1.017	1.003	0.994	1.004	0.983	0.991	0.980	0.985
	(0.2)	1.014	0.997	0.986	0.986	0.995	1.006	1.017	0.977	0.977	0.985
	(0.4)	1.041	1.026	0.980	0.994	1.020	1.040	1.011	0.979	0.978	0.969
	(0.8)	1.112	1.024	1.045	0.987	1.032	1.099	1.059	1.011	1.003	1.027
MA(1)	(-0.8)	1.060	0.989	0.972	0.973	1.001	1.020	1.013	1.021	1.008	0.982
	(-0.4)	1.017	1.030	1.015	0.996	1.006	1.012	1.000	0.991	0.980	1.003
	(-0.2)	0.999	1.009	1.007	1.037	1.018	1.030	1.003	1.043	0.997	0.972
	(0.2)	0.983	1.039	1.023	1.011	1.010	1.001	0.992	1.006	0.977	1.016
	(0.4)	0.991	0.975	1.003	0.999	1.014	1.012	0.996	0.996	0.980	0.996
	(0.8)	1.001	1.037	0.991	0.974	0.987	1.021	1.030	0.993	0.994	0.999
AR(2)	(0.1, -0.3)	1.062	1.001	1.017	1.026	0.993	1.023	1.011	1.005	1.029	0.970
	(-0.1, 0.3)	1.059	1.022	0.994	0.998	1.023	1.050	0.991	1.014	1.017	0.990
AR(3)	(0.1, -0.3, -0.2)	1.078	1.041	1.043	1.032	1.039	1.114	1.025	1.025	1.017	1.018
	(-0.3, -0.5, 0.1)	1.129	1.089	1.023	1.000	1.003	1.082	1.061	1.024	1.016	0.997
MA(2)	(0.1, -0.3)	1.040	1.017	0.986	1.016	0.988	1.070	1.010	1.005	0.992	0.994
	(-0.1, 0.3)	1.032	1.021	0.981	1.013	0.978	1.026	1.003	0.991	0.996	0.989
MA(3)	(0.1, -0.3, -0.2)	1.078	1.065	1.020	0.994	1.014	1.080	1.052	1.004	1.025	0.980
	(-0.3, -0.5, 0.1)	1.062	1.020	1.018	1.023	1.007	1.051	1.033	1.026	1.013	1.040
ARMA(1,1)	(0.4; -0.2)	1.037	1.010	1.033	1.024	0.962	1.046	0.998	0.991	1.003	1.040
	(-0.4; -0.4)	1.115	1.068	1.053	1.051	1.057	1.106	1.083	1.060	1.011	1.028
ARMA(2,1)	(-0.2, -0.3; 0.1)	1.048	1.025	0.992	0.991	1.003	1.071	1.018	0.982	0.960	0.991
	(0.2, -0.3; -0.1)	1.040	1.032	0.997	0.976	0.958	1.060	1.041	0.986	0.988	0.971
ARMA(1,2)	(0.3; 0.5, -0.1)	1.099	1.051	1.041	1.033	0.993	1.066	1.043	1.029	1.035	1.041
	(0.3; -0.5, 0.1)	1.069	1.010	1.022	0.992	1.007	1.041	1.029	1.016	1.002	1.008
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	1.153	1.045	0.977	0.980	0.970	1.111	1.082	0.985	0.988	0.957
	(0.2, -0.3; -0.5, 0.1)	1.114	1.038	1.056	0.992	1.004	1.117	1.051	1.037	1.020	0.991

Table 31

Relative Efficiency at $\sigma^2 = 1$ and $h = 1$

Models	Parameters	$\alpha = 1$					$\alpha = -1$				
		T=25	T=50	T=100	T=200	T=400	T=25	T=50	T=100	T=200	T=400
AR(1)	(-0.8)	1.053	1.040	1.011	0.979	0.999	1.069	1.046	1.030	0.985	1.003
	(-0.4)	1.074	1.014	1.041	0.992	1.011	1.107	1.024	1.042	0.994	1.011
	(-0.2)	1.086	1.023	1.020	1.008	0.976	1.024	1.036	0.993	1.018	0.998
	(0.2)	1.049	1.008	1.021	1.023	1.004	1.059	1.042	1.014	1.023	1.009
	(0.4)	1.075	1.025	1.035	1.024	0.991	1.098	1.026	1.034	1.013	0.996
	(0.8)	1.034	1.066	1.028	0.997	0.998	1.071	1.043	0.983	1.013	1.017
MA(1)	(-0.8)	1.073	1.044	1.023	1.063	1.036	1.069	1.026	1.033	0.989	1.007
	(-0.4)	1.086	1.065	0.982	1.011	1.013	1.104	1.002	1.058	1.000	1.015
	(-0.2)	1.114	1.037	1.006	1.022	0.967	1.093	1.046	0.997	0.996	0.990
	(0.2)	1.078	1.049	1.041	1.004	1.005	1.065	1.023	0.988	0.986	1.005
	(0.4)	1.038	1.036	1.024	1.002	0.996	1.075	1.055	1.019	1.015	1.016
	(0.8)	1.083	1.070	1.000	1.018	1.023	1.088	1.029	1.022	1.018	1.005
AR(2)	(0.1, -0.3)	1.113	1.046	1.025	0.995	1.022	1.103	1.045	1.006	1.013	1.003
	(-0.1, 0.3)	1.158	1.031	1.018	1.018	1.020	1.095	1.033	1.037	1.013	1.018
AR(3)	(0.1, -0.3, -0.2)	1.207	1.102	1.061	1.023	1.040	1.170	1.077	1.004	1.023	1.016
	(-0.3, -0.5, 0.1)	1.176	1.063	1.031	1.025	1.005	1.206	1.057	1.026	1.035	0.999
MA(2)	(0.1, -0.3)	1.171	1.101	1.042	1.006	1.009	1.162	1.109	1.025	0.991	1.006
	(-0.1, 0.3)	1.170	1.087	1.033	1.004	0.987	1.168	1.076	1.004	1.019	1.010
MA(3)	(0.1, -0.3, -0.2)	1.230	1.096	1.045	1.027	1.011	1.288	1.116	1.032	1.008	0.995
	(-0.3, -0.5, 0.1)	1.277	1.090	1.037	1.029	0.994	1.216	1.138	1.025	1.018	1.039
ARMA(1,1)	(0.4; -0.2)	1.115	1.042	1.024	0.993	1.015	1.110	1.060	1.007	1.030	1.012
	(-0.4; -0.4)	1.118	1.028	1.046	1.030	1.006	1.131	1.039	1.024	1.031	1.012
ARMA(2,1)	(-0.2, -0.3; 0.1)	1.199	1.051	1.041	1.021	0.955	1.222	1.131	1.056	1.024	1.034
	(0.2, -0.3; -0.1)	1.153	1.097	1.053	1.025	1.030	1.203	1.093	1.007	1.036	0.987
ARMA(1,2)	(0.3; 0.5, -0.1)	1.206	1.110	1.069	1.046	1.023	1.225	1.101	1.039	1.026	1.029
	(0.3; -0.5, 0.1)	1.222	1.059	1.045	0.994	1.017	1.242	1.125	1.042	1.006	1.012
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	1.320	1.157	1.067	1.041	1.019	1.302	1.166	1.057	1.026	1.016
	(0.2, -0.3; -0.5, 0.1)	1.334	1.127	1.054	1.016	1.049	1.324	1.188	1.102	1.074	0.996

Table 32

Relative Efficiency at $\sigma^2 = 1$ and $h = 2$

Models	Parameters	$\alpha = 1$					$\alpha = -1$				
		T=25	T=50	T=100	T=200	T=400	T=25	T=50	T=100	T=200	T=400
AR(1)	(-0.8)	1.150	1.027	1.033	1.017	1.007	1.069	1.033	1.016	1.034	1.023
	(-0.4)	1.016	1.008	1.001	1.007	0.997	1.033	1.038	1.011	1.031	0.992
	(-0.2)	1.050	1.004	1.026	1.009	1.015	1.061	1.000	1.034	1.006	1.005
	(0.2)	1.039	0.976	0.990	0.988	1.030	1.002	1.022	0.991	1.008	1.010
	(0.4)	1.025	1.051	1.020	1.001	0.981	1.027	1.013	1.016	1.004	1.024
	(0.8)	1.071	1.033	1.042	1.014	1.026	1.107	1.028	0.994	1.034	0.995
MA(1)	(-0.8)	1.010	0.993	1.033	1.030	1.010	1.070	0.988	1.030	0.991	1.008
	(-0.4)	1.052	1.005	1.008	1.018	1.014	1.053	1.004	0.984	1.015	1.001
	(-0.2)	1.017	1.014	0.993	1.036	1.023	1.006	1.007	1.027	1.019	0.984
	(0.2)	1.004	1.021	1.015	1.002	0.989	1.017	1.022	0.979	0.980	1.007
	(0.4)	1.027	1.030	0.988	0.991	0.999	1.004	1.014	1.011	1.004	0.999
	(0.8)	0.998	1.024	1.025	0.984	0.993	1.066	0.994	1.009	0.988	1.027
AR(2)	(0.1, -0.3)	1.094	1.027	1.003	0.993	1.023	1.063	1.061	1.002	1.019	0.988
	(-0.1, 0.3)	1.063	0.991	1.004	1.023	0.991	1.053	1.024	0.997	1.023	0.996
AR(3)	(0.1, -0.3, -0.2)	1.091	1.046	1.039	1.011	1.006	1.093	1.053	1.021	1.000	0.995
	(-0.3, -0.5, 0.1)	1.116	1.017	1.026	1.005	0.974	1.112	1.040	1.008	1.009	0.990
MA(2)	(0.1, -0.3)	1.121	1.055	0.996	0.998	1.008	1.107	1.029	1.033	1.000	1.005
	(-0.1, 0.3)	1.082	1.014	1.009	1.008	1.033	1.095	1.067	0.995	0.996	1.022
MA(3)	(0.1, -0.3, -0.2)	1.135	1.081	1.059	1.032	1.005	1.194	1.088	1.020	1.019	0.986
	(-0.3, -0.5, 0.1)	1.122	1.086	1.045	1.027	1.005	1.137	1.054	1.048	1.001	1.023
ARMA(1,1)	(0.4; -0.2)	1.020	1.024	1.024	1.013	1.021	1.039	1.065	0.999	1.001	0.989
	(-0.4; -0.4)	1.106	1.016	1.032	0.975	0.990	1.110	1.028	1.015	1.034	1.042
ARMA(2,1)	(-0.2, -0.3; 0.1)	1.155	1.039	1.043	1.024	1.037	1.160	1.080	1.044	1.026	0.995
	(0.2, -0.3; -0.1)	1.154	1.092	1.016	1.026	1.009	1.117	1.049	1.024	1.021	1.018
ARMA(1,2)	(0.3; 0.5, -0.1)	1.147	1.064	1.044	1.053	0.998	1.230	1.070	1.059	1.033	1.011
	(0.3; -0.5, 0.1)	1.159	1.081	1.060	1.006	1.017	1.159	1.062	1.032	1.043	0.990
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	1.209	1.120	1.026	1.014	0.967	1.206	1.078	1.037	1.008	1.000
	(0.2, -0.3; -0.5, 0.1)	1.239	1.125	1.049	1.004	1.009	1.230	1.097	1.049	1.008	1.025

Table 33

Relative Efficiency at $\sigma^2 = 1$ and $h = 3$

Models	Parameters	$\alpha = 1$					$\alpha = -1$				
		T=25	T=50	T=100	T=200	T=400	T=25	T=50	T=100	T=200	T=400
AR(1)	(-0.8)	1.154	1.072	1.061	1.011	0.997	1.073	1.055	1.059	0.988	1.047
	(-0.4)	1.004	1.005	0.985	1.024	1.008	1.058	1.013	1.016	1.016	1.004
	(-0.2)	1.058	0.973	1.027	1.036	1.015	0.991	1.008	1.023	0.996	0.987
	(0.2)	0.988	1.002	1.033	0.974	0.981	1.052	1.036	0.989	1.035	1.002
	(0.4)	1.025	1.001	1.035	1.007	0.984	1.035	1.008	1.014	1.038	1.033
	(0.8)	1.113	1.074	1.023	1.017	1.037	1.124	1.048	0.989	1.040	0.988
MA(1)	(-0.8)	1.035	1.011	0.988	1.004	1.001	1.033	1.017	1.009	1.013	1.016
	(-0.4)	1.032	1.037	0.990	1.006	1.002	0.985	0.999	0.996	1.022	1.013
	(-0.2)	1.007	1.002	1.016	0.999	1.037	1.048	0.988	1.025	0.999	0.976
	(0.2)	1.014	1.031	0.992	1.049	1.007	1.009	1.016	1.004	0.999	1.000
	(0.4)	1.003	1.052	1.000	0.994	1.014	1.039	0.983	1.023	1.014	0.991
	(0.8)	1.011	0.981	1.008	0.997	1.044	1.066	1.016	1.013	1.012	0.996
AR(2)	(0.1, -0.3)	1.060	1.034	1.036	1.014	1.020	1.067	1.040	0.994	1.001	1.023
	(-0.1, 0.3)	1.113	1.032	1.028	1.010	1.048	1.054	1.005	1.001	0.963	1.005
AR(3)	(0.1, -0.3, -0.2)	1.095	1.080	1.026	1.024	1.023	1.094	1.054	1.042	1.005	1.018
	(-0.3, -0.5, 0.1)	1.132	1.049	1.024	0.976	1.010	1.190	1.076	1.021	1.019	1.007
MA(2)	(0.1, -0.3)	1.024	1.028	1.026	0.995	1.020	1.068	1.024	0.985	0.983	1.018
	(-0.1, 0.3)	1.037	1.034	0.991	0.998	1.001	1.043	1.014	0.986	1.029	1.006
MA(3)	(0.1, -0.3, -0.2)	1.143	1.066	1.010	0.990	1.032	1.113	1.056	1.024	1.011	1.010
	(-0.3, -0.5, 0.1)	1.091	1.009	1.010	0.984	1.017	1.064	1.065	1.006	1.002	0.977
ARMA(1,1)	(0.4; -0.2)	1.061	1.044	1.008	1.009	1.016	1.037	1.028	0.996	1.024	0.987
	(-0.4; -0.4)	1.202	1.086	1.096	1.090	1.078	1.114	1.057	1.075	1.056	1.046
ARMA(2,1)	(-0.2, -0.3; 0.1)	1.063	1.021	0.999	0.978	0.971	1.077	1.040	1.003	1.002	1.011
	(0.2, -0.3; -0.1)	1.071	1.014	1.012	1.006	0.984	1.079	0.991	1.023	0.977	1.007
ARMA(1,2)	(0.3; 0.5, -0.1)	1.118	1.037	1.002	1.062	0.998	1.158	1.053	1.053	1.070	1.005
	(0.3; -0.5, 0.1)	1.092	1.010	1.028	1.018	0.993	1.077	1.005	0.993	0.999	1.021
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	1.219	1.094	1.021	1.005	0.981	1.190	1.090	1.042	0.967	0.970
	(0.2, -0.3; -0.5, 0.1)	1.191	1.097	1.049	0.987	0.990	1.148	1.067	1.013	0.972	0.987

Table 34

Relative Efficiency at $\sigma^2 = 5$ and $h = 1$

Models	Parameters	$\alpha = 1$					$\alpha = -1$				
		T=25	T=50	T=100	T=200	T=400	T=25	T=50	T=100	T=200	T=400
AR(1)	(-0.8)	1.224	1.012	1.032	1.008	0.960	1.106	1.038	1.022	1.132	0.960
	(-0.4)	1.195	1.041	1.102	1.028	0.991	1.173	1.066	0.994	1.051	0.977
	(-0.2)	1.111	1.043	1.054	1.021	0.988	1.268	1.079	1.038	1.032	0.983
	(0.2)	1.205	1.153	1.070	1.066	0.998	1.093	1.104	1.036	0.969	0.955
	(0.4)	1.007	1.042	1.073	0.984	1.012	1.311	1.080	1.038	1.041	0.981
	(0.8)	1.177	1.151	0.956	1.034	1.002	1.365	1.024	1.073	1.024	1.010
MA(1)	(-0.8)	1.283	1.007	0.999	1.010	1.030	1.187	1.034	1.020	1.014	0.987
	(-0.4)	1.107	1.085	1.030	1.013	1.036	1.202	1.101	1.081	1.065	0.981
	(-0.2)	1.222	1.171	1.009	1.004	1.013	1.865	1.089	1.033	1.008	1.039
	(0.2)	1.201	1.188	1.007	0.965	1.031	1.146	1.088	1.036	0.972	1.055
	(0.4)	1.150	1.092	1.020	1.032	1.015	1.111	1.044	1.063	0.979	0.985
	(0.8)	1.121	1.059	1.024	0.972	1.013	1.073	1.086	1.019	1.029	0.967
AR(2)	(0.1, -0.3)	1.251	1.063	1.059	1.008	1.039	1.264	1.065	1.058	0.977	1.042
	(-0.1, 0.3)	1.159	1.161	1.018	0.980	1.032	1.274	1.074	1.002	0.972	0.965
AR(3)	(0.1, -0.3, -0.2)	1.263	1.182	1.052	1.033	0.955	1.536	1.196	0.990	1.029	1.074
	(-0.3, -0.5, 0.1)	1.606	1.205	1.064	0.985	1.028	1.516	1.214	1.046	1.008	1.044
MA(2)	(0.1, -0.3)	1.318	1.140	1.114	1.008	1.017	1.977	1.180	1.029	1.054	1.039
	(-0.1, 0.3)	3.815	1.482	1.044	1.034	0.974	1.317	1.273	1.062	0.998	1.010
MA(3)	(0.1, -0.3, -0.2)	1.662	1.374	1.072	1.027	0.994	1.682	1.290	1.112	1.053	0.990
	(-0.3, -0.5, 0.1)	1.652	1.184	1.129	1.012	0.993	1.700	1.310	1.083	0.989	1.045
ARMA(1,1)	(0.4; -0.2)	1.475	1.144	1.017	1.081	0.978	1.388	1.073	1.063	1.040	0.995
	(-0.4; -0.4)	1.108	1.117	1.052	1.004	1.008	1.350	1.225	1.072	1.076	1.012
ARMA(2,1)	(-0.2, -0.3; 0.1)	1.483	1.223	1.036	0.996	0.989	1.487	1.102	1.020	1.137	0.999
	(0.2, -0.3; -0.1)	1.769	1.310	1.100	0.999	1.023	1.466	1.134	1.015	0.993	1.017
ARMA(1,2)	(0.3; 0.5, -0.1)	1.548	1.236	1.119	0.980	0.994	1.401	1.181	1.073	1.016	0.986
	(0.3; -0.5, 0.1)	1.406	1.129	0.999	1.070	1.087	1.766	1.170	1.035	0.971	0.970
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	1.648	1.194	1.271	0.980	0.994	1.954	1.269	1.051	1.249	1.004
	(0.2, -0.3; -0.5, 0.1)	1.766	1.282	1.134	1.109	1.041	1.619	1.209	1.070	1.066	1.030

Table 35

Relative Efficiency at $\sigma^2 = 5$ and $h = 2$

Models	Parameters	$\alpha = 1$					$\alpha = -1$				
		T=25	T=50	T=100	T=200	T=400	T=25	T=50	T=100	T=200	T=400
AR(1)	(-0.8)	1.143	1.042	0.973	1.011	1.042	1.190	1.544	1.028	0.949	1.000
	(-0.4)	1.229	1.043	0.987	0.939	1.011	1.128	1.046	1.005	1.012	0.997
	(-0.2)	1.110	1.145	0.981	0.988	0.991	1.084	1.011	0.985	0.984	0.980
	(0.2)	1.108	1.008	0.989	0.957	0.994	1.100	0.997	1.001	1.001	0.973
	(0.4)	1.168	1.096	1.005	0.994	0.997	1.110	1.019	1.051	1.027	0.996
	(0.8)	1.191	1.078	1.064	0.999	0.984	1.633	1.163	0.988	0.965	0.960
MA(1)	(-0.8)	1.176	0.977	0.978	0.971	1.004	1.253	1.018	1.287	1.124	1.007
	(-0.4)	1.153	1.054	1.009	0.992	0.986	1.113	1.246	0.953	1.013	1.012
	(-0.2)	1.100	1.053	1.005	0.965	0.956	1.022	1.021	1.007	1.020	0.996
	(0.2)	1.174	1.032	0.972	0.978	0.974	1.154	1.052	1.049	0.958	0.956
	(0.4)	1.132	1.045	1.079	1.014	0.995	1.114	1.075	1.066	1.059	0.979
	(0.8)	1.131	1.047	1.009	0.975	0.942	1.143	1.125	0.996	1.085	0.966
AR(2)	(0.1, -0.3)	1.185	1.065	1.087	1.062	0.987	1.303	1.101	1.068	0.980	1.016
	(-0.1, 0.3)	1.308	1.111	0.978	0.983	0.998	1.137	1.044	0.984	1.023	1.053
AR(3)	(0.1, -0.3, -0.2)	1.229	1.208	1.080	0.980	0.971	1.336	1.088	0.990	1.047	0.992
	(-0.3, -0.5, 0.1)	1.430	1.538	1.036	0.971	1.115	1.350	1.067	1.038	1.014	0.983
MA(2)	(0.1, -0.3)	1.198	1.034	1.005	1.097	0.985	1.484	1.138	1.104	1.151	0.998
	(-0.1, 0.3)	1.336	1.024	1.005	0.991	1.018	1.602	1.060	1.039	0.987	1.056
MA(3)	(0.1, -0.3, -0.2)	1.592	1.160	1.074	1.017	0.996	1.636	1.058	1.075	1.017	1.030
	(-0.3, -0.5, 0.1)	1.369	1.102	1.142	0.953	0.994	1.478	1.246	1.048	1.088	1.024
ARMA(1,1)	(0.4; -0.2)	1.215	1.144	1.018	1.083	0.982	1.108	1.116	1.208	1.000	1.028
	(-0.4; -0.4)	1.165	1.040	1.079	1.042	0.970	1.423	1.262	1.002	0.977	0.994
ARMA(2,1)	(-0.2, -0.3; 0.1)	1.343	1.192	1.032	1.034	1.001	1.437	1.122	1.051	1.061	1.038
	(0.2, -0.3; -0.1)	1.333	1.231	1.044	1.060	1.004	1.211	1.239	1.049	1.054	1.011
ARMA(1,2)	(0.3; 0.5, -0.1)	1.494	1.042	1.296	0.961	0.977	1.612	1.185	1.030	0.965	1.039
	(0.3; -0.5, 0.1)	1.417	1.097	1.010	1.042	1.013	1.547	1.257	1.031	1.021	1.019
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	1.538	1.198	1.198	1.007	0.998	3.349	1.116	1.056	1.058	1.004
	(0.2, -0.3; -0.5, 0.1)	1.520	1.240	1.115	1.053	0.943	1.565	1.207	1.035	1.051	1.032

Table 36

Relative Efficiency at $\sigma^2 = 5$ and $h = 3$

Models	Parameters	$\alpha = 1$					$\alpha = -1$				
		T=25	T=50	T=100	T=200	T=400	T=25	T=50	T=100	T=200	T=400
AR(1)	(-0.8)	1.332	1.349	1.196	1.185	0.986	1.756	1.352	1.019	1.018	0.948
	(-0.4)	1.292	1.034	1.009	1.002	1.018	1.175	1.100	1.011	0.985	1.030
	(-0.2)	1.132	1.089	1.018	0.999	1.016	1.098	1.060	1.059	1.001	1.009
	(0.2)	1.074	1.055	0.990	1.067	0.950	1.092	1.021	1.052	0.995	1.017
	(0.4)	1.212	0.994	0.975	1.077	0.943	1.112	1.179	1.027	1.044	1.000
	(0.8)	1.119	1.206	1.177	0.991	0.938	1.523	1.220	2.270	0.966	0.935
MA(1)	(-0.8)	1.093	1.064	1.360	1.026	0.976	1.186	1.065	1.258	1.008	1.016
	(-0.4)	1.100	1.056	0.996	1.001	1.005	1.097	1.081	1.021	0.986	1.023
	(-0.2)	1.080	1.036	1.001	0.986	0.945	1.126	1.047	1.162	0.995	1.094
	(0.2)	1.104	1.011	1.064	1.041	0.964	1.011	1.111	0.996	1.003	0.977
	(0.4)	1.250	1.102	1.018	1.004	1.003	1.116	1.094	1.023	0.981	1.003
	(0.8)	1.378	1.029	1.642	0.991	0.958	1.375	1.225	1.060	0.946	0.961
AR(2)	(0.1, -0.3)	1.153	1.038	1.036	0.979	1.022	1.048	1.041	1.005	0.975	1.015
	(-0.1, 0.3)	1.194	1.047	1.010	1.006	1.000	1.146	1.071	1.003	1.023	1.040
AR(3)	(0.1, -0.3, -0.2)	1.171	1.205	1.089	1.035	0.965	1.232	1.137	1.065	1.038	1.010
	(-0.3, -0.5, 0.1)	1.193	1.134	1.066	0.948	0.959	1.268	1.021	1.081	1.054	0.968
MA(2)	(0.1, -0.3)	1.087	1.064	1.035	1.002	1.006	1.243	1.072	0.997	0.989	0.980
	(-0.1, 0.3)	1.140	1.034	1.064	1.017	0.975	1.127	1.066	0.991	0.973	0.960
MA(3)	(0.1, -0.3, -0.2)	1.212	1.111	1.114	0.997	0.994	1.236	1.054	1.071	0.984	1.046
	(-0.3, -0.5, 0.1)	1.249	1.119	1.173	1.040	1.040	1.163	1.030	1.008	0.978	1.041
ARMA(1,1)	(0.4; -0.2)	1.175	1.083	1.039	0.955	0.995	1.134	1.140	1.045	1.034	1.036
	(-0.4; -0.4)	1.261	1.203	1.285	1.143	1.052	1.446	1.306	1.132	1.004	1.046
ARMA(2,1)	(-0.2, -0.3; 0.1)	1.128	1.034	0.946	0.971	0.946	1.159	1.070	0.989	0.949	0.969
	(0.2, -0.3; -0.1)	1.257	1.023	1.030	1.012	1.014	1.250	1.123	0.990	0.943	0.953
ARMA(1,2)	(0.3; 0.5, -0.1)	1.383	1.202	1.026	0.975	0.993	1.291	1.076	1.034	1.027	1.046
	(0.3; -0.5, 0.1)	1.118	1.064	0.980	0.986	0.979	1.139	1.101	1.001	1.007	0.989
ARMA(2,2)	(-0.2, -0.3; 0.5, -0.1)	1.370	1.150	1.003	1.005	1.036	1.410	1.138	1.028	1.016	0.973
	(0.2, -0.3; -0.5, 0.1)	1.661	1.308	1.057	1.026	0.927	1.733	1.053	1.011	1.103	0.954

APPENDIXE

R SIMULATION CODE

```
install.packages("forecast")
install.packages("xlsx")
library(forecast)
library(xlsx)

gg=  CLU( model=Model, T, h=3,  sigma=sigma, nsim,
          maxit=5000,
          reltol=sqrt(.Machine$double.eps),
          alpha=alpha)

#Function for CLU
CLU=function( model, T, h=3,  sigma, nsim,
              maxit=5000,
              reltol=sqrt(.Machine$double.eps),
              alpha ){
  n=T+h
  ar=model$ar
  ma=model$ma
  order=c(length(ar),0, length(ma))
  zhat<-matrix(rep(0,nsim*h), ncol=h)
  zlast<-matrix(rep(0,nsim*h), ncol=h)
  lzhat<-matrix(rep(0,nsim*h), ncol=h)

  #simulate time series
  for (i in 1:nsim){
    z<-(as.numeric(arima.sim(n=n,list(ar=ar, ma=ma),
                            sd=sigma)))
    zlast[i,]=z[(T+1):n]
    zfit<-Arima(z[1:T], order = order,
                include.mean = FALSE,
                method = c("ML"),
                optim.control = list(maxit=maxit,
                                     method="Nelder-Mead")
    )

    #compute LINEX unbiased predictor
    f<-predict(zfit, n.ahead = h)
    predmat<-f$pred
    semat<-f$se
    sigma2hat=(semat)^2
    zhat[i,]<- predmat
    lzhat[i,]<- predmat +(alpha/2)*sigma2hat
  }

  #ordinary unbiased prediction
```

```
  u<-apply(zlast-zhat , 2, mean)
  #LINEX unbiased prediction
  lu<-apply(exp(alpha*(zlast-lzhat)), 2, mean) -1
  #LINEX risk
  lzr<- lu-alpha*(apply(zlast-lzhat, 2 ,mean))
  return(list(u,lu, lzr))
}

# Array of Models
phi=c(-0.8, -0.4, -0.2, 0.2, 0.4, 0.8)
theta=phi
Ar<- vector(mode = "list", length = 6)
for( i in 1:6){
  Ar[[i]]= list(ar=phi[i])
}

Ma<- vector(mode = "list", length = 6)
for( i in 1:6){
  Ma[[i]]= list(ma=theta[i])
}

Model=vector(mode = "list", length = 28)

Model[1:6]=Ar
Model[7:12]=Ma

Model[13:28]=list(
  list(ar=c(0.1,-0.3)),
  list(ar=c(-0.1, 0.3)),
  list(ar=c(0.1,-0.3,-0.2)),
  list(ar=c(-0.3,-0.5,0.1)),
  list(ma=c(0.1,-0.3)),
  list(ma=c(-0.1, 0.3)),
  list(ma=c(0.1,-0.3,-0.2)),
  list(ma=c(-0.3,-0.5,0.1)),
  list(ar=c(0.4),ma=c(-0.2)),
  list(ar=c(-0.4),ma=c(-0.4)),
  list(ar=c(-0.2,-0.3),ma=c(0.1)),
  list(ar=c(0.2,-0.3),ma=c(-0.1)),
  list(ar=c(0.3),ma=c(0.5, -0.1)),
  list(ar=c(0.3), ma=c(-0.5, 0.1 )),
  list(ar=c(-0.2,-0.3),ma=c( 0.5,-0.1)),
  list(ar=c(0.2,-0.3),ma=c(-0.5,0.1))
)
```

```
# T=25, alpha=1, sigma^2=0.5 #
nsim<-10000
alpha=1
sigma=sqrt(0.5)
T=25

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=3000,
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=25, alpha= -1, sigma^2=0.5 #
nsim<-10000
alpha=-1
sigma=sqrt(0.5)
T=25

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=3500,
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}
```

```
write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=50, alpha=1, sigma^2=0.5 #
nsim<-10000
alpha=1
sigma=sqrt(0.5)
T=50

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=4000,
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=50, alpha=-1, sigma^2=0.5 #
nsim<-10000
alpha=-1
sigma=sqrt(0.5)
T=50

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=4000,
            reltol=sqrt(.Machine$double.eps),
```

```
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=100, alpha=1, sigma^2=0.5 #
nsim<-10000
alpha=1
sigma=sqrt(0.5)
T=100

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=4000,
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=100, alpha=-1, sigma^2=0.5 #
nsim<-10000
alpha=-1
sigma=sqrt(0.5)
T=100

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)
```

```
for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=4000,
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=200, alpha=1, sigma^2=0.5 #
nsim<-10000
alpha=1
sigma=sqrt(0.5)
T=200

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)


for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=3000,
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=200, alpha=-1, sigma^2=0.5 #
nsim<-10000
alpha=-1
sigma=sqrt(0.5)
```

```
T=200

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=3000,
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=400, alpha=1, sigma^2=0.5 #
nsim<-10000
alpha=1
sigma=sqrt(0.5)
T=400

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=3000,
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')
```

```
# T=400, alpha=-1, sigma^2=0.5 #
nsim<-10000
alpha=-1
sigma=sqrt(0.5)
T=400

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=3000,
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=25, alpha=1, sigma^2=1 #
nsim<-10000
alpha=1
sigma=sqrt(1)
T=25

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=4000,
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
```

```
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=25, alpha=-1, sigma^2=1 #
nsim<-10000
alpha=-1
sigma=sqrt(1)
T=25

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=4000,
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=50, alpha=1, sigma^2=1 #
nsim<-10000
alpha=1
sigma=sqrt(1)
T=50

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=5000,
```

```
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=50, alpha=-1, sigma^2=1 #
nsim<-10000
alpha=-1
sigma=sqrt(1)
T=50

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=5000,
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=100, alpha=1, sigma^2=1 #
nsim<-10000
alpha=1
sigma=sqrt(1)
T=100

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)
```

```
for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
             maxit=4000,
             reltol=sqrt(.Machine$double.eps),
             alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=100, alpha=-1, sigma^2=1 #
nsim<-10000
alpha=-1
sigma=sqrt(1)
T=100

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
             maxit=4000,
             reltol=sqrt(.Machine$double.eps),
             alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=200, alpha=1, sigma^2=1 #
nsim<-10000
alpha=1
sigma=sqrt(1)
```

```
T=200

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=3000,
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=200, alpha=-1, sigma^2=1 #
nsim<-10000
alpha=-1
sigma=sqrt(1)
T=200

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=3000,
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')
```

```
# T=400, alpha=1, sigma^2=1 #
nsim<-10000
alpha=1
sigma=sqrt(1)
T=400

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=3000,
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=400, alpha=-1, sigma^2=1 #
nsim<-10000
alpha=-1
sigma=sqrt(1)
T=400

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=3000,
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
```

```
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=25, alpha=1, sigma^2=5 #
nsim<-10000
alpha=1
sigma=sqrt(5)
T=25

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=5000,
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=25, alpha=-1, sigma^2=5 #
nsim<-10000
alpha=-1
sigma=sqrt(5)
T=25

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=5000,
```

```
              reltol=sqrt(.Machine$double.eps),
              alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=50, alpha=1, sigma^2=5 #
nsim<-10000
alpha=1
sigma=sqrt(5)
T=50

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
              maxit=6000,
              reltol=sqrt(.Machine$double.eps),
              alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=50, alpha=-1, sigma^2=5 #
nsim<-10000
alpha=-1
sigma=sqrt(5)
T=50

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)
```

```
for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=3500,
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=100, alpha=1, sigma^2=5 #
nsim<-10000
alpha=1
sigma=sqrt(5)
T=100

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=3000,
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=100, alpha=-1, sigma^2=5 #
nsim<-10000
alpha=-1
sigma=sqrt(5)
```

```
T=100

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=3000,
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=200, alpha=1, sigma^2=5 #
nsim<-10000
alpha=1
sigma=sqrt(5)
T=200

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=3000,
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')
```

```
# T=200, alpha=-1, sigma^2=5 #
nsim<-10000
alpha=-1
sigma=sqrt(5)
T=200

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=3000,
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=400, alpha=1, sigma^2=5 #
nsim<-10000
alpha=1
sigma=sqrt(5)
T=400

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=3000,
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
```

```
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')

# T=400, alpha=-1, sigma^2=5 #
nsim<-10000
alpha=-1
sigma=sqrt(5)
T=400

results1=matrix(0, 28,3)
results2=matrix(0, 28,3)
results3=matrix(0, 28,3)

for ( k in 1:28){
  model=Model[[k]]
  gg=  CLU( model=model, T, h=3,  sigma, nsim,
            maxit=3000,
            reltol=sqrt(.Machine$double.eps),
            alpha=alpha)
  results1[k,]=gg[[1]]
  results2[k,]=gg[[2]]
  results3[k,]=gg[[3]]
}

write.xlsx(results1,'Ordinary.xlsx')
write.xlsx(results2,'LINEX.xlsx')
write.xlsx(results3,'Risk.xlsx')
```