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UNIVERSITY OF NORTHERN COLORADO

Greeley, Colorado

The Graduate School

THE SOCIOCULTURAL MEDIATION OF METACOGNITION DURING PROBLEM SOLVING IN UNDERGRADUATE MATHEMATICS CLASSROOM COMMUNITIES OF PRACTICE

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

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College of Natural and Health Sciences
School of Mathematical Sciences
Educational Mathematics

May 2018
This Dissertation by: Emilie R. Hancock

Entitled: *The Sociocultural Mediation of Metacognition in Undergraduate Mathematics Classroom Communities of Practice*

has been approved as meeting the requirements for the Degree of Doctor of Philosophy in College of Natural and Health Sciences in School of Mathematical Sciences, Program of Educational Mathematics

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ABSTRACT

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Metacognition has long been identified as an essential component of the problem-solving process. While the language of metacognition has been conveyed in teaching, research, and policy, much of the research on metacognition does not describe the explicit role metacognition plays during students’ real-time problem-solving process. Moreover, metacognitive interventions are typically disconnected from the natural mathematical activity and discourse within a classroom community. Research concerning metacognition and metacognitive interventions has historically adopted an acquisition metaphor for learning. This qualitative study takes a participationist lens to consider metacognition as a problem-solving habit of mind, a normative way of thinking to which students become attuned by participating in authentic problem-solving situations.

This study explored one such situation, in which “portfolio” problem-solving sessions and write-ups were used to mediate metacognitive thinking in a first-year mathematics content course for pre-service elementary teachers. Six qualitative data sources were collected and analyzed: (1) recorded classroom sessions, (2) three individual interviews with 15 of the 24 students, (3) two interviews with the instructor of record, (4) students’ written artifacts, (5) recorded planning sessions with the instructor, and (6) journal reflections written by the instructor and myself, the researcher, after each
class session. Two levels of analysis were employed to characterize sociocultural complexity surrounding students’ problem-solving activity.

Results of micro-level analysis revealed a shift from product- to process-focused metacognitive norms. Through participation in authentic problem-solving situations, namely the portfolio problems, students problem-solving activity transformed in a way that afforded them opportunities to readily engage in process-focused metacognitive actions. Macro-level analysis utilized activity theory to operationalize the participation structure of the classroom and document the development of metacognitive norms, highlighting social mediators of activity and contradictions as catalysts for change. Results of macro-level analysis illustrated a correspondence between the shift in normative metacognitive actions identified in micro-level analysis, broader transformations of students’ problem-solving activity, and the teacher’s shifting goals and actions in response to students’ problem solving.

This work extends previous research on metacognitive interventions, demonstrating that “embeddedness” of metacognitive activity during problem solving is beyond just the content, but also embedded in the collective classroom culture. Moreover, activity theory captured students’ agency in negotiating their problem-solving activity, suggesting its continued use by researchers wishing to adopt an anti-deficit framing. This research has additional implications for teaching content courses for pre-service teachers. Students’ metacognitive activity was very much situated in the sociocultural context of the classroom, especially their dual identities as current mathematics students and future teachers. For the pre-service teachers to value mathematical problem-solving habits of mind, legitimate participation meant as students, not just as future teachers, of
mathematics. Finally, this study provides broader insight into how instructors can support undergraduate students’ process-focused metacognitive activity during problem solving through a combination of Inquiry-Based Learning (IBL) techniques and explicit reflection on real-time problem-solving processes.
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This work is dedicated to my mother, Mary Ann, and to my grandmother, Pat.
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CHAPTER I

INTRODUCTION

Technological tides are changing mathematics learning and cognition (Kaput & Thompson, 1994). Beyond just using calculators or computers to aid in computations, there is a more profound shift from procedural fluency to deeper mathematical understanding. This has manifested itself in mathematics education through the use of more active learning classrooms and student-centered learning environments. In fact, the Obama White House Office of Science and Technology Policy (OSTP) made a call to action to improve the teaching of Science, Technology, Mathematics, and Engineering (STEM) fields using active learning classrooms, which “have been shown to increase retention of knowledge; develop higher-order skills such as analysis, synthesis, and evaluation; and increase student retention in STEM fields” (White House OSTP, 2016). Moreover, “we can see an obvious interplay between pedagogical tides that are moving toward increased student control of their learning activities and the technological tide of ever more powerful computational and graphic processing” (Kaput & Thompson, 1994, p. 678, emphasis added). Research on technology-based learning environments to scaffold metacognition (Manlove, Lazonder, & de Jong, 2007), as well as research on self-regulated learning (Zimmerman, 2002) which has metacognition as a core factor, are evidence of such a trend.
Noting “there is a growing recognition that a serious mismatch exists (and is growing) between the low-level skills emphasized in test-driven curriculum materials and the kind of understanding and abilities that are needed for success beyond school” (Lesh & Zawojewski, 2007, p. 764), one result of the aforementioned changes seems to be a focus on mathematical habits of mind. Costa and Kallick (2000) define habits of mind as “dispositions displayed…in response to problems, dilemmas, and enigmas, the resolutions of which are not immediately apparent” (p. xvii). For example, Laursen, Hassi, Kogan, Hunter, and Weston (2011) found that successful Inquiry-Based Learning (IBL) classrooms were those where “course goals tended to emphasize development of skills such as problem solving, communication, and mathematical habits of mind, not just covering specific content” (p. 164). When investigating the motivations of undergraduate instructors who implemented a flipped classroom design, Naccarato and Karakok (2015) found that participants were “motivated by other learning goals, ‘spoken objectives’ as one participant phrased, such as metacognitive and critical thinking skills, perseverance, autonomy, and mathematical communication” (p. 974). In their 2015 curriculum guide to majors in mathematical sciences, the Mathematical Association of America (MAA) Committee on the Undergraduate Programs in Mathematics (CUPM) emphasized that “every mathematical sciences major should be designed to help students acquire ‘mathematical habits of mind’” (p. 10). The Conference Board of the Mathematical Sciences (CBMS) also recommended that “[a]ll courses and professional development experiences for mathematics teachers should develop the habits of mind of a mathematical thinker and problem solver, such as reasoning and explaining, modeling, seeing structure, and generalizing” (2012, p. 19).
Metacognition, or in the most general sense “thinking about thinking” that involves monitoring, control, and the associated judgment and decision-making processes, is a mathematical problem-solving habit of mind (Selden & Lim, 2010; Stacey, Burton, & Mason, 1982). The language of metacognition has been conveyed in teaching, research, and policy. Within their cognitive recommendations, the MAA CUPM (2015) lists “effective thinking” as a skill students should acquire; this centers on metacognitive skill. Further, the Standards for Mathematical Practices of the Common Core include “making sense of problems” and “using appropriate tools strategically” (Common Core State Standards Initiative (CCSSI), 2010), which both require an awareness of what is known and unknown, as well as awareness of possible strategies and when to use them; these are central components of metacognition (Flavell, 1979). Moreover, in describing the problem-solving process standard in their Principles to Actions (2014) publication, the National Council of Teachers of Mathematics (NCTM) also highlighted that effective mathematics teaching should provide students ample opportunities to “develop metacognitive awareness of themselves as learners, thinkers, and problem solvers, and learn to monitor their learning and performance” (p. 9). While metacognition is considered essential, there is an evident “gap between metacognitive awareness of cognition and real-time control of cognition/self-management” (Hsu, Iannone, She, & Hadwin, 2016, p. 244). A need to examine “real-time” metacognition is substantiated further in the following Statement of the Problem section, addressing the purpose of this research study.
Statement of the Problem

The importance of problem-solving practices has been emphasized and studied by earlier researchers (e.g., Carlson & Bloom, 2005; Pólya, 1957; Schoenfeld, 1992). Many teachers and researchers have recognized a need to foster conceptual understanding and skills beyond solely accumulation of facts or problem-solving procedures (e.g., Erickson, 2002; Skemp, 1987). Especially within Science, Technology, Engineering and Mathematics (STEM) related fields, skills such as metacognition, self-regulation, innovation, and creativity are gaining recognition (e.g., Ali, Abd-Talib, Ibrahim, Surif, & Abdullah, 2016). In particular, while metacognition has been identified as an essential 21st Century skill (Binkley et al., 2012; Saavedra & Opfer, 2012) it remains undertheorized and under-studied in its application to classroom communities of practice, especially the use and implementation of this construct at the undergraduate level (Dumford, Cogswell, & Miller, 2016). As mathematics is a fundamental component of Science, Technology, and Engineering (STE) fields, the development of mathematical problem-solving skills, which includes metacognitive skills (Schoenfeld, 1985), plays a crucial role in students’ critical and creative thinking even beyond the required mathematics courses for STE majors. It is reasonable to conclude that a focus on metacognition in mathematical problem solving has the potential to create a lasting impact on STEM majors, within both their academic and future careers.

Mathematical problem solving research studies abound, and a significant portion of these studies express the role of metacognition as an underlying component of the problem-solving process (e.g., Carlson & Bloom, 2005; Schoenfeld, 1985). Unfortunately, much of the research on metacognition in mathematics does not describe
the explicit role metacognition plays during students’ real-time problem-solving process. Moreover, metacognitive interventions are typically disconnected from authentic mathematical activity and natural discourse within a classroom community. As cognitive or information processing theory models account explicitly for metacognition (Silver, 1987), most research on metacognition within mathematics comes from these perspectives. This research has often been in the form of interventions in which students are taught isolated metacognitive skills. Common content-based exams or survey instruments such as the Motivated Strategies for Learning Questionnaire (MSLQ) are typically used to evaluate the interventions (e.g., Bol, Campbell, Perez, & Yen, 2013; Hoffman & Spatariu, 2008). Such studies suffer from notable issues related to transfer of learning and their inability to attend to the contextualization of metacognitive behavior.

These assessments (transfer tasks and self-report measures) do not account for the situated nature of learning dispositions. In fact, Carroll (2008) claimed that students have “on the whole, not benefited from at least 20 years of [such] metacognitive research” (p. 411). This begs the question: can such assessment really evaluate metacognition if one only looks at the product of metacognition – static, decontextualized metacognitive skills? Carlson and Bloom (2005) argue that although “literature supports that control and metacognition are important for problem-solving success, more information is needed to understand how these behaviors are manifested during problem solving, and how they interact with other problem-solving elements reported to influence the problem-solving process” (p. 46, emphasis added). Neisser (1976) argued that we must “understand cognition in the context of natural purposeful activity” (p. 7), and this suggestion has been echoed in later years by both Nelson and Narens (1994) and Carroll (2008).
As the natural, purposeful activity within a classroom community of practice creates a microculture of negotiated activities and interactions among students and the teacher (Lave & Wenger, 1991), over time normative behavior emerges. While some norms are related to general behavioral expectations and social norms, sociomathematical norms are taken-as-shared, mathematically-based activity (Yackel & Cobb, 1996). For example, a sociomathematical norm might be what it means to justify a mathematical argument or the types of metacognitive actions taken while problem solving. These norms are negotiated by participants in a classroom and may be different among various classroom communities. For metacognition to develop, it needs to be an explicit part of the classroom microculture and have opportunities to become established as normative activity. Levenson, Tirosh, and Tsamir (2009) found that students and teachers may not perceive the same norms, and Levenson, Tirosh, and Tsamir (2006) noticed that the taken-as-shared norms established by the majority of students may not be productive, both for students in the minority as well as norms that are contradictory to the teacher’s proposed norms. To investigate the metacognitive processes within a classroom community, particular attention must be paid to the way in which sociomathematical metacognitive norms are established and negotiated, as well as any potential contradictions among various participants in the classroom.

This study sought to contribute to a current dearth in the literature by attempting to understand metacognition in the context of natural, purposeful activity within a classroom community of practice. In particular, this study investigated how sociomathematical metacognitive norms were established, developed, utilized, and how they evolved in an undergraduate mathematics classroom for pre-service elementary
teachers. Understanding how metacognition manifests itself during the problem-solving process in a classroom environment could help in developing explicit ways to foster metacognition as normative activity, a habit of mind within the mathematics classroom, thus addressing the call for an emphasis on habits of mind described previously.

The remainder of this chapter includes research questions aligned with the study’s purpose, outlines definitions of key terms and assumptions utilized throughout the study, and provides a comprehensive review of the literature. Chapter I concludes with an overview of the methods and context of the dissertation study and a description of how the standalone manuscripts in Chapters II, III, and IV fit together into the overall study.

Purpose and Research Questions

The purpose of this phenomenological case study was to contribute to a current paucity of research on metacognition within the context of natural, purposeful classroom problem-solving activity, specifically at the undergraduate level. Because this study had “a focus on exploring how human beings make sense of experience and transform experience into consciousness, both individually and as shared meaning” (Patton, 2002, p. 104), a phenomenological approach was appropriate. Here, the relevant phenomenon was the development of sociomathematical metacognitive norms in undergraduate mathematics communities of practice. Phenomena are experienced through and can be studied in the context of specific cases, and so phenomena create the bounded system (Merriam, 2009) for a particular case. A first-year mathematics content course for preservice elementary teachers taught using Inquiry-Based Learning (IBL) techniques (Ernst, Hodge, & Yoshinobu, 2017) was chosen as an instrumental case (Stake, 1995) for the dissertation study to illustrate the larger phenomenon.
This research study attempted to answer the following research questions. The general phenomenon was addressed broadly through research question Q1, while the more specific research questions, Q1a through Q1d, relate to the particular case of one classroom community.

**Q1** How do metacognitive norms during problem solving evolve in an undergraduate mathematics community of practice?

**Q1a** What are the normative metacognitive actions taken by students in authentic problem-solving situations?

**Q1b** What contradictions or tensions catalyze changes in the object (problem solving) of the student activity system?

**Q1c** What is the relationship between the metacognitive norms identified in Q1a and changes in students’ problem-solving activity identified in Q1b?

**Q1d** What is the role of the teacher in negotiating students’ metacognitive development?

To address these research questions, data were collected in a first-year undergraduate mathematics course for future elementary teachers, Number Sense and Algebra, at a mid-size university in the Rocky Mountain Region of the United States over the 15-week Fall 2016 semester. Six qualitative data sources were collected: (1) video- and audio-recorded classroom sessions, (2) three videotaped, semi-structured individual interviews with 15 of the 24 students at the beginning, middle, and end of the course, (3) two audio-recorded interviews with the instructor of record, (4) students’ written artifacts (assignments, exams, and portfolio-problem submissions and scratch work) collected before grading, (5) recorded planning sessions with the instructor, and (6) journal reflections written by the instructor and myself, the researcher, after each class session.

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1 13 of the 15 students completed all three interviews.
Research questions were delineated and data sources were collected in light of the theoretical perspective adopted. In this study, a Vygotskian lens was adopted to adequately characterize metacognition within the context of a classroom community of practice. Situated cognition as delineated by Vygotsky (1978, 1986) provided a lens through which both the individual, internal, as well as external lines of development were considered. Higher psychological functions, including conscious awareness and voluntary control, were viewed as internalized by the individual from society via mediation by tools and signs (chiefly language). In addition to an emphasis on semiotic mediation, the reflexive relationship between the individual and larger community was also stressed (Ernest, 2010). All members of a classroom community, both the teacher and other students, were viewed as acting as more knowledgeable others that actively contributed to an individual’s learning and development (Goos, Galbraith, & Renshaw, 2002). In this way, students’ metacognitive behavior was negotiated as it developed into taken-as-shared mathematically-based activity, or a sociomathematical norm (Yackel & Cobb, 1996) of the classroom.

While Vygotsky himself did not explicitly delineate a framework for characterizing the interactions between individual and community, Leont’ev (1979) and subsequently Engeström (1987/2105) further developed Vygotsky’s notions of semiotic mediation and reflexivity into a theory of goal-driven activity, activity theory, which is described in detail in Chapter II. The taken-as-shared, normative mathematical behavior of a classroom was viewed as the general activity exhibited by the student activity system. Sociocultural factors such as the nature of mediating tools, implicit and explicit rules, community, and the division of labor all impacted the goal-driven, object oriented
actions of subjects within the classroom community. These factors transformed the normative activity of the students as a whole, which when mathematical in nature was precisely the transformation of sociomathematical norms. Thus, the constructs outlined by activity theory provided the theoretical lens and basis for analysis of sociomathematical metacognitive norms in this study.

**Significance of the Research**

Although problem-solving practices have been emphasized and heavily studied, research on problem solving has not sufficiently explained “how and why people made the choices they did. That is, [research to date] offered a framework for characterizing problem solving, but it did not yet offer a theory of problem solving” (Schoenfeld, 2007, p. 539, emphasis added). Through attending to the aforementioned research questions, this dissertation work explicated how and why pre-service teachers in a first-year mathematics content course engaged in particular metacognitive actions during authentic problem-solving situations. Results of analysis employed to answer research question Q1a revealed a shift in the normative metacognitive activity of students in the Number Sense and Algebra course (the “how”). In particular, students began assessing and considering various solution approaches or strategies, relying heavily on different representations to help them navigate their problem-solving attempts (see Chapter III). In addressing research questions Q1b, Q1c, and Q1d, my research offered insight into the contextual factors that afforded this transformation (the “why”).

As is presented in Chapter III, analysis using activity theory to situate the answer to research question Q1a indicated a correspondence between students’ changing normative metacognitive actions, broader transformations of their problem-solving
activity, and the teacher’s shifting goals and actions in response to students’ problem-solving activity. In answering research question Q1b, I found that a contradiction between students’ dual identities as current mathematics students and future teachers affected their problem-solving activity. Actions taken in service of their dominant teacher identity impeded their development as mathematical problem solvers themselves. The Number Sense and Algebra students were only able to develop as mathematical problem solvers when they realized the tools seemingly presented to them for teaching afforded them productive struggle in their own problem solving, merging their two identities. As such, my research has implications for the way mathematics teacher educators leverage pre-service teacher identities in their mathematics content courses. If pre-service teachers are to help their future students develop mathematical habits of mind, they need to value the usefulness of mathematical habits of mind in addition to building proficiency with them (Oesterle et al., 2016). This means aligning their problem-solving activity to be both in service of their development as future teachers and in service of their own problem-solving endeavors.

Addressing research question Q1c, the merging of students’ dual identities while problem solving identified in research question Q1b occurred in tandem with students’ adoption of a new metacognitive action, namely the assessment and consideration of various representations identified in research question Q1a. Previous researchers have argued that metacognitive instruction should be embedded in the mathematics content and take place for an extended period of time (Veenman, Van Hout-Wolters, & Afflerbach, 2006). My study extends this notion of “embeddedness” to suggest that the embeddedness of metacognitive instruction during problem solving is beyond just the
content, but also embedded in classroom culture and the “vital life activity” (Engeström, 2011) of students. Additionally, my dissertation suggests concrete classroom practices that can support metacognition as part of classroom culture. In answering research question Q1d, I found that the instructor of the Number Sense and Algebra course aided students in developing their metacognitive habits by leveraging a combination of Inquiry-Based Learning (IBL) techniques and individual student reflection. In Chapters III and IV, I outline how the instructor intentionally built a process-focused community of inquiry and provided students explicit opportunities for reflecting on their problem-solving process in authentic problem-solving situations via the portfolio problems and write-ups.

**Definition of Key Terms**

Throughout this study, certain terms are used with particular meanings specific to this research. To alleviate the potential for misinterpretation, key terms as they are utilized in this study are described here.

**(Mathematical) Problem**

Schoenfeld (1985) distinguishes between a *problem* and an *exercise*, and he emphasizes that this difference is highly dependent on the problem solver. Unlike an exercise, problems are difficult for the problem solver; an “intellectual impasse rather than a computational one” (p. 74). Problems are not easily solved with procedure to which the solver has “easy access” (p. 11). The National Council of Teachers of Mathematics (NCTM) (2010) further characterizes a mathematical problem, and these ten additional criteria are included in the definition of “problem” used here.
1. The problem has important, useful mathematics embedded in it.

2. The problem requires higher-level thinking and problem solving.

3. The problem contributes to the conceptual development of students.

4. The problem creates an opportunity for the teacher to assess what his or her students are learning and where they are experiencing difficulty.

5. The problem can be approached by students in multiple ways using different solution strategies.

6. The problem has various solutions or allows different decisions or positions to be taken and defended.

7. The problem encourages student engagement and discourse.

8. The problem connects to other important mathematical ideas.

9. The problem promotes the skillful use of mathematics.

10. The problem provides an opportunity to practice important skills. (p. 1-2)

(Mathematical) Problem Solving

The definition of problem solving in mathematics is highly dependent on the definition of a mathematical problem. Thus, solving problems means engaging in the activities as outlined in the definition of problem. Pólya (1957) separated the process of solving problems into four general stages: understand the problem, devise a plan, carry out the plan, and look back. Variations of this outline have been used by researchers (e.g., Batha & Carroll, 2007; Carlson & Bloom, 2005; Palingsar, 1990). Carlson and Bloom (2005) further explicated the cyclic nature of problem solving, noting that experienced problem solvers first conjecture a plan, imagine carrying out the plan, and then evaluate
its effectiveness before executing a strategy. Schoenfeld (1985) stressed the importance of heuristics, resources, beliefs, and control in mathematical problem solving.

**Metacognition**

Metacognition is most broadly defined as thinking about your thinking or an awareness of cognitive activity. Flavell (1979) defined this construct as “the active monitoring and consequent regulation and organization of these processes to the cognitive objects on which they bear” (p. 232). So, metacognition can be seen to consist of two main parts: monitoring and regulation (control) which operate simultaneously (Nelson & Narens, 1990). To adequately monitor and control one’s behavior, an individual must have sufficient metacognitive knowledge and strategies available to employ. Van Overschelde (2008) noted that the processes of metacognitive monitoring and control are goal-oriented and subject to both internal and external constraints. He highlighted the importance of the judgment and decision-making processes involved in control of one’s cognition. Thus, metacognition is not merely the aggregate *product* of metacognitive knowledge and strategies, but also the *process* by which these strategies and knowledge are chosen and employed.

This notion of judgment and decision making is also emphasized by mathematics education researchers in the context of problem solving (e.g., Carlson & Bloom, 2005; Schoenfeld, 1985), who tend to use any of ‘control’, ‘monitoring’, ‘regulation’, and/or ‘judgment and decision making’ interchangeably when describing metacognition. In mathematical problem solving, metacognition can appear within any phase of the problem-solving process (*orienting, planning, executing, checking*) and includes metacognitive activity such as planning, strategy or heuristic choice, reflection,
evaluation, assessment, checking, asking questions, or any actions that involve thinking about one’s cognitive actions during problem solving.

**Sociomathematical Norms**

Classroom norms, the “usual” or “typical” behavior of a classroom, evolve and are established by the classroom community over time. Unlike behavioral expectations and more general social norms, for instance expectations about the amount of student discussion in a classroom, sociomathematical norms are taken-as-shared mathematically-based activity (Yackel & Cobb, 1996). For example, the notion of mathematical difference, or what makes a solution different mathematically, is a sociomathematical norm because it is established as typical behavior and is mathematically-based. Bowers, Cobb, and McClain (1999) further made a distinction between sociomathematical norms and *mathematical practices*, which refer to specific activities related to a particular mathematical idea. For instance, thinking of numbers as units of tens and ones instead of just counting by ones is a mathematical practice, but not a sociomathematical norm.

**Mathematical Discourse**

Sfard (2001) defined *discourse* as “any specific instance of communicating, whether diachronic or synchronic, whether with others or with oneself, whether predominantly verbal or with the help of any other symbolic system” (p. 28). Such discourse is mediated through symbolic tools such as language and is subject to meta-discursive rules that encompass cultural norms, values, and beliefs. These meta-rules are “understood as expressing themselves in regularities observed in those aspects of communicational activities that are not directly related to the particular content of the exchange” (p. 30). As “discourses are analyzed as acts of communicating, anything that
goes into communication and influences its effectiveness – body movements, situational clues, interlocutor’s histories, etc. – must be included in analysis” (p. 28). Others have argued that mathematical communication can occur through the use of technology (Borba & Villarreal, 2006), gesture and other body language (Roth, 2001), representations (Arcavi, 2003), inscriptions (Roth & McGinn, 1998), and through diagrammatic reasoning (Dörfler, 2001). All such means of mathematical communication are included in the definition of mathematical discourse used in this study.

**Review of Selected Literature**

The literature review provided in this chapter supplies the reader with necessary information on problem solving and what it means to think mathematically, of which metacognition is a fundamental component. After motivating metacognition as instrumental in successful problem solving and mathematical thinking, a description of metacognition is provided with emphasis on the judgment and decision-making processes involved with choosing and utilizing metacognitive knowledge and strategies. This section is followed by an overview of metacognitive interventions that have been implemented in an attempt to describe or improve metacognition in educational settings. These interventions, whose foundations lie in cognitive and information processing theories, have largely been unable to account for the real-time, natural metacognitive judgments and decisions of students during the problem-solving process in authentic problem-solving situations. Finally, normative behavior is the central mechanism through which this dissertation study describes the activity of a classroom community. As such, literature is presented at the end of this section related to sociomathematical norms in general and specifically in the context of metacognition.
Problem Solving and Thinking
Mathematically

Noting that “the mathematician’s main reason for existence is to solve problems, and that, therefore, what mathematics really consists of is problems and solutions” (Halmos, 1980, p. 519), much emphasis in mathematics education has been placed on problem solving. The National Council of Teachers of Mathematics (NCTM) Principles and Standards for School Mathematics (2000) listed problem solving as one of five process standards, emphasizing that “[s]olving problems is not only a goal of learning mathematics but also a major means of doing so. It is an integral part of mathematics, not an isolated piece of the mathematics program” (p. 52). By learning mathematical problem solving, students “should acquire ways of thinking” (p. 52). Further, in their Principles to Actions, the NCTM (2014) stated as one of eight mathematics teaching practices that in every mathematics lesson, tasks should be implemented that promote reasoning and problem solving. In fact, the first of eight Standards for Mathematical Practice from the Common Core is to make sense of problems and persevere in solving them (CCSSI, 2010). The Conference Board of the Mathematical Sciences (CBMS) also recommended that “[a]ll courses and professional development experiences for mathematics teachers should develop the habits of mind of a mathematical thinker and problem-solver” (2012, p. 19), and the Mathematical Association of America (MAA) in their 2015 Curriculum Guide to Majors in the Mathematical Sciences highlighted that “[d]iscussing problem solving using authentic student work…helps both the teacher and the students gain insight into what is understood” (p. 85). Problem solving is at the heart of all of these documents. Stanic and Kilpatrick (1989) argued that “problem solving has become a slogan encompassing different views of what education is, of what schooling is, of what
mathematics is, and of why we should teach mathematics in general and problem solving in particular” (p. 1). Recalling the definition of problem solving in the previous section of this chapter, this section elaborates on the notion of problem solving at the heart of this study.

In his seminal book, *How to Solve It*, George Pólya (1957) described problem solving within the context of mathematics. For Pólya, the purpose of a (mathematical) problem was to challenge curiosity and elicit inventive behavior so that an individual may experience mathematical discovery. This means problems must be at the right level of difficulty and interesting for the problem solver. While sometimes the domain of mathematics may appear methodical and deductive, the process of mathematical problem solving is actually experimental and inductive in nature. It is this latter depiction of mathematics that Pólya wished to make explicit for students and teachers of mathematics by delineating a structured way of thinking during mathematical problem solving. Thinking “mathematically” or logically during problem solving consists of continued self-questioning or prompting during four stages of the problem-solving process: *understand the problem, devise a plan, carry out the plan, and look back* (Pólya, 1957).

Pólya (1957) further made the distinction between “problems to find” and “problems to prove”, where the goal of the former is to find a specific object or unknown and whose principal components are the unknown object, the information given in the problem (the data), and any restrictions or relationships between the given information and the unknown (the condition). “Problems to find” are traditionally found in elementary mathematics, and Pólya provided the following example:
Find the diagonal of a rectangular parallelepiped of which the length, the width, and the height are known. (p. 7) (Figure 1)

The unknown is the length of the diagonal; the data are the length, width, and height; and the condition is the fact that if the length, width, and height of a rectangular parallelepiped are all known, then the length of the diagonal is determined, and in fact equal to $\sqrt{a^2 + b^2 + c^2}$.

Figure 1. Images for the “problem to solve” (left) (Pólya, 1957, p. 11) and “problem to prove” (right) examples (p. 26).

Alternately, “problems to prove” are more noticeable in advanced mathematics. The goal of a “problem to prove” is to show a theorem or claim is either true or false and its key elements are typically the hypothesis and conclusion. Pólya (1957) provided an example of such a problem:

Two angles are in different planes but each side of one is parallel to the corresponding side of the other, and has also the same direction. Prove that such angles are equal. (p. 25) (Figure 1).

In this example, the first sentence is the hypothesis while “the angles are equal” is the conclusion. Claiming that the focus of his book, How to Solve It, is on “problems to find” rather than “problems to prove”, in describing the four stages of problem solving Pólya’s
suggestions and guiding questions focused on “problems to find”. In the following paragraphs, examples of questions to ask while solving “problems to find” are provided, but it should be noted that relatively isomorphic questions to those listed for “problems to find” are applicable to proof situations and so the same types of questions can be used for both settings. For example, ‘Do you know a related problem (with the same or similar unknown)?’ for a “problem to find” is essentially the same as ‘Do you know another, familiar theorem with the same or similar conclusion?’ for a “problem to prove”. The following paragraphs provide the details of Pólya’s four-step problem-solving heuristic and the guiding questions that help to drive the problem solver toward a solution.

When solving a problem, the solver must first understand the problem (Pólya, 1957), which can be aided by self-directed or teacher-prompted questions such as ‘What is the unknown?’, ‘What information is given?’, ‘How is the unknown related to the information given?’, and ‘Is the given information enough to find the unknown?’ The individual may also prompt themselves or be encouraged by a teacher to introduce a variable or draw a picture. After understanding the problem, the problem solver then needs to devise a plan to solve the problem. While this plan may not emerge immediately, it hinges upon finding a “bright idea” (p. 8) for the plan, which can develop either gradually or abruptly. To help in finding this key idea, Pólya suggested asking ‘Do you know a related problem (with the same or similar unknown)’ and if such a related problem is found, ‘Can you use it to solve this problem?’ If these questions are unsuccessful, then asking ‘Could you restate the problem?’ or ‘Could you solve a related problem first?’ may be useful. Through the process of finding a plan, the problem solver may become swept up experimenting by using various strategies or attempting variations
of the problem at hand, and so questions including ‘Did you use all of the given information?’ and ‘Have you used the relationship between the unknown and the information given?’, may help bring the problem solver’s attention back to the current problem.

With a sufficiently devised plan, the problem solver must then *carry out the plan* (Pólya, 1957). If the plan has been clearly outlined, then carrying out the plan is only a function of time and the most important feature of this portion of the problem-solving process is checking each step. Here, questions to aid the problem-solving process are related to checking work as the plan is completed and are of the form ‘Can you see clearly that this step is correct?’, or ‘Could you prove that this step is correct?’, the latter request emphasizing an “honest” understanding of the correctness of each step. The final step of the problem-solving process, *looking back*, also involves a global evaluation of the correctness of the solution to the problem: ‘Can you check the result?’ or ‘Can you check your argument?’ If not convinced of the validity of the solution, one may wish to ask ‘Can you derive this result differently?’ or ‘Can you see the quality of this solution at a glance?’ Further, looking back also allows the problem solver an opportunity to summarize the solution path and solution, and to generalize ideas from this result/method pair to use in future problem-solving endeavors. These connections can be made more obvious by asking ‘Can you use the result, or the method, for some other problem?’ In this way, the process of solving mathematical problems can be abstracted into Pólya’s four general stages with guiding questions. Pólya generally described such questioning as a way for teachers to aid students through problem solving, to catalyze the process for their students. Teachers can monitor student progress by asking these questions.
constantly while their students are problem solving and can model these questions when demonstrating problem solving for their students.

After even a casual reading of Pólya’s (1957) problem-solving progression, it is apparent that thinking explicitly about the problem at hand is imperative. Having guiding questions to ask is strategic knowledge that can push the problem solver toward a solution. However, the shortcomings of Pólya’s steps to problem solving in practice are related to the fact that having these questioning tools is not the same as knowing when and how to use them. Pólya argued that through imitation and practice students may assimilate both the questions and eventually “discover the right use of these questions and suggestions, and doing so he [or she] will acquire something that is more important than the knowledge of any particular mathematical fact” (p. 5). The “something” described by Pólya in this quote appears similar to what Schoenfeld (1992) described as “thinking mathematically”. Becoming a successful problem solver or, more generally, an individual who thinks through problems “mathematically”, involves

(a) developing a mathematical point of view – valuing the processes of mathematization and abstraction and having the predilection to apply them, and
(b) developing competence with the tools of the trade, and using those tools in service of the goal of understanding structure – mathematical sense-making. (p. 335)

How, specifically, the act of thinking mathematically during problem solving develops was not made explicit by Pólya (1957), and he was reluctant to “make problem solving a science but rather considered it, like teaching, a practical art [that] often gets lost amid the homage and the disparagement” (Kilpatrick, 1987, p. 299). This is not to say that teachers do not know how to think mathematically themselves, but that this type of thinking is not automatically transferred to students through direct instruction of
problem solving heuristics, just as direct instruction is not sufficient for students to understand or transfer mathematical concepts (Lew, Fukawa-Connelly, Mejía-Ramos, & Weber, 2016). Some of this difficulty may also be due to what constitutes a problem. As opposed to an exercise, a student solving a problem has “no easy access to a procedure for solving the problem” (Schoenfeld, 1985, p. 11). Moreover, a problem is relative to the problem solver. A problem, unlike an exercise, is difficult for the problem solver: an “intellectual impasse rather than a computational one” (p. 74). What is difficult for one person may not be for the next. Recalling Pólya’s (1957) description of problem solving, Schoenfeld’s definition of problem is in line with Pólya’s focus, and Pólya’s detailed description of problem solving heuristics appeared to be in service to problem, not exercise, solving. Considering the ostensibly more complex nature of problem solving, the instruction of “thinking mathematically” (Schoenfeld, 1985) may be much more intricate and nebulous than what Pólya (1957) assumed.

Pólya’s (1957) model has become seemingly commonplace when discussing problem solving, often presented in textbooks and even guiding the design of instructional interventions (e.g., Lee, Yeo, & Hong, 2014), but “[a]lthough mathematics teachers have been quick to adopt what they understand to be Pólya’s approach to solving problems…few have been able to alter their instruction and recast the curriculum to reflect his challenging pedagogical ideas” (Kilpatrick, 1987, p. 300). Direct instruction of problem-solving heuristics could be viewed as an attempt to help students “think mathematically”, but by reducing strategies and heuristics to a step-by-step guide for problem solving, there is no accountability for students to arrive at these general principles by themselves (Schoenfeld, 1985) and “[t]he potential benefits of heuristics
can be diluted to the point where their impact is negligible” (p. 73). Direct instruction of problem solving has typically involved the presentation of a heuristic with several worked examples at the board to passive learners, followed by assigned exercises on what has just been presented (Smith, 1996; Stigler, Fernandez, & Yoshida, 1996; Stigler & Hiebert, 1999). Teachers can require their students to write down relevant quantities, define variables, and check their solutions, but often there is no decision about which strategy or heuristic should be carried out because the necessary tool has been demonstrated to the student immediately prior to solving the current problem, turning problems into exercises. Superficial familiarity and memorization “dominate over reasoning based on ‘deeper’ mathematics, even when the latter can lead to considerable progress” (Lithner, 2003, p. 29).

As an example, consider an excerpt from *Mathematics for Elementary Teachers: A Conceptual Approach* (Bennett, Burton, & Nelson, 2015) in which Pólya’s four stages are provided as a guideline for problem solving. In building up to these steps, the book’s authors define a problem as “a situation you want to resolve but no solution is readily apparent” (p. 3) and problem solving as “the process by which the unfamiliar situation is resolved” (p. 3). The authors highlight that problems are dependent on the problem solver, just as Schoenfeld (1985) emphasized. However, this textbook suffers from precisely the concerns described here. After describing the four stages of problem solving, the textbook provides examples solving problems using these four stages. The book highlights various strategies to implement in the ‘making a plan’ and ‘carrying out the plan stages’, such as making a picture or table, guess and check, making a model, or working backwards. This advice is incontestably valuable, but the homework provided at
the end of the section in this textbook reduces the problem solving to a series of closed-ended exercises in which the student is told precisely which strategy to employ and scaffolded (linearly) through each of the four stages (e.g., Figure 2).

Figure 2. Sample homework with Pólya’s four stages (Bennett et al., 2015, p. 14).

Similar findings exist in other subject areas of mathematics. In one College Algebra textbook, Coburn (2009) included a problem-solving guide at the end of a section on equations and inequalities (Figure 3). While not explicitly Pólya’s four stages, the first two guidelines are encompassed by Pólya’s understanding of the problem, the third guideline is one specific devised plan or problem-solving heuristic, and the fourth guideline includes both carrying out the plan and looking back. After introducing the four guidelines, four different examples of problem solving are provided. Following each example is the statement “Now try exercises ___ through ___”, with a handful of more problems in an identical context. In this way, problems can be reduced to exercises where students are able to memorize one procedure for a particular context.
**Problem-Solving Guide**

- **Gather and organize information.**
  Read the problem several times, forming a mental picture as you read. Highlight key phrases. List given information, including any related formulas. Clearly identify what you are asked to find.

- **Make the problem visual.**
  *Draw and label a diagram* or create a table of values, as appropriate. This will help you see how different parts of the problem fit together.

- **Develop an equation model.**
  *Assign a variable* to represent what you are asked to find and build any related expressions referred to in the exercise. Write an equation model from the information given in the exercise. Carefully reread the exercise to double-check your equation model.

- **Use the model and given information to solve the problem.**
  Substitute given values, then simplify and solve. State the answer in sentence form, and check that the answer is reasonable. Include any units of measure indicated.

*Figure 3.* Problem-solving guide following Pólya’s stages (Coburn, 2009, p. 78).

In examining PreK-6 mathematics textbooks used in Mexico, Santos-Trigo (2007) also found that generally

> all the lessons have a similar format that describes the context of each lesson and presents relevant data. The students are asked to respond a series of questions that require the use of the contents. In this context, students have no opportunities to pose and discuss their own questions since their work is reduced to answer the posed questions using only the information provided (p. 529).

He further noticed that all the problems had the same level of complexity, necessitated only short answers, did not require reflection on the connections between them, and focused on relevant content rather than processes to be developed and utilized during problem solving.

Rather than memorizing classes upon classes of problem contexts and solution methods, authentic problem-solving settings that relate to the definition of problem outlined in the previous section are necessary if students are to “think mathematically” (Schoenfeld, 1985). Moreover, the issue of control, one of the mechanisms of
metacognition defined earlier, moves to the foreground since many times “[t]he number of useful, adequately delineated techniques is not numbered in the tens, but in hundreds…The question of selecting which ones to use (and when) becomes a critical issue” (p. 73). If not implemented appropriately, there is limited need for students to engage in mathematical sense-making or develop a mathematical point of view as they go through the motions. For Schoenfeld, problem solving required more than just a vast knowledge of heuristics. Thinking mathematically during problem solving meant being “resourceful, flexible, and efficient in your ability to deal with new problems in mathematics” (p. 12).

In his own problem-solving courses, Schoenfeld (1985) realized that even high performing, mathematically inclined students who appeared confident in their abilities had far fewer resources and heuristics than expected, limited efficiency with and control of their resources and heuristics, and what students did know many times was not used during problem solving because of their mathematical world views in which they did not see their mathematical knowledge as useful. When investigating the problem-solving efforts of research mathematicians and mathematics Ph.D. candidates, DeFranco (1996) found that while both groups had similar levels of content knowledge, the research mathematicians made better control decisions. He asserted,

[i]t is apparent that university mathematics departments train students in subject matter but not in problem solving skills. To the extent that solving problems is important . . . the mathematics community needs to rethink the culture in which students are trained to be mathematicians. (p. 209)

As has been evidenced with recent textbooks in the previous paragraphs, the nature of mathematical problem-solving education must be reexamined to create successful
problem solvers. This may require attending to other constructs related to the problem-solving process such as metacognition, which appear currently to be overlooked.

Schoenfeld (1983, 1985) argued that to describe the problem-solving process, in addition to heuristics attention must be paid to resources, control, and beliefs. Schoenfeld’s (1985) framework for problem solving incorporated these four components. *Resources* include mathematical knowledge such as facts and routine procedures, as well as informal knowledge, “intuitions based on empirical experience” (p. 54), and an understanding of the normative rules for discourse in the content domain where problem solving occurs. *Heuristics* are the problem-solving strategies and techniques that Pólya (1957) described in great detail, for example using related problems, drawing pictures and diagrams, rephrasing the problem, or checking steps and solutions. *Control* involves decisions about the use (or nonuse) of strategies and heuristics, including planning, monitoring, assessment, and other metacognitive acts. Note that while Schoenfeld (1985) placed monitoring as a subset of control, both monitoring and control have been considered equally important, simultaneous processes of metacognition (Nelson & Narens, 1990, 1994). This idea is described further in the next section of this chapter, and the reader is encouraged to refer to the definition of metacognition provided earlier in this chapter. *Belief systems* or mathematical world views include beliefs about one’s self, the environment, the topic at hand during the problem-solving process, and the broader beliefs about mathematics in general. For example, Schoenfeld’s students in his problem-solving course did not always utilize their resources and heuristics and instead used “naïve empiricism” (p. 174), not because they were not capable, but because they may
have believed that making a strong, informal justification instead of a more formal argument was sufficient.

Heuristics and resources are deeply related in that the former depends heavily on the latter (Schoenfeld, 1985). For instance, simplifying a problem to first work with a less complex version may result in solving a less complicated problem, but this problem may be so distant from the original problem that this simplification is not useful. An understanding of the content domain is a necessary condition to efficiently utilize heuristics. But while having a vast pool of resources and heuristics is necessary, this is not sufficient for success in problem solving. In fact, there is evidence that “attempts to teach students to use heuristic strategies have consistently produced less than what was hoped for” (p. 70), and these strategies do not transfer to new situations. Schoenfeld argued that this insufficiency may be due to the fact that the complexity of problem solving heuristics have not been adequately described in a way that allows them to be meaningfully prescriptive. For instance, the strategy ‘using sub-goals’ could have different meanings in different contexts and to different students.

Schoenfeld (1985) also asserted that emphasis must also be placed on the aforementioned process of control to aid in the effective selection of the many available heuristic strategies while problem solving. Successful problem solving depends on sufficient subskills and on an individual’s ability to control strategy use with executive decision-making, while ineffective control is detrimental for productive problem solving (Schoenfeld, 1985, 1992). For example, if a problem solver uses the strategy of ‘easier, related problems’, then Schoenfeld (1985) stressed that effective use of this particular strategy relies heavily on all of the following:
1. knowing to use the right strategy,

2. knowing the appropriate versions of it for that problem,

3. generating appropriate easier, related problems,

4. assessing the likelihood of being able to solve and then exploit each of the easier problems,

5. choosing the right one,

6. solving the chosen problem, and

7. exploiting its solution (p. 95-6).

Schoenfeld’s (1985) claim that effective problem solvers *assess the likelihood* that various strategies and subskills would be useful is also emphasized in more recent characterizations of problem solving. In interviews with research mathematicians and mathematics Ph.D. candidates, Carlson and Bloom (2005) found that experienced problem solvers do not solve problems in a linear fashion. Here, Carlson and Bloom also used Schoenfeld’s (1985) definition of problem as compared to an exercise. Carlson and Bloom (2005) utilized an analogy of a skier to describe the way mathematicians solve problems:

In one split second as she pushes off, the skier analyzes the possible paths before her, assaying factors such as moguls, slope, snow conditions, and her own skills and limitations. With little hesitation, she accesses a vast reservoir of techniques, knowledge, and past experiences to imagine the moves required to navigate each possible path, and – employing *transformational reasoning* – assesses the likely outcome of selecting each one. So too with our mathematicians: their ability to play out possible solution paths to explore the viability of different approaches appears to have contributed significantly to their efficient and effective decision making and resultant problem-solving success (p. 69).

Successful problem solvers do not just move through the steps of problem solving linearly, but rather cyclically, as seen in Figure 4. Rather than spending wasted time
chasing wild geese and pursuing dead ends for too long (Schoenfeld, 1983, 1985, 1987, 1992), practitioners of mathematics are able to imagine, conjecture, and evaluate hypothetical trajectories efficiently during their problem-solving process (Carlson & Bloom, 2005). Moreover, these experienced problem solvers utilize monitoring and control in each phase of problem solving (orienting, planning, executing, and checking).

Figure 4. The problem-solving cycle (Carlson & Bloom, 2005, p. 54).

Other researchers have also emphasized the need for metacognition through the acts of monitoring, control, and judgment and decision making throughout the problem-solving process (Carlson, 1999; Garofalo & Lester, 1985; Goos et al., 2002; Kuhn, 2000; Lester, 1994; Schneider & Artelt, 2010; Silver, 1987). Carlson and Bloom (2005) argue that
While the literature supports that control and metacognition are important for problem-solving success, more information is needed to understand how these behaviors are manifested during problem solving, and how they interact with other problem-solving attributes reported to influence the problem-solving process (e.g., resources, heuristics, affect). (p. 46)

As will be described in the following sections, there has been significant research attempting to delineate and foster metacognition and the associated behaviors related to monitoring, control, and judgment and decision-making processes in problem solving. Although much theoretical work has been done, the goal of metacognitive research has not been, but should be, “not only to construct more complete theoretical models but also to better meet the needs of our students” (Carroll, 2008, p. 411). The goal of this study is to contribute to this call to translate metacognitive research from theory to practice.

**What is Metacognition?**

As was argued in the previous section, only focusing on the heuristics of problem solving is not sufficient to understand the process of problem solving. Mathematics education researchers have discussed the importance of metacognition in problem solving. In describing this concept, researchers have used terms such as monitoring, control, as well as judgment and decision-making processes to broadly characterize the actions related to metacognition. In this section, a definition of metacognition is advanced that consists of both monitoring and control. This definition emphasizes a process view of metacognition related to judgment and decision making rather than a static, product view of metacognition focusing on decontextualized metacognitive knowledge and skills.

The construct ‘metacognition’ is discussed as it has been characterized in the context of cognitive psychology and work on memory and metamemory. Whereas cognition concerns one’s understanding of information and related thought processes, in
the broadest sense metacognition is thinking about one’s thinking. Other terms are related to this idea, for example Skemp’s (1987) notion of reflective intelligence, which was itself borrowed from Piaget (Skemp, 1961). A definition of metacognition is generally attributed to Flavell (1979), which refers, “among other things, to the active monitoring and consequent regulation and organization of these processes to the cognitive objects on which they bear” (p. 232). Flavell characterized metacognition or cognitive monitoring as being composed of two main mechanisms: metacognitive knowledge and metacognitive experiences, with goals (tasks) and actions (strategies) both subsumed within each of metacognitive knowledge and experiences.

*Metacognitive knowledge* consists of one’s knowledge or beliefs about person, task, and strategy factors that influence cognitive activity (Flavell, 1979). For example, metacognitive knowledge of oneself or others as cognizing agents might be the belief that one is better at taking tests than their classmates, or that it is easier to complete proofs using the method of contradiction as opposed to implementing other proof strategies. Knowledge about tasks involves both what information is available to the individual during cognitive activity, as well as information about the task itself. In the case of mathematical problem solving, this may include knowledge of what information is available to solve a particular problem and one’s ability to manage the perceived difficulty of the problem. Finally, metacognitive knowledge also includes strategies for completing cognitive tasks, which include techniques such as creating organizational tables, rereading a problem statement to glean relevant information, or checking one’s work.
Metacognitive experiences are, most often, the conscious utilizations of metacognitive knowledge to manage cognitive or other metacognitive activity (Flavell, 1979). Flavell offered the example of a problem solver working on a problem and suddenly remembering another, similar problem s/he had solved before. The solver can then use this information to inform current problem solving. Metacognitive experiences can happen before cognitive activity (e.g., through planning), during cognitive activity (e.g., recognizing a similar problem), or after cognitive activity (e.g., through reflection or verification), which in turn revise one’s metacognitive knowledge base. Some metacognitive experiences may not transform the metacognitive knowledge base. If the solver becomes consciously aware of her/his progress towards accomplishing a current cognitive goal, this may inform future cognitive or metacognitive behavior, but this conscious awareness does not necessarily become part of his/her metacognitive knowledge.

While Nelson and Narens’ (1990) focus was specifically on the monitoring and control of memory, their general model of metacognition built on Flavell’s (1979) notions of metacognitive knowledge and experiences, expanding on the interactions between these two mechanisms. Nelson and Narens (1990) described the process of metacognition in their work on metamemory and characterized their metacognitive model as having three fundamental principles. As seen in Figure 5, the first principle asserts that cognitive processes are divided into two distinct levels: an object-level of cognition typically associated with external objects such as formulae or proofs and a meta-level which involves cognition about cognition. These two levels operate simultaneously (Nelson & Narens, 1994). The second principle states that the meta-level includes a
“dynamic” (Nelson & Narens, 1990, p. 126), subjective model or representation of cognition at the object-level that the meta-level can use to inform decisions related to the object-level.

![Diagram of metacognitive model](image)

*Figure 5. Nelson and Narens’ (1990) metacognitive model (p. 126).*

The third principle claims that the meta-level acquires information from the object-level to potentially alter its representation of the object-level through the process of *monitoring*, while the meta-level sends information to and potentially alters the current state of the object-level through the process of *control*. These notions are similar to that of Schoenfeld (1985), where the problem solver must make decisions about strategy choice and use (control), plan (monitoring and control), and make assessments (monitoring). Further examples of processes involved with monitoring include judgments of how easy it would be to learn a particular topic, how well one understands a concept, whether one is able to recall information about a given concept, and confidence that one’s answer is correct. Examples of processes involved with control include selecting a particular strategy or beginning, altering, or terminating a cognitive process or strategy.
Pulling from the work of Nelson and Narens (1990, 1994; Nelson, 1996), Van Overschelde (2008) outlined elements within the meta-level or metamodel (Figure 6). Recall from the three aforementioned principles outlined in Nelson and Narens’ (1990) model that, at the meta-level, an individual creates a dynamic model of their object or cognitive level. In addition to this model of an individual’s cognition at the object-level, the meta-level also holds metacognitive knowledge and knowledge of strategies to control the object-level (Van Overschelde, 2008). Metacognitive knowledge is “explicit, factual knowledge about how the mind works” (p. 53) and is similar to Flavell’s (1979) conception of metacognitive knowledge. Metastrategic knowledge is “implicit, procedural knowledge about how one can use the mind to accomplish goals” (Van Overschelde, 2008, p. 53), for example, correcting errors as you evaluate an expression. Metacognitive knowledge and strategies are then used to control the object-level cognitive behavior, but their implementations are subject to both intrinsic and extrinsic constraints. Intrinsic constraints include beliefs and expectations, for example expectations that a given strategy will be successful, beliefs about ability, or beliefs about how external constraints affect cognitive activity (Van Overschelde, 2008). Extrinsic constraints might be the amount of study time available or the amount of time allotted for solving a particular problem in class or for homework.
The use of metacognitive knowledge and strategies, along with the construction of the metamodel of the object-level, are done in light of an individual’s goals or goal states (Van Overschelde, 2008). For instance, a student holding a mastery goal orientation focuses on “developing competence, with an emphasis on improvement, learning, and understanding” (Linnenbrink-Garcia et al., 2012, p. 282). Alternatively, a student with a performance goal orientation strives to “demonstrate or validate competence, often in comparison to others” (p. 282). Mastery and performance goal orientations (Wentzel & Brophy, 2014), as well as other more specific goals such as minimizing effort, can influence an individual’s perception of cognition at the object-level and the way in which metacognitive knowledge and strategies are used (Van Overschelde, 2008). For example,
a performance goal orientation, specifically a performance-avoidance goal orientation, is often associated with low self-efficacy, high test anxiety, avoidance of help seeking, and disorganized study strategies (Wentzel & Brophy, 2014). These consequences of a performance-avoidance goal orientation can then, in this case negatively, affect the cognitive and subsequent metacognitive activity of the individual. The constraints described here by Van Overschelde (2008) related to goal orientations and specifically internal factors are similar to the individual beliefs accounted for in Schoenfeld’s (1983, 1985) problem-solving framework in the previous section.

After monitoring of cognitive activity at the object-level informs the meta-level, judgment and decision-making processes are used to determine what metacognitive knowledge or strategies to utilize and how to use them, moving the individual from the current state of cognitive endeavors in the direction of the overarching goal or goal orientation (Van Overschelde, 2008). Just as the model of the current state of the object-level is made through the subjective perception of the individual, “metacognitive judgments and control actions are made not on the object level per se, but on one’s interpretation or assessment of the accessible information about the object level, along with a host of goal-relevant information” (p. 65, emphasis added). Thus, metacognition is not merely the aggregate product of metacognitive knowledge and strategies, but also the process by which these strategies and knowledge are chosen and employed within navigation of the current state of the metamodel.

Learners make predictions about their cognitive activity with some level of uncertainty. For example, making a subjective assessment of how well one has learned something immediately after learning the content is a more unreliable prediction of one’s
knowledge than making a judgment of learning sometime longer after having originally learned the content (Van Overschelde, 2008). One’s judgment and decision-making processes may also involve estimations of the probability that one has seen something related earlier or the frequency with which one has seen current material previously. These judgments and decision-making processes seem to be what made mathematicians effective and efficient problem solvers for Carlson and Bloom (2005) while characterizing the problem-solving process of research mathematicians and mathematics Ph.D. candidates. As described in the previous section, mathematicians did not solve problems linearly, but rather in a cyclical fashion where, after planning, engaged in a conjecture cycle to assess solution paths and relevant information before executing a particular strategy. Additionally, the mathematicians also engaged in these judgment and decision-making processes in the orienting, planning, executing, and checking phases of problem solving. For example, they might stop mid-execution of a strategy to assess the probability that what they were generating was reasonable and/or productive. Schoenfeld (1985) also emphasized this process of judgment and decision making, as opposed to focusing on just the products of metacognitive skill and knowledge, preferring “to reserve the term control for discussions of active decision making” (p. 20). The regulation of cognition is to some degree more important than metacognitive knowledge itself (Palingsar, 1990; Van Overschelde, 2008; Veenman, Elshout, & Busato, 1994).

**Metacognitive Interventions**

Thus far, metacognition has been discussed by highlighting the importance of judgment and decision-making processes within the context of navigating the metamodel to decide when and how to use metacognitive knowledge and strategies. Instantiations of
metacognition in mathematical problem solving are often characterized within delineated stages of problem solving similar to Pólya’s (1957) four phases (understanding the problem, devising a plan, carrying out the plan, and looking back) (Batha & Carroll, 2007; Palingsar, 1990). For example, Carlson and Bloom (2005) identified acts of metacognition, specifically monitoring, within their proposed four stages of problem solving (Figure 7), which they noted sometimes influenced mathematicians’ control decisions.
Figure 7. Problem-solving actions within the problem-solving cycle (Carlson & Bloom, 2005, p. 67).
Likewise, many metacognitive interventions target a specific aspect of problem solving and provide direct instruction on this particular skill. Metacognitive interventions often consist of direct instruction related to one or more of the following:

- monitoring the reading and understanding of the question or problem,
- assessing what information is necessary to reach a conclusion,
- using a strategy or plan to organise the information so gathered,
- monitoring the use of this strategy, and
- calculating a solution, followed by re-checking every step of the strategy use to ensure the right decision has been made (Batha & Carroll, 2007, p. 65).

For example, Bond and Ellis (2013) conducted an intervention using reflective assessments through written “I learned” statements and follow up “think aloud” small group discussions about the “I learned” statements during the last 5 minutes of class. Desoete, Roeyers, and De Clercq (2003) provided verbal instruction on the metacognitive skill of prediction of task difficulty. Hoffman and Spatariu (2008) used metacognitive prompts (guiding questions) while students solved mathematics problems to encourage students to make connections to various strategies aiding in strategy choice and execution. Lee et al. (2014) focused on the understanding and planning phases of problem solving using a checklist of guiding questions based on those of Pólya (1957) (Figure 8).
This feature of metacognitive interventions, explicit practice with specific metacognitive skills, is likely a consequence of the fact that since cognitive or information processing theory models account explicitly for metacognition (Silver, 1987), most research on metacognition within mathematics comes from these perspectives. Nelson and Narens’ (1990) model of metacognition described previously takes this view as well. By likening cognition to that of computers, an individual’s cognitive behavior controls the flow of information through different levels of memory (Figure 9). Within this context, emphasis is on procedural fluency so that basic facts and procedures do not consume too much working memory and processing speed, allowing individuals the ability to deal with more cognitively demanding, complex mathematical activities (Caron, 2007; Pellegrino & Goldman, 1987; Poncy, Skinner, & Jaspers, 2007; Ramos-Christian, Schleser, & Varn, 2008; Woodward, 2006). Just as grade school students practice basic multiplication facts separately from problem solving, one must
practice basic metacognitive skills for use in later problem solving, moving these skills from working memory to long term memory. This has translated to the use of metacognitive processing interventions (Morris & Mather, 2008) to increase student’s metacognitive skills.

Figure 9. An information processing view of cognition (Silver, 1987, p. 37).

Consequently, while metacognition is described as occurring within the process of problem solving, metacognition is often discussed in interventions by practicing and evidencing the product of metacognitive knowledge and strategies rather than describing the process by which these strategies and knowledge are chosen and used. Metacognitive interventions in which students are taught one or two specific metacognitive skills have been heavily utilized, but these interventions lack instruction on navigating the metamodel during the process of problem solving, helping students understand when and how to use metacognitive knowledge and strategies. This type of metacognitive instruction is disconnected from the authentic problem solving practiced by successful problem solvers when dealing with the mathematics problems described by Schoenfeld
Schoenfeld highlights that “any positivist view that considers teaching problem solving to be equivalent to providing a set of prescriptions for students’ productive behavior” (p. 14) is too narrow.

Unfortunately, most metacognitive interventions have suffered from such issues, either disconnected from the regular mathematics instruction and classroom norms or too short in length (or both). For example, Batha and Carroll (2007) administered metacognitive instruction following steps outlined by Cardelle-Elawar (1995) by first drawing students’ attention to comprehending the situation, gathering information, using multiple strategies and monitoring strategy use and its appropriateness, and finally checking and re-checking the utilized strategy. Notably, this instructional intervention lasted only 10 to 15 minutes, was unrelated to regular classroom content, and immediately following this instruction the experimental group was given two new tasks and asked explicitly to use the information about metacognition they had just received. The reflective assessment intervention run by Bond and Ellis (2013) took place in only the last five minutes of class for four weeks of the semester. Desoete et al.’s (2003) intervention took place outside of the regular classroom over five, 50-minute sessions in the span of two weeks. Hoffman and Spatariu (2008) conducted an intervention during one computer-guided session with a two-minute break halfway through to “avoid cognitive failure” (p. 883), while Lee et al. (2014) provided explicit metacognitive instruction via metacognitive prompting for students one hour each week for six weeks, replacing regular class instruction.

Kramarski (2004) also used metacognitive questions to aid students in graph interpretation and construction, though unlike Lee et al. (2014), she focused on prompts
during all of the following phases of problem solving: comprehending the problem, making connections to prior problem solving, using appropriate problem-solving strategies, and reflecting on errors during problem solving as well as on the final solution.

Even though this intervention occurred within the context of a Linear Graph unit, this intervention only took place for ten class periods over a two-week span. Development of metacognitive skills requires years of practice and cannot be achieved through short-term interventions (Pressley, Borkowski, & Schneider, 1989). There is also evidence that such intervention effects have low long-term retention (Bond & Ellis, 2013). Instead, metacognitive instruction should be embedded in the mathematics content and take place for an extended period of time (Veenman et al., 2006).

Rather than focusing one or two specific metacognitive skills, Lester, Garofalo, and Kroll (1989) conducted a twelve-week instructional intervention to help students become more aware of and practice metacognitive strategies and procedures in order to monitor and assess their own problem-solving activity. On some days, students worked on problem-solving activities guided by the instructor. Other days students watched videos of an individual solving a problem and discussed the good and poor aspects of the individual’s problem-solving attempt, including a discussion about self-monitoring questions such as “what am I doing?” Finally, some instructional days involved students watching the instructor solve a problem modeling metacognitive behavior by thinking aloud. When designing the instructional intervention, Lester et al. (1989) had seven fundamental assumptions that heavily influenced their study design, four of which directly related to metacognition. The first and third of these assumptions are particularly pertinent:
(1) There is a dynamic interaction between mathematical concepts and the processes (including metacognitive ones) used to solve problems involving those concepts. That is, control processes and awareness of cognitive processes develop concurrently with the development of an understanding of mathematical concepts;

(2) Metacognition instruction is most effective when it takes place in a domain-specific context (in the case of this study, problems were related to mathematics content appropriate for grade seven students) (p. 27).

Both of these assumptions express the need for metacognitive behavior and instruction to be considered in the context of learning mathematics concepts themselves. As students are learning new mathematical concepts, facts, and skills, they should also learn how to manage and regulate the application of this new knowledge. Lester et al. (1989) argued that although their metacognitive instructional intervention did take place in the context of mathematical problem solving, this instruction occurred as a separate entity from regular class instruction with two instructors in addition to the regular classroom teacher. Furthermore, this instruction involved small group work, which was not a typical strategy employed by the regular classroom instructor and new, at least in a mathematics classroom, for many of the students. Intervention instruction also only took place an average of one hour and 20 minutes a week for a total of 16.1 hours over a twelve-week period. Moreover, the instruction was largely isolated from the regular mathematics curriculum, and it probably did not take place over a long enough period of time. For the most part, the problem-solving sessions had little or no direct relation to the regular mathematics instruction and many students did not view them as being a central part of their mathematics class. Any future effort of the sort undertaken in this study should insure that the instruction was truly consistent with its guiding principals (Lester et al., 1989, p. 118).

In addition to issues related to the intervention designs, the assessment of these interventions has been critically imprecise, lacking sufficient explanatory power
(Schoenfeld, 2000) of students’ metacognitive processes during problem solving. Typically, interventions are considered “successful” if students perform significantly better at the end of the intervention than a control group on a common content-based test measuring academic achievement (e.g., Bond & Ellis, 2013; Cardelle-Elawar, 1995; Hoffman & Spatariu, 2008; Hudesman et al., 2013; Zan, 2000), and/or on a survey instrument to assess metacognitive skill such as the Motivated Strategies for Learning Questionnaire (MSLQ) (e.g., Bol et al., 2013), the Metacognitive Awareness Inventory (MAI) (e.g., Batha & Carroll, 2007), or others (e.g., Desoete, Roeyers, & Buysse, 2001; Gomes, Golino, & Menezes, 2014). These tools evaluate the transfer of metacognitive skill using a definition of transfer in which metacognitive skills are utilized successfully or unsuccessfully, regardless of context. There are a significant number of lurking variables that could account for success on an assessment, and these factors are not explained when evaluating the interventions. For example, Bond and Ellis (2013) evaluated their metacognitive intervention within the context of a larger curriculum pilot of the Connected Mathematics Project (CMP), but all conclusions were attributed to the metacognitive intervention alone. Alternate views of transfer differ from this “traditional” view in that one must consider transfer as actor-oriented and situated (Lobato & Siebert, 2002; Wagner, 2006), and this more contextualized view of transfer appears warranted to explore the nuanced transfer of metacognitive skills.

Another weakness of utilizing metacognition survey instruments is related to their self-report nature. While transfer tasks provide evidence of metacognition through the lens of the researcher or teacher, self-reported survey measures provide evidence through the lens of the individual being assessed. Students are not typically able to accurately
assess their own metacognitive monitoring and awareness (Hofer & Sinatra, 2010), and when post hoc, retrospective reflection is not done in “real time of students’ metacognitive use… it is very difficult to discern whether the[y] were epistemically aware during the task or became aware upon reflection” (p. 117). For example, the Metacognitive Awareness Inventory is a fifty-two question self-assessment to characterize students’ domain general metacognitive knowledge and regulation. Sample items include ‘I know when each strategy I use will be most effective’ and ‘I use different learning strategies depending on the situation’. This prompting could evoke metacognition that may or may not have transpired otherwise (Greene, 2015), further supporting the notion that “cognitive engagement is not a stable characteristic of either a learner or a learning environment but rather a fluid set of processes that can be influenced by learners themselves and by the environment” (p. 27).

Even if both forms of assessment (transfer/content-based tasks and self-report measures) are utilized, the nature of these measures is not such that they can be easily combined to provide a situated picture. These two concerns are essentially the same problem: can assessment really evaluate metacognition if one only looks at the product of metacognition – static, decontextualized metacognitive skills? Lester et al. (1989) and Veenman et al. (2006) both highlight the contextual and prolonged nature of metacognitive instruction. With this in mind, we must “understand cognition in the context of natural purposeful activity. This would not mean an end to laboratory experiments but a commitment to the study of variables that are ecologically important” (Neisser, 1976, p. 7, as cited in Carroll, 2008, p. 411).
Sociomathematical Norms

In line with Neisser’s (1976) comment that cognition must be explained within the “context of natural purposeful activity” (p. 7), one must also consider that classroom mathematical behaviors are created by a community and influence individual construction of knowledge (Cobb, Wood, Yackel, & McNeal, 1992). The decisions and actions of both the teacher and students in a classroom create a microculture of taken-as-shared activities and interactions within a community of practice (Cobb, Stephan, McClain, & Gravemeijer, 2011; Lave & Wenger, 1991). Classroom norms evolve and are established by the classroom community over a period of time. Unlike behavioral expectations and more general social norms, for instance expectations about the level and type of student participation, sociomathematical norms are taken-as-shared mathematically-based activity (Yackel & Cobb, 1996). Bowers et al. (1999) further made a distinction between sociomathematical norms, for which mathematical dispositions are associated with the individual, and mathematical practices, for which specific problem-solving activities related to a particular mathematical idea are associated with the individual. They offered an example of thinking of numbers as units of tens and ones instead of just counting by ones as a mathematical practice that became taken-as-shared in a third grade classroom. As an example of a sociomathematical norm (as opposed to a mathematical practice or more general social norm), in describing how sociomathematical norms are established within a second grade classroom Yackel and Cobb (1996) identified the sociomathematical norm of “mathematical difference”, in other words delineating what makes solutions different mathematically. They provided an
example of students sharing solutions to the problem ’16 + 14 + 8 = ____’. Through classroom discussions,

the children learned that the teacher legitimized solutions that involved decomposing and recomposing numbers in differing ways but not those that were little more than restatements of previously given solutions. At the same time, the teacher furthered his pedagogical agenda by guiding the development of a taken-as-shared understanding of what was mathematically significant in such situations (p. 463).

The meaning of this sociomathematical norm was negotiated between the teacher and students and this norm regulated mathematical activity in the classroom.

Although mathematical justifications and explanations are ideally instantiated as a joint, taken-as-shared activity, such communication breaks down if what counts as a mathematical justification or explanation is not taken-as-shared (Cobb et al., 1992). As an example, communication between students where one student provides procedural justifications and explanations and the other has a goal to learn for deeper understanding may not be effective because there are two different types of justifications/explanations being used and no taken-as-shared interpretations of activity may be feasibly reached. For this reason, sociomathematical norms may or may not be established depending on the actions and beliefs of the members participating in a given classroom community.

Moreover, a disjointedness of expectations and preferences among the participants within a classroom community can even establish sociomathematical norms that may not be productive for all of the community members.

Levenson et al. (2006) investigated student and teacher preferences for mathematically-based (MB) and practically-based (PB) explanations and found that teachers negotiate their expectations based on what actually happens in the classroom. While the teacher in this study preferred mathematically-based explanations, students’
strong preferences for practically-based explanations established a sociomathematical norm for explanations that were more practical than mathematical. However, while this sociomathematical norm was established, one student, Dan, held on to mathematically-based explanations. This drastically affected his classroom behavior, as he stopped participating in class entirely. Consequently, this lack of participation was harmful to both his and the classroom’s mathematical development:

For himself, Dan loses the opportunity to further develop his mathematical reasoning skills. For the class, they lose the opportunity to engage in mathematical discourse. What would happen over time to students like Dan in classes that choose PB explanations as the normative explanations? Would they give up their individual preferences for MB explanations? (p. 341)

In this way, not only are sociomathematical norms negotiated by members within a community of practice, their development and use can subsequently influence the actions of the members of the community.

Sociomathematical Metacognitive Norms

Cobb and Yackel (1996) argued that students become more autonomous through the process of coming to know and negotiating sociomathematical norms. For Cobb and Yackel, autonomous students are those who use mathematical argumentation to justify claims rather than looking to the instructor as the sole arbitrator and authority in the classroom. However, achieving this autonomy requires students to know what an acceptable form of argumentation is and what makes an efficient and acceptable solution. These are exactly the sociomathematical norms negotiated within a classroom community. In this way, autonomy of the individual is developed through participation in a community of practice. Yackel and Cobb also described how the development of
sociomathematical norms affects higher-level cognitive activity. Consider the following example from their study, taken from a second grade classroom:

*Example 2:* The problem $'78 - 53 ='$ was written on the chalkboard and posed as a mental computation activity.

*Dennis:* I said, um, 7 and take away 50, that equals 20.
*Teacher:* All right.
*Dennis:* And then, then I took, I took 3 from that 8 and then that left 5.
*Teacher:* Okay. And how much did you get?
*Dennis:* 25....

*Teacher:* Ella?
*Ella:* I said the 7, the 70, I said the 70 minus the 50 ... I said the 20 and 8 plus 3,...
Oh, I added I, said 8 minus the 3, that’d be 5.
*Teacher:* All right. It’d be what?
*Ella:* And that’s 75 ... I mean 25.
*Dennis:* (Protesting) Mr. K, that’s the same thing I said. (p. 463)

Yackel and Cobb asserted that this exchange required Dennis to compare and contrast his and Ella’s solutions in order to determine if they were mathematically different. Thus, the establishment of the sociomathematical norm of mathematical difference necessitates sufficient metacognitive activity. Just as Cobb et al. (1992) described difficulties in establishing taken-as-shared activity when two members of classroom community have different beliefs and behaviors during their interactions, it appears that metacognition is also a necessary (though not necessarily sufficient) condition to establish sociomathematical norms within a classroom. Hence, metacognition may act as a “meta”-norm: a norm about sociomathematical norms. Conversely, this situation could also be viewed as an opportunity for the teacher to build on Dennis’ comparison of solutions in order to establish this act of metacognition as normative behavior in itself.

This leads to a question as to whether or not metacognitive norms are sociomathematical. Some researchers have advocated for the domain-generality of
metacognition (Schraw, 1998; Schraw, Dunkle, Bendixen, & Roedel, 1995; Schraw & Nietfeld, 1998), implying that any metacognitive norms in the classroom would subsequently be general social norms rather than sociomathematical norms directly tied with mathematical dispositions and activities. If metacognition acts as a potential “meta”-norm, then it appears that metacognition may in fact be domain general. However, there is also evidence to support a domain-specific view of metacognition (Jacobse & Harskamp, 2012; Kelemen, Frost, & Weaver, 2000; Veenman & Spaans, 2005; Vo, Li, Kornell, Pouget, & Cantlon, 2014). For this reason, it may be the case that (at least some) metacognitive norms are in fact specific to mathematics and thus sociomathematical.

Considering that mathematical problem solving in general is domain specific and different from problem solving in other fields, as well as that metacognition is a fundamental component of mathematical problem solving (Schoenfeld, 1985), it is reasonable to suppose that, transitively, metacognition must be domain specific. For the purpose of this study, the term “sociomathematical metacognitive norm” is used under the assumption that the metacognitive norms witnessed in a mathematics classroom are, in fact, sociomathematical.

More notably, this situation provided by Yackel and Cobb (1996) could be seen as a missed opportunity to establish comparing solutions as a metacognitive norm. If there is no discussion as to why the two solutions provided by Dennis and Ella are different, then this particular metacognitive act does not have the opportunity to become taken-as-shared by other students and the classroom community as a whole. Yackel and Cobb (1996) evidence this concern explicitly when describing a discussion surrounding another number sentences example:
In Example 1, many children may have interpreted the teacher’s enthusiastic response (“Yeah!”) following Rodney’s solution as an indication that this solution was favored. However, because the issue did not become an explicit topic of conversation, the children were left to decide in what sense the solution was special. (p. 464)

Because Yackel and Cobb (1996) were concerned with establishing student autonomy as opposed to developing metacognition (though these are not mutually exclusive), they viewed this lack of conversation by the teacher as an opportunity for students to behave independently and determine on their own what aspect of the given solution method was special: “Events of this type are occasions for the children to infer what aspects of their mathematical activity the teacher values. In the process, the teacher both elaborates his own interpretative stance toward mathematics and inducts students into that stance” (p. 464). Citing Voigt (1995), Yackel and Cobb (1996) proposed that “[s]tudents can develop a sense of the teacher’s expectations for their mathematical learning without feeling obliged to imitate solutions that might be beyond their current conceptual possibilities” (p. 465).

The difficulty with this position is that the teacher’s “interpretive stance”, their proposed norm concerning what is mathematically important, is not necessarily well-defined to students. Levenson et al. (2009) found that “even when the observed enacted norms are in agreement with the teachers’ endorsed norms, the students may not perceive these same norms” (p. 171), and others have argued that students’ mathematical noticing is subjective to an individual and situation and can significantly impact their reasoning (Lobato, Hohensee, & Rhodehamel, 2013). Moreover, a basic tenet of the metacognitive interventions described in a previous section is to bring to the overt forefront metacognitive activities such as this. Without making such conversations explicit, how
can students be expected to independently develop metacognitive behavior and a normed understanding of what is mathematically relevant? If classroom norms are cultivated in the context of an ever-changing microculture, it follows that the teacher and other students play significant roles in shaping both individual and the normed classroom metacognitive behavior within the classroom community. For metacognition to develop, it needs to be an explicit part of the classroom microculture.

Unfortunately, there is limited research on the influence that teachers and students can have during natural classroom activities on metacognitive behavior. With respect to teacher influence, Veenman et al. (2006) found that not only is there minimal information about the way in which the teacher can model metacognitive behavior for students, but “many teachers lack sufficient knowledge about metacognition: When they interviewed teachers about metacognition, responses did not go beyond “independent learning...,” while a further query about how teachers applied metacognition in their lessons “only resulted in blanks” (p. 10). This observation from Veenman et al. (2006) only furthers the claim made by Carroll (2008) that metacognitive research has not sufficiently transferred to classroom practices.

In addition to teachers facilitating metacognitive activity, Goos et al. (2002) widened the scope of metacognitive research beyond that of cognitive psychology, taking a sociocultural view and investigating metacognition in the context of “collaborative conversations between peers of comparable expertise that made the processes of monitoring and regulation overt” (p. 219). Through peer interaction, metacognition is facilitated and mediated. Here, metacognitive prompts are generated spontaneously by students during problem solving, provoking reflection and assessment. This work
contrasts intervention studies described previously in which metacognitive questioning was only utilized by the instructor as the sole authority in the classroom (Hoffman & Spatariu, 2008; Lee et al., 2014). In this way, both the teacher and students contribute to metacognition during problem solving. Consequently, the establishment of metacognitive norms should be considered a dynamic interaction between teacher and students, rather than as separate endeavors (teacher influence or students influence). This study sought to contribute to research on metacognition by investigating the negotiation between teacher and students with respect to metacognitive norms during problem solving within a natural classroom context. In the next section, I describe the methods of my dissertation study to document metacognitive norms in one such classroom context, where “portfolio” problems were used to mediate metacognitive thinking in a first-year mathematics content course for pre-service elementary teachers.

Methods

In this section, I describe the setting of this dissertation research by characterizing study participants and classroom context. I then provide details of my data collection procedures. Data analysis procedures are outlined in the manuscript in Chapter III. Institutional Review Board (IRB) approval for this study can be found in Appendix A.

Setting

Participants. Participants were 24 students and the instructor, Dr. Arkadash (pseudonym), in a freshman-level mathematics course at a mid-size university in the Rocky Mountain Region of the United States during the Fall 2016 semester. Number Sense and Algebra is required for elementary education majors with mathematics emphasis. These students made up the entire class population save for one student with a
music concentration who was considering a mathematics emphasis. Most students in the course were first-time freshman, except for a small handful of students who had recently changed majors. Most students had taken differential calculus in high school. All students agreed to participate in classroom data collection, and everyone was invited to participate in up to three individual interviews at the beginning, middle, and end of the semester. Of the 24 students in the course, 20 students were willing to participate in individual interviews. 15 students completed the first interview and 13 students completed all three interviews.

**Classroom context.** Students in the Number Sense and Algebra course met for 75-minutes twice a week for 15 weeks. The one-semester course was designed as the concatenation of two courses required for all elementary education majors. Course topics included:

- properties of natural numbers, counting, and place value (Unit 1),
- the meaning and interpretation of addition, subtraction, multiplication, and division for both integers (Unit 2) and fractions and decimals (Unit 4),
- factors, multiples, prime factorization, greatest common factor, and least common multiple (Unit 3),
- expressions, equations, and solving equations (Unit 5), and
- ratio, proportion, and functions (Unit 6).

The course was focused primarily on mathematical content knowledge, but also aimed to foster confidence in future elementary teachers. In appealing to both mathematical content goals and the pedagogical development of pre-service teachers, one overarching objective for the Number Sense and Algebra course was to improve mathematical
reasoning through analysis of student work, both from students in the Number Sense and Algebra course itself, as well as hypothetical elementary student work.

Typically, class time began with a warm-up problem related to the previous class session, followed by a mixture of active, whole class discussion and small group work on activities that guided the day’s discussions. Group work constituted a significant portion of class time, with Dr. Arkadash moving from group to group asking students to explain their thinking. As needed, Dr. Arkadash would provide a brief introduction to a new topic, introduce new vocabulary, or wrap up small group work by facilitating a whole class summary or student presentations.

Content was presented within the context of exploration and problem solving. Emphasis was on the problem-solving process, as opposed to the final product or answer. When investigating ideas and solving problems, students were encouraged to utilize multiple representations and solution paths, look for and analyze relationships and patterns, and make conjectures. Their mathematical arguments were to include explanations of the mathematics underlying procedures and be communicated in a clear and precise way. Additionally, mathematical arguments, conjectures, and solutions were evaluated for reasonableness, and an attempt was made to make connections between various solutions. Based on classroom observations and interviews, it was evident that Dr. Arkadash did not want students to look to her as the sole authority in the classroom. Her focus was on empowering students to engage in the aforementioned objectives. During group work, she attempted to push students’ thinking forward by deflecting questions asked of her to other group members or by responding with another question.
While students presented, only the presenter stood at the front of the room. Other students were requested to address the presenter, not Dr. Arkadash, with their questions. As the semester continued, students became much more comfortable with this format, offering to come to the board to explain their thinking without being prompted by Dr. Arkadash. Further, there was an expectation from Dr. Arkadash that all voices be heard as often as possible. During whole class discussions, she consistently made an effort to explicitly ask for contributions from students who had stayed quiet, and during group work she requested input from such students. While there were still times when small group work was less collaborative than what Dr. Arkadash might have considered ideal, all students were becoming more conversational and willing to voluntarily share ideas, either in small group work or in whole class discussions. Small groups were also shuffled often to maximize collaborative efforts and provide students opportunities to share with and listen to ideas from many different individuals.

Course participation accounted for 10% of students’ final grades and weekly homework assignments were 20%. Approximately once a week, students spent roughly twenty minutes working together on a portfolio problem, a problem typically more open-ended than usual course work with key ideas related to current content (see Appendix F for the problems used in the course and Chapter III for examples of student work). While working on portfolio problems during class time, students were provided with scratch paper and each member of a small group used a different colored pen. Students engaged in some individual think time, but then shared ideas and worked on the problem together. Any scratch work was scanned and emailed to group members after class for them to reference while continuing to work on the portfolio problem on their own outside of
class. Students then wrote up a revised solution at least three pages in length. This final draft could include further investigation of a solution path begun during class or exploration of new ideas or conjectures. Write-ups needed to include mathematically justified arguments, but also an explanation of thinking and reasoning as they related to scratch work revisions and the choice to pursue specific conjectures or problem-solving strategies. Further, while two midterm exams and one final exam were worth the remaining 55% of students’ course grades, one question on each of the first midterm and final exam was given and assessed as a “portfolio”-style problem, looking at students’ problem-solving processes and reflections of real-time thinking.

Data Collection Procedures

I was present in the course as an observer participant (Gold, 1958). While I was in the course daily and supported Dr. Arkadas in asking questions to small groups and very occasionally student presenters, my primary role was for data collection and observation. My membership in the classroom community was peripheral.

Six qualitative data sources were collected and are summarized in Table 1. Recorded in-class portfolio problem-solving sessions were utilized in micro-level analysis to address research question Q1a, while all data sources were used in macro-level analysis to address research questions Q1b, Q1c, and Q1d, and to triangulate micro-level findings in addressing Q1a. The first and second student interviews had three parts. First, students were asked questions targeting their beliefs about mathematics, mathematical problem solving, and perceptions of the course. Students then worked, thinking aloud, on problems related to course content. This portion of the interview served as a reference point for students to discuss their problem-solving activity more
generally. Finally, students compared their problem-solving attempts during the interview with their “typical” problem-solving activity in the course, the problem-solving activity of Dr. Arkadash, other students in the course, and other courses. The third and final interview did not include problem solving, but was a series of questions asking students to reflect on their experiences in the course.
### Table 1

**Summary of data sources**

<table>
<thead>
<tr>
<th>Data Source</th>
<th>Time Collected</th>
<th>Purpose</th>
</tr>
</thead>
<tbody>
<tr>
<td>Video- and audio-recorded portfolio problem sessions</td>
<td>In-class portfolio sessions, roughly once a week with two sessions for each of the six portfolio problems (The first problem had one session.)</td>
<td>Used in micro-level analysis to identify normative metacognitive actions in authentic problem-solving situations</td>
</tr>
<tr>
<td>Video- and audio-recorded classroom sessions</td>
<td>Daily</td>
<td>Used in macro-level analysis to identify student and teacher goal-directed actions</td>
</tr>
<tr>
<td>Video-recorded, semi-structured individual interviews with 15 of 24 students (Questions provided in Appendix B)</td>
<td>Weeks 2, 8, and 15</td>
<td>Used in macro-level analysis to identify motives/goals and beliefs/preferences</td>
</tr>
<tr>
<td>Students’ written artifacts (portfolio-problem write-ups)</td>
<td>Collected before grading</td>
<td>Used in macro-level analysis to triangulate students’ portfolio problem-solving activity</td>
</tr>
<tr>
<td>Audio-recorded interviews with Dr. Arkadash (Questions provided in Appendix C)</td>
<td>Weeks 1 and 15</td>
<td>Used in macro-level analysis to identify motives/goals and beliefs/preferences</td>
</tr>
<tr>
<td>Recorded planning sessions</td>
<td>Weekly</td>
<td>Used in macro-level analysis to identify changing goals</td>
</tr>
<tr>
<td>Journal reflections from Dr. Arkadash and myself</td>
<td>Daily</td>
<td>Used in macro-level analysis to identify Dr. Arkadash’s changing goals and my memos for noteworthy classroom episodes</td>
</tr>
</tbody>
</table>

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2 13 of the 15 students completed all three interviews.
Organization of the Dissertation

This chapter introduced the dissertation research related to the development of metacognition during authentic problem-solving situations. Chapters II, III, and IV are standalone manuscripts. Chapter II discusses the theoretical perspective and analytic framework utilized to analyze the collected data for the study. Chapter III provides a brief literature review related to the teaching and learning of metacognition and work with pre-service teachers. This manuscript addresses the main research questions with results from both micro- and macro-level analysis. Chapter III also briefly outlines data collection and provides a detailed description of data analysis methods. Footnotes are included in this chapter for additional reference to appendices D and E that provide “codebook” information related to micro- and macro-level coding and data analysis. Chapter IV presents the results of the study from a practitioner perspective, providing suggestions for classroom practices to both mathematics teachers and teacher educators. Chapter V provides a discussion of the research study, including a summary and discussion of major findings, implications for teaching and research, limitations and delimitations of the study, and future avenues for research related to metacognition and pre-service mathematics teachers.
CHAPTER II

SHIFTING THE PROBLEM-SOLVING PARADIGM:
RECASTING THE ROLE OF THEORY IN THE
PRACTICE OF PROBLEM SOLVING

Abstract In this theoretical piece, I suggest that addressing non-content skills and habits of mind in the context of mathematical problem solving requires the adoption of a participation metaphor for learning. I discuss methodological advantages of utilizing activity theory to examine students’ participatory activity within a classroom community, specifically documentation of the non-linear development of and attunement to classroom problem-solving norms. Finally, I contend that in addition to research-based implications, activity theory offers practical leverage for interventionist-motivated design research.

There has been a persistent swing in mathematics education, both in research and practice, from a focus on the memorization of static facts, procedures, and tools, to a focus on the dynamic use of these skills. This trend parallels broader societal goals to prepare 21st-century learners for success in an increasingly connected and technology-dependent world, providing students with a global readiness transcending classroom walls. Advances in pedagogy reflecting these motives have helped make overt a more sophisticated notion of what it means to do and be successful in mathematics. Active learning environments and inquiry communities have been designed to foster mathematical problem-solving mindsets, attempting to improve students’ higher-order
thinking skills or habits of mind such as creative and flexible thinking, communication, perseverance, autonomy, and metacognition (see Cuoco, Goldenberg, & Mark, 1996; Costa & Kallick, 2000).

Our current, complex world demands non-routine problem solvers, though the importance of teaching and learning mathematical problem solving, “the mathematician’s main reason for existence” (Halmos, 1980, p. 519), is not new. In his highly influential book, How to Solve It, Pólya (1957) wished to illuminate the experimental and inductive nature of mathematical problem solving by outlining a semi-structured way of thinking, consisting of continued self-questioning during a four-stage process: understand the problem, devise a plan, carry out the plan, and look back. After even a casual reading of Pólya’s work, it becomes clear that sustained monitoring and control of one’s problem-solving attempt (i.e., metacognition) is fundamental to problem-solving success. In mathematics education research, metacognition has long been identified as an essential component of the problem-solving process. Toward developing a mathematical “point of view” (Schoenfeld, 1985), metacognition is a “tool of the trade” with which students must be competent. A 21st-century problem solver equipped to handle non-routine problems must be inclined to employ metacognitive thinking.

Despite evidence of its importance, metacognition has become considerably minimized as an explicit part of problem-solving teaching practices. While “mathematics teachers have been quick to adopt what they understand to be Pólya’s approach to solving problems…few have been able to alter their instruction and recast the curriculum to reflect his challenging pedagogical ideas” (Kilpatrick, 1987, p. 300). Indeed, the problem-solving paradigm still considerably encompasses proceduralized, heavily
scaffolded instruction of problem-solving heuristics, which can be observed in the “problem-solving” section of many textbooks. Often, problem solving is reduced to a series of closed-ended exercises (see Schoenfeld, 1985 for this distinction). Students are told precisely which strategy to employ and then scaffolded (linearly) through each of the four stages (e.g., Bennett, Burton, & Nelson, 2015; Coburn, 2009). Teachers might require students to record relevant quantities, define variables, and check solutions, but frequently quantities and variables are plainly identifiable, answers are in the back of the book, and student-generated strategy or heuristic choice is unnecessary because an appropriate tool has been demonstrated for the student immediately prior to solving the current problem. Just as direct instruction is not sufficient for students to understand mathematical concepts (e.g., Lew et al., 2016), students do not become metacognitive, creative, and flexible thinkers through only the direct instruction of problem-solving heuristics.

Moreover, although educators seek to encourage student autonomy during problem solving, metacognition and other higher-order thinking skills contributing to such autonomy are habitually produced by the teacher and perceived as the teacher’s responsibility. In identifying metacognition as an integral part of problem solving, the NCTM (2000) described the role of the teacher as “helping to enable the development of these reflective habits of mind by asking questions” (p. 54-5). Even if students actively engage with mathematical content, they are not typically given authentic opportunities to participate in metacognitive thinking. In a sincere attempt to streamline the teaching of problem solving, higher-order thinking skills and habits of mind have been inadvertently removed from students’ problem-solving activities. In sum, while successful problem
solving necessitates abilities, like metacognition, beyond the accumulation of facts and memorized procedures, current problem-solving practices do not necessarily reflect these motives.

The main purpose of this paper is to call into question a foundational assumption in much of the research and teaching of mathematical problem-solving, namely an acquisition metaphor for learning (Sfard, 1998). I contend that a shift in perspective is advantageous for addressing the aforementioned difficulties related to metacognition (and other habits of mind) in the practice of problem solving. At the heart of this conversation is a consideration of the role of theory as a mediator between the practice, problems, and research of mathematical problem solving. Employing Silver and Herbst’s (2007) theory-centered scholarship triangle, the remaining discussion is structured into three sections (Figure 10). Sfard’s (1998) framing of existing learning theories in terms of two metaphors – acquisition and participation – leverages the transformation of the practice of teaching and learning metacognition and other problem-solving habits of mind into a researchable problem (arrow (1) in Figure 10). This transformation brings to the fore a need to document the long-term, nonlinear processes of classroom norm development. Activity theory (Engeström, 1987/2015) provides a theoretical lens to bridge the problem of analyzing the negotiation process of classroom norms to research on this problem (arrow (2) in Figure 10). Finally, I expand on the connection between research and practice (arrow (3) in Figure 10), highlighting the additional benefit of activity theory to guide interventionist-based research studies, which can lead to “best practices” for promoting problem-solving habits of mind like metacognition as normative classroom activity.
Can one “Acquire” Habits of Mind?

Sfard (1998) describes two metaphors for learning: learning as acquisition of concepts and learning as participation in a community. With an acquisition metaphor, learning is the amassing of knowledge units, objects able to be possessed by an individual or individuals. Teaching means transmitting or mediating the attainment of this commodity. With a participation metaphor, “the permanence of having gives way to the constant flux of doing” (p. 6). Knowing is equated to belonging to or participating in a “community of practice” (Lave & Wenger, 1991). Knowledge is no longer a static product, and learning is an active, context-dependent process of emerging membership.

Research and teaching related to metacognition has historically adopted the acquisition metaphor, where the use of metacognitive interventions to transmit metacognitive knowledge or skills stems from cognitive information processing theory. Within this context, emphasis is on procedural fluency so that basic facts and procedures do not consume too much working memory and processing speed, allowing individuals the ability to deal with more cognitively demanding, complex mathematical activities.
Just as grade school students might practice basic multiplication facts separately from problem solving, one can practice basic metacognitive skills for use in later problem solving, moving these skills from working to long-term memory. With this perspective of learning, problem-solving habits of mind could be seen as checklists of demonstrable or re-constructible behavior. From experimenting to generalizing to persisting, habits of mind could be treated in the same way as the mathematical constructs upon which they act. For example, just as one can characterize understanding of quadratic functions with a list of static attributes and procedures, one could attempt to characterize habits of mind with inventories of decontextualized actions (Table 2).

Table 2

*Viewing concepts and habits of mind as static, decontextualized objects*

<table>
<thead>
<tr>
<th>Quadratic Function Actions/Skills</th>
<th>Communication Actions/Skills</th>
</tr>
</thead>
<tbody>
<tr>
<td>Represent with a parabolic graph</td>
<td>Use at least one mathematical representation</td>
</tr>
<tr>
<td>Represent with a polynomial of degree two</td>
<td>Show work</td>
</tr>
<tr>
<td>Complete the square</td>
<td>Use correct mathematical notation and terms</td>
</tr>
<tr>
<td>Use the Quadratic Formula</td>
<td>Listen to others and explain your reasoning</td>
</tr>
</tbody>
</table>

Various forms of metacognitive strategy training, including self-reflection modeled by a teacher, metacognitive strategies and prompts delivered through handouts, metacognitive skills presented as heuristics, or a combination of direct instruction and exercise (Baten, Praet, & Desoete, 2017), all comprise acquisitionist language and assumptions in their efforts to transmit skills. Likewise, assessments of metacognitive knowledge and skill are existence proofs. An intervention is successful if learners
evidence particular metacognitive knowledge or skills on measures designed to evaluate their accumulation. This approach has seemed valuable, but students have not benefited from years of such metacognitive research (Carroll, 2008). While the importance of prolonged metacognitive instruction embedded in content matter has been emphasized (Lester et al., 1989; Veenman et al., 2006), most metacognitive research has overlooked the crucial impact of sociocultural contexts and learning environments in the development of metacognition (Larkin, 2015).

By having the acquisition metaphor for learning as its premise, current metacognitive research and instruction has largely overlooked the process characteristics of metacognitive thinking. When problem solving, it is not enough to have the techniques, as “[t]he number of useful, adequately delineated techniques is not numbered in the tens, but in hundreds…The question of selecting which ones to use (and when) becomes a critical issue” (Schoenfeld, 1985, p. 73). Similarly, metacognitive thinking employed by expert problem solvers is not merely the aggregate products of knowledge and strategies, but also the way in which these strategies and knowledge are chosen and used during real-time problem solving. There is a distinction between “knowing about” and “knowing to act in the moment” (Mason & Spence, 1999). Sfard (1998) warns of relying on only one metaphor for learning, and the use of only the acquisition metaphor for habits of mind presents limitations in moving students from possessors of procedures and facts to “genuine” mathematical thinking. Becoming a skillful problem solver means coming to know the nuanced patterns of thinking of skillful problem solvers during authentic problem-solving situations.
To this end, including a participation metaphor is advantageous. In this perspective, becoming a skillful problem solver means both “communicat[ing] in the language of the community and act[ing] according to its particular norms” (Sfard, 1998, p. 6, emphasis added). Problem-solving habits of mind can be viewed as normative ways of thinking or acting within the “skilled problem solver” community of practice (ways of doing instead of having). Through legitimate peripheral participation (Lave & Wenger, 1991), students become attuned to these normative, habitual tendencies or dispositions, eventually transforming their own habits of mind as they become full participants in this community (i.e., skilled problem solvers).

Teaching problem-solving habits of mind necessitates providing students with opportunities to authentically practice using these habits of mind. This requires understanding the process of student participation, with attention to the contexts that afford or constrain such dispositional transformations toward full participation. As such, viewing habits of mind through a participation lens means careful documentation of the emergence of normative classroom problem-solving activity, attending to both student and teacher contributions in this negotiation. Thus, the practice of teaching habits of mind such as metacognition is transformed into the important, researchable problem of documenting the development of and attunement to classroom problem-solving norms.

**Documenting the Development of and Attunement to Classroom Problem-Solving Norms**

While attention to context exists within both aforementioned metaphors for learning, attunement to community norms is a notion stemming exclusively from the participation metaphor. Thus, to document problem-solving habits of minds as community norms to which students become (or do not become) attuned, a framework is
needed to describe and examine the process of participatory activity of students within a classroom community. Moreover, while teachers may intend for students to develop “sophisticated” ways of thinking, as a classroom community becomes consolidated the activity that becomes normative may or may not be in the image of the community for which the teacher is the cultural representative. Thus, any framework or theory used in the context of norm development needs to account for these possibilities. I propose that activity theory (Engeström, 1987/2015; Leont’ev, 1979) provides both a theoretical foundation and analytic framework conducive to these objectives.

**Overview of Activity Theory**

Activity theory is based on Vygotsky’s (1978, 1986) notions of semiotic mediation and the reflexive influence of the individual and the community in which the individual resides. As seen in Engeström’s model (Figure 11), activity theory accounts for the complex interaction between the individual and community by expanding Vygotsky’s notion of mediated activity to include additional social mediators. Individuals or groups of individuals form an activity system. The six components describing human activity systems are defined as follows (Engeström, 1993):

- **Subject**: The individual or subgroup whose agency is chosen as the point of view in the analysis.

- **Object**: The “raw material” or “problem space” at which the activity is directed and which is molded or transformed into outcomes with the help of physical and symbolic, external and internal tools.

- **Instruments**: The tools and signs used to mediate activity.

- **Community**: Multiple individuals and/or subgroups who share the same general
object.

**Division of Labor:** Both the horizontal division of tasks between the members of the community and the vertical division of power and status.

**Rules:** The explicit and implicit regulations, norms and conventions that constrain actions and interactions within the activity system. (p. 67)

The bidirectional arrows in Figure 11 indicate the transformative, mediating relationship between components. Each element of the system can change and be changed by other elements.

*Figure 11.* Vygotsky’s mediated activity embedded within Engeström’s (1987/2015) expanded activity triangle.

Summarized here are three key principles of activity theory. First, there exists a structured hierarchy of object-oriented activity (Leont’ev, 1979). Human *activity* is catalyzed and propelled forward by a sociohistorical, overarching *motive* (activity $\leftrightarrow$ motive). An activity system is created by the participants within a given setting who engage in *actions* subject to conscious *goals* that can be feasibly accomplished (actions $\leftrightarrow$ goals). These actions are composed of working *operations,*
which are constrained by given *conditions* (operations\(\leftrightarrow\)conditions). Elements in each level transform fluidly between levels. For example, motives that become conscious and operationalized become goals.

Second, no individual or action is viewed in isolation. The unit of analysis is the collective activity system: “We may well speak of the activity of the individual, but never the *individual activity;* only actions are individual” (Engström, 1987/2015, p. 54).

Defining the six components of the activity system and identifying a hierarchy of activity are only part of analysis. The activity system is in constant flux, rather than the aggregate of discrete parts. As such, the true objective of analysis “is always to grasp the systematic whole, not just separate connections” (p. 62).

Third, an activity system dynamically transforms, expanding or changing qualitatively over (relatively long periods of) time through adaptation to contradictions or tensions. Engeström (1987/2015) delineates four types of contradictions:

1. Within each component of the activity system: human activity has dual nature, subject to the dichotomy of specific versus general. Products are simultaneously “independent of and subordinated to” (p. 66) society;
2. Between components of the activity system;
3. Between the object/motive pair of the central activity system and the object/motive pair of a “culturally more advanced form” (p. 71) of the central activity system introduced by a cultural representative (e.g. a teacher); and
4. Between the central activity system and another neighboring activity system: a neighboring system may be connected to the central activity system for a period of time in one of the following four ways, with conflicts appearing as a result of
these associations: sharing the same object activity, producing instruments used by the central activity system, shaping the subjects of the central activity system (e.g. school), or producing rules that govern the central activity system.

Activity theory is gaining recognition (Roth, 2004), but its mainstream use in mathematics education is still relatively minimal. In the context of documenting the process of classroom norm development, I contend there are, at minimum, four ways activity theory is advantageous in this endeavor.

**Operationalization**

Activity theory operationalizes the participation structure of a classroom community for detailed, systematic investigation. Two levels of analysis can be used to account for sociocultural complexity of an activity system: a micro-analysis of language-mediated discourse (the upper boxed portion of the Activity theory triangle in Figure 11), followed by macro-analysis using Engeström’s expanded activity triangle to highlight tensions within the activity system (Jaworski & Potari, 2009). Micro-analysis serves to describe the language-based interactions between student and teacher cognitions, while macro-analysis subsumes the micro-scale classroom interactions, accounting for the influence of cultural factors that mediate learning. Jonassen and Rohrer-Murphy (1999) further delineate six steps that can guide macro-level analysis of the activity system (Table 3), where contradictions and tensions are identified in Step 6. With respect to habits of mind, this means documenting both the interpretation and function of a particular habit of mind on a micro-level, as well as the broader sociohistorical features that give meaning to possible interpretations and functions of the habit of mind.
Table 3

*Six Steps for Analyzing an Activity System (Jonassen & Rohrer-Murphy, 1999)*

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Clarify the purpose of the activity system. Describe the motives and conscious goals of the activity system.</td>
</tr>
<tr>
<td>2</td>
<td>Analyze the activity system. Define the subject, object, community, division of labor, and rules.</td>
</tr>
<tr>
<td>3</td>
<td>Analyze the activity structure. Delineate the hierarchy of activity, concrete actions, and automatized operations.</td>
</tr>
<tr>
<td>4</td>
<td>Analyze tools and mediators. Describe the tools, rules, and roles of participants that mediate activity within the system.</td>
</tr>
<tr>
<td>5</td>
<td>Analyze the context. Characterize the internal, subject-driven and external, community driven contextual bounds.</td>
</tr>
<tr>
<td>6</td>
<td>Analyze activity system dynamics. Step back from the delineated activity system to describe and assess how components affect each other.</td>
</tr>
</tbody>
</table>

**Reflexivity**

In helping students become attuned to habits of mind as problem-solving norms, one must consider that the natural, purposeful activity within a classroom community of practice creates a microculture of negotiated activities and interactions among students and the teacher (Lave & Wenger, 1991). Over time, normative behavior emerges, but the interpretation and function of these norms also change through iterations of negotiation. Not only does the classroom collective influence individual students, but individual students can modify tools and signs, potentially influencing others in the larger classroom community. This reflexivity of sign (and tool) use can be captured and documented using Ernest’s (2010) model, which represents Vygotsky’s developmental theory “in a cycle of
appropriation, transformation, publication, conventionalization” (p. 44) (Figure 12).

Activity theory provides explicit language with which to document this nuanced, reflexive interaction over time. For example, the unit of analysis could be the student activity system which changes over time as a result of interactions with the teacher activity system or by Type III contradictions introduced by the instructor.

![Figure 12. Cycle of appropriation, transformation, publication, and conventionalization (Ernest, 2010, p. 44).](image)

**Expansion and Horizontal Expansion**

The reflexive negotiation of problem-solving norms implies that norms are not pre-established, unchanging concepts. While the teacher has a particular understanding of habits of mind and aspires for students to develop these habits, students are coming to know ways of thinking that, for them, do not yet exist. How, then, do students expand their ways of thinking and become attuned to problem-solving habits of mind? At its core, activity theory is a theory of expansive transformation. Through adaptation to contradictions, expansive transformation “is accomplished when the object and motive of the activity are reconceptualized to embrace a radically wider horizon of possibilities than in the previous mode of the activity” (Engeström, 2001, p. 137). In other words, students’ increased participation in the mathematical problem-solving community of practice can
be viewed as an expansion of their system of activity. By providing opportunities for students to reconceptualize their problem-solving activity to include attention to habits of mind, and by making the improvement of habits of mind an explicit object of activity, students are more likely to embrace these habits of mind. When developed, the norms within a classroom community become the implicit rules within students’ expanded activity system. Eventually, students (subjects) are eventually transformed themselves, expanding their beliefs about what it means to do and come to know mathematical problem solving, developing productive problem-solving dispositions and habits of mind.

In addition to opportunities for student expansion, the reflexive negotiation of classroom problem-solving norms implies that the interpretation and function of a particular habit of mind may develop in a way so that the resulting normative activity is not necessarily identical to the norm intended by the teacher. Students may not become attuned to a particular habit of mind, students may become attuned to nonproductive interpretations of a habit of mind, or there may be multiple, equally viable ways to interpret and employ a habit of mind. Students learning and development is not necessarily vertical, from “lower” to “higher” levels of competence. Activity theory’s integral notion of expansion can capture this horizontal, “sideways” learning and development (Engeström, 2001). Instead of viewing the development of problem-solving habits of mind as teachers “transmitting” culture, “expansive learning increasingly involves horizontal widening of collective expertise by means of debating, negotiating and hybridizing different perspectives and conceptualizations” (Engeström, 2000, p. 960). Students’ perspectives and conceptualizations of habits of mind are just as integral in their development as the teacher’s.
Process-Focused Interaction of Social Activity and Individual Actions

Citing Lerman (2001) that an “integrated account, one that brings the macro and micro together,” (p. 89) is necessary to situate intersubjectivity between individuals, Jaworski and Potari (2009) suggest that activity theory can evade the dichotomy between social and individual. Activity theory includes a “middle level analysis” (p. 221) to illuminate the “hidden curriculum” (Engeström, 1998) at the boundary of activity systems/structures and everyday classroom actions/practices (i.e. rules, community, and division of labor). With respect to the development of classroom norms, this hidden curriculum becomes essential. When viewing the negotiation of and attunement to norms through a participation lens, focus is precisely on the transformational process of reflexive interaction between covert social activity affecting participation and individual actions of participation. This process-focused integration of social activity and individual actions provides the context for both the reflexivity and horizontal expansion aspects of norm development, which are both “process” conceptions.

With this process focus in mind, it should be noted that activity theory is not the only possible method to account for the social dimension when considering norm development in general and habits of mind in particular. For example, Lim and Selden (2009) have suggested using the Emergent Perspective to investigate habits of mind and the most recent, expanded Emergent Perspective framework includes “disciplinary practices” (a similar construct to habits of mind) as an object of investigation (Rasmussen, Wawro, & Zandieh, 2015). Cobb and Yackel’s (1996) Emergent Perspective has been a dominant framework for documenting mathematically-based classroom
norms, using a combination of cognitive constructivist and interactionist analysis methods to coordinate the individual and collective. Both the (expanded) Emergent Perspective and activity theory recognize the need for simultaneous framing of classroom products as individual and social, use similar constructs to characterize these dimensions, and are pragmatically motivated by tenets of design research. Both frameworks include normative, collective activity as a central element.

With respect to norm development, the difference between the two frameworks is a dichotomy between product and process objectives. For the Emergent Perspective, stability is assumed (Roth, 2016). Researchers document progress toward this structure, cataloguing the intersubjective products that arise. This approach limits the ability of the Emergent Perspective to capture the “dynamic nature of learning, a continuing change process” (p. 88). Indeed, coordinating the individual and collective products (a synthesis reaction) does not illuminate the continuous transformation of norms through reflexive interaction between individual and collective. The process of norm negotiation is not revertible from the product. Norms are in constant flux, rather than the aggregate of discrete parts. Thus, activity theory, whose true objective of analysis “is always to grasp the systematic whole” (Engeström, 1987/2015, p. 62), is preferred for documenting the process through which the interpretation and function of norms, such as problem-solving habits of mind, are negotiated.

Harnessing the Power of Contradictions as Catalysts for Change

A fifth advantage of activity theory, the power of contradictions or tensions, brings this discussion full circle within the theory-centered scholarship triangle (Silver & Herbst, 2007). At the beginning of this paper, the practice of teaching metacognition
during problem solving, as well as other habits of mind related to problem solving, was transformed into a researchable problem by framing the learning of metacognition (or other problem-solving habits of mind) as participation in a mathematical problem-solving community of practice. Viewing habits of mind as desired normative activity means acknowledging that classroom norms are negotiated among classroom participants, and the reality of this negotiation may not match its ideal preconceived form (e.g., Levenson et al., 2009). This necessitates an appropriate framework to document the nonlinear development of classroom problem-solving norms. Activity theory can answer this call by offering concrete language with which to document the participatory unit, providing a framing for the reflexive negotiation of norms and the potential for students to “horizontally” develop ways of thinking, and integrating social activity and individual actions through its focus on the process of their interaction over time. Thus far, these benefits have portrayed activity theory as a useful descriptive tool, but not necessarily prescriptive. Characterizing system dynamics through contradictions and tensions is a powerful means for interventionist-motivated design research in search for best practices in teaching habits of mind.

Consider the following example of research utilizing activity theory to foster metacognition as a habit of mind during problem solving. In a recent research study (Hancock, YEAR), the instructor of a first-year content course for pre-service elementary teachers wanted to promote students’ metacognitive thinking during problem solving as a habit of mind by implementing “portfolio” problems and write-ups as an intervention to make explicit students’ judgement and decision-making processes. The instructor’s intent was to create a purposeful contradiction between the object/motive pair (problem
solving) of the student activity system and the object/motive pair of a culturally more advanced form (the teacher). Students were forced to adapt to non-routine problem-solving sessions followed by individual write-ups documenting their thinking. The portfolio problems and write-ups afforded a semester-long negotiation of what it meant to think about thinking during problem solving. By the end of the semester, students’ normative interpretation of metacognition shifted from retroactively checking final answers (e.g., “Did you get 2 miles?”), to actively thinking about strategy and representation choice throughout the problem-solving process (e.g., “Would it be better to solve this problem with a picture?”). But it was not just the isolated introduction of this mediating instrument that led to the aforementioned change. Social mediators of activity, like the grading mechanisms (explicit rules) attached to the portfolio problems, affected the form and use of the mediating tool, which subsequently influenced students’ constraints, affordances, and attunements in coming to know a “culturally more advanced” notion of metacognitive thinking.

Students bring with them preconceptions concerning the nature of mathematical problem solving and their role in this process. Teachers, as representatives of the larger mathematical community, seek to increase legitimate student participation in this community of practice, often through the development of problem-solving habits of mind. As such, formulating best practices for teaching metacognitive thinking cannot be conceptualized as only adding isolated portfolio problems (or other tools) into the sphere of students’ daily actions. Documenting the process of transforming, expanding activity systems over the course of the semester becomes crucial to understand how students overcome (or do not overcome) the catalyzing contradictions. Generating best practices is
only possible by understanding the context influencing transformation and the (advantageous or constraining) adaptations to the contradiction created by the introduced tool. Emphasizing this process to produce ecologically valid studies requires attention to social mediators of activity and longitudinal documentation of norms.
CHAPTER III

DEVELOPING THE METACOGNITIVE HABITS
OF PRE-SERVICE TEACHERS DURING
PROBLEM SOLVING

Abstract Metacognition has long been identified as an essential component of the problem-solving process and is a habit of mind of mathematical thinkers and problem solvers. Mathematics teachers must develop this habit of mind to provide such experiences for students, therefore mathematics courses designed for pre-service teachers should foster metacognitive thinking employed in all parts of the problem-solving process. Metacognitive actions can be viewed as normative ways of thinking to which students become attuned by participating in authentic problem-solving situations. This study explored one such situation, where “portfolio” problems were used to mediate metacognitive thinking in a first-year mathematics content course for pre-service elementary teachers. Analysis utilized activity theory to operationalize the participation structure of a classroom and document the nonlinear development of metacognitive norms during problem solving. Micro-analysis revealed a shift from product- to process-oriented metacognitive actions. Macro-analysis situated these results, highlighting social mediators of activity and contradictions as catalysts for change.

Keywords Problem solving · Metacognition · Habits of mind · Activity theory · Participation metaphor for learning · Pre-service teachers
**Introduction**

Mathematical habits of mind have been well-established as an important component of mathematical understanding (CCSSI, 2010; National Council of Teachers of Mathematics (NCTM), 2000) and mathematically powerful classrooms (Schoenfeld, 2014). Habits of mind are present at all levels of mathematical problem solving, including elementary grades (Goldenberg, Shteingold, & Feurzeig, 2003) and secondary grades (Cuoco & Levasseur, 2003). Costa and Kallick (2000) defined habits of mind generally as “dispositions displayed…in response to problems, dilemmas, and enigmas, the resolutions of which are not immediately apparent” (p. xvii). Within the context of mathematics, Cuoco et al. (1996) described habits of mind as the mental habits of skilled mathematical problem solvers, such as using examples, generalizing, taking multiple points of view, and mixing deduction and experiment. Others have similarly described the practices employed by mathematicians when solving problems (e.g., *disciplinary practices* (Rasmussen et al., 2015)). Lim and Selden (2009) highlight the habitual characteristic of habits of mind as related to what Mason and Spence (1999) consider ways of *knowing-to-act in the moment*.

Proficiency with mathematical habits of mind is especially important for mathematics teachers. Watson and Barton (2011) discuss a capacity for teachers to enact *modes of enquiry* in the classroom. Similar to Stockero and Van Zoest’s (2013) conception of a *pivotal teaching moment*, this component of Mathematical Knowledge for Teaching (Ball, Thames, & Phelps, 2008) requires teachers to know-to-act in the moment, “‘sparking off’ a student comment to make a wider point about mathematics and extend the students’ thinking when the moment was ripe” (p. 68). Watson and Barton
(2011) argue that modes of enquiry enacted in mathematics teaching are often limited “by teachers’ own mathematical experiences and by the ways they were taught” (p. 80). As such, there is a need to explicitly include mathematical habits of mind in the mathematical experiences of pre-service teachers. To provide these opportunities for students, not only do pre-service teachers need to develop mathematical habits of mind themselves, but they should value their usefulness (Oesterle et al., 2016). The Conference Board of Mathematical Sciences (CBMS) recommends all pre-service and in-service mathematics teachers have experiences to develop the “habits of mind of a mathematical thinker and problem solver” (2012, p. 19).

Metacognition is one mathematical problem-solving habit of mind (Selden & Lim, 2010; Stacey et al., 1982), broadly defined as thinking about one’s thinking or awareness of cognitive activity. Metacognition consists of monitoring and regulation (control) (Flavell, 1979) which operate simultaneously (Nelson & Narens, 1990). To adequately monitor and control one’s problem solving, an individual must have sufficient metacognitive knowledge and skills to actively use the knowledge. Metacognition is not merely the aggregate product of metacognitive knowledge and strategies, but also the process by which these strategies and knowledge are chosen and employed (Van Overschelde, 2008). For skilled mathematical problem solvers, metacognitive actions are chosen and employed in all stages of the problem-solving process (Carlson & Bloom, 2005; Pólya, 1957; Schoenfeld, 2011). In sum, metacognition is a fundamental mathematical habit of mind with which students, and thus pre-service teachers, must become proficient.
There is a significant, growing body of literature concerning the improvement of students’ metacognition during problem solving. Previous research on metacognition has helped identify and define metacognition components, grapple with the assessment of metacognition, correlate metacognitive skills with performance and psychological concepts such as motivation, and identify the importance of metacognitive strategy training (see Baten et al., 2017 for a detailed overview). Stemming from cognitive information processing theory, most of the original metacognition research was laboratory-based. Carroll (2008) argued that students were “on the whole, not benefit[ing] from at least 20 years of [this type of] metacognitive research in cognitive psychology” (p. 411), and that the object of research on metacognition should be the “natural” setting of classroom practice. In classroom settings, there is a large body of intervention research with teachers providing explicit metacognitive training through direct instruction. There also exists research concerning the metacognitive influence of peers during group work (e.g., Vorhölter, 2018). However, these settings often do not capture metacognitive development within “natural” classroom settings.

Viewing metacognition as a habit of mind necessitates an investigation of students’ habitual metacognitive tendencies (norms) and how these tendencies change over time. Developing proficiency with the metacognitive actions of skilled mathematical problem solvers (i.e., using metacognition as a mathematical habit of mind) is a long process of coming to know-to-act metacognitively in the moment. Moreover, the in-the-moment aspects of metacognitive monitoring and control are goal-oriented and subject to both internal and external constraints (Van Overschelde, 2008). Most of the aforementioned studies have been grounded in an ‘experimental paradigm’ that ignores
the sociocultural factors mediating an individual’s (or small group’s) metacognitive development (Larkin, 2015). As such, they are limited in characterizing the long-term process of shifts in normative metacognitive classroom activity and how this process is affected by sociocultural context.

The purpose of this study is to investigate the development of students’ habitual metacognitive tendencies. Rather than either isolating the teacher’s instruction or students’ small-group problem solving, this research aims to provide a detailed account of the semester-long negotiation between teacher and students as they participate in an authentic, “natural” classroom activity. To understand the process of student participation in a mathematical problem-solving community of practice, the research questions for this study are as follows:

Q1 What metacognitive actions taken by pre-service elementary teachers in authentic problem-solving situations are normative?

Q2 How do system tensions catalyze the development of these metacognitive norms?

Q3 How does the teacher influence the development of these metacognitive norms?

Theoretical Perspective and Research Questions

This study views learning mathematical problem-solving habits of mind as legitimate peripheral participation in a community of practice (Lave & Wenger, 1991). Sfard (1998) distinguishes this participation metaphor for learning from an acquisition metaphor for learning, where “the permanence of having [acquiring] gives way to the constant flux of doing” (p. 6). Through this lens, learning to become a full member of the mathematical problem-solving community of practice (i.e., learning to “think mathematically”) means both “communicat[ing] in the language of the community and
act[ing] according to its particular norms” (p. 6). Through participation in authentic problem-solving situations, students become attuned to mathematical problem-solving habits of mind, eventually transforming their own habits of mind as they become full participants in this community.

As students come to know the language and norms of the larger mathematical community for which the teacher is a cultural representative, the natural, purposeful activity within the classroom creates a microculture of negotiated activities and interactions among students and the teacher (Lave & Wenger, 1991). Ernest (2010) emphasizes the reflexive nature of language-based negotiation. The activity that emerges is a result of an expansive process of transformation and creation, as opposed to transmission of culture. Engeström and Sannino (2010) argue for an “expansion” metaphor for learning as an extension of the participation metaphor to emphasize non-vertical, horizontal movement and hybridization features of this process.

Activity theory provides language to operationalize the participation metaphor for learning (Barab, Evans, & Baek, 2004) and was used as an analytic framework in this study. Activity theory expands Vygotsky’s (1978) conception of mediated activity to include additional social mediators (Engeström, 1987/2015) (Figure 13). The unit of analysis is collective, goal-motivated, and object-oriented activity (Leont’ev, 1979). Sociohistorical, overarching motives guide activity (activity → motive). Motives for which the subject becomes consciously aware are goals that can be achieved through concrete actions (actions → goals). Actions are comprised of operations constrained by conditions (operations → conditions). In this study, the subject of the central activity system was the collective student-group of pre-service elementary teachers in a first-year
mathematics content course (referred to by the term “students”). Students’ problem solving, the object of the system to which activity is directed, was mediated by instruments such as the instructor and classroom activities. Additionally, students’ problem solving was influenced by other sociocultural factors: implicit and explicit rules (e.g., grading systems, social and sociomathematical norms), the communities to which the students belong (e.g., future mathematics teachers), and the division of labor that determines how actions are taken in the system (e.g., small-group work).

Figure 13. Vygotsky’s mediated activity embedded within the expanded activity triangle.

While identifying components of the system and delineating the hierarchy of activity are important steps in analyzing an activity system, the focus of analysis is on “the systematic whole, not just separate connections” (Engeström, 1987/2015, p. 62) to characterize system dynamics. The activity system changes over time through adaptations to tensions: (1) within components, (2) between components, (3) between the object/motive pair of the central activity system and the object/motive pair of a “culturally more advanced form” (p. 71) of the central activity system introduced by a
cultural representative (e.g., a teacher), and (4) between two activity systems. The use of activity theory to guide data analysis is provided in the methods section.

**Data Sources and Context**

Number Sense and Algebra is a one-semester, 15-week course for pre-service elementary teachers with mathematics emphasis at a mid-size public university in the Rocky Mountain region of the United States. Students met twice a week for 75-minutes, and most of the 24 students in the course were first-time freshman who had taken differential calculus in high school. Course topics include:

- properties of natural numbers, counting, and place value (Unit 1),
- the meaning and interpretation of addition, subtraction, multiplication, and division for both integers (Unit 2) and fractions and decimals (Unit 4),
- factors, multiples, prime factorization, greatest common factor, and least common multiple (Unit 3),
- expressions, equations, and solving equations (Unit 5), and
- ratio, proportion, and functions (Unit 6).

**Data Sources**

This qualitative work assumed a naturalistic approach to inquiry (Lincoln & Guba, 1985). Six qualitative data sources were collected and are summarized in Table 4. Details of micro- and macro-level analyses are outlined in the Analysis and Results section. During interviews one and two, students were asked questions targeting their beliefs about mathematics, mathematical problem solving, and perceptions of and expectations for the course. Students then worked, thinking aloud, on non-routine problems to provide a reference point for students to discuss their problem-solving
activity more generally and compare their process with their typical problem solving in
class, the problem-solving activity of Dr. Arkadash, and that of other students. The final
interview was a series of questions asking students to reflect on their experiences in the
course.

Classroom Context

Class time typically began with a warm-up problem, followed by a mixture of
small-group work and active, whole-class discussion related to activities guiding the
day’s discussions. It was important for the instructor, Dr. Arkadash, that students did not
look to her as the sole authority in the classroom. As she facilitated classroom activities,
she pushed student thinking forward through probing and clarifying questions and
encouraged other students to ask similar questions. She utilized student presentations for
students to share their own thinking, and as students became comfortable with this format
they began offering to present and explain their thinking without being prompted. Dr.
Arkadash expected all voices to be heard and purposefully sought input from all students.
Small groups were shuffled often to maximize collaborative efforts and provide students
opportunities to share with and listen to ideas from many different individuals.
Table 4

Summary of data sources

<table>
<thead>
<tr>
<th>Data Source</th>
<th>Time Collected</th>
<th>Purpose</th>
</tr>
</thead>
<tbody>
<tr>
<td>Video- and audio-recorded portfolio problem sessions</td>
<td>In-class portfolio sessions, roughly once a week with two sessions for each of the six portfolio problems (The first problem had one session.)</td>
<td>Used in micro-level analysis to identify normative metacognitive actions in authentic problem-solving situations</td>
</tr>
<tr>
<td>Video- and audio-recorded classroom sessions</td>
<td>Daily</td>
<td>Used in macro-level analysis to identify student and teacher goal-directed actions</td>
</tr>
<tr>
<td>Video-recorded, semi-structured individual interviews with 15(^3) of 24 students (Questions provided in Appendix B)</td>
<td>Weeks 2, 8, and 15</td>
<td>Used in macro-level analysis to identify motives/goals and beliefs/preferences</td>
</tr>
<tr>
<td>Students’ written artifacts (portfolio-problem write-ups)</td>
<td>Collected before grading</td>
<td>Used in macro-level analysis to triangulate students’ portfolio problem-solving activity</td>
</tr>
<tr>
<td>Audio-recorded interviews with Dr. Arkadash (Questions provided in Appendix C)</td>
<td>Weeks 1 and 15</td>
<td>Used in macro-level analysis to identify motives/goals and beliefs/preferences</td>
</tr>
<tr>
<td>Recorded planning sessions</td>
<td>Weekly</td>
<td>Used in macro-level analysis to identify changing goals</td>
</tr>
<tr>
<td>Journal reflections from Dr. Arkadash and myself</td>
<td>Daily</td>
<td>Used in macro-level analysis to identify Dr. Arkadash’s changing goals and my memos for noteworthy classroom episodes</td>
</tr>
</tbody>
</table>

\(^3\) 13 of the 15 students completed all three interviews.
The course was designed to focus primarily on improving students’ mathematical content knowledge, but Dr. Arkadash believed in teaching content through problem solving (Stein, Boaler, & Silver, 2003) and aimed to also help pre-service teachers develop as mathematical problem solvers. She believed mathematics is more about problem solving than building a toolkit and told the pre-service teachers they needed to experience authentic mathematical problem solving so they could provide these opportunities for their future students. To help students develop the skills of mathematical thinkers and problem solvers, Dr. Arkadash introduced six “portfolio problems” and write-ups (adapted from Omar, Karakok, & Savic, YEAR). About once a week, students worked together in class on a problem typically more open-ended than usual course work with key ideas related to current content (Figure 14). These problems were chosen to maximize the likelihood they were, in fact, problematic for students, aligning as much as possible with the NCTM’s (2010) worthwhile-problem criteria.
Portfolio Problem 2: (Driscoll, 1999) Take a three-digit number, reverse its digits, subtract the smaller from the larger. Reverse the digits of the result and add it to the original result. For example,

- 123 becomes 321, and 321 − 123 = 198
- 198 becomes 891, and 198 + 891 = 1089

Try this process with several numbers. What do you observe? Why?

Portfolio Problem 3: (Liljedahl, Chernoff, & Zazkis 2007) A pentomino is a shape that is created by joining five squares such that every square touches at least one other square along a full edge. There are 12 such shapes, named for the letters they most clearly resemble.

Now consider a 100’s chart! If a pentomino is placed somewhere on a 100’s chart, will the sum of the numbers be divisible by 5? If not, what will the remainder be? Explain how you can know “quickly”!

Portfolio Problem 5: (Dorichenko, 2011) At sunrise, two old women started to walk towards each other. One started from point A and went towards point B while the other started at B and went towards A. They met at noon but did not stop; each one continued to walk maintaining her speed and direction. The first woman came to the point B at 4:00 pm, and the other one came to point A at 9:00 pm. At what time did the sun rise that day?

Figure 14. Portfolio problems two, three, and five.

In class, students worked together and recorded scratch work with different colored pens to identify individual contributions. Scratch work was emailed to group members to reference as they continued working on the problem outside of class. Groups came back together once more in class to continue working on the problem and generate
more scratch work. Students then wrote and submitted a revised solution at least three pages in length. This final draft could include further investigation of a conjecture from class work, or exploration of new ideas or conjectures. In addition to an explanation and justification of the conjecture, students were asked to provide an explanation of their judgement and decision-making process throughout their problem-solving attempt. For example, write-ups included questions students asked themselves and how they attempted to make sense of them (Figure 15) or comments about strategy choice, such as Kerri’s in Portfolio Problem Three:

*I thought it would benefit me to understand the Pentominos with no remainder first and then use that understanding to explain the sideways U’s with remainders.*

Figure 15. A portion of Alexis’ write-up for portfolio problem three.
Analysis and Results

As suggested by Jaworski and Potari (2009), two levels of analysis were employed to characterize sociocultural complexity surrounding the collective student problem-solving activity (referred to by the term “students”). Micro-level analysis of language-mediated discourse (the upper boxed portion of the activity triangle in Figure 13) addressed research question one (Q1) by identifying normative metacognitive actions present during the in-class portfolio problem-solving sessions (i.e., authentic problem-solving situations). Macro-level analysis using activity theory as an analytic framework situated micro-analysis results, addressing research questions two (Q2) and three (Q3).

Micro-Level Activity

Recalling that a participation metaphor for learning operationalized by activity theory was adopted in this study, the focus of micro-level analysis was on students’ real-time actions (doing as opposed to having). Thus, while student write-ups of their problem solving during portfolio problem-solving sessions were collected, only students’ real-time actions during in-class portfolio problem-solving sessions were used to evidence students’ normative metacognitive activity.

Rasmussen et al. (2015) describe normative activity as that which is taken “as if it is a mathematical truth in the classroom…as if everyone has similar understandings, even though individual differences in understanding may exist” (p. 262-3). Micro-level analysis of the in-class portfolio problem-solving sessions began with a list of metacognitive actions during the problem-solving cycle taken from Carlson and Bloom (2005). This list provided an organizing framework or “start list” for an initially deductive approach to coding (Miles & Huberman, 1984) the in-class portfolio problem-
solving sessions. As the “prefigured” codes were applied to the data, they were reduced, combined, and revised to avoid restricting analysis and better reflect students’ actions\(^4\) (Creswell, 2013). This process resulted in a final list of six metacognitive actions relevant to the data set (Table 5). In-class portfolio problem-solving sessions were re-coded using this list\(^5\). Looking across data revealed a shift in function of metacognitive thinking\(^6\), from retroactively assessing final answers for correctness (MA4), to a proactive focus on evaluating the problem-solving process in earlier phases of the problem-solving cycle, especially the consideration of various solution approaches and strategies (MA2).

This contrast is illustrated by comparing a summary of student problem solving from Portfolio Problem-Solving Session 2 (PPS 2, Days 4 and 5) with problem solving from Portfolio Problem-Solving Session 5 (PPS 5, Days 24 and 25). The problems for each of these episodes were provided in the ‘Introduction’ section. The following transcript excerpts, taken from Lance, Kerri, and Paula’s discussions over two in-class problem-solving sessions, illustrate the archetypal problem-solving of the collective student activity system during PPS 2. Students immediately divided-and-conquered, trying the procedure outlined in the problem statement with different three-digit numbers and searching for patterns.

\(^4\) See Appendix D for memos and edits to the “start list”.
\(^5\) Sample coding available in Appendix D.
\(^6\) See Appendix D for summary of counts for each metacognitive action.
Table 5

Metacognitive actions identified during portfolio problem-solving sessions

<table>
<thead>
<tr>
<th>Metacognitive Action</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>MA 1</td>
<td>Mathematical concepts, knowledge, tools, and facts are assessed and considered</td>
</tr>
<tr>
<td></td>
<td>Data Example: <em>So, does that mean it's a quadratic relationship?</em></td>
</tr>
<tr>
<td>MA 2</td>
<td>Various solution approaches or strategies are assessed and considered</td>
</tr>
<tr>
<td></td>
<td>Data Example: <em>I wonder if there’s a way we could work backwards.</em></td>
</tr>
<tr>
<td>MA 3</td>
<td>Validity/reasonableness of solution process is assessed/considered/tested</td>
</tr>
<tr>
<td></td>
<td>Data Example: <em>I know you can plug in the numbers, but is there a reason why that works or why you found that, besides just plugging in the numbers?</em></td>
</tr>
<tr>
<td>MA 4</td>
<td>Results (answers) are assessed/tested/considered for their reasonableness-validity</td>
</tr>
<tr>
<td></td>
<td>Data Example: <em>But we don’t know for sure sunrise is at 6.</em></td>
</tr>
<tr>
<td>MA 5</td>
<td>Reflects on the efficiency and effectiveness of cognitive activities</td>
</tr>
<tr>
<td></td>
<td>Data Example: <em>I feel like it should be harder than this, you know?</em></td>
</tr>
<tr>
<td>MA 6</td>
<td>Manages emotional responses to problem-solving situation</td>
</tr>
<tr>
<td></td>
<td>Data Example: <em>Ok we can just get up and walk away, take a break.</em></td>
</tr>
</tbody>
</table>

Note: MA – Metacognitive Action
Lance: Complicated.
Kerri: Yeah, but I think there’s gonna be some bizarre pattern that shows up.
Kerri: OK. I’m gonna try the number 145. Everybody pick a number and then we’ll come back together again.
:
Paula: So we all want different numbers?
Lance: I’m doing 999.
Kerri: OK. Tell me what you get.
Lance: I got 0.
Kerri: That’s weird.
Paula: Well I think it depends on the number you choose.
Lance: 999 minus 999 is 0. Flip 0. 0 minus 0… I’ll try 542.

The problem-solving strategy, using guess-and-check to search for patterns, guided students to a list of rules.

Kerri: Yeah, so it’s either 198 or 1089 were the two? Is that right?
Lance: So, 198 or 1089, and 0.
Kerri: Which one ended up to 0?
Lance: Any palindrome. So, 999 and 878 [Pause.] And we were just working on the difference between getting 198 and 1089. And I think it has to do with if the first number is within 1 of the second, er, the third number. If a is within 1 of c. I think that’s when you get the 198.
Kerri: So, [writing and talking] ‘a’ plus or minus 1 is equal to ‘c’, then – what do you get then? [Figure 16]
Lance: Then I think that’s when you get 198. But I have to investigate it a little bit. Cause this is 198, this is not. This is 198 and it’s the opposite. That’s what I think.
Kerri: Hm. Very good. Then 198. [writing and talking] If ‘a’ plus or minus 1 does not equal ‘c’ then it’s 1089 [Pause.] If ‘a’ is equal to ‘c’ [Stops talking.]
Lance: Perfect. We have our three rules.
:
Kerri: **I’m gonna look at 315 again.** I’m glad we figured – that’s really good. Those are our rules. Rules! We like rules! Rules are fun. Yay!
Like other students in the class, Kerri employed MA4 (bolded), testing the reasonableness of her rules with more examples. Students’ exact rules to get 1089, 198, or 0 varied, but all groups convinced themselves their rules were true by repeatedly testing them with more examples (i.e., repeatedly utilizing MA4). Students were satisfied with their rules as a ‘final’ answer to the problem and did not question their use of guess-and-check as the only strategy to support their conclusion. Without Dr. Arkadash intervening, they did not engage in the process-focused metacognitive actions MA1, MA2, or MA3.

Dr. Arkadash continually pushed students to develop as problem solver by using process-focused metacognitive tools (MA1, MA2, MA3). To trigger these actions, Dr. Arkadash frequently and unambiguously encouraged students to think about the current content unit of place value and integer addition and subtraction, and she suggested representations such as manipulatives and variables to help them explain the rules they generated.

Dr. Arkadash: [To class] So I heard multiple conjectures, but I really want you to be at a point where you’re investigating why and when that’s happening. For example, if I give you a number, any number I wish. I’m gonna tell you my number is ‘abc’. You don’t know what ‘a’ is. You don’t know what ‘b’ is. You don’t know what ‘c’ is. I’m gonna tell you ‘abc’ and you’re gonna tell me what happens to
‘abc’. OK? In other words, show the process with ‘abc’. ‘abc’ minus whatever, plus whatever, and look through that process. Generalize your process, whatever conjecture you’re working on. ... You have manipulatives at your table. I’m not selling these things [holding up manipulatives], but think about how it works, how the process works. I have ‘abc’, meaning I have ‘c’ many ones, ‘b’ many tens, ‘a’ many hundreds. What does that look like when I’m subtracting? Remember, we are on the adding and subtracting unit.

Throughout PPS2, there was a disconnect between Dr. Arkadash’s and the students’ goals for their problem-solving activity. Students relied on guess-and-check, justified their reasoning with more examples, and only made connections to course content or representations (manipulatives, variables, etc.) as an afterthought to their problem-solving attempt. Alternatively, Dr. Arkadash wanted students to consider different representations such as variables ‘abc’ or base blocks as strategies to make connections to the mathematical meaning of objects and operations. This disconnect resulted in students attending to the alternative strategies in their three-page write ups, but only superficially and not integrated into their problem-solving process. For example, rather than reason about the problem by manipulating variables ‘abc’ (i.e., working within the representation), students used the variables to state their rules such as Kerri did in Figure 16.

A transition in students’ reasoning and related process-focused metacognitive actions can be seen in students’ problem-solving activity during PPS 5, in which students considered various solution approaches and strategies (MA2), specifically various mathematical representations. This type of problem-solving is illustrated in Paula, Skylar, and Jordan’s debate about strategies and the use of a double number line to help them
reason about the problem, connecting the representation to mathematical features of the problem.

*Paula:* So, this goes from A to B and this goes from B to A. [Drawing.] I’m gonna look at the times first. They met at noon.

*Skylar:* They start at sunrise.

*Paula:* Did they start at the same time?

*Skylar:* Yeah. They both start at sunrise.

*Jordan:* See? This is like – we have to do a double number line. They both meet at noon.

*Paula:* They have to meet at the same time, but it doesn’t mean they went the same distance. Cause they didn’t – they were different speeds.

*Jordan:* They met at noon, but that doesn’t mean they met halfway.

[Pause.] So, should we try to figure out how far they’re going every hour? Then backtrack until like –

*Skylar:* But we don’t know how long this is.

*Paula:* It just says A and B.

*Jordan:* So, from the time they met –

*Paula:* But that might not be in the middle. [Pause.] I’m trying to – does she have a higher speed even though she finished early? I’m trying to figure out her speed and then backtrack the time. I was thinking. You were saying the distance is the same distance, but the difference between here is 4 hours [distance on upper horizontal line between 12:00 and 4:00], and then 9 hours [distance on lower horizontal line between 12:00 and 9:00], so 9 and 4.

*Skylar:* But then I was like it only works if they met exactly in the middle, because then we know – but then it wouldn’t work since they started at the same time.

*Paula:* Maybe we should do a graph?

Throughout the portfolio problem, the group considered multiple solution approaches (MA2) via different representations and reasoned *within* the representations.
themselves. They considered a variety of other representations, attempting to reason within strip diagrams and ratio tables. Throughout the problem, the representations allowed them to connect the context of the problem to mathematical reasoning supporting their reasoning. They rejected the use of a strip diagram because it did not properly represent ratios. Even though students struggled to solve the portfolio problem and get a correct answer, their problem-solving process was ostensibly more similar to that of a skilled mathematical problem solver than in portfolio problem two.

Micro-level analysis revealed a shift in students’ normative metacognitive actions and these metacognitive norms were connected to students’ normative reasoning and justification. This is possibly unsurprising, as cognition is the problem space of metacognition. Process-focused reasoning allows for and necessitates the evaluation of problem solving in earlier phases of the problem-solving cycle. As such, if students attend to their problem-solving process, they are more likely to use process-focused metacognition. A noteworthy aspect of this transition is the specific way in which students engaged in process-focused metacognitive actions, namely considering and assessing various representations as strategies/approaches (MA2) to help them make connections to the mathematical meaning of the problem context. There are many other forms of MA2 or other metacognitive actions a problem solver can take during the problem-solving process. Dr. Arkadash’s goal-oriented actions, and her use of mediating classroom tools such as the portfolio problem-solving sessions, guided students to use and reason with different representations to help them make connections to mathematical meaning. Her actions influenced the type of metacognitive activity students engaged in at the end of the semester, but not without pushback from students. Understanding and
situating the (nonlinear) process of negotiation between students and Dr. Arkadash was the focus of macro-level analysis.

**Macro-Level Activity**

Recalling the theoretical lens of activity theory guiding this research, the purpose of macro-level analysis was to understand and describe the sociocultural factors influencing the development of students’ metacognitive activity from heavy use of product-focused MA4 to process-focused MA2. Specifically, the aim of analysis was to outline the role of tensions catalyzing change to the collective student activity system over the course of the semester. The object of interest in the student activity system was their mathematical problem solving. As tensions influenced the student activity system, students’ problem-solving activity transformed as an outcome of their responses to system disturbances, affording a shift in normative metacognitive activity.

Characterizing the initial student activity system followed the methods outlined by Jonassen and Rohrer-Murphy (1999) to systematically describe various components of the system (Table 6). Steps one through five utilized a process of open and axial thematic coding (Strauss & Corbin, 1990) of student classroom actions and interview remarks to conceptually order the data components in each step. Actions taken in the first three weeks of the course were triangulated with student comments from initial interviews and written reflections in this same time period. The components of the initial student activity system and features that emerged as discernably relevant to students’ problem-solving activity are presented in Table 7.

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7 See appendix E for summary of student actions and statements organized by activity system component.
Table 6

Six Steps for Analyzing an Activity System (Jonassen & Rohrer-Murphy, 1999)

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Clarify the purpose of the activity system. Describe the motives and conscious goals of the activity system.</td>
</tr>
<tr>
<td>2</td>
<td>Analyze the activity system. Define the subject, object, community, division of labor, and rules.</td>
</tr>
<tr>
<td>3</td>
<td>Analyze the activity structure. Delineate the hierarchy of activity, concrete actions, and automatized operations.</td>
</tr>
<tr>
<td>4</td>
<td>Analyze tools and mediators. Describe the tools, rules, and roles of participants that mediate activity within the system.</td>
</tr>
<tr>
<td>5</td>
<td>Analyze the context. Characterize the internal, subject-driven and external, community driven contextual bounds.</td>
</tr>
<tr>
<td>6</td>
<td>Analyze activity system dynamics. Step back from the delineated activity system to describe and assess how components affect each other.</td>
</tr>
</tbody>
</table>

After identifying components of the initial activity system, the final phase (step 6) of macro-analysis involved stepping back from the system to understand dynamics as catalyzed by tensions within system components, between system components, and between the motives/goals of the student activity system and that of the teacher. These dynamics were identified by first looking across daily teacher and student actions over the course of the semester and then coordinating these actions with student and instructor interviews, as well as recorded planning sessions. From this evaluation, there were noticeable shifts in students’ problem-solving activity during portfolio problem-solving sessions. Moreover, these occurred in tandem with shifts in Dr. Arkadash’s goals and

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8 Summary of relevant actions and statements can be found in Appendix E.
related actions in response to daily classroom activity between sessions. Her actions “stirred the pot”, bringing student tensions to surface as potential opportunities to drive students’ problem-solving activity forward. The reflexive relationship between Dr. Arkadash’s goals and students’ problem-solving activity generally followed the phases outlined in Table 8. The cycle of tensions and student adaptations are presented in the following section.
Table 7

*Initial Student Activity System*

<table>
<thead>
<tr>
<th>Component</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Object</td>
<td>Problem solving activity</td>
</tr>
<tr>
<td>Subject</td>
<td>Pre-service elementary teachers</td>
</tr>
</tbody>
</table>

Motives and Goals:
- Become a good elementary mathematics teacher
  - Build a teaching “toolkit” of different procedures and pedagogical strategies to help the diverse needs of their future students (primary, “teacher hat”)
  - Get better at math (secondary, “student hat”)
- Get a good grade
  - Students told what to think and do by the teacher (playing school)

Beliefs:
- As students: value correct answers and most “efficient” method to get there
- As future teachers: visual representations help students who don’t “get it” right away

Community
- Future elementary teachers (“teacher hat”)
- Students in a college mathematics classroom (“student hat”)

Division of Labor
- Active, student-centered classroom environment

Mediating Instruments
- Daily, problem-based learning activities
- Dr. Arkadash
  - Teacher asks questions to move student thinking forward
  - Portfolio problem-solving sessions and write ups
  - Student presentations

Rules
- Teacher is authority: tell the teacher what they want to hear; do what the teacher wants (implicit)
- Grading components: tests, homework, class participation, portfolio problems (explicit)
- Social and sociomathematical norms related to previous “traditional” mathematics classroom experiences (implicit)
Table 8

*Correspondence between students’ problem-solving activity and Dr. Arkadash’s goals*

<table>
<thead>
<tr>
<th>Dr. Arkadash’s Goal/Action Shifts</th>
<th>Students’ Problem-Solving Activity</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Problem-Solving Activity 0</strong></td>
<td>• Just “do” the problem (no explanation of thinking)</td>
</tr>
<tr>
<td>Explain your thinking process</td>
<td></td>
</tr>
</tbody>
</table>
| **Problem-Solving Activity 1**    | • When stuck, pattern-find with examples  
• Telling thinking process is afterthought  
  Provide more examples  
  Representations illustrate these examples |
| Justify your process with “mathematical” meaning |                                   |
| **Problem-Solving Activity 2**    | • When stuck, pattern-find with examples  
• Mathematically-based justification is afterthought  
  Attend to “superficial” features  
  Representations illustrate these features |
| Focus on language ("connect to definitions") to justify key mathematical features |                                   |
| **Problem-Solving Activity 3**    | • When stuck, appeal to emotion  
• Mathematically-based justification is afterthought  
  Attend to language and definitions of key mathematical features  
  Representations illustrate these features |
| Representations can mediate these connections and help you get “unstuck” |                                   |
| **Problem-Solving Activity 4**    | • Reason *within* representations to arrive at answer (*Strategy Shift*)  
• Justification via connection to mathematical meaning is part of the problem-solving process |
Tensions and Adaptations

Analysis of system dynamics coordinated three aspects of student-instructor negotiation: (1) students’ problem-solving activity as the object of the student activity system (right column, Table 8), (2) Dr. Arkadash’s shifting goals and actions, carried out through mediating instruments, that initiated student tensions (left column, Table 8), and (3) student tensions and goal-directed adaptations to these tensions, resulting in shifting students’ problem solving activity. The dynamics between (2) and (3) lead to the creation of a new, “expanded object” (Engeström & Sannino, 2010, p. 7) of problem-solving activity and a “pattern of activity oriented to the object” (p. 7). These relationships are summarized in the following discussion.

Problem-Solving Activity 0 → Problem-Solving Activity 1. Students’ transition from no explanation of thinking to sharing how they thought about a problem was rather immediate. Dr. Arkadash’s primary goal was for students to explain their thinking, rather than “just doing the problem” as one student described in a class reflection after the second day. Her actions brought to the fore a tension between students’ implicit understanding of participation norms within a “traditional” classroom setting and a new, active and student-centered division of labor. Students arrived on the first day and began working on small-group activities. They worked mostly in silence, but were tirelessly prompted to share ideas, both in activity directions and frequently by Dr. Arkadash:

Dr. Arkadash: What do we have? What are we thinking?
Delia: A dollar.
Dr. Arkadash: I’m not looking for an answer. I’m looking for your thinking.
Dr. Arkadash: [Turns to another group.] Kelly, can you explain your thinking to them?
:
Ronnie: So, is number four just asking how much [Pauses.]
Sharon: I think it’s asking for the answer and, like, how did you get the answer.

Kim: [Reading.] Discuss what happens. What does she mean, ‘discuss’? Is that what she means – should we just write that?

Paula: Which one?

Kim: Like this. What you’re writing.

Paula: I guess so. Cause that’s your actual thought process, right?

Kim: But I did it differently. I divided it. [Continues explaining her thinking.]

As students discerning the “hidden curriculum” (Engeström, 1998) of what is required for a good grade in the course, they engaged in the required actions and explained their thinking. Although they adapted quickly to this emerging rule, students did not necessarily understand why explaining their thinking was necessary. Sharon speculated: “I wonder if it’s like this because we’re, like, elementary education majors”.

Dr. Arkadash appealed to students’ “teacher hat” for buy in: “You need to practice explaining to each other; You’re a teacher”, but her focus was on students’ personal mathematical development. She discussed in the planning meeting after the first day that students needed to compare and contrast different solutions. There is not just one way to solve a problem, and comparing and contrasting solutions places more emphasis on the solution process rather than the answer. In the following class periods, students were asked in activities to generate multiple solutions to the same problem, and Dr. Arkadash intentionally presented different student solutions.

These mediating actions created a tension between her goal of students’ valuing alternative solutions as helpful for them in their personal mathematical development, and students’ value of the correct answer supported by their belief that there is one, most efficient method to get to the correct answer. For example, while converting from base-10 to base-6, Keith suggested dividing the base-10 number by 6. Dr. Arkadash asked
groups to discuss if they could use Keith’s proposed method for the conversion. In small-group conversations, Lance was sure Keith’s method would not work. Other groups commented that Keith’s method was “ridiculously long” and their method was much easier, going back to their discussion of their more “efficient” method. When Dr. Arkadash asked groups to share out, Lance suggested using his group’s method. Dr. Arkadash redirected students’ focus to Keith’s method: “Keith’s process is totally correct”, and she walked the class through this process. “Faster doesn’t always mean the best way”.

Through engagement with different solution paths in presentations and in small-group work, students realized why they were explaining their thinking, but this explanation was not in service of their own mathematical development as students. Rather, they saw value in alternative views to help them explain concepts to their future students. Students’ tension between the two communities to which they belonged, future teachers and current students, was made visible in the way they resolved the tension between Dr. Arkadash’s goals for them as students and their student beliefs about correct answers and efficient methods. Appealing to their “teacher hat” allowed them to circumvent resolution of this tension while wearing their “student hat”.

Focusing pre-service teachers’ attention to their thinking as students of mathematics was the primary goal of the portfolio problem-solving sessions and write-ups. The problems were, on the whole, problematic for students and they turned off their “teacher hat” while solving the problem. To reflect on their problem solving, they were required to explain their thinking process, but they associated as part of teaching. As a separate endeavor from their problem solving, they explained their steps to a hypothetical
student. Dr. Arkadash’s purposeful action to implement the portfolio problem-solving sessions yet again required students to navigate an internal contradiction between their identities as mathematics students in this course and as future mathematics teachers. Students resolved this tension, as was seen in PPS 2, by problem-solving with the only problem-solving strategy they had been using, namely pattern-finding with examples to arrive at an answer. As an afterthought, appealing to their role as future teachers and Dr. Arkadash’s explicit rule to explain their thinking, students explained their process of guess and check with examples and utilized additional examples to further illustrate their answer. Appealing to themselves as future teachers manifested itself as MA4, where the reasonableness of their solution was assessed by providing an explanation of their thinking. Students’ normative metacognitive actions were influenced by their changing problem-solving activity, as influenced by their adaptations to tensions and contradictions initiated by Dr. Arkadash.

**Problem-Solving Activity 1 → Problem-Solving Activity 2.** Dr. Arkadash continued to utilize classroom activities and discussions to highlight multiple solution paths. “You are peculiar,” she told students, to emphasize that they all have different ways of thinking and learning and that their students will too. She encouraged them to continue practicing explaining and understanding their and each other’s thinking. Concurrently, she noticed that while students were getting better at explaining their thinking, these explanations did not include mathematically-based justifications. In a planning meeting, she identified a goal to emphasize mathematical justifications and “generalization” beyond examples. During class, the purpose of student presentations changed. While students still presented alternate solution methods, Dr. Arkadash pushed
students to make connections to mathematical meaning and encouraged other students to
push each other in this way. This instructional move elicited a tension between students’
current problem-solving activity and the implicit rule that they need to do what the
teacher says in service of their goal to get a good grade in the course. More noticeably,
Dr. Arkadash’s mediating actions conflicted with students’ current sociomathematical
norm for justification through telling the steps they took. This tension can be seen in the
following episode in which Sasha presents her solution for the problem of modeling the
subtraction two minus four with two-colored integer chips.

*Dr. Arkadash:* OK. We have a teacher, Sasha. Sasha is our teacher and she’s
gonna tell us what she’s thinking about this question.
*Sasha:* You want me to go up there?
*Dr. Arkadash:* You’re the teacher.
[Sasha walks to the document camera.]
*Sasha:* OK. So you start with two positive [yellow] chips and then you
add zeros. [Adds two yellow and two red pairs of chips to the
document camera.]
*Dr. Arkadash:* Sasha, I missed that. Why are we adding zeros?
*Sasha:* We have to take away four so we need more chips. So, these
cancel out [pairing red and yellow chips] and then you have to take
away four positive chips [removes yellow chips]. Then you end
with negative two [two red chips].
*Dr. Arkadash:* Questions? Comments? Concerns? [To Sasha as Sasha walks
toward her chair.] These are your fourth graders. They might have
a question. You’re not done. [Sasha and class laugh.]
*Sasha:* [To class.] Do you have any questions?
*Kerri:* [Imitating child.] Why can I just add zero? [Sasha and class laugh.]
*Dr. Arkadash:* Thank you. Right? This is what’s gonna happen. Why?
*Sasha:* You need more chips to take away. You have to take away four. So
if you add zeros, you make it equivalent to two still and you just
get more positive chips to take away.
*Dr. Arkadash:* OK. So teacher, are you saying that if I have two chips [pause.]
Are you saying when I add zero chips I’m not changing the value
that I started with? Is that what you’re saying?
*Sasha:* Yes.
*
*Dr. Arkadash:* [To class.] If you forget about this method, what kind of things
could help you to remind yourself about this model and this
method? What are some mathematical ideas that we are using in
this method? What is the main mathematical idea in this whole process?

*Lance:* Zero is hero.

*Dr. Arkadash:* I love it. Zero is hero. What is zero’s superpower that helps us in this situation?

*Kim:* It doesn’t change the value.

Note that Dr. Arkadash again mitigated the severity of this tension for students by appealing to students as future teachers. This action alleviated students’ need to grapple with these changes while wearing their “student hat”, allowing them to more quickly adapt to this form of justification. Students began adapting their problem solving to this new rule. Instead of supporting their answers with more examples or re-telling the steps as Sasha did, they attempted to connect to mathematical meaning. Even so, students were not yet attuned to finding key mathematical ideas on their own. They often attended to superficial features of the problem and used representations to illustrate these superficial features.

For instance, while students were discussing how to model the distributive property of multiplication over addition for two binomials, Mary presented a Punnett square to help her keep track of all the terms when using the rule ‘FOIL’. Amie added this method could be helpful for students: “When it’s FOIL – cause it’s ‘a’ times ‘c’ and then ‘a’ times ‘d’. If a kid didn’t know what FOIL was, this would be an easier way to teach them.” Dr. Arkadash wondered aloud what the components of Mary’s drawing represented, but Mary and Amie were stuck. Kerri raised her hand to share how her group thought about the problem. Dr. Arkadash put Lance’s drawings on the document camera as she talked (Figure 17):

*Kerri:* So, when we started this we were trying think about what shape to draw to figure this out. Lance was over here drawing all kinds of shapes. And that was getting us nowhere, cause how do you – like,
they weren’t holding any value. They were just variables without any numbers. You could plug in numbers to figure out that they did in fact equal each other, but there wasn’t any combination squares and triangles. Without this we couldn’t have gotten to the square, cause I was like none of these hold any value for us. We can’t represent, we can’t do anything with that!

Dr. Arkadash: It does not have meaning! It does not have mathematical meaning.

Kerri: And because there were four of them [numbers/variables], we had to figure out how to stick them in a square so we could line them up to get the right thing. So, we have – one side is like the length.

Dr. Arkadash: This side is the length. So that is the mathematical meaning that we’re after here.

Kerri: The other side is the height. And you can split it up, and you have a chunk of the height and another chunk of the height. And another length, and another chunk of the length. And then you just go from there.

Dr. Arkadash: I love this idea of ‘how can I put these things together? What kind of picture can I create?’ I can create random pictures, but that’s not gonna stick because it’s not gonna hold mathematical meaning of what we’re trying to do here.

![Figure 17. Lance’s drawings (Redrawn for discernibility).](image)

Connecting to mathematical meaning came easier to some students than others, but this tool was integrated into students’ authentic problem-solving process. In portfolio problem-solving session three (Figure 14), students continued to look for patterns with examples when they were stuck, rather than connect to mathematical meaning as they had been learning to do in class. While problem solving as students during the portfolio
problem, they did not think to use tools they were learning as a future mathematics teachers. Making connections to mathematical meaning was a tool for explaining concepts to future elementary students, not to help them in their own problem-solving endeavors. This disconnect in actions, in service of different goals, meant that justification of their answers was still an afterthought to their problem-solving attempt.

While justifying their answers in class and in write-ups, students switched goals to put on their “teacher hat”, now employing the tools they had learned for teaching to complete the portfolio-problem write-up. Here, many students attempted to connect to mathematical meaning, but attended to superficial features of the problem. For example, after looking for patterns for the pentomino ‘F’, Paula, Alexis, Anna, and Lucy noticed combinations of even and odd numbers depending on the location and orientation of the letter (Figure 18). This line of thinking generates a list of 16 possible combination-based rules (eight orientations, two even-odd combinations), and the group explained a subset of these rules. While these students explicitly connected to mathematics via even and odd numbers, this connection was not fruitful in solving the problem. They could determine if the resulting sum was even or odd, but this did not help them connect to the key idea, divisibility by five. The superficial connections did not push their problem solving forward.
Problem-Solving Activity 2 → Problem-Solving Activity 3. To help students who were not connecting to key mathematical ideas, Dr. Arkadash explicitly shifted goals again to a focus on language precision to mediate this connection, especially connecting to definitions. As students learned about new operations and concepts, during student presentations and in small-group work Dr. Arkadash purposefully and consistently asked students to make connections to the meaning (definition) of operations. With Dr. Arkadash’s guidance, students slowly developed proficiency with thorough attention to language. For instance, while working with divisibility rules, students were asked to determine if the following statement was true or false:

*If B is divisible by A and C is divisible by A, then B+C is divisible by A.*

*Alexis:* I explained it by saying that if A divides B and C evenly then B and C are both multiples of A. And then when you add multiples together, you just get another multiple of A.

*Dr. Arkadash:* Did you all hear that explanation? That’s a very nice way of – that’s mathematically a nice way of explaining it. We can even put more details into that explanation. I will try to paraphrase what you said. Please correct me if I’m not saying the right thing. So, you’re saying that if B is divisible by A, then B is a multiple of A. And then you said C is a multiple of A too. So, what do these two
things mean? I can write B as A times some number \([B = AM]\), and then I can write C as A times some number \([C = AN]\). In other words, if we use our multiplication definition – I’m going back to the meaning of multiplication. This means that B is a number of total objects we have and A, if you think about the number of groups. And then M is how many objects in each group. The same thing for C. C is the object, total product number and A is how many groups we have. Then N is the number of objects in each group. We can think about that using the definition or meaning of multiplication. What is the next step you took Alexis?

**Alexis:** You would take B plus C, which in your case would be AM plus AN.

**Dr. Arkadash:** So, you’re just gonna put these things together. That’s what we’re adding up. The investigation question asks if this is a multiple of A.

**Alexis:** You can factor out the A. And it would be A times M plus N \([A(M+N)]\). And so then, since you’re multiplying it by A, it’s a multiple of A.

**Dr. Arkadash:** So, in each group I had M objects and N objects. I add them together and then I still have A groups. That is going to give me the total number of objects B plus C. So, we can use Alexis’ argument. The only detail I brought in is the meaning of multiplication at each step. What does multiple mean? What does multiplication mean? That’s all I did, but that is the explanation we are looking for. So, examples give us ideas to see where we are heading, but we can’t really use just examples to justify this is true. We need one step further to generalize and then the justification that Alexis provided as to why that generalization is working.

While students were increasingly paying attention to language and the meaning of operations during daily class activities, these were still utilized as an afterthought to their real-time problem-solving activity. Simultaneously, Dr. Arkadash, as seen in her statements in the previous transcript, continued to move students away from merely relying on the problem-solving strategy of pattern finding with examples. Previously, students could solve problems using this guess and check strategy, and then afterwards use tools from the course to help them justify their answer. With Dr. Arkadash’s explicit rule to move away from pattern finding, students now had language-based tools to help
them solve problems, but did not know how to use these tools in their real-time problem-solving process.

The tension between students’ identities as students of mathematics and as future mathematics teachers had ostensibly reached a breaking point. When students became stuck and were asked to justify beyond guess and check via examples to generate answers (rules), they became frustrated. Kerri described being stuck and frustrated in her mid-semester interview:

*Emilie:* So, you were talking about portfolio problems, like maybe you’re not completely justifying. What does it mean to completely justify? What would it have meant in the Pentominos problem to have completely justified things?

*Kerri:* Well, I would have been able to say ‘If it faces this way the remainder is 2, because this’ and like mathematically be able to say all the reasoning behind it. So, I reached that answer just from trying it like several times, and I could kind of make sense of it as far as like: ‘Well there’s probably a reason for this and it’s probably related to this and this and this.’ But I can’t always tie all of those together…. I never know how to get past that. Like if I don’t know, how can I then just know? You know what I mean? How do you get past that point of being stuck?

*Emilie:* Do you feel like you tend to be one-track minded in the way that you solve problems? You jump on something and then that’s what you –

*Kerri:* I think so… I try to do that more and push out from my one path. But yeah, I totally go down one path.

*Emilie:* Is it something that you feel – You said in the last few portfolio problems you’ve tried to push yourself out of that –

*Kerri:* I really want to be able to solve it. I’d like to get to that lightbulb and explanation – have it be, feel good about it afterward.

*Emilie:* It’s interesting, because when you were talking about [Dr. Arkadash], you were saying she has this knowledge so she’s able to pick the best path. She just has more to look at. Do you feel like if she were to get stuck that is the thing that would help her get unstuck? Or do you feel there’s something else that helps her do that?

*Kerri:* No. I think it’s discerning which method might work. So, it’s not that she has a bunch of knowledge. She has a bunch of knowledge
of math, but more than that she has the ability to know what to apply when.

**Problem-Solving Activity 3 → Problem-Solving Activity 4.** Dr. Arkadash was becoming aware of this tension, sharing her observations and new goals in planning meetings. She felt that students were good self-starters, but she needed to “push them to the next level”. Students could pattern find well and make observations, but were “limited” by these patterns and misapplying them.

*Dr. Arkadash:* I feel very confident that they are self-problem solvers now. They start. They ask questions…I need to push them to the next level…Representations. I’m gonna push that, because I noticed that they are ready for the whole “I’m reasoning”, but they need to push themselves for “Now I’m going to do more writing down with different representations” … The next step, I think. How to represent what we’ve found in different ways. I think they are good with sentences, but now connecting those – like inserting certain representations. It could be a picture. It could be a symbolic representation.

While representations had been a central component of the course, their purpose was now to help students move their problem solving forward. Further, Dr. Arkadash wanted students to be using tools such as representations when they were stuck, rather than questioning their confidence. This was made poignant after a conversation with Lucy and Taryn during portfolio problem four:

*Lucy:* I’m concerningly confident.
*Taryn:* I know. It’s like almost too easy.
*Lucy:* I don’t like it.
*Dr. Arkadash:* You don’t like what?
*Lucy:* Because portfolio problems are my biggest challenge in this class. I have a hard time wrapping my head around what we’re given. And so, when I feel good about something, like I know it has to be wrong somewhere.
Dr. Arkadash was worried that students’ emotional triggers were causing them to question their confidence, and she wanted them to turn this emotional trigger into “mathematical action”.

_Dr. Arkadash:_ For them, it is not triggering “let me check my work”. It’s triggering their confidence. I would like it to trigger the expert version of things: “let me ask myself questions and justify things.” … I want them to go there instead of going into the confidence questioning… I don’t want to say “I want to build your confidence. Be confident.” I want them to notice that you can do something quickly. You can get something not quickly. Regardless, you need to not be questioning your confidence. You should be questioning “What else can I do.” … It’s not about getting something quickly. It’s when you get stuck.”

In class, Dr. Arkadash played a video about Andrew Wiles’ work on Fermat’s Last Theorem to help her make this point to students. She told them that frustration was part of the problem-solving process. This was why she was giving them portfolio problems. The portfolio problems were intended to be situations where they would not immediately know how to solve the problem.

Over the next few weeks, Dr. Arkadash was intentional with highlighting representations, making connections between representations, and using representations to make sense of problems, especially when students were stuck.

_Dr. Arkadash:_ Alright. You’re stuck. What do you do?
_Amie:_ Cry. [laughs.]
_Dr. Arkadash:_ Totally. I do that too. What other things do you do when you are stuck? Seriously. It happens.
_Lucy:_ I usually move on to something else and then come back.
_Dr. Arkadash:_ Exactly. So, you just give yourself time. What else? [Pause.] First of all, are you convinced that you’ve got the problem in your head? Did you orient yourself to the problem enough? Or do you feel like ‘Yes. I’m really, absolutely stuck.”
_Taryn:_ No…
_Dr. Arkadash:_ So, what kind of things do you have? I see a couple of things. I see John and Mary. I see 350. My question is, when I look at 350, I cannot identify John’s and Mary’s money separately. This problem
is asking us to identify them separately, giving us information to look at them separately. So, when I look at your strip diagram right there, it looks like John and Mary have the same amount of money. So, immediately in my head, I’m dividing 350 into two and I’m getting 175 for each of them. OK. Nice. So, you’re strip diagram changes. How is it changing? What is the story telling us about John and Mary’s money? [She guides them to match the strip diagram to the problem.]

This goal-shift catalyzed the biggest shift in students’ problem-solving activity and consequently their metacognitive activity. What the pre-service teachers were learning in the course about representations was helpful for their personal problem solving. They now had explicit tools to help them make connections to mathematical meaning and visualize the language and definitions they were trying to use. Instead of getting stuck pattern finding, getting frustrated, and stopping, students reasoned within representations and justified their reasoning as part of their problem-solving process. Representational strategies allowed students to bridge their identities as students and teachers, merging their personal problem solving (“student hat”) with their justification (“teacher hat”). This merging is what was seen in Paula, Skylar, and Jordan’s problem solving while working on portfolio problem five. Consequently, students now utilized MA2 heavily. Considering and assessing the relevance of various representations allowed them to persevere in their problem solving and provide mathematically-based justification of their process.

**Discussion and Implications**

In reflecting on the course, Lance noted that he had “never really thought to ask [him]self ‘Why did you do this?’” prior to taking the class. “You were supposed to reflect on even your very first thoughts when you see a problem. No matter how wrong they were.” To prepare for teaching mathematics, pre-service teachers need to develop
mathematical knowledge for teaching, but also mathematical knowledge as problem solvers themselves. Skilled mathematical problem solvers use metacognition throughout the problem-solving cycle (Carlson & Bloom, 2005), taking process-focused metacognitive actions long before they get to an answer. The pre-service teachers in this course did not use this type of “expert” metacognition during problem solving. Micro-level analysis, addressing Q1, found that the pre-service teachers began the course relying on the product-focused metacognitive action MA4, and that this was tied to their problem-solving practices. Through participation in authentic problem-solving situations, namely the portfolio problems, students problem-solving activity shifted in a way that afforded them opportunities to readily engage in the process-focused metacognitive action MA2. Kerri recognized this change in her final interview:

I’ve just been able to be actively engaged in the problem, realizing what I’m doing. Rather than just like, ‘Well, this is the first step and second step,’ and then afterwards I’m like, ‘Oh, that was wrong, and that was wrong.’

Macro-level analysis, addressing Q2 and Q3, situated this change in students’ metacognitive activity during problem solving as a reflexive process of negotiation between students and Dr. Arkadash. As seen in Table 8, Dr. Arkadash shifted her goals in response to students’ current problem-solving activity. Her intentional actions in service of these goals stimulated tensions for students, providing opportunities for students to expand their problem solving and engage in a new pattern of activity oriented to this expanded conception of problem solving. Dr. Arkadash’s timing in goal-shifting was fundamental to students’ transformation. During portfolio problem two, she heavily encouraged students to consider different strategies (MA2) via different representations (variables ‘abc’ or base blocks), but students did not yet value process over answers.
Instead, she needed to scaffold her long-term objectives to meet students where they were in their problem solving. Students’ metacognitive activity was very much situated in the sociocultural context of the classroom. Just as mathematical content is carefully planned through backwards planning to align curriculum and classroom activity with learning objectives, designing learning experiences for students to develop metacognition, or other habits of mind, requires the same careful attention.

Finally, pre-service teachers wore two “hats” in the course, appealing to their dual identities as a future mathematics teacher and as a students of mathematics. Dr. Arkadash intended for pre-service teachers’ own thinking to be the primary object of their learning so they could develop the habits and practices of a mathematical thinker and problem solver. However, students’ competing teacher identity impeded this development. Students believed the primary purpose of the class was for them to develop as mathematics teachers, so while they engaged in mathematical practices, this engagement was not in service of their personal development (“student hat”). Although the portfolio problems focused students’ attention on their own thinking, they engaged with the problems wearing their “teacher hat”. The portfolio problems only catalyzed the tension between their identities. While students’ justifications improved and became more “mathematical” (PSA 0 through PSA 3), this tool did not help their real-time problem-solving process. They continued to solve problems the same way they always had and then, in a separate endeavor, justified their answers. Even though they were developing “teacher” tools that could help them in their own problem solving, they were only able to develop as problem solvers when they realized the tools seemingly presented to them for teaching, multiple representations, proved useful to them as problem solvers themselves.
This finding has implications for the how mathematics teacher educators prepare future teachers, adding to the body of knowledge for teaching mathematics teachers. Oesterle et al. (2016) emphasized that to help their future students develop mathematical habits of mind, pre-service teachers need to value the usefulness of mathematical habits of mind in addition to building proficiency with them. The pre-service teachers in this course were only able to change their metacognitive habits when they valued MA4 to help them get unstuck. MA4 afforded them productive struggle. Mathematics teacher educators must build learning environments where pre-service teachers are legitimately participating in the mathematical problem-solving community of practice to become full participants (i.e., skilled problem solvers). For pre-service teachers to value mathematical problem-solving habits of mind, legitimate participation means as students, not as teachers, of mathematics.
CHAPTER IV

SUPPORTING THE DEVELOPMENT OF “PROCESS-FOCUSED” METACOGNITION DURING PROBLEM SOLVING

Abstract As students learn to problem solve in authentic problem-solving situations, they must also develop metacognitive tools to manage and regulate their problem-solving process. To foster “process-focused” metacognition utilized by mathematical thinkers and problem solvers, Inquiry-Based Learning classroom practices and an adapted version of “portfolio” problems were implemented in a content course for pre-service elementary teachers. In this article, I describe how a process-focused (instead of product-focused) classroom culture and explicit reflection on student thinking mediated by in-class portfolio problem-solving sessions and write-ups supported students’ process-focused thinking while problem solving. The problems, portfolio structure, and student interview reflections are shared.

Keywords problem-solving process, metacognition, inquiry-based learning

Introduction

Problem solving is “the mathematician’s main reason for existence” (Halmos, 1980, p. 519), and metacognition is a fundamental component of the mathematical problem-solving process (Carlson & Bloom, 2005; Lester, 1994; Schoenfeld, 1985). Students can acquire many problem-solving heuristics, but “the number of useful, adequately delineated techniques is not numbered in tens, but in the hundreds…The
question of selecting which one to use (and when) becomes a critical issue” (Schoenfeld, 1985, p.73). The recent MAA Instructional Practices Guide (MAA, 2017) suggested that undergraduate course design practices should include metacognitive support for students’ deep, lifelong learning, and that including metacognitive strategies in discussions about problem solving can help students persist while working on complex tasks.

Metacognition is a mathematical habit of mind (Selden & Lim, 2010; Stacey et al., 1982), a dispositional tendency or normative way of thinking or acting of mathematical thinkers in problem-solving situations. Mathematical thinkers and problem solvers know how to act (metacognitively) in the moment (Mason & Spence, 1999), utilizing metacognitive actions throughout all phases of the problem-solving cycle (Figure 19) (Carlson & Bloom, 2005). The metacognitive actions employed by mathematicians are both “product-focused”, occurring at the end or Checking phase of the problem-solving cycle, as well as “process-focused”, occurring in the earlier phases of problem solving. Moreover, mathematicians often rely on a Conjecture Cycle during the Planning phase of the problem-solving process, which includes the consideration and assessment of various strategies or solution approaches, a “process-focused” metacognitive strategy. Students’ metacognitive habits do not necessarily develop on their own, and there is a large body of research concerning the improvement of students’ metacognition during problem solving (see Baten et al., 2017). If metacognitive activity is a normative habit of mind of the mathematical problem-solving community, then students should develop these normative ways of thinking or acting while problem solving (i.e., habitually using both “product-focused” and “process-focused” metacognitive actions).
As such, students require opportunities to legitimately participate in authentic problem-solving situations where metacognition is necessary, and to come to know the product- and process-focused metacognitive language in these settings. The ultimate goal is for students to self-initiate metacognitive actions in all phases of the problem-solving process, rather than use them only in response to instructor prompting. Researchers emphasize the importance of prolonged metacognitive instruction embedded in content (Lester et al., 1989; Veenman et al., 2006), but research on metacognition has largely overlooked the influence of socio-cultural context on this development (Larkin, 2015).
This paper is an attempt to demystify the transition from teacher-initiated metacognitive activity to students’ independent use of metacognitive actions during problem solving within a classroom setting. In particular, I conducted a qualitative study investigating the development of students’ “process-focused” metacognitive activity in authentic problem-solving situations (Hancock, YEAR). Based on results from this study, the purpose of this paper is to provide practical suggestions for classroom practices to support students’ “process-focused” metacognitive activity.

In this paper, I first summarize results of an analysis of normative metacognitive actions during small-group problem solving in a mathematics course for pre-service elementary teachers. Results indicated that students’ collective metacognitive activity in authentic problem-solving situations changed from a retroactive “product-focus” on checking final answers, to a more proactive focus on their problem-solving process (see the next section). Subsequently, I identify related classroom practices that afforded this shift and supported process-focused metacognition. In the ‘Building a Process-Focused Community of Inquiry’ section, I describe how an Inquiry-Based Learning (IBL) classroom environment helped students attend to their problem-solving process. In the ‘Explicit Support for Reflection on Process-Focused Thinking’ section, I explain how “portfolio problems” (Omar, Karakok, & Savic, YEAR) were adapted to support process-focused reflection while participating in authentic problem-solving situations. In the ‘Student Feedback’ section, I share student feedback on the course and how the course supported their metacognition during problem solving. I conclude in the final section with some suggestions on classroom design strategies based on the classroom practices presented in earlier sections.
**Shifted Metacognitive Habits**

In a first-year Number Sense and Algebra course for pre-service elementary teachers at a mid-size public university in the Rocky Mountain region of the United States, 24 students and the instructor, Dr. Arkadash, met twice a week for 75-minutes over a 15-week semester. The course was taught with Inquiry-Based Learning (IBL) classroom instructional practices (Ernst et al., 2017) to support Dr. Arkadash in teaching through problem solving (Stein et al., 2003). She described these goals in her pre-semester interview:

For me to deliver that content goal, how I deliver that is through problem solving, and that brings another goal for me...I want [students] to be thinking about their problem-solving skills too. ‘So, you learn all these problem-solving skills. Which ones are applicable now? Which ones are not?’ Or, ‘Here’s another problem-solving skill – or a strategy I should say – that you haven’t used before. Let’s see what’s that and then when it’s gonna be applicable.’...Some people might say that these content goals and problem-solving skill goals, or process goals, are kind of separate from each other. So, that’s another thing that I would like to highlight and emphasize in class a little bit. I don’t want [students] to think, ‘I’m doing problem solving, but I’m not really learning the content.’ So, I just want to make sure that they tie in together nicely too.

Dr. Arkadash wanted to provide more realistic problem-solving opportunities for students. She noted in her post-semester interview that problem solving in school mathematics can be particularly inauthentic when students get stuck:

The classroom environment has weird dynamics. [Students] know that some of the things we are doing will be answered, resolved... They are hoping that if they pose a question or if they sit there long enough, somebody will come in [to help].

When listening to classroom recordings of the Number Sense and Algebra course, I noticed that students often verbalized this sentiment, stopping small-group discussions to wait for Dr. Arkadash to tell them how to continue working on the problem. To provide a more authentic problem solving experience for students, Dr. Arkadash adapted a version
of “portfolio problems” (Omar, Karakok, & Savic, YEAR) (described in detail in the ‘Explicit Support for Reflection on Process-Focused Thinking’ section). Periodically throughout the semester, students worked in small groups on problems intended to be problematic, an “intellectual impasse rather than a computational one” (Schoenfeld, 1985, p. 74). These problems were not easily solved with procedures to which students had “easy access” (p. 11).

As part of a larger qualitative study (Hancock, YEAR), multiple data sources were collected to characterize students’ collective problem-solving activity: audio- and video-recorded classroom sessions; three semi-structured individual interviews with 159 of the 24 students; pre- and post-semester interviews and audio-recorded weekly course planning sessions with Dr. Arkadash; students’ written artifacts collected before grading; and daily written reflections from Dr. Arkadash, as well as myself as an observer participant (Gold, 1958) in the classroom. Detailed analysis procedures and results with data-based examples may be found in (Hancock, YEAR), but are very briefly summarized here. Micro-level analysis of students’ in-class, small-group “portfolio” problem-solving sessions was conducted to assess students’ normative metacognitive activity in authentic problem-solving situations throughout the course. For a metacognitive action to be normative, I mean the metacognitive action is taken “as if it is a mathematical truth in the classroom…as if everyone has similar understandings, even though individual differences in understanding may exist” (Rasmussen et al., p. 262-3).

Starting with a list of metacognitive actions during the problem-solving cycle drawn from Carlson and Bloom (2005), this list was applied to students’ verbal data and

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9 13 of the 15 students completed all three interviews.
refined to create a final list of six metacognitive actions relevant to the data set (Table 9). After coding small-group portfolio problem-solving sessions with these six codes and looking across the data, there was a shift in students’ normative metacognitive activity in authentic problem-solving situations. Students’ transitioned from a retroactive “product-focus” on checking final answers (MA4), to a more proactive focus on their process earlier in the problem-solving cycle, especially the assessment and consideration of various solution approaches and strategies (MA2).

Coming into the class, the normative metacognitive activity of students while problem solving was a focus on assessing and checking final answers while their problem-solving process was hidden: “Did you get 1089 too?” By the end of the semester, students spent most of their time together in class debating which strategy or representation was most appropriate for the problem at hand: “Maybe we should do a graph [instead of a double number line]?” So, students were beginning to employ, and self-initiate, metacognitive actions earlier in their problem-solving process, as mathematical thinkers and problem solvers do. Dr. Arkadash’s IBL classroom structure and practices, as well as her explicit inclusion of more “authentic” problem-solving situations, supported students’ collective “process-focused” metacognition. In the following sections, I describe aspects of the classroom context that helped shape students’ process-focused metacognition during problem solving, highlighting specific teacher actions and instruments utilized in the course that may prove beneficial in other undergraduate mathematics classroom settings. These suggestions are grounded in detailed analysis of all aforementioned data sources (beyond just the micro-level results presented here), which can be found in (Hancock, YEAR).
Table 9

*Metacognitive actions identified during portfolio problem-solving sessions*

<table>
<thead>
<tr>
<th>Metacognitive Action</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>MA 1</td>
<td>Mathematical concepts, knowledge, tools, and facts are assessed and considered</td>
</tr>
<tr>
<td></td>
<td>Data Example: <em>So, does that mean it's a quadratic relationship?</em></td>
</tr>
<tr>
<td>MA 2</td>
<td>Various solution approaches or strategies are assessed and considered</td>
</tr>
<tr>
<td></td>
<td>Data Example: <em>I wonder if there’s a way we could work backwards.</em></td>
</tr>
<tr>
<td>MA 3</td>
<td>Validity/reasonableness of solution process is assessed/considered/tested</td>
</tr>
<tr>
<td></td>
<td>Data Example: <em>I know you can plug in the numbers, but is there a reason why that works or why you found that, besides just plugging in the numbers?</em></td>
</tr>
<tr>
<td>MA 4</td>
<td>Results (answers) are assessed/tested/considered for their reasonableness/validity</td>
</tr>
<tr>
<td></td>
<td>Data Example: <em>But we don’t know for sure sunrise is at 6.</em></td>
</tr>
<tr>
<td>MA 5</td>
<td>Reflects on the efficiency and effectiveness of cognitive activities</td>
</tr>
<tr>
<td></td>
<td>Data Example: <em>I feel like it should be harder than this, you know?</em></td>
</tr>
<tr>
<td>MA 6</td>
<td>Manages emotional responses to problem-solving situation</td>
</tr>
<tr>
<td></td>
<td>Data Example: <em>Ok we can just get up and walk away, take a break.</em></td>
</tr>
</tbody>
</table>

Note: MA – Metacognitive Action

**Building a Process-Focused Community of Inquiry**

Dr. Arkadash employed Inquiry-Based Learning (IBL) classroom instructional practices (Ernst et al., 2017). Every day, students worked together in small groups on discovery and problem-based learning activities. While having students actively engage with content is essential in helping students effectively learn mathematics (Freeman et al.,
active and inquiry-based learning classrooms encompass a broad range of classroom activity, some more productive than others. At the beginning of the Number Sense and Algebra course, students were actively engaging with mathematical content through rich mathematical tasks, but their view of mathematical problem solving was very “product-focused”. Delia described mathematical problem solving to me in her initial interview:

Emilie: What is problem solving in mathematics?
Delia: I picture it as you giving me – it depends on the math class. But you giving me, say like algebra and you’re giving me an equation, and then I have to solve for ‘x’ or whatever.

Emilie: You said it kind of depends on the class. So, what would it look like if you weren’t in algebra?
Delia: Like trig or something would be like a triangle and you’re doing the sine, cosine, or tangent to find each angle.

Emilie: Are you always given some sort of equation?
Delia: Yeah.

While working on problems in class, students worked to find the most “efficient” set of steps or rules to get to the correct answer as quickly as possible. Schoenfeld (1985) has previously described this type of problem solving. Students in the Number Sense and Algebra course expected to spend no more than five minutes on any given problem and often “embarked on a series of computations without considering their utility and failed to curtail those explorations when (to the outside observer) it became clear that they were on a wild goose chase” (p. 316). Their normative metacognitive activity was simply checking answers retroactively, in service of their product-focused problem solving.

One of Dr. Arkadash’s primary goals was for students in the course to value the process of their problem-solving attempt as much as (if not more than) getting an answer. By the end of the semester, students’ collective view of mathematical problem solving
transitioned to a “process-focused” view. Compare Delia’s response in her first interview with her response to the same question in her final interview:

Emilie: What is problem solving in mathematics?
Delia: Problem solving is understanding what you’re doing to get the answer instead of just having the fact or the trick, I guess, to get the answer. Cause that’s not really solving the problem; that’s just getting an answer. But solving it would actually be going through the process of understanding the problem and why it’s happening.

Over the semester, the classroom dynamic changed. Similar to the beginning of the semester, students were actively engaging with rich mathematical tasks, but the way they engaged became “process-focused”. Dr. Arkadash took daily actions to help students realize there is more than one mathematically valid way to solve a problem. She had students do the following.

**Explain your Thinking Process**

From the first day of class, Dr. Arkadash persistently made this rule for discourse explicit. Students were not being asked to check answers. Instead, they were being asked to share their process to get the answers. In class activities, she included written directions such as ‘Discuss what happens.’ Students were not used to these prompts: “What does she mean, ‘Discuss’? Is that what she means – should we just write that?” (Kim, Week 1 Classroom Transcript). In this way, even if Dr. Arkadash was not working directly with a group of students, they were still expected to share their ideas with each other. As she facilitated small-group work, she reiterated: “I’m not looking for an answer. I’m looking for your thinking.” Eventually, this explicit rule became an implicit norm for classroom activity, but not without her persistent encouragement.
**Evaluate and Connect Different Solution Processes**

Dr. Arkadash purposefully chose student presenters who had taken a different approach to the same problem, or had students evaluate hypothetical student work for methods they may not have initially thought of on their own. She asked students to identify the “key mathematical idea” that connected the different solutions. This was particularly effective in combating students’ belief that the most efficient method is the “best” method (conducive with a product-focused view). For example, it is mathematically valid for students to compare fractions with common numerators (same number of pieces) instead of common denominators (same size pieces), but most small groups would have been unexposed to this alternative, equally valid view without Dr. Arkadash’s intentional choice to include a presentation of a fraction-comparison method using common numerators.

**Generate Multiple Solution Paths and Strategies**

In addition to exposing students to different solution paths, Dr. Arkadash encouraged students to, themselves, generate multiple solution paths for the same problem. For instance, students were frequently required to solve the same problem using a different representation and then identify the key mathematical connections between the two representation-based strategies. Together, the class also discussed benefits and drawbacks of their problem-solving strategies. For example, students shared that trying examples to look for patterns was helpful to orient themselves to the problem, but it was not a useful tool for trying to “generalize” and show that their working conjecture would hold for all possible examples.
Through these practices, students began attending to their problem-solving process. For many students, this felt very different from their previous experiences. In her mid-semester interview, Alexis compared this aspect of Number Sense and Algebra to her high school calculus course:

[Dr. Arkadash] talks about different ways to do things and how she sees problems and how different people see different things. In calculus, that’s not really how that math class works. It’s kind of just like ‘I did it this way and that’s the right way.’… Usually calculus problems, you can do it different ways but there’s usually only one way to get the solid answer, the right answer. So, I haven’t had much exposure to ‘You can do it this way, but you can also do it that way and that way.’

Dr. Arkadash helped students develop process-focused thinking. For students to develop process-focused metacognition (thinking about thinking), this was a necessary condition. Students cannot develop metacognitive skills to be utilized during their problem-solving process if they are not attuned to their problem-solving process in the first place. Nevertheless, process-focused thinking is not a sufficient condition for students to develop the metacognitive habits of mathematical thinkers and problem solvers. In the next section, I describe how Dr. Arkadash aided students in this development by providing them explicit opportunities for reflecting on their problem-solving process in authentic problem-solving situations.

**Explicit Support for Reflection on Process-Focused Thinking**

There is a difference between students engaging in metacognitive actions in routine class work prompted by the teacher, and students self-initiating these actions on their own in authentic problem-solving situations. Students need explicit opportunities to develop the latter. To provide such opportunities, Dr. Arkadash implemented an adapted version of “portfolio” problems (Omar, Karakok, & Savic, YEAR) which have been used
previously in an upper-level combinatorics course for mathematics and computer science majors to support students’ technical writing and creative exploration of challenging mathematics problems. In the combinatorics course, open mathematics problems and theorems from mathematics research journals were used as homework problems. Students submitted their scratch work, a “clean” write-up of their work on the problem, and a three-page reflection on their thinking process using the Creativity-in-Progress Rubric on Proving (Savic, Karakok, Tang, El Turkey, & Naccarato, 2017).

Dr. Arkadash saw the portfolio-problem structure as an opportunity to help students in the Number Sense and Algebra course explicitly reflect on their problem-solving process. She modified the portfolio problems from the combinatorics course to better help the first-year undergraduates in her course. Within the following subsections, I first describe how Dr. Arkadash picked problems appropriate for students in the Number Sense and Algebra course, and I provide the problems she used in the Appendix. I then explain how she added in-class portfolio problem-solving sessions to support students in “authentic” problem solving. Finally, I share Dr. Arkadash’s portfolio–problem written assignment and the relationship between the in-class and out-of-class aspects of the portfolio problems.

Selecting Appropriate Portfolio Problems

The six portfolio problems used by Dr. Arkadash in the Number Sense and Algebra course are provided in the Appendix. In her pre-course interview, Dr. Arkadash highlighted that she wanted students to work on problems, not exercises (Schoenfeld, 1985). She considered the National Council of Teachers of Mathematics (NCTM) “worthwhile” problem criteria (NCTM, 2010, p. 1-2) when picking portfolio problems,
especially:

- The problem has important, useful mathematics embedded in it.
- The problem requires higher-level thinking and problem solving.
- The problem can be approached in multiple ways using different solution strategies.
- The problem has various solutions or allows different decisions or positions to be taken and defended.
- The problem encourages student engagement and discourse.

Aligned with her view of teaching through problem solving, Dr. Arkadash emphasized that she wanted portfolio problems to be directly connected to content, rather than having students engage in problem solving for the sake of problem solving. The timing of portfolio problems with unit content is provided in Table 10. Even though the problems selected were not open in the field of mathematics, relative to the students in Number Sense and Algebra, the portfolio problems were, on the whole, problems.
Table 10

Schedule of in-class portfolio problem-solving sessions related to unit content

<table>
<thead>
<tr>
<th>Unit</th>
<th>Week</th>
<th>Tuesday</th>
<th>Thursday</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Counting, Natural Numbers, Place Value</td>
<td>1</td>
<td>PPS 1</td>
<td>PPS 1</td>
</tr>
<tr>
<td>2. Meaning/Interpretation of Four Basic</td>
<td>2</td>
<td></td>
<td>PPS 2</td>
</tr>
<tr>
<td>Arithmetic Operations</td>
<td>3</td>
<td>PPS 2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>PPS 3</td>
<td></td>
</tr>
<tr>
<td>3. Factors, Multiples, Prime Factorization, GCF, LCM</td>
<td>5</td>
<td></td>
<td>PPS 3</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Figure 22. Meaning/Representation of</strong></td>
<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fractions, Decimals</td>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>9</td>
<td></td>
<td>PPS 4</td>
</tr>
<tr>
<td>5. Expressions, Equations, Solving Equations</td>
<td>10</td>
<td></td>
<td>PPS 4</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6. Ratio and Proportion, Functions</td>
<td>12</td>
<td></td>
<td>PPS 5</td>
</tr>
<tr>
<td></td>
<td>13</td>
<td></td>
<td>PPS 5</td>
</tr>
<tr>
<td></td>
<td>14</td>
<td></td>
<td>PPS 6</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td></td>
<td>PPS 6</td>
</tr>
</tbody>
</table>

Note: PPS – Problem-Solving Session

**In-Class Problem-Solving Sessions**

Typically, one in-class portfolio problem-solving session lasted approximately 20-25 minutes of the 75-minute class period, and students were usually given two in-class sessions to work on the problems together. Students were not expected to finish solving the portfolio problem together in class, but were expected to work on the problem further outside of class. Dr. Arkadash encouraged students to record scratch work, questions, and observations, and group members used different colored pens to identify individual contributions to the group’s collective scratch work (Figure 20). Scratch work was emailed to group members to be used outside of class.

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10 Portfolio problem write-ups were submitted one week after the final in-class session.
Incorporating the portfolio problems in class allowed problem solving to be an integral part of the classroom culture. The students, who had not experienced this type of authentic problem solving before, needed to practice using appropriate language to talk about their problem-solving process, and students could talk with each other and Dr. Arkadash about their thinking processes. To help with language, Dr. Arkadash paused the class during the second portfolio problem to make a list of problem-solving strategies and clarify their meaning. For example, as part of this class discussion students decided that ‘guess and check’ meant trying examples to look for patterns. Dr. Arkadash also suggested they use each other to practice posing questions as a problem-solving strategy to help them understand the problem. Before the third portfolio problem, Dr. Arkadash shared student examples describing the strategies they had tried, such as guess and check, finding counterexamples, using cases, working backwards, or using a representation (or manipulative) to organize thinking or generalize an observation. Note that the problem-solving strategies used by the class were student-generated, giving everyone common
language with which to talk about their problem-solving processes. By the later portfolio problem-solving sessions, Dr. Arkadash was essentially “hands off”, as students could communicate more fluidly with each other about what they were thinking.

**Out-of-Class Write-Ups**

Similar to the portfolio problems used in (Omar, Karakok, & Savic, YEAR), students submitted individual portfolio write-ups, providing their scratch work, a revised solution that included both mathematical justification and reasoning, as well as their judgement and decision-making processes during the entire problem-solving attempt. So, students were not just writing about what they did with mathematical justification for each step, but they were asked to, as best they could, document their real-time, messy, non-linear thinking process – right or wrong. Students might include questions they asked themselves like Alexis did in Figure 21, or a discussion of why they tried or abandoned a problem-solving strategy.

Students’ portfolio problems write-ups were graded using the following scheme, for a total of 15 points per write-up.

**Scratch work (3 points).** Students submitted all group and/or individual scratch work, including dead ends and errors. Scratch work, like in Figure 20, did not need to include full sentences (or even words) and was graded for submission only. Dr. Arkadash explained to students that the scratch work would show her their problem-solving process: making sense of the problem, what they tried, planned, conjectured, and so forth.
I asked myself “if the shape is made of 5 squares, then why wouldn’t it be divisible by 5?” The answer to the questions lies within the shape of the pentomino and the numbers it is placed over. If the chart had the same number in every box, instead of increasing numbers, then the number would be divisible by 5 every time.

Example:

![Diagram of a 5x5 grid with numbers and notes]

*No matter where you put the letter, the answer will always be divisible by 5, because there are 5 groups.

*Because the one’s place changes in each column, the numbers may not be divisible by 5 even though there are 5 squares.

Figure 21. A portion of Alexis’ write-up for portfolio problem three.

**Revised solution (8 points).** This three-page minimum “essay” included work on a solution path, conjecture, or idea, but this did not mean that students had to solve the problem. Students were required to address two aspects of their problem solving in this essay. First, students needed to provide a “complete” solution (4 points) with mathematical reasoning, justification, and related computations. Students also needed to explain their problem-solving process (4 points), including which solution strategies and ideas from scratch work were used or abandoned, and why. Students could also include any other observations about their thinking they made while solving the problem (similar to Alexis in Figure 21).
**Accuracy of mathematical work (4 points).** All work in the revised solution (not scratch work), was graded for mathematical accuracy. Again, this did not mean a correct solution to the problem. Dr. Arkadash informed students they may not find “THE ANSWER” because there could be multiple “answers” based on different observations. So, accuracy also did not mean checking the “final” answer. Instead, she attended to arithmetic calculations, use of mathematical notation, mathematical properties, and so forth. For example, students needed to use the equal sign accurately, instead of writing something such as ‘7 + 5 = 12 + 6 = 18 − 2 = 16’ in their revised solution.

**Student Feedback**

In addition to the micro-level analysis results of students’ real-time problem solving described in the ‘Shifted Metacognitive Habits’ section, students reflected on their experiences with the portfolio problems in their post-semester interviews. They identified all aspects of the portfolio problems (in-class problem-solving sessions, scratch work, and submitted write-ups), as well as aspects of the IBL course design, as contributing to thinking about their problem-solving process. Isley expressed this sentiment:

I think I learned more about recognizing my problem-solving process…Just the basis of my thinking, I’ve learned to recognize a little bit better. A lot better. I haven’t had to analyze my own thinking very much before, at least not in math. I feel like I’m saying that a lot. This math class is different from all the others. It’s true though, because I haven’t had to think about the way I’m thinking. It’s just, I thought about it, and it’s done. This class has helped me with analyzing that.

Lance, too, felt that reflecting on his thinking, even if it was wrong, was very much emphasized in the course:

You were supposed to reflect on even your very first thoughts when you see a problem. No matter how wrong they were you’re still supposed to mention them
and talk about them. And then say when did you realize that it was not the right
tought and why wasn’t it the right thought.

In fact, Lance was using this new skill in his English course to help him write better
essays. Further, part of reflecting is asking questions, and Lance described how asking
himself questions was a new experience:

It’s all about the asking of questions that made this course successful… You
could be asking yourself the question, and it’s understanding that you should ask
yourself those questions that really pushes you, yourself, to explain things in a
way, in just a way that’s more universally seen and understood…I’ve never really
thought of to ask myself ‘Why did you do this?’ I’ve never really done that
before…You’re basically being the person you’re explaining the issue to, when
you’re asking yourself a question.

Over the semester, Lance began turning the questions that Dr. Arkadash or other students
asked back on himself.

Kerri also talked about asking more questions, highlighting that in the last
portfolio problem her group found an equation, “and we were like, ‘Why this equation?
We have to know why!’ I don’t know if it’s just more interest in the problem, or just
because it’s gotten more challenging. We’ve been able to ask more questions.”
Persevering and overcoming challenging problems was a transformation for Kerri
throughout the semester. She lamented her experience with what Schoenfeld (1985)
would call a “wild goose chase” in portfolio problem four and how hard it is to break way
from this type of thinking:

We had the answer written down the first day in class, but there was a
computation error. So, we spent the next two days going around in a circle, only
to come back to that same answer. It was like, ‘Oh, man. Just cause I had
multiplied wrong.’…It’s so hard, once you’re in your little track...To break away
from that, because you may not, you feel like you may not know of another way.
Because the first way you think of feels like the right way. And it’s the way that
you know how to solve a problem. And so, to back up and think of it differently
feels weird sometimes.
The structure of the portfolio problems helped alleviate some of Kerri’s frustration. The write-ups gave her a space to reflect on her problem-solving process and generate new ideas. Kerri believed this was because “you’ve got to look for all the different things that you were maybe even subconsciously thinking and say those out loud more.” Further, because there was no expectation for the answer, she felt she could relax and could spend more time exploring:

A lot of the time that’s what’s frustrating with math. You’re like, ‘I am on a deadline and I have not gotten the right answer yet.’ But with [the portfolio problems], it was like ‘Okay. You can relax about it.’…If you’re stressed and you’re on a deadline, your brain’s gonna clench up and be like ‘Try it again. Try the same thing over and over and over.’ And you’re like ‘Agh!’ But if you can relax about it, you’re like, ‘I have time. I don’t need a correct answer so I might as well just try this over here and just do that and see what that does.’ And sometimes that gives you the correct answer.

What she was learning while working on the portfolio problems also translated to other parts of the course. Because emphasis on the problem-solving process instead of just the answer “parallel[ed] the whole class”, she started applying strategies from the portfolio problems on her homework exercises. Kerri “got better about sitting there, and actually really thinking about it, and solving through those, and explaining those better.” Just as she did with the portfolio problems, she would step away from homework exercises to “subconsciously process” them before coming back to work again.

Alexis also described a similar transformation, learning to stay “actively” engaged with her problem solving, instead of waiting until after problems were graded to reflect on her process:

I’ve just been actually actively thinking about it as I’m going. Before the class, I think I’ve done that kind of stuff, but more after the fact. I’m like, ‘Oh, I solved the problem, awesome.’ And then I get it back and it’s wrong… ‘Why did I write that?! What am I doing?!’… I’ve just developed that. I’ve just been able to be actively engaged in the problem, realizing what I’m doing [as I’m doing it].
Alexis felt that writing down questions in her scratch work helped her keep track of her ideas: “I could see the exact questions I had at that time.” She added that the act of writing down her in-the-moment questions “makes them more important…I feel like writing it down makes me realize like, ‘Oh I probably had a reason that I wanted to answer it. It would get me to another step or get me to understand something else.’”

**Conclusion**

The Number Sense and Algebra course assisted students in reflecting on their problem-solving process and developing “process-focused” metacognition. The IBL structure of the course allowed them to practice the communication skills to describe their problem-solving process, and the portfolio problems gave them an explicit setting within which to reflect on their thinking in authentic problem-solving situations. In light of the ideas discussed in this paper, I conclude by advocating for classroom design strategies to help students, at any level, develop “process-focused” metacognition while problem solving.

**Build classroom norms that value process over product (or at least equal to).**

This takes time and persistence, and the suggestions provided in the ‘Building a Process-Focused Community of Inquiry’ section can help. Through class activities, your own facilitation, or student presentations, encourage students to (1) explain their thinking processes, (2) evaluate and connect different solution processes, and (3) generate multiple solution paths and strategies themselves. Additionally, holding students accountable for their process, such as with portfolio problem write-ups, can support this norm more explicitly.
**Bring students’ process into the classroom.** For process-focused norms to develop, process needs to be integrated into classroom culture. Even if students are working in small groups instead of listening to a lecture, they may just be sharing answers. Instead of having students present final products, proofs, or answers, have them explain their thinking process. How did they orient themselves to the problem? Why did they pick that particular strategy? What did they try that didn’t work? How did they get back on track? Have multiple students present different thinking processes and solution strategies, and have students evaluate the appropriateness of these strategies.

**Have students reflect on, and write about, their “real-time” process.** Experience, without learning, is nothing. Reflective writing can be useful in many aspects of the mathematics classroom (e.g., Karaali, 2015), but to help students *know to act in the moment* (Mason & Spence, 1999), they should reflect on their in-the-moment or “real-time” problem-solving process. Portfolio problem write-ups helped Dr. Arkadash in her Number Sense and Algebra class, but incorporating writing into in-class problem-solving sessions may support this type of reflection even further.

**Provide authentic problem-solving situations.** For students to develop process-focused metacognition towards becoming independent thinkers and problem solvers, they need opportunities to practice and develop these tools. Note that the open mathematics problems for upper-level math majors in a combinatorics course (Omar, Karakok, & Savic, YEAR) may not be suitable problems for all freshman, non-majors. However, Dr. Arkadash was still able to find “worthwhile” problems (NCTM, 2010) to provide these opportunities for her students.
Everyone can problem solve. Reiterating the previous point, as Paul Lockhart would argue in *A Mathematician’s Lament* (Lockhart, 2009), all students should have opportunities to paint on a blank canvas:

If everyone were exposed to mathematics in its natural state, with all the challenging fun and surprises that that entails, I think we would see a dramatic change both in the attitude of students toward mathematics, and in our conception of what it means to be “good at math.” We are losing so many potentially gifted mathematicians—creative, intelligent people who rightly reject what appears to be a meaningless and sterile subject. (p. 7)

Students do not need to be math majors in an upper-level proof course to engage with true, authentic mathematics problems and consequently develop the habits of mind of a mathematical thinker and problem solver. So, I end this paper with a challenge and a question: How can you provide access for all students to authentic problem-solving situations? Are we limiting students by not providing authentic problem-solving experiences?

**Appendix**

**Portfolio Problem 1:** (Dr. Steven Leth, personal communication) The last digit of a number is a 0 when it is represented in base 5 and a 1 when represented in base 2. What is the last digit when it is represented base 10?

**Portfolio Problem 2:** (Driscoll, 1999) *Take a three-digit number, reverse its digits, subtract the smaller from the larger. Reverse the digits of the result and add it to the original result. For example,*

123 becomes 321, and $321 - 123 = 198$

198 becomes 891, and $198 + 891 = 1089$

*Try this process with several numbers. What do you observe? Why?*
Portfolio Problem 3: (Liljedahl et al., 2007) A pentomino is a shape that is created by joining five squares such that every square touches at least one other square along a full edge. There are 12 such shapes, named for the letters they most clearly resemble.

Now consider a 100’s chart! If a pentomino is placed somewhere on a 100’s chart, will the sum of the numbers be divisible by 5? If not, what will the remainder be? Explain how you can know “quickly”!

Portfolio Problem 4: (Northern Colorado Math Circles, 2013) Find four different digits a, b, c, d so that the sum \(\frac{a}{b} + \frac{c}{d} < 1\) and the sum is as close to 1 as possible. Justify why your answer is the largest such number less than 1. (When we say a, b, c, d digits, we mean that they can be any whole number between 0 and 9.)

Portfolio Problem 5: (Adapted from Dorichenko, 2011) At sunrise, two old women started to walk towards each other. One started from point A and went towards point B while the other started at B and went towards A. They met at noon but did not stop; each one continued to walk maintaining her speed and direction. The first woman came to the point B at 4:00 pm, and the other one came to point A at 9:00 pm. At what time did the sun rise that day?

Portfolio Problem 6: (Adapted from Mathematics Achievement Partnership, 2002) Below is a triangle formed with numbers.
• What are the first and last numbers in the \( n \text{th} \) row? (E.g., the first number in 3\(^{rd} \) row is 7 and the last number in the 3\(^{rd} \) row is 11). Justify your answer.

• What is the sum of the numbers in this \( n \text{th} \) row? (E.g., the sum of the numbers in the 3\(^{rd} \) row is \( 7 + 9 + 11 = 27 \).) Justify your answer.

• What is the sum of all the numbers up to and including the \( n \text{th} \) row? (E.g., the sum of the numbers up to and including the 3\(^{rd} \) row is 36). Justify your answer.

• What other patterns do you notice in this triangle? Justify your answer.

CHAPTER V

DISCUSSION

Chapter V begins with a summary of the dissertation study that sought to capture the development of metacognitive norms during problem solving, followed by a brief discussion of major findings explicating how the results presented in the standalone manuscripts in Chapters III and IV directly address the research questions. Research and teaching implications of these findings are then presented, followed by limitations and delimitations of the study. Finally, I make suggestions for future research to promote process-focused metacognition during mathematical problem solving.

Summary of the Study

Students and teachers should develop the habits of mind of mathematical thinkers and problem solvers (CBMS, 2012; CUPM, 2015). Metacognition is one such habit of mind (Stacey et al., 1982; Selden & Lim, 2010). However, research concerning students’ learning of metacognition has not necessarily translated to students’ natural, purposeful classroom activity (Carroll, 2008), largely ignoring sociocultural factors (Larkin, 2015) that mediate metacognitive actions taken during problem solving. Much of the previous metacognition research has used an “acquisition” metaphor for learning (Sfard, 1998). The purpose of this research study was to investigate the teaching and learning of metacognition in a natural classroom setting using a “participation” metaphor for
learning, which emphasizes the role of language-based interactions and norms. This research sought to address the following research questions:

Q1  How do metacognitive norms during problem solving evolve in an undergraduate mathematics community of practice?

Q1a  What are the normative metacognitive actions taken by students in authentic problem-solving situations?

Q1b  What contradictions or tensions catalyze changes in the object (problem solving) of the student activity system?

Q1c  What is the relationship between the metacognitive norms identified in Q1a and changes in students’ problem-solving activity identified in Q1b?

Q1d  What is the role of the teacher in negotiating students’ metacognitive development?

This qualitative research was grounded in Vygotsky’s (1978, 1986) conception of language-based, mediated action and utilized activity theory (Leont’ev, 1979; Engeström, 1987/2015) as an analytic framework to operationalize the participation metaphor for learning. Further, it is assumed that semiotic mediation of activity is a reflexive process (Ernest, 2010), with both students and the teacher negotiating the development of classroom activity. To address the aforementioned research questions, I collected data as an observer participant (Gold, 1958) in an undergraduate mathematics content course for pre-service elementary education teachers with mathematics emphasis. Six data sources were collected in the 15-week semester: (1) video- and audio-recorded classroom sessions, (2) three videotaped, semi-structured individual interviews with 15 of the 24 students at the beginning, middle, and end of the course, (3) two audio-recorded interviews with Dr. Arkadash, (4) students’ written artifacts (assignments, exams, and portfolio-problem submissions and scratch work) collected before grading, (5) recorded
planning sessions with Dr. Arkadash, and (6) journal reflections written by Dr. Arkadash and myself. I conducted micro-level analysis of students’ real-time problem solving during in-class portfolio problem-solving sessions to identify normative metacognitive activity (Q1a). In subsequent macro-level analysis based on all data sources, I used activity theory as an analytic framework to delineate the initial student activity system and identify contradictions or tensions influencing the development of students’ problem-solving activity (Q1b). The last phase of macro-level analysis also coordinated students’ problem solving with the normative metacognitive actions during in-class portfolio problem-solving sessions (Q1c) and the role of the teacher’s goals and actions in shaping this development (Q1d).

Summary and Discussion of Major Findings

Major findings were described and synthesized in Chapters III and IV. The purpose of this section is to explain how those findings address the research questions outlined in Chapter I. Further, due to the nature of theoretical perspective (see Chapter II) and methodological tools utilized to analyze real-time metacognition in the context of mathematical problem solving, this section includes discussions evaluating the methodology employed in this dissertation study.

Documenting Metacognitive Participation

In Chapter II, I asserted that only relying on an “acquisition” metaphor for learning (Sfard, 1998) can be limiting in understanding and characterizing students’ development of metacognitive skills as a mathematical habit of mind, for use in authentic problem-solving situations. A “participation” metaphor for learning can better capture the in-the-moment, habitual nature of mathematical habits of mind, where the emphasis is on
actions taken or “doing” (Sfard, 1998). From the lens of participation in a community of practice, learning mathematical habits of mind means coming to know the language and norms of the “expert” mathematical problem-solving community in authentic problem solving situations. To evaluate students’ collective, normative activity in authentic problem-solving situations, answering research question Q1a, language-based social interactions were analyzed within the context of in-class portfolio problem-solving sessions. Based on a process of iterative coding (see Chapter III and Appendix D), I analyzed in-class portfolio problem-solving sessions using a list of six metacognitive actions. These sessions were chosen to maximize the likelihood that the portfolio problems were, in fact, problems for the collective student activity system, giving students opportunities for legitimate peripheral participation (Lave & Wenger, 1991). To document collective participation, I sought to determine which metacognitive actions were taken “as if it is a mathematical truth in the classroom…as if everyone has similar understandings, even though individual differences in understanding may exist” (Rasmussen et al., 2015, p. 262-3).

Answering research question Q1a, micro-level analysis of students’ collective problem-solving activity during in-class portfolio problem-solving sessions revealed that the normative activity of students in the Number Sense and Algebra course during authentic problem-solving situations changed over the course of the semester. At the beginning of the course, students were largely focused on checking final answers, heavily relying on Metacognitive Action 4 (MA4): Results are assessed/tested/considered for their reasonableness/validity. By the end of the semester, as students became more attuned to their thinking processes, they began relying on more “process-focused”
metacognitive actions, especially Metacognitive Action 2 (MA2): Various solution approaches or strategies are assessed and considered.

In Chapter III, I illustrated students’ shift from product-focused to process-focused metacognitive actions (MA4 to MA2) by contrasting students’ problem-solving activity while working on portfolio problem two with their problem-solving during portfolio problem five. Lance and Kerri’s work on portfolio problem two highlighted students’ reliance on using guess-and-check to search for patterns in order to generate a list of rules. Their problem solving at the beginning of the semester could best be characterized with Schoenfeld’s (1985) notions of “wild goose chases” and 5-minute limits (see Chapter IV). Consequently, students’ collective metacognitive activity focused on checking answers retroactively (MA4) in service of their product-focused problem solving. A lack of process-focused metacognition was especially evident when students were stuck. Dr. Arkadas encouraged them to consider alternative problem-solving strategies such as using manipulatives or variables (e.g., ‘abc’ in portfolio problem two), but students were focused on finding the most “efficient” way to get to an answer or rule. This focus prevented them from moving their own problem solving forward, relying on the instructor to help them overcome struggles and roadblocks while problem solving.

There was a disconnect between students’ current problem-solving activity and Dr. Arkadas’s goals and expectations.

Students’ view of problem solving had changed by the end of the semester, from a focus on answers (realized through facts and tricks) to a focus on the process of understanding and thinking (e.g., Delia’s pre- and post-semester definitions of mathematical problem solving provided in Chapter IV). Paula, Skylar, and Jordan’s
problem-solving activity in portfolio problem five (see Chapter III) demonstrates a
transition in students’ problem solving and related metacognitive activity. The group spent most of their time together in class considering and assessing different representations (MA2) to help them solve the problem. Although students struggled to solve the portfolio problem, the use of different representations allowed them to self-initiate mathematical action toward solving the problem (e.g., changing representations), rather than wait for Dr. Arkadash to move their thinking forward.

Because students’ collective metacognitive activity at the end of the semester was process-focused, students spent more class time attending to the earlier phases of the problem-solving cycle and worked outside of class to finish the problem, often together. Classroom recordings of the problem-solving sessions were unable to capture the later phases of students’ collective problem-solving activity. The product-focused MA4 was not recorded heavily during in-class problem-solving sessions, but was most likely still present. Students’ written portfolio submissions offered evidence of this type of metacognitive activity. For example, in Jordan’s write-up for portfolio problem five, she described having a text conversation with Paula and Skylar:

That afternoon, Paula sent a message to the group to see if any of us had made any progress on the problem. Minutes later, Skylar sent a message saying that she might have solved it, but she wasn’t completely sure it was correct...Although I saw how Skylar got her answer, I [was] still a little confused on why she did some of the equations she did.

Jordan described how she used MA4, but this was not captured in real time. There is no way to know how distorted her portrayal of what actually happened may be.

By investigating students’ metacognitive actions taken in real time, in the natural context of day-to-day classroom activity, this study addresses the call made by Carroll
She lamented that students were not benefiting in the classroom from years of metacognition research coming from cognitive psychology, citing and agreeing with Neisser (1976) who contended that we must “understand cognition in the context of natural purposeful activity” (p. 7). Moreover, documenting students’ actual problem-solving activity during portfolio problem-solving sessions through video- and audio-recordings, and triangulating this activity with recordings of daily classroom activity, allowed me to more effectively determine which actions were initiated by Dr. Arkadash and which were self-initiated by students. Actions initiated by students as opposed to actions taken by students in response to teacher prompting can provide evidence of students’ increased participation in the mathematical problem-solving community of practice.

As such, documenting student participation in metacognitive thinking requires data collection and analysis tools to help distinguish between self-initiated student actions and teacher-initiated actions as part of “playing school”. For instance, at the beginning of the semester, students typically did not consider various solution approaches unless they were prompted by the instructor. In portfolio problem two, Dr. Arkadash encouraged students to consider using variables ‘abc’ or base-block manipulatives. In their portfolio write-ups, students wrote as if these considerations were their own idea. Even though students were instructed to write about their own thinking process, the product-nature of the mediating tool was unable to recover students’ agency in this aspect of their problem-solving process.
Metacognition as a Sociocultural Construct

Micro-level analysis addressing research question Q1a documented a shift in students’ metacognitive activity in authentic problem-solving situations within a natural, purposeful classroom context. However, only answering this research question does not provide insight into how the classroom context afforded such a transformation. In this section and the next section (‘The Role of the Teacher’), I describe how macro-level analysis was used to answer research questions Q1b, Q1c, and Q1d and shed light on what can be done in other settings to help students transition from only product-focused metacognitive actions to adopting more process-focused metacognition during problem solving. In particular, activity theory (Leont’ev 1979; Engeström, 1987/2015) was utilized as an analytic framework to operationalize the participation structure of the student activity system and account for social mediators of activity (see Chapter II). Detailed results of macro-level analysis are presented in Chapter III and discussed in Chapters III and IV.

To answer research questions Q1b and Q1c, I used a six-step procedure outlined by Jonassen and Rohrer-Murphy (1999) to describe the activity system and identify catalysts for changes to the object of the system. The first five steps of this process systematically delineated various components of the initial student activity system (Table 6), and the sixth step identified contradictions or tensions that catalyzed changes in the object (problem solving) of the student activity system. There were two key features of the student activity system that notably shaped the development of their problem solving and related metacognitive activity. First, students had beliefs about mathematics and the
norms of mathematical activity consistent with the archetypal “traditional” mathematics classroom, similar to Ernest’s (1989) Platonist or Instrumentalist conceptions of mathematics. Dr. Arkadash was viewed as the authority of mathematical knowledge (concepts) and thinking (processes). She was keeper of the “right” way to think about and explain mathematical concepts, the “correct” answers, and the most “efficient” methods to get there. When students worked together, they were trying to decipher the “right”, “correct”, “efficient” answer, and their individual processes was effectively irrelevant in this endeavor. Moreover, it was Dr. Arkadash’s job to ask questions to move student thinking forward. For instance, on the first day of class, Dr. Arkadash asked students to discuss group norms for working together in the course. Mary stated she wanted Dr. Arkadash to walk around and ask students questions to elicit student thinking, since she could “do it, but not explain it – and that’s the role of the teacher.”

The second relevant characteristic of the student activity system was related to how students balanced two competing identities in the course, appealing frequently to two different communities to which they belonged. While participating in the course, they primarily appealed to themselves as future elementary school teachers. As future teachers, students identified wanting to learn (1) to teach the mathematics concepts they would be required to teach as elementary school teachers, (2) multiple methods or approaches for problems to appeal to the diverse needs of their future students, and (3) to improve explaining and showing their steps to help struggling students understand material. To achieve these teaching goals, students recognized a secondary course goal, appealing to themselves as students of mathematics. Students wanted to improve their mathematics skills, the “fundamental”, elementary concepts. Several students also
described a desire to build a growth mindset, referencing a video from Jo Boaler that Dr. Arkadash sent them before the first day of class. Students sought to build confidence, resilience, and not be afraid to make mistakes.

In the final step of analysis, I identified changes to the object of the student activity system (problem solving) over the course of the semester (research question Q1b), which is outlined in detail in Chapter III (see Table 8 for a summary). Students’ collective metacognitive activity transitioned from product-focused to process-focused, developing concurrently with transformations in their problem-solving activity (research question Q1c). At the beginning of the semester, students’ “traditional” beliefs about mathematics and mathematics classrooms influenced their problem solving. When faced with an Inquiry-Based Learning (IBL) classroom setting, they were forced to resolve a tension between the explicit IBL classroom norms and their current views of mathematics and classroom norms. Students adapted to new classroom rules to focus on their thinking processes, but now had to navigate their own thinking, which they had not necessarily attended to meaningfully prior to the Number Sense and Algebra course. In Chapter IV, I identified actions taken by Dr. Arkadash to prompt and support this change, which are also summarized in the next section.

In day-to-day classroom activities, students were learning methods to navigate their thinking. Specifically, they learned to make “mathematical” justifications by connecting to definitions and using multiple representations. In the context of authentic problem-solving situations, seen in the in-class portfolio problem-solving sessions, students were struggling to apply these tools. Navigating both student and (future) teacher identities created tensions for students during the course, resulting in a gap
between the tools they were using while wearing their “teacher” hat and using these tools for their own development as students. For example, on the fourth day of class Molly told her group she did not understand why they had to learn to add and subtract in different bases since they would never teach this to their future elementary school students. Flexibly working with different representations of numbers was in service of their development as students, and it was not until the end of the semester that students merged their day-to-day problem solving as “teachers” with their portfolio problem solving as “students”. Eventually, while engaging in the portfolio problems, students began to assess and consider various representations as strategies to help them solve problems (MA2).

The Role of the Teacher

In Chapter III, I described how Dr. Arkadash intentionally shifted her short-term goals and actions to help students more independently navigate their problem solving in authentic problem-solving situations (summarized in Table 8), directly addressing research question Q1d. Her goals and actions were in response to students’ current problem-solving activity. In the first few days of class, students would just “do” the problem, picking the first problem-solving strategy that came to mind without reflecting on its appropriateness. After getting an answer, they collectively engaged in retroactive metacognition by checking answers with each other. Dr. Arkadash responded by soliciting students’ thinking to create a process-focused community of inquiry. As outlined in Chapter IV, using Inquiry-Based Learning (IBL) techniques (Ernst et al., 2017) she encouraged students to explain their thinking processes to her and each other, evaluate and connect different solution processes presented by different students or from sample student work, and generate multiple solution paths and strategies themselves.
Students began sharing their thinking processes, but they were unable to push their own problem solving forward when they were stuck. Dr. Arkadash aided students in this endeavor through a sequence of goal shifts. First, she requested that students justify their processes with “mathematical” meaning. When students struggled with this justification, she encouraged them to focus on language, “connect to definitions”, as a method for mathematically justifying their reasoning. Finally, Dr. Arkadash noticed that although students were developing these tools, they were not using them in the portfolio problem-solving sessions when they were stuck. In response, she highlighted the use of multiple representations to mediate connections. Students were finally able to break through when they were stuck, shifting representations to better navigate their problem-solving process. Consequently, considering multiple representations was a concrete tool to support students’ process-focused metacognition while problem solving.

At the end of the semester, students had made progress towards becoming more self-sufficient problem solvers, using process-focused metacognition in addition to checking final answers, and problem solving in a way that more closely resembled “expert” mathematical thinkers and problem solvers. However, students in the Number Theory and Algebra course did not initially have tools to support their process-focused metacognition. Research on the teaching and learning of metacognition has evidenced that “[s]chool teachers and mathematics educators should explicitly instruct metacognitive knowledge and model and teach metacognitive skills to their children about mathematics learning” (Baten et al., 2017, p. 8). This dissertation study supports the claim by Baten et al., but provides information concerning the situated nuances of what instruction supporting metacognition can look like.
Attention must be given to the way in which metacognitive skills are adopted and adapted by students in the context of classroom dynamics. In particular, Lave and Wenger (2002) caution against ‘didactic caretakers’, such as classroom teachers, “assum[ing] responsibility for motivating newcomers” (p. 122):

In such circumstances, the focus of attention shifts from co-participating in practice to acting upon the person-to-be-changed…Where there is no cultural identity encompassing the activity in which newcomers participate and no field of mature practice for what is being learned, exchange value replaces the use value of increasing participation. The commodization of learning engenders a fundamental contradiction between the use and exchange values of the outcome of learning, which manifests itself in conflicts between learning to know and learning to display knowledge for evaluation. (p. 122, emphasis added)

Thomas (2012) proposed the use of metacognitive conflict, analogous to cognitive conflict, to present a motivating need for students to consider new conceptions of thinking about their thinking. Mediated by the portfolio problems, Dr. Arkadash challenged students to consider an alternate conception of their current mathematical problem solving, namely a focus on the process rather than the answer or product. A need to be successful in this new space of problem solving facilitated students’ alternate conception of metacognitive thinking, specifically an emphasis on process-focused metacognitive actions.

Students’ Increased Agency

Results of analysis using activity theory to situate findings and delineate the participation structure of the classroom revealed a non-vertical transformation of student problem solving. By non-vertical, I mean that while Dr. Arkadash had a conception of process-focused mathematical problem solving and metacognition consistent with the community of mathematical problem solvers for which she is a cultural representative, the result of semester-long negotiation did not illustrate vertical movement from students’
“everyday” (Vygotsky, 1978) conception of mathematical problem solving and related metacognition to her “scientific” or “expert” conception of mathematical problem solving and related metacognitive activity. On one hand, this research describes how students made progress towards habitually using process-focused metacognition consistent with “expert” problem solvers, implicitly painting a picture of enculturation into the mathematical thinking community of practice. On the other hand, students’ resulting metacognitive activity, while looking more similar to that of a mathematical thinker and problem solver, was “off.” Students relied heavily on MA2 to help them get unstuck, as opposed to other potential metacognitive actions or combinations of actions. Further, students utilized a form of MA2 based on mathematical representations, instead of other possible problem-solving strategies.

Rather than theorizing learning as vertical, Engeström (2001) proposed a complementary theory of emergence and horizontal expansion. Catalyzed by contradictions or tensions, the object and related goals of the student activity system were “reconceptualized to embrace a radically wider horizon of possibilities than in the previous mode of the activity” (p. 137). In the Number Sense and Algebra course, students’ problem solving transformed as a series of new conceptions of problem solving emerged (Table 8). Moreover, these conceptions of problem solving can be viewed as “sideways” moves as a result of negotiation and hybridization between Dr. Arkadash’s “scientific” view of mathematical problem solving and students’ “everyday” conceptions. Engeström (2003) described this process:

Multiple competing ideas often emerge and collide as candidates for the new concept. In such contexts, concept formation typically occurs as stepwise two-dimensional negotiation and hybridization. The first step may be a debate between an administratively given pre-articulated (‘scientific’) concept and situated
articulations of (‘everyday’) experience. This may lead to a proposal for an alternative ‘scientific’ concept, again contested by some participants on experiential grounds, etc. (p. 3)

Instead of merging the everyday experiences of students with Dr. Arkadash’s conception of mathematical problem solving, alternate conceptions of problem solving emerged. Students’ problem solving (and resulting metacognitive activity) by the end of the semester can be viewed as a sequence of sideways negotiations of problem solving.

Considering students’ development of problem solving and metacognition as non-vertical and “sideways” implies a second, horizontal dimension of learning. In this study, student agency in negotiating their problem-solving activity impacted students’ trajectory of participation (Boaler & Greeno, 2000) in mathematical problem solving (Figure 22). Boaler and Greeno found that students in discussion-based classrooms were afforded more agency, as they were “required to contribute more of themselves” (p. 189). They argued that students in traditional classrooms have less agency because they do not participate in judgement and decision-making processes. This dissertation study illustrates what “more” means in the context of mathematical problem solving and how Dr. Arkadash supported students in this development. As is discussed in Chapter IV, even though students in the Number Sense and Algebra course were in an active, inquiry and “discussion”-based learning environment, at the beginning of the semester their product-focused view of mathematical problem solving limited their ability productively deal with struggle while problem solving. Dr. Arkadash first had to build a process-focused community of inquiry, and the portfolio problems and write-ups provided explicit support for students’ reflection on process-focused thinking.
Anderson, Nashon, and Thomas (2009) highlight that investigating natural, purposeful metacognition necessitates sufficient data collection and analysis methods, and they argued that different metacognitive skills may require different assessment methods. For example, questionnaires and stimulated recall techniques are not necessarily effective tools for assessing real-time task performance, as they are subject to memory distortions (Baten et al., 2017). This study has taken steps to address these methodological concerns, by investigating students’ real-time metacognitive activity in the natural context of a mathematics classroom, as well as by utilizing video- and audio-recorded problem-solving sessions to capture in-the-moment activity. Students’ verbal data during in-class portfolio problem-solving sessions provided a more accurate representation of students’ real-time metacognitive actions, minimizing the potential for
memory distortion that can occur with methods such as questionnaires or stimulated recall. While other researchers have successfully utilized videotaping of real-time activity with coding (e.g., Goos, 2002; Stillman, 2011), Vorhölter (2018) argued that scaling this method beyond a few tasks to be used more generally “would be costly in terms of time and money” (p. 5) and cannot measure nonverbalized metacognitive actions. However, in the context of describing collective metacognitive activity, the combination of micro-level analysis of tasks throughout the semester with macro-level analysis of classroom activity seemed effective in documenting change for a class over a semester.

In this study, I was able to document a more complete problem-solving cycle at the beginning of the semester, because the collective activity was focused on the “checking” phase (Carlson & Bloom, 2005) of the problem-solving cycle. Thus, I could capture students’ entire problem-solving attempt. However, data collection tools are needed to capture students’ real-time, authentic (i.e., not visibly distorted) metacognitive activity outside of class. At the end of the semester, students used in-class sessions engaging in earlier phases of the problem-solving cycle. As such, classroom recordings of the problem-solving sessions were unable to capture the later phases of students’ collective problem-solving activity. As mentioned in the previous Major Findings section, MA4 was still present in students’ problem solving, but was not visible in classroom problem-solving sessions. A recording of this conversation and work via a smart pen or online collaboration tool could provide better insight into students’ entire problem-solving process.

In addition to the micro-level analysis tools to document metacognitive actions, this research provided insight into how metacognition manifested itself during students’
problem solving in authentic problem-solving situations, as called for by Carlson and Bloom (2005, p. 46). The “expert” problem solvers studied by Carlson and Bloom utilized metacognitive actions throughout the problem-solving cycle, but students in the Number Sense and Algebra course did not engage in metacognition as a habit of mind of mathematical thinkers and problem solvers. This work provides evidence of one possible trajectory for students’ development of process-focused metacognitive actions during problem solving (MA4 to MA2), though these outcomes were very much grounded in the student population of pre-service teachers and the Inquiry-Based Learning classroom environment. To understand how students’ metacognition manifests in problem-solving situations, and how students develop the metacognitive tools of mathematical thinkers and problem solvers, more research is needed across different contexts. Related to the idea that the metacognitive activity of students in the Number Sense and Algebra course was situated is the domain-specific nature of the metacognition that was employed by students. Students in the course engaged in a particular form of Metacognitive Action 2. Their assessment and consideration of different problem-solving strategies was a consideration of different mathematical representations. As such, this research supports the notion of domain-specific aspects of metacognition (Scott & Berman, 2013; Vo, Li, Kornell, Pouget, & Cantlon, 2014).

Viewing results of this study as a sequence of horizontal or sideways moves resulting from negotiation and hybridization of the teacher’s and students’ conceptions of problem solving has implications for future research on metacognition during problem solving. This dissertation research was originally conceptualized as a design experiment (Cobb, Confrey, DiSessa, Lehrer, & Schauble, 2003) in which the portfolio problems
acted as a possible intervention tool to help transform students’ metacognitive activity during problem solving. However, the outcomes of using this tool were not as conceptualized by myself as the researcher, or as Dr. Arkadash implementing this tool to help students develop problem solving tendencies of mathematical thinkers. In design experiment research, there is an “assumption that researchers know what they want to implement, how they want to change the educational practice” (Engeström, 2011, p. 599). Engeström argues that this type of research implicitly treats learning and interventions to improve learning as a linear process. “Design experiments aim at closure and control…The implication is that the researchers have somehow come up with a pretty good [theoretical] model which needs to be perfected in the field” (p. 601). Agency in design experiments is given to the researchers and minimizes the agency of participants in shaping the outcomes of such research. However, in this study the resulting problem solving activity and related metacognitive actions were greatly influenced by the students’ agency and resistance in the class. Building on Vygotsky’s “experimental-genetic method” (Vygotsky, 1978), Engeström (2011) proposed “Formative Interventions” as an alternative to design research, describing the differences between these two methods in Table 11.
Table 11

*Comparison of Design Experiment Research and Formative Intervention (Engeström, 2011, p. 606, emphasis added)*

<table>
<thead>
<tr>
<th>Feature</th>
<th>Design Experiment Research <em>(variable-based research)</em></th>
<th>Formative Intervention <em>(process-oriented research)</em></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Starting Point</strong></td>
<td>The contents and goals of the intervention are known ahead of time by the researchers, and the intervention itself is commonly detached from vital life activities of the participants</td>
<td>Participants...face a problematic and contradictory object, embedded in their vital life activity, which they analyze and expand by constructing a novel concept, the contents of which are not known ahead of time to the researchers.</td>
</tr>
<tr>
<td><strong>Process</strong></td>
<td>The participants, typically teachers and students in school, are expected to execute the intervention without resistance. Difficulties of execution are interpreted as weaknesses in the design that are to be corrected by refining the design.</td>
<td>The contents and course of the intervention are subject to negotiation and the shape of the intervention is eventually up to the participants... [T]he participants gain agency and take charge of the process.</td>
</tr>
<tr>
<td><strong>Outcome</strong></td>
<td>The aim is to complete a standardized solution module, typically a new learning environment, that will reliably generate the same desired outcomes when transferred and implemented in new settings.</td>
<td>The aim is to generate new concepts that may be used in other settings as frames for the design of locally appropriate new solutions. <em>A key outcome of formative interventions is agency among the participants.</em></td>
</tr>
<tr>
<td><strong>Researcher’s Role</strong></td>
<td>The researcher aims at control of all the variables.</td>
<td>The researcher aims at provoking and sustaining an expansive transformation process led and owned by the practitioners.</td>
</tr>
</tbody>
</table>

A significant theme in this dissertation research was the agency of students in negotiating their problem-solving activity and consequently shaping their metacognitive activity while problem solving. When faced with the initially problematic and contradictory object (portfolio problems) embedded in their problem-solving activity,
students analyzed and expanded by reconstructing a novel conception of problem solving. While not a researcher, Dr. Arkadash responded to students’ expansion to support and sustain their transformation. The situated nature of the portfolio-problem intervention, and the influence of students in negotiating their problem solving with Dr. Arkadash, means that students’ resulting problem solving and related metacognitive actions was not something to be controlled in future refinements for “standardized” implementation, but rather a conception of problem-solving activity that could frame implementation in “locally appropriate” situations. Additionally, by centering students’ agency, formative interventions have the potential to better support an advancement narrative, rather than a deficit or “gap” narrative of research (Gutiérrez, 2008). Gutiérrez suggests the use of more contextualized intervention studies in this endeavor, and formative interventions have the potential to provide such a framing.

**Implications for Teaching**

**Metacognitive Training and Interventions**

Related to the previous discussion in the ‘Implications for Research’ section, the metacognitive interventions described in Chapter I generally had top-down, vertical qualities, where a teacher or researcher defined and facilitated the practice of metacognitive behaviors that were to be consumed and utilized by students. This research provides evidence that this assumption may not be warranted, at least for aspects of metacognition related to mathematical problem solving. For students in the Number Sense and Algebra course, the shift in students’ metacognitive norms during problem solving was modest and took an entire semester to be adopted by students. This work calls into question the lasting effects of metacognitive interventions that were either
disconnected from the regular mathematics instruction and classroom norms or too short in length (or both). As such, this study corroborates findings from Bond and Ellis (2013) that such intervention effects have low long-term retention and supports the idea that metacognitive training needs to be embedded in content for an extended period of time (Lester et al., 1989; Veenman et al., 2006).

Moreover, this research suggests that the “embeddedness” is beyond just the content, but also embedded in the collective classroom culture, as well as students’ normative problem-solving activity. While research in instructional design to foster metacognition is increasingly attending to social metacognition, collaborative learning environments (largely via reciprocal peer tutoring), are “assumed not only to encourage children in the processes of adopting and refining their personal metacognitive skills, but are also assumed to engage them in social forms of regulation skills as well” (Baten et al., 2017, p. 3, emphasis added). Students in the Number Sense and Algebra course were working together at the beginning of the semester, but this was not enough to help them develop metacognitive skills. Creating an environment that values process-focused problem solving and reflection on this process was critical in effecting change in this study.

Creating this environment was achieved partly through Inquiry-Based Learning classroom practices and is discussed in the next subsection. Additionally, Dr. Arkadash used the portfolio problems as an explicit setting for students to build awareness of their problem-solving processes in authentic problem-solving situations, as well as an assessment to extrinsically motivate students to focus on their thinking. By incorporating authentic problem-solving situations to have students reflect on, and write about, their
“real-time” process, the portfolio problems provided an explicit mechanism for building process-focused metacognitive skills that was embedded in the day-to-day culture of the classroom.

While other researchers have emphasized the use of writing to promote deep learning and problem solving in mathematics (e.g., Hensberry & Jacobbe, 2012; Jaafar, 2016), this study emphasizes the use of writing to not only help students document their problem solving, but also as an explicit tool to reflect on their real-time problem-solving process. Jansen, Cooper, Vascellaro, and Wandless (2017) describe “rough draft talks” as a way to support classroom discourse that values “talking to learn” (p. 304) instead of correct or complete final products or drafts. The portfolio problems used by Dr. Arkadash are in a similar vein, providing students with a space to take risks and reflect on their thinking, whether correct or incorrect, final or incomplete.

Finally, students’ metacognition during problem solving in the Number Sense and Algebra course was significantly influenced by classroom norms related to problem solving. For example, the use of Metacognitive Action 2, various solution approaches or strategies are assessed and considered, was deeply tied to classroom emphasis on using different representations to support mathematical justification during problem solving. The classroom problem-solving activity influenced the nuances of how students thought about their thinking. Thus, teachers should be aware of the intimate relationship between problem solving norms and their effect on metacognition. Teachers can use this information to plan problem solving discussions and activities to support various aspects of metacognitive activity.
In Chapter IV, I discussed the role of IBL classroom practices to support students’ process-focused metacognition. Dr. Arkadash was able to help students in the Number Sense and Algebra course develop process-focused metacognition while engaging in authentic problem-solving situations. One necessary condition in guiding students to habitually relying on process-focused metacognitive actions was shifting students’ focus on products or answers, to a focus on their problem-solving processes. If students do not attend to their problem-solving processes, there is no need for them to take metacognitive actions related to their problem-solving process. As outlined in Chapter IV, Dr. Arkadash utilized Inquiry-Based Learning techniques (Ernst et al., 2017) to build classroom norms in which students valued process over product. However, just having students actively engage in rich mathematical tasks is not necessarily sufficient for an emphasis on thinking processes. Often, students work together but only to share answers, just as the students in the Number Sense and Algebra course at the beginning of the semester. In this study, those actions were tied to beliefs that there was one, “best” procedure to solve problems.

Dr. Arkadash intentionally brought students’ thinking processes into the classroom, making students’ diverse ways of thinking and solving problems a significant portion of class discussion. To explicitly support these process-focused norms, she encouraged students to explain their thinking processes to her and each other, evaluate and connect different solution processes presented by different students or from sample student work, and generate multiple solution paths and strategies themselves. Her focus on student thinking empowered students to engage in difficult mathematics. In an
interview, Kristy described her excitement when Dr. Arkadash emphasized Kristy’s own thinking while working together.

I thought it was so funny cause after my meeting yesterday with Dr. [Arkadash], I went back to my dorm room and all my suitemates were like, ‘How did it go?’ And I was like, ‘My life has changed. I understand math.’ And I was just like freaking out and I busted out the three pages of the portfolio…We worked on stuff and it was weird that I understood because she didn't even tell me what to do. She made me do it myself and figure it out myself. And I thought that was weird because usually, like teachers in high school would be like ‘Oh, yeah, this is how you do it. Now go work on it yourself.’ But she was pushing me to think and try to find the process of how it works. And when I did it... I think that was why I was like so excited. Because I figured it out myself, with help obviously, but it was my own thinking.

If we want students to become mathematical thinkers and problem solvers, then students need sufficient opportunity to “participate in a legitimately peripheral way entails that newcomers have broad access to arenas of mature practice” (Lave & Wenger, 1991, p. 110).

Liljedahl (2016) would characterize the learning environment that Dr. Arkadash was creating as a “thinking classroom”:

[A] thinking classroom is a classroom that is not only conducive to thinking but also occasions thinking, a space that is inhabited by thinking individuals as well as individuals thinking collectively, learning together, and constructing knowledge and understanding through activity and discussion. It is a space wherein the teacher not only fosters thinking but also expects it, both implicitly and explicitly. (p. 4)

Inquiry-Based Learning classrooms are settings with the potential to be thinking classrooms, but while they are conducive to thinking, they do not always occasion thinking. Dr. Arkadash engaged in specific teaching practices to build and maintain a thinking classroom. Her use of portfolio problems in this setting further supported this endeavor. Specifically, Liljedahl identified nine elements of thinking classrooms, the ninth being the inclusion of assessment with a “focus on the processes of learning more
so than the products, and it needs to include both group work and individual work” (p. 23).

**Pre-Service Teachers’ Identities and Mathematical Habits of Mind**

This dissertation study documented pre-service teachers’ development of metacognition as a mathematical habit of mind. Even beyond metacognition, opportunities for teachers (pre- or in-service) to develop mathematical habits of mind is critical (CBMS, 2012), especially if they are to provide these opportunities for students. Watson and Barton (2011) describe this type of teaching ability as being able to enact *modes of enquiry*, where a teacher uses a student comment to “make a wider point about mathematics and extend the students’ thinking when the moment was ripe” (p. 68). They argue that this skill is often limited by teachers experiences as students themselves. Thus, there is a need to explicitly include mathematical problem-solving habits of mind in the mathematical experiences of pre-service teachers.

As discussed in Chapter III, the pre-service teachers in the Number Sense and Algebra course wore two different “hats” throughout the course, and these dual identities competed with each other in authentic problem-solving situations like the portfolio problems. Students viewed themselves as teachers first (“teacher hat”), with the goal of building a teacher “toolkit” of different procedures and pedagogical strategies to help the diverse needs of their future students. Their second identity was as students of mathematics (“teacher hat”). The pre-service teachers understood that to be a quality math teacher they needed to understand the fundamental mathematics concepts they would teach. The hierarchy of identities and related goals of the pre-service teachers was
the opposite order of what Dr. Arkadash intended for the course. She wanted pre-service teachers to develop as students of mathematics, so that their own thinking was the primary object of their learning. The pre-service teachers not only needed to improve content knowledge, but they also needed authentic problem-solving opportunities to develop the habits of mind of mathematical thinkers and problem solvers.

In the Number Sense and Algebra course, the pre-service teachers’ dominant teacher identity impeded their development as mathematical problem solvers themselves. Because they believed their primary purpose was to develop as mathematics teachers, their engagement with mathematical practices was not in service of their development as students of mathematics. The portfolio problems focused the pre-service teachers’ attention on their own thinking, which catalyzed a tension between their dual identities. In day-to-day classroom activity, the pre-service teachers were learning tools such as connecting to definitions and using different representations, and these tools were perceived to be in service of teaching. These tools could help provide more “mathematical” justifications, but they were not being utilized by students in their own problem-solving endeavors. As was seen in Table 8, the pre-service teachers continued to solve problems the way they always had (“student hat”) and then retroactively provide the necessary “teacher” explanation of their process (Problem-Solving Activity 0 through 3). They were only able to develop as problem solvers when they realized the tools seemingly presented to them for teaching proved useful to them as problem solvers themselves, i.e. when the use of MA4 afforded them productive struggle (Problem-Solving Activity 4). These findings support Oesterle el al.’s (2016) claim that pre-service
teachers need to develop mathematical habits of mind, but also value their usefulness if they are to provide such opportunities for their future students.

**Limitations and Delimitations**

While the current research provides the findings and implications outlined in the previous sections, there were design choices delimiting the parameters of current investigation, as well as limitations from conducting this research. Here, I comment on (de)limitations from the choice of sample, the data sources for micro-level analysis, a distributed analysis of small-group metacognitive actions, and the subset of students willing to participate in interviews.

In this study, I chose to work with pre-service elementary teachers. One major finding from this study involved the situated nature of the development of metacognition, and the nuances of this development were very much grounded in the sample, potentially limiting generalizability beyond this population. The dual identities of the pre-service teachers in the Number Sense and Algebra course, both as future mathematics teachers and as current mathematics students, presented themselves as a constant tension for the pre-service teachers throughout the course. As such, there was a separation between their activity while appealing to themselves as future teachers and while appealing to themselves as students. Though Dr. Arkadash navigated this tension and ultimately married the two identities, subsequently pushing students’ metacognitive activity forward, the trajectory for students’ problem-solving activity may not be “typical” of a course not designed for pre-service teachers.

Generalizability was also delimited by the design choice for micro-level analysis to assess students’ metacognitive actions only during in-class portfolio problem-solving
sessions. This choice was intentional, as students would be engaging with authentic problems. Explicitly in interviews and more implicitly during recorded classroom sessions, students identified the portfolio problems as problems instead of exercises (Schoenfeld, 1985). In light of findings related to students’ dual identities, this choice ensured that students were wearing their “student hat” while solving these problems, so that the portfolio problem-solving sessions provided legitimate participation for students as mathematical problem solvers themselves, rather than as future mathematics teachers. Nevertheless, this choice reduced the amount of data with which to identify students’ normative metacognitive actions. Students were faced with problem-solving situations throughout the course, and many of these involved the pre-service teachers appealing to their identities as students. However, what makes an exercise a problem is subjective (Schoenfeld, 1985). Such analysis of all daily classroom activity, with students self-identifying problems, was not a feasible endeavor for this study.

Adopting a participation metaphor for learning was beneficial as it better aligned with the habitual, in-the-moment nature of mathematical habits of mind employed in real time during authentic problem-solving situations. However, this metaphor for learning delimited the study by necessitating a focus on language and norms as evidence for learning. Students increase their participation in “expert” problem solving by learning to communicate in the community’s language and acting according to the community norms. In the context of this study, that means students are learning language to communicate their problem-solving process and using metacognitive actions in all phases of the problem-solving cycle. Assessing student participation in problem-solving norms could be done with different units of analysis: the individual, the collective, or interaction
analysis between the two. The in-class portfolio problem-solving sessions, within the context of an Inquiry-Based Learning classroom setting, were completed by groups of students. As such, the metacognitive activity in these sessions could best be described as distributed among group members, and the unit of analysis in this study was the collective student group. Thus, the data I collected was not conducive to analysis of individual cognition and metacognitive activity, and results were presented to describe the collective student group. Different units of analysis may provide additional insight into students’ development of metacognition during problem solving.

Portions of macro-level analysis were limited by the students’ willingness to participate in interviews throughout the semester. Although the problem-solving actions of all students were identified during micro-level analysis, just 15 of the 24 students participated in any interviews, with only 13 participating in all three. Specifically, student interviews were utilized heavily to describe student goals and motives while in the course, as well as their beliefs about mathematics, problem solving, and mathematics teaching. This informed the delineation of the ‘Subject’ component of the initial student activity system. By reviewing video- and audio-recorded classroom sessions, I could glean information concerning the goals/motives and beliefs of those students who did not participate in interviews. Nevertheless, inferences made based on collective beliefs were somewhat incomplete.

**Future Research**

My dissertation work has provided evidence of positively changing classroom metacognitive norms and offered insight into how we can help students move closer towards utilizing the habits of mind of a mathematical thinker and problem solver.
Research exploring the sociocultural mediation of metacognition during mathematical problem solving is still sparse, so there are many possible avenues to explore.

As described at the end of the Limitations and Delimitations section, the unit of analysis for my current investigation was the collective student group. In addition to this unit of analysis, more insight into the development of metacognition during problem solving could be attained by garnering information about individual experiences in their process of development with respect to metacognition during authentic problem-solving situations. For example, Levenson et al. (2006) noticed that the taken-as-shared norms established by the majority of students may not be productive for students in the minority. It may be useful to identify and understand the experiences of these “minority” students, or “negative cases” (Lincoln & Guba, 1985) to investigate how to support their metacognitive activity during problem solving. Ernest (2010) described the reflexive relationship between the individual and collective. Not only is the individual shaped by the social environment, but the individual transforms the internalized signs and symbols from the collective, subsequently influencing and potentially altering the socially negotiated beliefs and behavior of the collective: “both change mutually, the members and their culture” (Bauersfeld, 1995, p. 281).

In my study, I was able to capture the reflexive relationship between the collective student activity system and Dr. Arkadash, but was limited in understanding the relationship between individual students and the collective group of students. Ernest outlined this reflexive process of negotiation, representing Vygotsky’s developmental theory in a cycle of appropriation, transformation, publication, conventionalization (Figure 12). In a future study, this cyclical process could be captured in data analysis by
coding all data with the stages of Ernest’s cycle of appropriation, transformation, publication, conventionalization: *Public/Individual, Public/Collective, Private/Collective,* and *Private/Individual.* Such analysis would require more data from individual students. Activity theory was useful for guiding analysis of relationships on a macro-level, and the addition of Ernest’s (2010) cycle could prove useful in micro-level analysis. The combination of these two analysis frameworks may provide a better description of the complex interaction between the individual and community, while accounting for differences among potentially diverse communities of practice.

Inquiry-Based Learning classroom practices greatly influenced the adoption of process-oriented metacognitive norms. Active learning techniques gave students opportunities to publicize their problem-solving processes, and the portfolio problems supported reflection on these processes. I would like to explore how various classroom ecologies afford or constrain process-focused metacognition in mathematical problem solving. Research on metacognition highlights the benefits of collaboration to foster metacognition (e.g., De Backer, 2015), but there is little research investigating how classroom environments shape metacognition, especially in mathematics classrooms. For example, undergraduate instructors who implemented a flipped classroom design were motivated to use this pedagogy to help students develop metacognition (Naccarato & Karakok, 2015), but there is not currently sufficient evidence to support that flipped classroom environments support metacognition, especially within the context of mathematical problem solving. In science education, Thomas (2003) developed the Metacognitive Orientation Learning Environment Scale – Science (MOLES-S) to assess the metacognitive orientation of a classroom by looking at metacognitive demands,
student-student discourse, student-teacher discourse, student voice, distributed control, encouragement and support, and emotional support. Thomas and Mee (2005) used a version of this tool to describe a primary school classroom. A tool such as the MOLES-S, adapted for mathematical problem solving, may prove useful in comparing the impacts of different classroom environments on metacognitive actions in authentic problem-solving situations. Studies across settings could lead to “best practices” for integrating metacognition into classroom goals and culture, especially in shifting the metacognitive authority from teacher to students.

Mathematical thinkers and problem solvers use specific metacognition actions throughout the problem-solving cycle (Carlson & Bloom, 2005). My dissertation research provides insight into supporting students in developing metacognition as a mathematical problem-solving habit of mind. I believe the consideration of metacognition as a mathematical practice, a problem-solving habit of mind, can provide a lens through which to better coordinate instructional coherence of problem solving across courses, helping bridge transitions such as those (a) between courses before calculus and calculus courses, (b) between introductory and upper-level mathematics courses, or (c) between mathematics content and methods courses for pre-service teachers. Cuoco et al. (1996) envisioned mathematics curriculum organized around mathematical habits of mind, and Cuoco and McCallum (2018) emphasize that curricular coherence requires coherence of practice in addition to coherence of content: “The way students do mathematics, their mathematical practice, may have an effect on their ability to take advantage of a coherent curriculum” (p. 252). Currently, the Common Core Standards for Mathematical Practice (CCSSI, 2010) address mathematical dispositions and habits of mind for K-12 students.
However, there is a need to bridge these efforts for post-secondary mathematics students (Lockwood & Weber, 2014). Even beyond mathematics, an explicit focus on habits of mind has the potential to more coherently bridge STEM education than content integration alone (Bennett & Ruchti, 2014).

In future research, I would like to investigate curricular coherence of practice over the aforementioned transitions (items a, b, and c in the previous paragraph), as well as in courses that bridge STEM education. With respect to courses for pre-service teachers (item c), research is needed to investigate the role of “student” and “teacher” identities of pre-service teachers in shaping their experiences as mathematical problem solvers. While the Conference Board of Mathematical Sciences (CBMS) recommends all pre-service and in-service mathematics teachers have experiences to develop the “habits of mind of a mathematical thinker and problem solver” (2012, p. 19), pre-service teachers need to value their usefulness if they are to provide these opportunities for students (Oesterle et al., 2016). In the Number Sense and Algebra course, students’ “teacher” identities interfered with their problem-solving activity as students during the portfolio problem sessions. It was not until they reached a critical point of frustration that they valued the use of process-focused metacognitive activity to help them move their own problem solving forward to productively get past struggle. Future research is needed to investigate how pre-service teacher identities influence productive struggle (Warshauer, 2015). This fluctuation of identities in the course may also warrant investigation into how pre-service teachers develop other problem-solving habits of mind in mathematics content courses. Moreover, research is needed to investigate how mathematics classrooms with students
who are *not* pre-service teachers afford buy-in into process-focused thinking and metacognition, where there are no “teacher” identities to leverage.
REFERENCES


Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics, 46*(1), 13-57.


APPENDIX A

INSTITUTIONAL REVIEW BOARD MATERIALS
DATE: June 14, 2016
TO: Emilie Naccarato
FROM: University of Northern Colorado (UNCO) IRB
PROJECT TITLE: [920575-1] Development of Metacognitive Norms in Undergraduate Mathematics Classrooms
SUBMISSION TYPE: New Project
ACTION: APPROVED
APPROVAL DATE: June 13, 2016
EXPIRATION DATE: June 13, 2017
REVIEW TYPE: Expedited Review

Thank you for your submission of New Project materials for this project. The University of Northern Colorado (UNCO) IRB has APPROVED your submission. All research must be conducted in accordance with this approved submission.

This submission has received Expedited Review based on applicable federal regulations.

Please remember that informed consent is a process beginning with a description of the project and insurance of participant understanding. Informed consent must continue throughout the project via a dialogue between the researcher and research participant. Federal regulations require that each participant receives a copy of the consent document.

Please note that any revision to previously approved materials must be approved by this committee prior to initiation. Please use the appropriate revision forms for this procedure.

All UNANTICIPATED PROBLEMS involving risks to subjects or others and SERIOUS and UNEXPECTED adverse events must be reported promptly to this office.

All NON-COMPLIANCE issues or COMPLAINTS regarding this project must be reported promptly to this office.

Based on the risks, this project requires continuing review by this committee on an annual basis. Please use the appropriate forms for this procedure. Your documentation for continuing review must be received with sufficient time for review and continued approval before the expiration date of June 13, 2017.

Please note that all research records must be retained for a minimum of three years after the completion of the project.

If you have any questions, please contact Sherry May at 970-351-1910 or Sherry.May@unco.edu. Please include your project title and reference number in all correspondence with this committee.
Hello Emilie,

Thank you for your most outstanding IRB application. I concur with Dr. Collins as to the thoroughness and specificity of your application. What a pleasure to review and approve.

Good luck with this important research.

Sincerely,

Nancy White, PhD, IRB Co-Chair

This letter has been electronically signed in accordance with all applicable regulations, and a copy is retained within University of Northern Colorado (UNCO) IRB's records.
DATE: March 2, 2017
TO: Emille Naccarato
FROM: University of Northern Colorado (UNCO) IRB
PROJECT TITLE: [920575-2] Development of Metacognitive Norms in Undergraduate Mathematics Classrooms
SUBMISSION TYPE: Continuing Review/Progress Report
ACTION: APPROVED
APPROVAL DATE: March 2, 2017
EXPIRATION DATE: June 13, 2018
REVIEW TYPE: Expedited Review

Thank you for your submission of Continuing Review/Progress Report materials for this project. The University of Northern Colorado (UNCO) IRB has APPROVED your submission. All research must be conducted in accordance with this approved submission.

This submission has received Expedited Review based on applicable federal regulations.

Please remember that informed consent is a process beginning with a description of the project and insurance of participant understanding. Informed consent must continue throughout the project via a dialogue between the researcher and research participant. Federal regulations require that each participant receives a copy of the consent document.

Please note that any revision to previously approved materials must be approved by this committee prior to initiation. Please use the appropriate revision forms for this procedure.

All UNANTICIPATED PROBLEMS involving risks to subjects or others and SERIOUS and UNEXPECTED adverse events must be reported promptly to this office.

All NON-COMPLIANCE issues or COMPLAINTS regarding this project must be reported promptly to this office.

Based on the risks, this project requires continuing review by this committee on an annual basis. Please use the appropriate forms for this procedure. Your documentation for continuing review must be received with sufficient time for review and continued approval before the expiration date of June 13, 2018.

Please note that all research records must be retained for a minimum of three years after the completion of the project.

If you have any questions, please contact Sherry May at 970-351-1910 or Sherry.May@unco.edu. Please include your project title and reference number in all correspondence with this committee.
Thank you for the thorough submission of IRB continuation application materials. This project is approved to proceed with the documents submitted in this package.

Best wishes with the continued work on this interesting and relevant research.

Sincerely,

Dr. Megan Stellino, UNC IRB Co-Chair

This letter has been electronically signed in accordance with all applicable regulations, and a copy is retained within University of Northern Colorado (UNCO) IRB’s records.
Title: Development of Metacognitive Norms in Undergraduate Mathematics Classrooms
Principal Investigator: Emilie Naccarato, University of Northern Colorado
Research Advisor: Dr. Gulden Karakok, University of Northern Colorado

A. Purpose
1. The importance of problem solving practices and conceptual understanding have been recognized and studied by earlier researchers (Schoenfeld, 1992; Carlson & Bloom, 2005). Currently, many teachers and researchers recognize a need to foster skills beyond just mere accumulation of facts or problem solving procedures. Especially within Science, Technology, Engineering and Math (STEM) related fields, skills such as metacognition, self-regulation, innovation, and creativity are gaining recognition. In particular, while metacognition has been identified as an essential 21st Century skill, it remains undertheorized and under-studied at the undergraduate level. As mathematics is a fundamental component of Science, Technology, and Engineering (STE) fields, the development of mathematical problem-solving skills together with metacognitive skills plays a crucial role in students’ critical and creative thinking even beyond the required mathematics courses for STE majors. It follows that metacognition in mathematical problem solving has the potential to create a lasting impact on STEM majors, within both their academic and future careers. Unfortunately, much of the research on metacognition in mathematics does not describe the explicit role metacognition plays during problem solving. Moreover, metacognitive interventions are typically disconnected from natural mathematical activity and discourse within a classroom community.

The purpose of this dissertation study is to characterize sociomathematical metacognitive norms within the context of an undergraduate mathematics classroom and their potential influence on students’ problem-solving processes. In particular, this study is designed to address the following research questions:

Q1 How do sociomathematical metacognitive norms develop in an undergraduate mathematics community of practice?
Q1a What contradictions or tensions exist between the metacognitive activity of different participants (teacher and students) within the classroom community?
Q1b How do these contradictions or tensions transform the normative metacognitive activity of the classroom community over time?
Q1c In what ways does the normative metacognitive activity of the classroom community influence students’ metacognitive processes during problem solving?
Q1d In what ways does the normative metacognitive activity of the classroom community influence the teacher’s role in guiding students’ metacognitive activity during problem solving in the classroom?
In this study, I (the Principal Investigator, Emilie Naccarato) will video and audio record all class sessions, as well as collect students’ ungraded assignments, exams and quizzes, and in-class work artifacts from MATH 185, Number Sense and Algebra, at the University of Northern Colorado during the Fall 2016 semester. Focus of audio and video recordings will be on student-to-student and student-to-instructor interactions. At multiple points throughout the semester, students will be asked to participate in individual interviews, where students will work on problems related to their course. I will also work with the instructor of the course to develop course materials and aid in facilitating classroom discourse during the semester. The instructor will be interviewed throughout the semester, and planning meetings between myself and the instructor will be audio recorded. The instructor will not share any students’ grades nor I will do any grading of student work.

2. This research falls under the expedited review category because the research activities present no more than minimal risk to human participants (see section C for details) and data collection will be in the form of video and audio recordings of MATH 185 students’ interactions with each other and/or the instructional team (instructor and myself as an aide) in class, students’ ungraded assignments, quizzes, exams, and in-class work artifacts, as well as video and audio recording of interviews. I will not be involved with any grading throughout the semester and will not have access to any course grades. Furthermore, this research is designed to describe group characteristics from a population who is not vulnerable.

B. Methods

1. Participants
Participants of this project will be students and the instructional team (the instructor and myself as an aide) in the MATH 185, Number Sense and Algebra, course in the mathematics department at the University of Northern Colorado during the Fall 2016 semester. This course was chosen because the instructor uses student-centered curriculum materials and asks students to work in groups during most of the course. The permission letter from the instructor to conduct the study in her classroom is attached in this proposal (Appendix F). I (the principal investigator) will only have the role of aide in the course, helping to design class activities and facilitate discourse in the classroom. I will not grade any assignments, quizzes, or exams, or have access to any grades.

All students in the MATH 185 course who are 18 years or older will be invited to participate in this study. At the beginning of the semester, Emilie Naccarato (the principal investigator) will invite all students to participate by explaining the study and the data collection process. They will be informed that I will be video and audio recording all class sessions (see Appendix B-Classroom recruitment script) until the end of semester. During this recruitment time in class, the instructor will not be in the classroom. I will provide two copies of informed consent forms (see Appendix A- Consent for classroom data) for the classroom
video and audio recording. After addressing all questions from students, I will collect one copy of the informed consent form. The instructor will not be informed who agrees or does not agree to participate in the study. Only the principal investigator, Emilie Naccarato, will know who participates in the study until after final grades have been submitted. Students who wish not to participate in the study will not be recorded during small group work or during whole classroom discussion recordings. This will be done by having those students who do not wish to participate sit together in a group in a location of their choice in the classroom. The principal investigator, Emilie Naccarato, will be the person recording, editing and working with the cameras, and she will make sure these students are not recorded. Since she will be the only person to know who these students are, she will communicate with students regarding their choice of location in the classroom and handling the video-taping to honor these students’ requests. However, if some of these students are accidentally captured in the videos (because they were moving around or discussing ideas with another group), then non-participants’ communication will not be used in transcripts and their faces will be blurred if these videos are to be used for publication purposes.

To investigate students’ problem-solving processes and metacognitive skills individually, during the classroom data collection I will invite all students to participate in up to three 90-minute, semi-structured, individual interviews. These interviews will be conducted in the fall semester of 2016. I will send an email invitation to these students during the Spring 2016 semester (see Appendices C and D for a sample invitation email and interview consent form).

2. Data Collection Procedures
There will be four sources of data: video and audio recorded classroom observations, collection of students’ ungraded homework, quizzes, exams, and in-class artifacts, videotaped interviews with students, and audio recording of interviews and planning meetings with the instructor.

Because the sequencing of events in class, the topics and the activities of the course, and how the instructor designs the activities and teaches the content will most likely provide the basis for the social learning environment and stimulate the interaction among students, the instructor will be interviewed up to 3 times throughout the semester (see Appendix E for a consent form). These interviews will take at most 90 minutes and will be audiotaped. The instructor will be asked to sign an informed consent form at the beginning of the first interview. A sample of potential interview questions is in the Appendix G. Additionally, the instructor’s weekly meeting with myself, the principal investigator, will be audiotaped to document the decisions made about the teaching of the course. Permission from the instructor to audio record these sessions is also included in the consent form found in Appendix E.

The main data source will be the classroom video-recordings since they will help me to address the question of sociomathematical metacognitive norms.
Conducting classroom observations will not require any extra time from the participants. The purpose of the classroom observation data is to document the student-to-student and student-to-instructor interactions. In other words, videotaping of the course will capture the social learning environment so that I can document the development of classroom norms throughout the data collection period. Videotaping will result in stronger research because it allows me to “retain a rich record of behavior that can be reexamined again and again” (Clement, 2000, p. 577). It will also free myself up to document field notes and on the spot interpretations.

In order to document the students’ development of metacognitive behavior, their written work will also be collected before grading. These artifacts will provide insight to how students further develop ideas from classroom activities and how they communicate their thinking through writing. Such practices will help the researcher to understand what ideas from classroom activities can be observed in these written work. Students ungraded assignments, quizzes, exams, and in-class work will be collected and scanned (as a pdf file) after removing students’ names, and will be returned to the instructor. All students’ work will be collected so that the instructor would not know who agreed to participate in the research study. However, only the work of students who agreed to participate will be scanned for the research study. I, the principal investigator, will get the collected artifacts from the instructor as soon as they are collected, make copies of the participating students’ work, remove the names from the copies by using a black marker and write the assign code number of students on the copies, and return the original work to the instructor on the same day. Thus, the written work will be given back to the instructor within the same day of collection.

At the beginning of the fall semester all students will be invited to participate in up to three 90-minute-semi-structured interviews (see Appendix H for a sample of interview questions). The purpose of collecting interview data is to explore individual students’ personal metacognitive activities and beliefs to document both how they individually influence and are influenced by classroom metacognitive norms. Students will work on questions related to their course. All students will be invited to in participate the initial interviews via email. Any student participating in the first interview is eligible to participate in subsequent interviews. The interview invitation will explain the interview process and students will be given opportunities to ask questions before they make any decisions to participate. Students will be explained that they will be compensated with a $30.00 gift card if they complete all three interviews, a $20.00 gift card if they complete only the first and last interviews, and a $10.00 gift card if they complete only the first and second interviews. If students miss two or more interviews (i.e., did not attend the second and the third interviews), no compensation will be provided. The interviews will be scheduled and conducted according to the students’ schedule at a seminar room of the School of Mathematical Sciences. The interviews will be videotaped and audio recorded. I, the principal investigator, will conduct the interviews. Students will be asked to
sign the consent form at the beginning of the first interview and the research study will be explained to them again before the interview starts. Again, only I, the principal investigator, will know who agreed to participate (or not) and have access to the collected data. It is possible that the principal investigator could ask her research advisor’s assistance on some portion of the data. Thus, the research advisor might view some portion of the video-data, but this will only take place after final course grades have been submitted.

On the first day of the course I, Emilie Naccarato, will explain the study to students. The classroom video and audio recording of the course, collection of ungraded assignments and artifacts, and the interview process will be explained in this visit and the instructor will not be present in the classroom. I will inform the students of the purpose of the research as described in Appendices B & C. I will also disclose that I am interested in the social learning environment of the classroom. I will explain that students’ interactions with each other and the instructor will be recorded to understand how such interactions help them develop understanding of the course material. Thus, I will be requesting that the participants allow me to video and audio record them in the course and while they work in groups and collect any ungraded artifacts they create. They will be informed that their participation is voluntary and that they can decide to participate in different parts of this study. The importance of video and audio recording (i.e. gathering rich data that can be observed multiple times) to help substantiate any possible conjectures will be explained to the students. Also, collecting ungraded assignments and artifacts will help to better understand their thinking processes throughout the course.

Students will also be told that only myself, Emilie Naccarato, will know who agrees to participate or not, and the collected data will be viewed only by myself, the Principal Investigator, and occasionally my research advisor, but only after the final grades have been submitted. Finally, students will be informed about the interview process and the compensation for the interviews (see section D for full details on compensation). Students will be informed that it is voluntary to participate and their instructor will not know who participates or not. They will be informed that if they don’t wish to participate but the video-recording captures them then data from them will not be used from transcriptions and if the videos are used their faces will be blurred.

All the students will be informed orally and through the consent forms that they are not required to participate in the research and that their instructor will not know if they decide to participate or not. Students will also be informed that if they are willing to participate in classroom videotaping, only the researcher will view the collected data. Furthermore, if students are selected and willing to participate in the interviews, only the researcher will know their involvement and their participation will not be disclosed to the instructor. Students will be told that if they decide to participate they can stop at any time. They just need to inform the researcher via email so that they are not video-taped in the course.
C. Data Analysis Procedures

Given that the main purpose of this research is to characterize metacognitive norms from a social learning perspective and the research is qualitative in nature, the researcher will use qualitative methods to analyze the data. The data analysis of the classroom videos and field notes of observations will start at the beginning of the Fall 2016 semester and continue until it is completed. Data analysis will start by assigning participating students code names and these code names will be used throughout the research. After assigning code names, the analysis of the videos will start. The analysis of the classroom video data will be done by implementing a modified version of established qualitative methods for video analysis, specifically the **whole-to-part inductive approach** (Erickson, 2006). According this approach, the researcher will watch the whole video without stopping and record notes as it is watched. Next, the researcher will review the video again, and stop and review parts that seem significant to the research questions. Once the significant portions are marked, the researcher will transcribe these clips. Only these sections will be transcribed. These transcriptions will help the researcher to identify students and the ways that they engage in classroom discourse. Data analysis of transcribed episodes will consist of discourse analysis to establish existing metacognitive classroom norms. Students’ collected artifacts will be analyzed using the developed codes from the classroom video data and new codes will be generated for different metacognitive activities. Only participating students’ data will be transcribed and used.

All interviews will be transcribed by the researcher and the whole interviews will be analyzed by using the Qualitative Hypothesis-Generating process outlined by Auerbach and Sliverstein (2003). In their work, the authors describe a way of analyzing data that begins with identifying *relevant text*. Relevant Text is defined as “…passages of your transcript that express a distinct idea related to your research ideas” (p.46). The next step in the process involves organizing this text into *repeating ideas*, or ideas that appear in the text from two or more sources. Third, these repeating ideas are combined into *themes*, and then the research builds a *theoretical construct* from the themes (Auerbach & Sliverstein, 2003). This interview data will be used to describe individual students’ personal metacognitive activities and beliefs to document both how they individually influence and are influenced by classroom metacognitive norms throughout the data collection period.

The interviews with the instructor will help to understand the ways in which the instructor designs and runs the course, as well as describe her personal beliefs about metacognition in problem solving. The researcher will summarize the interviews and only selected parts will be transcribed. This data will help to see the specific goals that the instructor had for the course and what she thinks she managed during the data collection period.
Overall, the analysis of these four sources of data will help the researcher to address research questions and characterize sociomathematical metacognitive norms within the context of an undergraduate mathematics classroom.

4. Data Handling Procedures
Only I, the researcher (Emilie Naccarato) will have access to the data, and all the data (videos/audios and pdf copies of student artifacts) will be stored on the researcher’s password-protected computer. I will back-up all the data on a password-protected jump-drive and store this jump-drive in a locked file-cabinet in my UNC office (Ross 2061). Signed consent forms and any hard copies of students’ work will be stored in the researcher’s locked file cabinet in my UNC office. Participants will be assigned code names and an excel file which has the corresponding names and the codes will be saved on my password-protected computer in a folder separate from the data and the analysis files. My research advisor, Dr. Gulden Karakok, will retain copies of consent forms in a locked file-cabinet in her UNC office for three years, and will receive these forms after she submits the final grades at the end of the Fall 2016 semester.

Most of the data will be synthesized and portrayed as group results, but excerpts from classroom and/or interviews will be used to substantiate any hypotheses and a theoretical model. When students’ quotes are used, pseudonyms will be given and only I, the principal investigator, will know the corresponding names and the pseudonyms. All the signed consent forms and the data will be kept for three years. The researcher will destroy the data files (both electronic and hard copies) three years after data collection. Participants will be informed about this process in the informed consent forms.

C. Risks, Discomforts and Benefits
The risks inherent in this study are no greater than those normally encountered during classroom participation. Such minimal risks include participants being embarrassed about their responses, insecure about sharing their work or ideas, or worried that they will say something incorrect. I, the principal investigator, will attempt to mitigate these risks by assuring the students that I am not concerned about whether or not their responses, work, or remarks are incorrect, but rather I am interested in how they work together to understand the ideas presented. I will ensure that their names will not appear in our analysis or future paper publications and that to protect their confidentiality I will use code names. Students will be told that their participation is voluntary and their participation (or non-participation) will not affect their grade.

Participants may not benefit directly from participating in this project. The results from this study will help the future undergraduate students by improving the social learning environments of such courses to improve the mathematical metacognitive knowledge and skills.

D. Costs and Compensations
Students who agree to be videotaped in the classroom and/or share ungraded assignments and artifacts will not be compensated.

Students who agree to participate in interviews will be compensated for their time and commitment. The researcher will buy gift cards and give these cards to students at the end of the final interview. Students will be compensated with a $30.00 gift card if they complete all three interviews, a $20.00 gift card if they complete only the first and last interviews, and a $10.00 gift card if they complete only the first and second interviews. If students miss two or more interviews, no compensation will be provided.

E. Grant Information (if applicable):
Not applicable.

Attachments:
- Appendix A: Consent form for classroom video recording and artifact collection
- Appendix B: Classroom recruitment script
- Appendix C: Email invitation to students for interviews
- Appendix D: Consent form for interviews
- Appendix E: Consent form for interviews with the instructor
- Appendix F: Permission letter from the instructor
- Appendix G: Sample interview questions for the instructor interviews
- Appendix H: Sample interview questions for students’ interviews
Appendix A: Consent form for Classroom Data Collection

UNIVERSITY of
NORTHERN COLORADO

CONSENT FORM FOR HUMAN PARTICIPANTS IN RESEARCH
UNIVERSITY OF NORTHERN COLORADO

Project Title: Development of Metacognitive Norms in Undergraduate Mathematics Classrooms

Researchers: Emilie Naccarato, Ph.D. Candidate, School of Mathematical Sciences, University of Northern Colorado
Gulden Karakok, Ph.D., School of Mathematical Sciences, University of Northern Colorado

Phone: 970-351-2907; 970-351-2215
E-mail: emilie.naccarato@unco.edu; gulden.karakok@unco.edu

You are invited to participate in a research study that I am conducting on understanding students’ ways of learning in classroom settings in which social interactions are highlighted. In order to explore this phenomenon, I request that you allow me to videotape you during the Fall 2016 semester in this course and that you allow me access to your completed in-class work, quizzes, exams and assignments before grading. The results of this study could help researchers and teachers improve mathematics courses in general. There is no need to worry if you say something that is incorrect because I am solely interested in how you learn while you interact with each other in this course. In other words, the correct answers are not the focus of this study but the way you reason and communicate with others is the focus.

By videotaping this class, I will be able to observe the ways all of you interact with each other and the instructor to learn the topics of this course. The recordings will allow me to watch the episodes on multiple occasions and will also free me up to take notes while I observe the classroom. Your class work will help me to confirm some of my interpretations of the classroom observations. I will be present as an aide every class and videotape the course. I will be the only person to have access to the videotapes and I will transcribe some of the sessions from videotaping during the analysis of the collected data. I will be the only person to watch the videotapes. At the beginning of my data analysis, I will assign pseudonyms (code names assigned by me) and use these throughout the research. It is possible that during data analysis I will ask for my research advisor’s assistance, but this will only take place after final grades have been assigned and my research advisor will only see verbal data that has already been assigned a pseudonym.

Given the purpose of my research, it is possible that I might use some portions of the videos and your written work during presentations or in publications (using pseudonyms). Thus, I am requesting permission to do so, but if you would prefer that your video image
or written work image not to be used, I will honor that request. Your transcriptions will be used in such cases.

Please note that you are not under any obligation to participate in this research and your decision to not participate in this research will not impact your grade in this course. In fact, your instructor will not know who participates in this research study. Please note that all of the collected data will be kept on my password protected computer and external drive, which will be kept in a locked file cabinet in my UNC office.

All the names from the written work will be removed after making copies and before making pdfs of them. Your name will be blacked out using a marker and the assigned code numbers will be used instead of your names throughout all data analysis. Any hardcopies of data will be stored in a locked file cabinet in my office. The collected data will be analyzed by transcribing some portions of the videotapes. You will be given a code name and only I will know the assigned code names and the corresponding names. Your identity will not be made public. In publication I will use these code names. I will destroy all of the collected data (both hard copies and electronic copies) three years after the data collection. My advisor will also retain a copy of your consent forms for three years and will then destroy these files. You have the following options for participation in this research. You may choose to:

1. participate in the videotaping, allowing me to use your verbal remarks AND episodes showing your face and using your written work (using a code name),
2. participate in the videotaping, allowing me to use ONLY your verbal remarks and written work (with a code name); I am NOT allowed to use episodes showing your face,
3. not participate in the videotaping, allowing me to use ONLY your written work (with a code name),
or
4. not participate in the research at all.

There is no compensation for classroom data collection, however if you agree to participate in this study you will have the opportunity to participate in the interview portion of this research study, where you will be compensated with up to a $30.00 gift card at the end of the interviews. If you would like to participate in the interview portion of the study, I will contact you via email. For this reason, I ask you to provide me your email address at the end of this form. Your email address will only be used to contact you if you are selected for the interview portion of the study.
There are no foreseeable risks to participating in this study other than some discomfort if you do not feel comfortable being videotaped or are embarrassed by your work. It is possible that I may accidentally videotape you, especially if you are working closely with someone who has agreed to be videotaped. In such circumstances, I will attempt to edit the video accordingly. There are no direct benefits for you in this research however your participation may help future students in improved mathematics courses.

Participation in this research is voluntary. You may decide to not participate in this study and if you begin participation, you may still decide to stop and withdraw at any time. I, the researcher will be the only person to know that you participate in this study and I will be the only one to view the collected data. I will respect your decisions and lack of participation will not result in loss of benefits to which you are otherwise entitled. Your instructor will not know your decision and your grade will not be affected due to your decision.

Page 2 of 4 ________________ (Participant Initials)

Having read the above and having had an opportunity to ask any questions, please fill out the following information and sign if you would like to participate in my study. I will provide you with a copy of this form for you to retain for your records. If you have any concerns about your selection or treatment as a research participant, please contact Sherry May, IRB Administrator, in the Office of Sponsored Programs, Kepner Hall, University of Northern Colorado Greeley, CO 80639; 970-351-1910.

Please feel free to contact me via phone or email if you have any questions. Retain one copy of this letter for your records. Thank you for assisting me with my research.

Please choose ONE of the following options:

(1) If willing to participate in classroom videotaping and willing to provide written work and willing to disclose your identity (using a code name) i.e., agreeing to have your comments and/or video shared with others at conference presentations, classes, publications, etc. please complete the following.

____________________________________________________________________

Name (please print)  Signature  Date

Emilie Naccarato
(2) If willing to participate classroom videotaping and willing to provide written work but prefer to have identity protected (only verbal remarks will be used with a code name) please complete the following.

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<td>Emilie Naccarato</td>
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(3) If not willing to participate classroom videotaping but willing to provide written work, please complete the following.

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(4) If not willing to participate in the research, please complete the following.

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<tr>
<td>Emilie Naccarato</td>
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If you have chosen either Option (1) or Option (2) above and would like to be considered for the interview portion of the study, please provide your email so that I can contact you.

Email Address:
Appendix B: Classroom Recruitment Script

I am here to invite you to participate in a research project designed to understand the ways students learn mathematics in a classroom setting that is designed to have group activities and interactions among students. As part of this research I would like to video tape the class sessions throughout the semester and collect your ungraded in-class work, assignments and exams. I will observe each class session and take notes, and videotaping will help me to focus on observing you. Videotaping will allow me to collect data that I can watch multiple times and your class work will allow me to substantiate my conjectures. No individual is the focus of this study and I am interested in groups’ interactions and learning in a classroom like this one.

Given the purpose of my research, I would like to share portions of your video-clips during presentations and in publications and it is possible that I may want to incorporate photos that have your written work in a publication. Thus, I am requesting permission to do so, but if you would prefer that I protect your video image and written work image, then I will honor your request. In such a case, I will only use your responses (from transcripts).

Please note that you are not under any obligation to participate in this research and your decision to not participate in this research will not impact your participation in this course. In fact, I will be the only one to know your decision to participate (or not). Also, I will be the only one to have an access to the collected data. I will conduct the analysis and through this analysis all participants will be given code numbers. I will be the person that assigns the codes and knows the corresponding names and code numbers. The collected data will be saved on my password protected computer and the external drive. Your names will be removed from any hardcopies of written work and will be stored in a locked file cabinet in my office. You also have the option to participate in different aspects of the research. You may choose to:

1. participate in the video-taping where allow me to use episodes showing your face and use your written work (using pseudonyms),
2. participate in the video-taping where I am NOT allowed to use episodes showing your face but where I am allowed to use your remarks and your written work (using pseudonyms),
3. not participate in the video-taping but allow me to use your written work(using pseudonyms),
   or
4. not participate in the research at all.

There is no compensation for classroom data collection, however I am happy to help you with your questions during my office hours. Furthermore, if you agree to participate in this study you might be selected to participate in the interview portion of this research study that will be conducted during this semester. The interview portion of the study will compensate you with up to a $30.00 gift card at the end of the interviews. Participating in
options (1) or (2) above will make you eligible to participate in the interview portion of
the study.

There are no foreseeable risks to participating in this study other than some discomfort if
you do not feel comfortable being video-taped or are embarrassed by your work. If you
do n’t want to participate, it is possible that I may accidentally videotape you, especially if
you are working closely with someone who has agreed to be videotaped. In such
circumstances, I will attempt to edit the video accordingly.

There are no direct benefits for you in this research however your participation may help
future students in improved mathematics courses. Please note that your decision will not
affect your grade in this course. Your instructor will not know your decision.

Please take the time to read over the consent form and sign it to indicate your decision. If
you have any questions, please let me know. Thank you for your time.
Appendix C: Email Invitation to Participate in Interviews (Students)

Dear ________________,

As you know I will be videotaping your MATH 185 course over the next few weeks. You agreed to participate in this videotaping portion of the study. Thank you for your participation.

The purpose of this letter is to invite you to participate in the interview portion of the study. I am interested in investigating further your understanding of course related topics and the impact of the classroom environment on your learning in this course. To do so I will conduct at most three 90-minute individual interviews with you.

The interviews will take place throughout the next few weeks and all of them will be scheduled according to your available time in one of the seminar rooms of Mathematics Department. They will be video and audio recorded. I will be the only person to conduct the interviews, view and analyze them. During the interview I will ask you to complete some mathematics tasks. Here are the important facts regarding the interview.

- There will be at most three interviews and each will take at most 90 minutes of your time.
- You do not need to prepare anything for the interview.
- The interviews will be scheduled according to your schedule and be completed before the end of the Fall 2016 semester.
- You will be compensated for participating in all three interviews with a $30 gift card.
- If you miss the second interview but participate in the first and third interview, you will be compensated with a $20 gift card.
- If you miss the third interview but participate in the first two interviews, you will be compensated with a $10 gift card.
- If you miss two or more interviews, there will be no compensation.

Given the purpose of this research, I would like to share portions of your video-clips during presentations and/or publications. Thus, I am requesting permission to do so, but if you would prefer that your image not to be used, then I will honor your request. In such a case, I will only use your responses from transcripts. Also, note that I will be the only one to know your decision about the participation. I will destroy the collected data three years after collection. Further, your decision to participate or not will not affect your grade in any course.

I hope you will be willing to participate in this study, especially since the results of this study could inform improved teaching of mathematics courses in general. Please do not hesitate to contact me if you have any questions regarding the study or the protocol for the study. You may contact me at emilie.naccarato@unco.edu, or 970-351-2907.

Sincerely,

Emilie Naccarato, PhD Candidate, University of Northern Colorado
Appendix D: Consent Form for Interview Participants

CONSENT FORM FOR HUMAN PARTICIPANTS IN RESEARCH
UNIVERSITY OF NORTHERN COLORADO

Project Title: Development of Metacognitive Norms in Undergraduate Mathematics Classrooms

Researchers: Emilie Naccarato, Ph.D. Candidate, School of Mathematical Sciences, University of Northern Colorado
Gulden Karakok, Ph.D., School of Mathematical Sciences, University of Northern Colorado

Phone: 970-351-2907; 970-351-2215
E-mail: emilie.naccarato@unco.edu; gulden.karakok@unco.edu

You are invited to participate in a study investigating the impact of the classroom learning environment of the MATH 185 course you are currently taking in the Fall 2016 semester. I am interested in understanding the way you work on mathematical tasks within the context of this course. So, you will be asked to work on mathematical tasks during (up to) three interviews, each of which will take at most 90 minutes. I will video and audio record the interviews and during the interview you will be asked to think aloud. The interviews will be conducted at times that are convenient for you during the concurrent classroom video recording sessions and before the end of the Fall 2016 semester in one of the seminar rooms of the Mathematics Department. The results of this study could inform improved teaching of mathematics courses in general and help us understand how learning takes place in social environments.

There is no need to worry if you say something that is incorrect because I am solely interested in how you reason. Also, I will be the only person who will know that you participate in this study and I will conduct the interviews. Moreover, I will view, transcribe all of the interviews, and analyze the collected data. You will be given a code name and your name won’t be used throughout the analysis and in any publication.

Given the purpose of my research, I would like to share portions of your video-clips during presentations and/or publications. Thus, I am requesting permission to do so, but if you would prefer that your videos not to be used, then I will honor your request by using transcriptions of the interviews.

Please note that you are not under any obligation to participate in this research and your decision to not participate in this research will not impact your participation in any courses you take.
You also have the option to participate in different aspects of the research. You may choose to:

1. participate in the videotaping, allowing me to use your verbal remarks AND episodes showing your face (using a code name),
2. participate in the videotaping, allowing me to use ONLY your verbal remarks (with a code name); I am NOT allowed to use episodes showing your face,
3. not participate in the research at all.

If you participate in the interview, then I will compensate you with a $30.00 gift card after the completion of all interviews. If you miss the second interview but participate in the first and third interview, you will be compensated with a $20 gift card. If you miss the third interview but participate in the first two interviews, you will be compensated with a $10 gift card. If you miss two or more interviews, there will be no compensation.

Please note that all data will be stored on my UNC computer and external drive, both of which are password protected, thus no one will have access to this data other than me. I will destroy all of the collected data (both hard copies and electronic copies) three years after the data collection.

There are no foreseeable risks to participating in this study other than some discomfort if you do not feel comfortable being video-taped or are embarrassed by your work. You may benefit from participating in this research as reflecting on your work allows you to gain a new perspective of solving mathematical tasks which you can use later in your program.

Participation in this research is voluntary. You may decide to not participate in this study and if you begin participation, you may still decide to stop and withdraw at any time. I will respect your decisions and lack of participation will not result in loss of benefits to which you are otherwise entitled.

Having read the above and having had an opportunity to ask any questions, please fill out the following information and sign if you would like to participate in my study. I will provide you with a copy of this form for you to retain for your records. If you have any concerns about your selection or treatment as a research participant, please contact Sherry May, IRB Administrator, in the Office of Sponsored Programs, Kepner Hall, University of Northern Colorado Greeley, CO 80639; 970-351-1910.

Please feel free to contact me via phone or email if you have any questions and retain one copy of this letter for your records. Thank you for assisting me with my research. Please
choose one of the options from the following list on the next page and sign under that option.

Page 2 of 3 ________________ (Participant Initials)

Please choose ONE of the following options:

(1) If willing to participate in interviews with videotaping and willing to disclose your identity i.e., agreeing to have your comments and/or video shared with others at conference presentations, classes, publications, etc. (using a code name) please complete the following.

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<thead>
<tr>
<th>Name (please print)</th>
<th>Signature</th>
<th>Date</th>
</tr>
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<tbody>
<tr>
<td>Emilie Naccarato</td>
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Researcher’s Name | Research’s Signature | Date
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(2) If willing to participate interviews with videotaping but prefer to have identity protected (only verbal remarks will be used with a code name) please complete the following. Note that you will still be compensated with a $20.00 gift card after completion of all interviews.

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<th>Name (please print)</th>
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(3) If not willing to participate in the research, please complete the following.

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<th>Name (please print)</th>
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<td>Emilie Naccarato</td>
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Page 3 of 3 ________________ (Participant Initials)
Appendix E: Consent form for interviews with the instructor

CONSENT FORM FOR HUMAN PARTICIPANTS IN RESEARCH UNIVERSITY OF NORTHERN COLORADO

Project Title: Development of Metacognitive Norms in Undergraduate Mathematics Classrooms

Researchers: Emilie Naccarato, Ph.D. Candidate, School of Mathematical Sciences, University of Northern Colorado
Gulden Karakok, Ph.D., School of Mathematical Sciences, University of Northern Colorado

Phone: 970-351-2907; 970-351-2215
E-mail: emilie.naccarato@unco.edu; gulden.karakok@unco.edu

As you know I am investigating the ways students reason during problem solving, specifically looking at their metacognitive behavior, in your MATH 185 course. As part of my data collection, I would like to conduct three interviews with you, before, midway through, and after the Fall 2016 semester. The first interview is to understand your decisions about how to develop, run the course and how you set up the social learning environment. I also wish to know your beliefs about problem solving and the role of metacognition within problem solving and within the classroom environment. The second and third interviews will be conducted midway through and at the end of the semester, and I would like to get your overall opinion on how the activities and social learning environment went during the course and any changes you would make. The results of this study will help us to develop better social learning environments fostering metacognition that students can then take to other courses regardless of the mathematical content.

I will conduct the interviews, each taking at most 90 minutes. They will be scheduled according to your availability and they will be conducted in a seminar room in the Mathematics Department or your office, whichever is more convenient for you. I will audio record the interviews. These recordings will only be listened to by me. I will transcribe these recordings and analyze data so that I have a better understanding of metacognitive norms in the social environment of a classroom community.

All collected audio recording will be stored on my password protected computer and it will be destroyed three years after the collection of data. Your name will not be made public and if I need to refer to you in any publication I will use a pseudonym.

Participation is voluntary. You may decide not to participate in this study and if you begin participation you may still decide to stop and withdraw at any time. Your decision
will be respected and will not result in loss of benefits to which you are otherwise entitled.

There are no foreseeable risks for you in this study. You may benefit from this study when you reflect about your course at the end of the semester.

Page 1 of 2 ________________ (Participant Initials)

Having read the above and having had an opportunity to ask any questions, please fill out the following information and sign if you would like to participate in my study. I will provide you with a copy of this form for you to retain for your records. If you have any concerns about your selection or treatment as a research participant, please contact Sherry May, IRB Administrator, in the Office of Sponsored Programs, Kepner Hall, University of Northern Colorado Greeley, CO 80639; 970-351-1910.

Sincerely,

Emilie Naccarato

__________________________________________  ______________________________________
Instructor’s Name (please print)                Instructor’s Signature
   Date

__________________________________________  __________________________
Researcher’s Signature                      Date

Page 2 of 2 ________________ (Participant Initials)
Appendix F: Permission letter from the Instructor

UNIVERSITY of
NORTHERN COLORADO

College of Natural and Health Sciences
School of Mathematical Sciences

June 10, 2016

To whom it may concern;

As the instructor of Math 185 (Section 003-Number Sense and Algebra) course, I give permission to Emilie Naccarato to observe the course throughout the semester, collect audio and video data of student-teacher and student-student (subject to students’ consent) interactions in class, and collect any course curriculum design materials (e.g., lesson plans, revised lesson plans) for her dissertation study. I understand that I will not know who participates and does not participate in the study until I submit students’ grades.

Please do not hesitate to contact me if there are any questions/concerns.

Sincerely,

Gulden Karakok

University of Northern Colorado
School of Mathematical Sciences
501 20th Street Campus Box 122
Greeley CO 80639

Phone: 351-2215
Email: gulden.karakok@unco.edu
Appendix G: Sample Interview Questions (Instructor)

First Interview Questions:
(1) What are your overall teaching and learning goals for the MATH 185 course?
   a. How do these goals relate to your personal philosophy about mathematical
      problem solving?
      i. What is mathematical problem solving?
(2) Do you have any more specific goals for the content that will be covered during
   the data collection period?
(3) How will these goals be addressed by the problems that you plan to have students
   work on in class during this time?
   a. How did you select these problems?
   b. What, specifically, do you hope students gain from working on these
      problems?

Second and Third Interview Questions:
(1) Do you think the classes during the data collection period were in line with your
    overall teaching and learning goals for this course? Why or why not?
(2) Do you think your specific goals for this recent course content were addressed?
    How so?
(3) How were these goals addressed by the problems that you had students work on in
    class during this time?
    a. What, specifically, did students gain from working on these problems?
    b. What specific problems and related class discussions do you think were
       successful? Why?
    c. What specific problems and related class discussions do you think were
       unsuccessful? Why?
(4) [While watching a short clip from the course] I want you to watch this video –
    how would you improve this scenario if at all? What would happen if you said
    something else instead? Do you think this would improve what you were trying to
    get at?
Appendix H: Sample Interview Questions (Students)

Initial Interview Questions:

1. What is your
   a. Major?
   b. Year?
   c. How far are you in your program? [What courses are you taking now? Have taken?]

2. Why are you taking this course?

3. What are your overall goals for this course?
   a. How do these goals relate to your personal philosophy about mathematics?
      i. What is mathematics (and its purpose)?

4. Do you have any more specific goals for the content that will be covered during this course?

5. What is problem solving? How does this relate to your beliefs about math?

6. Please work on the following problem(s) while “thinking aloud”.
   *Problems will be chosen by the Principal Investigator with the help of the course instructor to align with current class activities.

7. How do you think your problem solving in this course relates to the problem solving in other courses? How does it relate to that of the instructor? Other students in this course?

All Interviews:

1. What went well in class this week (during a portion of the data collection period between interviews)? Why?

2. What did not go well in class this week (during a portion of the data collection period between interviews)? Why?

3. Please work on the following problem(s) while “thinking aloud”.
   *Problems will be chosen by the Principal Investigator with the help of the course instructor to align with current class activities.

4. How do you think this problem relates to what you’ve been working on in class?

5. How do you think your problem solving here relates to how you’ve been solving problems during class?

6. How do you think your problem solving here relates to how you solve problems in other classes?
APPENDIX B

STUDENT INTERVIEW QUESTIONS
Student Interview 1

Remind students that they shouldn’t discuss this interview with the instructor or with other students in the class.

1. What is your
   a. Major?
   b. Year?
   c. How far are you in your program? [What math courses are you taking now? Have taken?]
2. Why are you taking this course? (How did you get to be here?)
3. What are your overall goals for this course?
   a. How do these goals relate to your personal philosophy about mathematics?
      i. What is mathematics (and its purpose)?
4. Do you have any more specific goals for the content that will be covered during this course?
5. What has gone well in class so far? Why?
6. What has not gone well in class so far? Why?
7. What is problem solving? How does this relate to your beliefs about math?
8. Please work on the following problem(s) while “thinking aloud”.

   Problem 1: Use the five digits 1, 3, 5, 7, and 9 exactly once to build two numbers A and B so that AB is as large as possible. Then build two numbers C and D so that CD is as small as possible.

   Problem 2: How many ways could you tile an m x n rectangle using 1 x a tiles?

9. How do you think the problem solving you’ve done here relates to what you’ve been working on in class? (content)
10. How do you think the problem solving you’ve done here relates to how you’ve been solving problems in class?
11. How do you think your problem solving in this course relates to the problem solving in other math courses you’ve taken?
12. How do you think the problem solving you’ve done here relates to that of the instructor?
13. How do you think the problem solving you’ve done here relates to other students in this course?
14. What’s the difference between a problem and an exercise? Do you think you’ve been working on problems or exercises in this class so far? (Give examples)
Student Interview 2

Remind students that they shouldn’t discuss this interview with the instructor or with other students in the class.

1. What is going well in class? Why?
2. What is not going well in class? Why?
3. Do you think your process of solving problems has changed over this course so far? Why or why not? In what ways?
4. Please work on the following problem while “thinking aloud”.

Here are the steps of this method for computing 37×23:

Step 1: Start with writing 37×23, creating two “columns”, one underneath 37 (left column) and one underneath 23 (right column).

Step 2: Progressively halve the numbers in the left column (ignoring the remainders) while doubling the numbers in the right column. Continue this process until you get 1 in the left column.

Step 3: Delete all rows with an even number in the left column. Add all the numbers that survive in the right column. This sum is the desired product.

\[
\begin{array}{cc}
37 & \times \ 23 \\
18 & \times \ 46 \\
9 & \times \ 92 \\
-\ & \gets \ 184 \\
-\ & \gets \ 368 \\
-\ & \gets \ 736 \\
\hline
851 \\
\end{array}
\]

\[37 \times 23 = 851\]

Does this algorithm work all the time? Why or why not?

5. How do you think the problem solving you’ve done here relates to what you’ve been working on in class? (content)
6. How do you think the problem solving you’ve done here relates to how you’ve been solving problems in class? (process)
7. How do you think your problem solving in this course relates to the problem solving in other math courses you’ve taken? (other than math too, e.g. science)
8. How do you think the problem solving you’ve done here relates to that of the instructor?
9. How do you think the problem solving you’ve done here relates to other students in this course?
Student Interview 3

1. General questions about the class:
   a. If you were to describe this class (honestly) to a friend/student who was going to take it, what would you say about it? What’s the point? What will they learn?
   b. What did you learn this semester?
   c. What did you learn about mathematics?
      i. Was there any change over the course of the semester?
   d. What did you learn about yourself as a student of mathematics (not as a future teacher)?
      i. Was there any change over the course of the semester?
   e. What is problem solving in mathematics?
      i. Has your definition changed over the course of the semester?
      ii. What does “successful” problem solving look like?
      iii. What does “unsuccessful” problem solving look like?
      iv. Does being successful or unsuccessful have anything to do with being right or wrong? Is there always a right or wrong answer? Has this view changed over the course of the semester?
      v. Do you think you have been a successful problem solver in this class? Why or why not?
      vi. What do you think Dr. G values when it comes to problem solving? Why?
   f. What did you learn about yourself as a mathematical problem solver?
      i. Was there any change over the course of the semester?
   g. What was the point of the portfolio problems?
      i. Do you think they related at all to course content?
      ii. Did the way that you completed the portfolio problems change at all over the course of the semester?
      iii. Did the way that you completed your final write ups change at all over the course of the semester?
      iv. What successes did you have in the portfolio problems?
      v. How did you deal with challenges with the portfolio problems?
         1. Did this change over the course of the semester?
   h. Do you feel like you’ve been creative in this class? Why or why not? When?
   i. Would you say that before this class you were confident in your mathematical ability?
      i. Are you still confident in your mathematical ability?
      ii. Have you faced challenges related to confidence this semester? How did you try to overcome these challenges?
   j. In what ways were you successful in this course?
   k. What challenges did you face in this course?
   l. Did your expectations for the course change at all during the course of the semester?
      i. How do you think your expectations compare to Dr. G’s?
      ii. Do you think her expectations changed over the course of the semester?
m. What do you think you will take with you to MATH 186 next semester or other future math courses?

2. Think about yourself as a student of mathematics (not as a teacher).
   a. Do you think it was important for you to generate multiple solutions? Why or why not? [Do you think it was helpful?]
   b. Have you ever heard of steps for problem solving?
      i. [Give handout with steps from course textbook] How do you think these steps relate, if at all, to the problem solving we did and talked about in MATH 185 this semester?
   c. What is the role of the group in problem solving?
      i. Do you think you could be successful without your group?
      ii. How did your different groups work this semester? What role did you have? Did this affect your problem solving?
      iii. Did you work with others outside of class? Who? In what capacity?
   d. What is the role of pattern finding during problem solving?
   e. What is a counter example?
      i. When you are problem solving, do you look for counter examples? When? Why?
   f. In this class, we used lots of different representations (manipulatives, pictures, words, equations, etc.).
      i. Do you think this has been helpful for you in learning mathematics? Why or why not?
      ii. Do you think it’s valuable to be good at using all the different representations? Why or why not?
   g. One thing that seems to have come up for some students this semester is being able to explain why something works instead of just doing it.
      i. Do you think this has been the case for you this semester?
      ii. Has this changed at all over the course of the semester?
      iii. Do you think this is important for you in learning mathematics?
   h. Some students mentioned a tendency to be one track minded or see things only their way.
      i. Do you think this has been the case for you this semester?
      ii. Has this changed at all over the course of the semester?
      iii. Does it matter if you are one track minded in mathematics? Why or why not?

3. Think about your role as a future teacher (not a student) and your future students.
   a. What did you learn about yourself as a teacher of mathematics (not as a student)?
      i. Was there any change over the course of the semester?
   b. Do you think it is important for them to generate multiple solutions? Why or why not?
   c. Do you think it is important for your students to understand why something works instead of just being able to do it? Why or why not?
      i. Are there certain situations where this is more important? Why or why not?
   d. Do you think it is important for your students to use multiple representations (manipulatives, pictures, words, equations, etc.)? Why or why not? [Not just
because they may understand one method better, but for knowing how to use all these representations in their own right]

e. Do you think it’s OK for students of mathematics to be one track minded when solving problems in mathematics? Why or why not?

4. Ask specifically about metacognition stuff
   a. Give them the list of Carlson and Bloom’s metacognitive actions. [next page] For each item…
      i. Do you understand what this means? [I can clarify if they ask]
      ii. Do you think you did this when you problem solved this semester?
         1. Did this change at all over the course of the semester?
         2. Anything new this semester?
         3. Anything that stands out you do more than the others?
      iii. Do you think this was emphasized in the 185 class this semester?

5. Questions about the interview setting
   a. Did you reflect at all this semester on what we talked about during the interviews?
      i. Do you think these discussions influenced you in any way?
      ii. Did you learn anything from the interviews?
1. Effort is put forth to read and understand the problem
2. Information is organized
3. Evidence of sense making
4. Goals and givens are established
5. Goals and givens are represented
6. Relates problem to a parallel problem
7. Exerts conscious effort to access resources/mathematical knowledge
8. Mathematical concepts, knowledge, and facts are assessed and considered
9. Various solution approaches are considered
10. Generates conjectures
11. Strategies and tools are devised, considered, and selected
12. Effort is put forth to construct logically connected statements
13. Validity of conjecture is considered
14. Reflects on the efficiency and effectiveness of the selected methods
15. Results are tested for their reasonableness
16. Refines, revises, or abandons plans as a result of solution process
17. Verifies processes and results
18. Reflects on the efficiency and effectiveness of cognitive activities
19. Effort is put forth to stay mentally engaged
20. Manages emotional responses to problem-solving situation
APPENDIX C

INSTRUCTOR INTERVIEW QUESTIONS
1. What are your overall teaching and learning goals for the MATH 185 course?
   a. How do these goals relate to your personal philosophy about mathematical problem solving?
      i. What is mathematical problem solving?
2. Do you have any more specific goals for the content that will be covered during the data collection period?
3. How will these goals be addressed by the problems that you plan to have students work on in class during this time?
   a. How did you select these problems?
   b. What, specifically, do you hope students gain from working on these problems?
Instructor Interview 2

1. General questions about the class:
   a. If you were to describe this class (honestly) to another instructor who was going to teach it, what would you say about it? What’s the point? What will students learn?
   b. In what ways were you successful in this course?
   c. What challenges did you face in this course?
   d. Did your expectations for the course change at all during the course of the semester?
      i. How do you think your expectations compare to student expectations?
   e. What did your students learn this semester?
   f. What did your students learn about mathematics?
      i. Was there any change over the course of the semester?
   g. What did your students learn about themselves as students of mathematics (not as a future teacher)?
      i. Was there any change over the course of the semester?
   h. What is problem solving in mathematics?
      i. What does “successful” problem solving look like?
      ii. What does “unsuccessful” problem solving look like?
      iii. Does being successful or unsuccessful have anything to do with being right or wrong? Is there always a right or wrong answer? Has this view changed over the course of the semester?
      iv. What do you value when it comes to problem solving?
      v. What do you think your student believe problem solving is? Has this changed for them over the course of the semester?
      vi. Do you think your students have been successful problem solvers in this class? Why or why not?
      vii. What do you think your students value when it comes to problem solving? Why?
      i. What did your students learn about themselves as mathematical problem solvers?
         i. Was there any change over the course of the semester?
   j. What was the point of the portfolio problems?
      i. Do you think they related at all to course content?
      ii. How did students complete the portfolio problems?
         1. Did this change over the course of the semester?
      iii. How did students complete the final write ups?
         1. Did this change over the course of the semester?
         2. What kind of feedback did you give?
      iv. What successes did you have in the portfolio problems?
      v. What challenges did you have in the portfolio problems?
         1. Did this change over the course of the semester?
      vi. What successes did students have in the portfolio problems?
      vii. What challenges did students have in the portfolio problems?
1. Did this change over the course of the semester?

k. Do you feel like students were creative in this class? Why or why not? When?
l. Do you think students in this class were confident in their mathematical ability?
   i. Have you faced challenges related to student confidence this semester? How did you try to overcome these challenges?
m. What do you think students will take to MATH 186 next semester or other future math courses?

2. More about problem solving

a. Do you think it was important for students to generate multiple solutions? Why or why not? [Do you think it was helpful?]
b. How do you think the way we talked about problem solving in 185 with compares, if at all, to Pólya’s steps for problem solving?
c. What is the role of the group in problem solving?
   i. Do you think students could be successful without your group?
   ii. How did the different groups work this semester? What role did students have? Did this affect their problem solving?
d. What is the role of pattern finding during problem solving? Was this emphasized in the course?
e. What is a counter example?
   i. When problem solving, did students look for counter examples? When? Why?
f. In this class, we used lots of different representations (manipulatives, pictures, words, equations, etc.).
   i. Do you think this has been helpful for students in learning mathematics? Why or why not?
   ii. Do you think it’s valuable to be good at using all the different representations? Why or why not?
g. One thing that seems to have come up for some students this semester is being able to explain why something works instead of just doing it.
   i. Do you think this has been the case this semester?
   ii. Has this changed at all over the course of the semester?
   iii. Do you think this is important for learning mathematics?
h. Some students mentioned a tendency to be one track minded or see things only their way.
   i. Do you think this has been the case for this semester?
   ii. Has this changed at all over the course of the semester?
   iii. Does it matter if you are one track minded in mathematics? Why or why not?
i. What did students learn about themselves as a teacher of mathematics (not as a student)?
   i. Was there any change over the course of the semester?

3. Ask specifically about metacognition
a. Give them the list of Carlson and Bloom’s metacognitive actions [next page].
   For each item…
   i. Do you think students did this when they problem solved this semester?
      1. Did this change at all over the course of the semester?
      2. Anything new this semester?
      3. Anything that stands out you do more than the others?
   ii. Do you think this was emphasized in the 185 class this semester?

4. Questions about the study
   a. Do you think participating in this research influenced the way you taught the course? In what ways?
   b. Did you learn anything from participating in this research?
1. Effort is put forth to read and understand the problem

2. Information is organized

3. Evidence of sense making

4. Goals and givens are established

5. Goals and givens are represented

6. Relates problem to a parallel problem

7. Exerts conscious effort to access resources/mathematical knowledge

8. Mathematical concepts, knowledge, and facts are assessed and considered

9. Various solution approaches are considered

10. Generates conjectures

11. Strategies and tools are devised, considered, and selected

12. Effort is put forth to construct logically connected statements

13. Validity of conjecture is considered

14. Reflects on the efficiency and effectiveness of the selected methods

15. Results are tested for their reasonableness

16. Refines, revises, or abandons plans as a result of solution process

17. Verifies processes and results

18. Reflects on the efficiency and effectiveness of cognitive activities

19. Effort is put forth to stay mentally engaged

20. Manages emotional responses to problem-solving situation
APPENDIX D

MICRO-ANALYSIS CODING INFORMATION
Micro-level Analysis Methods

Micro-level analysis of the in-class portfolio problem-solving sessions began with a list of metacognitive actions during the problem-solving cycle taken from Carlson and Bloom (2005). This list provided an organizing framework or “start list” for an initially deductive approach to coding (Miles & Huberman, 1984) the in-class portfolio problem-solving sessions. As the “prefigured” codes were applied to the data, they were reduced, combined, and revised to avoid restricting analysis and better reflect students’ actions (Creswell, 2013). In this appendix, the ‘Start List with Modifications and Memos’ illustrates this process, which resulted in a final list of six metacognitive actions relevant to the data set. These actions are provided in this appendix, both with hypothetical examples utilized while conceptualizing the nature of the metacognitive actions, as well as with examples from the Number Sense and Algebra course in this research study.

In-class portfolio problem-solving sessions were re-coded using this final list of six metacognitive actions. Sample codes from in-class portfolio problem-solving sessions two and five are provided in this appendix. To make sense of these codes, I first looked at counts of metacognitive actions both for individual students in the course, as well as for the collective student group (see tables and chart provided at the end of this appendix). This data were then triangulated with classroom video and students’ write-ups to determine which metacognitive actions became normative per the definition provided by Rasmussen, Wawro, and Zandieh (2015).
Start List with Modifications and Memos

1. Effort is put forth to read and understand the problem [cognitive activity]
2. Information is organized
3. Evidence of sense making
4. Goals and givens are established
5. Goals and givens are represented
6. Relates problem to a parallel problem [problem-solving strategy]
7. Exerts conscious effort to access resources/mathematical knowledge
8. Mathematical concepts, knowledge, tools, and facts are assessed and considered
   - "Assess" or "consider" the relevance, etc. is metacognitive
   - "what math fact do I know that could help me?"
9. Various solution approaches or strategies are assessed and considered
   - "Assess" or "consider" the relevance, etc. is metacognitive
   - Problem solving strategies like working backwards, drawing a picture, etc.
   - different representations
10. Generates conjectures
11. Strategies and tools are devised, considered, and selected
12. Effort is put forth to construct logically connected statements
13. Validity/reasonableness of conjecture solution process ['Process'] is assessed/considered/tested
   - Valid = logically or factually sound (based on true premises), well grounded, justifiable, pertinent
     i. Justification or explaining why would happen here
   - Reasonableness = appropriate, based on good sense, rational
     i. seems more based on intuition
14. Reflects on the efficiency and effectiveness of the selected methods [18 is broader and captures these]
   - Processes, not products
15. Results ['Products'] are assessed/tested/considered for their reasonableness/validity
   - result = claim/conclusion/product made as a result of the executing phase
     • Valid = logically or factually sound (based on true premises), well grounded, justifiable, pertinent
     • Reasonableness = appropriate, based on good sense, rational
       i. seems more based on intuition
     • does it make sense in this context?
     • e.g., getting width of -2 would not reasonable
     • e.g., testing an expression with examples
16. Refines, revises, or abandons plans as a result of solution process
17. Verifies processes and results
   - e.g., plug number back into an equation or proposed expression to test if it works
18. Reflects on the efficiency and effectiveness of cognitive activities
   - Efficient = achieve max productivity with min wasted effort
   - Effective = successful in producing a desired or intended result
   • #14 is an example of this
19. Effort is put forth to any mentally engaged
20. Manages emotional responses to problem-solving situation
21. Engages in internal dialogue
   - How would I code for this – It's not specific
Final Code List with Hypothetical Examples

1. **Mathematical concepts, knowledge, tools, and facts are assessed and considered**
   - E.g.: “What math fact do I know that could help me?”

2. **Various solution approaches or strategies are assessed and considered**
   - E.g.: Problem solving strategies like working backwards, drawing a picture, etc.
   - different representations

3. **Validity/reasonableness of solution process ['Process' that can be executed] is assessed/considered/tested**

4. **Results ['Products'] are assessed/tested/considered for their reasonableness-validity**
   - E.g.: “Does it make sense?”, width of -2 not reasonable, testing expression with examples

5. **Reflects on the efficiency and effectiveness of cognitive activities**
   - Efficient = achieve max productivity with min wasted effort
   - Effective = successful in producing a desired or intended result

6. **Manages emotional responses to problem-solving situation**

<table>
<thead>
<tr>
<th>MA 1</th>
<th>Mathematical concepts, knowledge, tools, and facts are assessed and considered</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Data Example: <em>So, does that mean it’s a quadratic relationship?</em></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>MA 2</th>
<th>Various solution approaches or strategies are assessed and considered</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Data Example: <em>I wonder if there’s a way we could work backwards.</em></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>MA 3</th>
<th>Validity/reasonableness of solution process is assessed/considered/tested</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Data Example: <em>I know you can plug in the numbers, but is there a reason why that works or why you found that, besides just plugging in the numbers?</em></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>MA 4</th>
<th>Results (answers) are assessed/tested/considered for their reasonableness-validity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Data Example: <em>But we don’t know for sure sunrise is at 6.</em></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>MA 5</th>
<th>Reflects on the efficiency and effectiveness of cognitive activities</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Data Example: <em>I feel like it should be harder than this, you know?</em></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>MA 6</th>
<th>Manages emotional responses to problem-solving situation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Data Example: <em>Ok we can just get up and walk away, take a break.</em></td>
</tr>
</tbody>
</table>
## Sample Excerpts from Codebook

### Jordan, Paula, Skylar PPS 5 – Day 24

<table>
<thead>
<tr>
<th>Time</th>
<th>Speaker</th>
<th>Speech</th>
<th>Written Work</th>
<th>Metacognitive Action</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>01:00</td>
<td>Jordan</td>
<td>[Reads problem aloud]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:02:17</td>
<td>Paula</td>
<td>So, this goes from A to B and this goes from B to A.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:03:02</td>
<td>Paula</td>
<td>I'm gonna look at the times first. They met at noon, so (insinuates)...</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:04:52</td>
<td>Paula</td>
<td>I know they're both going different speeds.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:05:00</td>
<td>Skylar</td>
<td>They start at sunrise.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:05:25</td>
<td>Paula</td>
<td>Did they start at the same time?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:05:27</td>
<td>Skylar</td>
<td>Yeah, they both start at sunrise.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:06:17</td>
<td>Jordan</td>
<td>See this is like we have to do a double number line. They both meet at noon.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:06:34</td>
<td>Skylar</td>
<td>So, this is 4• hours in between. This is 9• hours in between.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:07:16</td>
<td>Paula</td>
<td>They have to meet at the same time, but it doesn't mean they went the same distance. They were different speeds.</td>
<td>MA 2-1</td>
<td>Assesses double number line approach proposed by Jordan – location of noon (solution process not yet started with this strategy)</td>
<td></td>
</tr>
<tr>
<td>01:07:38</td>
<td>Jordan</td>
<td>They met at noon, but that doesn't mean they met halfway... So, should we try to figure out how far they're going every hour? Then backtrack until like -</td>
<td>MA 2-2</td>
<td>Considers alternative approach to replace her previous suggestion (based on Paula’s questioning of the method) - find speed (unit rate) [algebraic representation instead of double number line] Extension of Paula’s previous MA 2</td>
<td></td>
</tr>
<tr>
<td>01:08:00</td>
<td>Skylar</td>
<td>But we don't know how long this is.</td>
<td>MA 2-3</td>
<td>Assesses new approach proposed by Jordan (solution process not yet started with this strategy) - find speed (unit rate) Extension of Paula’s previous MA 2</td>
<td></td>
</tr>
<tr>
<td>01:08:06</td>
<td>Paula</td>
<td>It just says A and B.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:08:33</td>
<td>Jordan</td>
<td>So, from the time they met...</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:08:35</td>
<td>Paula</td>
<td>But that might not be in the middle... I'm trying to do - does she have a higher speed even though she finished early? I'm trying to figure out her speed and then backtrack the time.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:09:42</td>
<td>Paula</td>
<td>But I was thinking, like you were saying the distance is the same distance, but the difference between here is 4 hours, and then 9 hours, so 9 and 4.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:09:54</td>
<td>Skylar</td>
<td>But then I was like it only works if they met exactly in the middle, because then we know - but then it wouldn't work since they started at the same time.</td>
<td>MA 3</td>
<td>Assess validity/ reasonableness of speed (unit rate) strategy</td>
<td></td>
</tr>
<tr>
<td>01:10:11</td>
<td>Paula</td>
<td>Maybe we should do a graph?</td>
<td>MA 2</td>
<td>Considers alternative approach - graph</td>
<td></td>
</tr>
<tr>
<td>01:10:23</td>
<td>Jordan</td>
<td>So, this is 4 hours. So, they started at the same time, so there's a 3-hour difference. OK. And then - I'm trying to think. If they pass at noon, then she's walking 5 hours ahead. She will not be faster now. And so, we divide this. This is 8. I don't want to write, because I don't know... And then you figure out where the 3-hour difference is and that's where they keep meeting. If that makes sense. So, like 8, 7, 6, 5, 4.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:12:53</td>
<td>Paula</td>
<td>I'm saying - kind of what you were saying. If it took her 4 hours, we figure out how much she went in 1 hour and then how long it took her to get to that 1 hour. But I don't even know if that would even help at all. They're moving at a constant rate, right?</td>
<td>MA 3-2</td>
<td>Assesses validity/ reasonableness of speed (unit rate) strategy Extension of Skylar’s previous MA 3</td>
<td></td>
</tr>
<tr>
<td>Time</td>
<td>Speaker</td>
<td>Speech</td>
<td>Written Work</td>
<td>Metacognitive Action</td>
<td>Notes</td>
</tr>
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<td>--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------</td>
<td>----------------------------------------------</td>
<td>---------------------------------------------</td>
<td>---------------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>57:42</td>
<td>Sharon</td>
<td>I don't understand this question.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>57:45</td>
<td>Alexis</td>
<td>So, we just have to start with a 3-digit number and just do it.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>57:49</td>
<td>Lucy</td>
<td>What's gonna be our 3-digit number? 467.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>58:29</td>
<td>Alexis</td>
<td>So, we have 297 and that becomes 792. And add it to the original result. Wait which original result? This one?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>58:45</td>
<td>Lucy</td>
<td>Yeah 792 plus 297.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>59:08</td>
<td>Lucy</td>
<td>792 plus 287 is 1089.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>59:14</td>
<td>Alexis</td>
<td>Oh, I did this in a class last year. Every time you do it the answer will always be 1089. I don't know why, but that's what happens. Cause like if you even look at the example, that becomes that, subtract it, and then you get to 1089.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>59:34</td>
<td>Lucy</td>
<td>I don't have any idea why. I'm gonna do it again. You're welcome to join me. 249.</td>
<td></td>
<td>MA 4</td>
<td>Answer of 1089 assessed for reasonableness/validity continues strategy of finding patterns with guess and check</td>
</tr>
<tr>
<td>59:50</td>
<td>Alexis</td>
<td>249 goes to 942. 942 minus 249 is 693. So then 693 is 396 and you -</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:00:15</td>
<td>Lucy</td>
<td>693 plus 396. I'm guessing 1089.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:00:22</td>
<td>Alexis</td>
<td>It is. Shocked, 1089.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:00:27</td>
<td>Lucy</td>
<td>Why? Why does it matter?</td>
<td>Questioning why they are doing the problem in the first place, not anything with problem solving</td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:00:32</td>
<td>Sharon</td>
<td>Who figures this stuff out? That's what I want to know.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:00:34</td>
<td>Lucy</td>
<td>People who get paid to think about stuff.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:01:06</td>
<td>Alexis</td>
<td>Yeah, so I don't know. This is confusing.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:01:25</td>
<td>Lucy</td>
<td>Why is that happening?</td>
<td>MA 4</td>
<td>MA 4</td>
<td>Considers why 1089 could make sense as an answer continues strategy of finding patterns with guess and check</td>
</tr>
<tr>
<td>01:01:44</td>
<td>Alexis</td>
<td>Will it work with a number that's like...</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:01:48</td>
<td>Lucy</td>
<td>Oh, like a number that's like 999? I want to try that.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:01:51</td>
<td>Alexis</td>
<td>Yeah, what if we did like 111? And then when you subtract it you get 0, you know? And then it wouldn't work.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:02:03</td>
<td>Lucy</td>
<td>Then we'd have 0, and then we're done. It stops there.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:02:09</td>
<td>Alexis</td>
<td>So, it has to be 3 different digits.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:02:19</td>
<td>Lucy</td>
<td>Well I don't know about 3 different digits. I think, because if you were to do like 211, that would work fine.</td>
<td></td>
<td>MA 4</td>
<td>MA 4</td>
</tr>
<tr>
<td>01:02:29</td>
<td>Alexis</td>
<td>Would it? 211 minus 112 equals 99. Wait no -</td>
<td></td>
<td>MA 4</td>
<td>MA 4</td>
</tr>
<tr>
<td>01:02:44</td>
<td>Lucy</td>
<td>Wait no, then it wouldn't work anymore... So then at that point you're stuck again. Does that happen with like 122? Would be 221. 221 minus 122...</td>
<td></td>
<td>MA 4</td>
<td>MA 4</td>
</tr>
<tr>
<td>01:03:11</td>
<td>Sharon</td>
<td>Numbers are too much.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:03:13</td>
<td>Lucy</td>
<td>999!</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:03:15</td>
<td>Alexis</td>
<td>99. Which then wouldn't work.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:03:17</td>
<td>Lucy</td>
<td>Exactly. So, we're having that same thing. I think you might be on to something with the whole 3 different digits. Cause even if you have 2, you end up getting to a stopping point.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01:03:32</td>
<td>Alexis</td>
<td>I just don't know why this works with like 3 different – like so we found out when it doesn't work, but we need to know why it happens, right?</td>
<td></td>
<td>MA 5</td>
<td>MA 5</td>
</tr>
</tbody>
</table>
Summary of Codes

Individual Codes for each Portfolio Problem

<table>
<thead>
<tr>
<th></th>
<th>PPS 1</th>
<th>PPS 2</th>
<th>PPS 3</th>
<th>PPS 4</th>
<th>PPS 5</th>
<th>PPS 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Courtney</td>
<td>2, 5, 2, 4</td>
<td>2</td>
<td>3, 4</td>
<td></td>
<td></td>
<td></td>
</tr>
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<td>Alexis</td>
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Class Totals for each Metacognitive Action

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Visualization of Class Totals for each Metacognitive Action
APPENDIX E

MACRO-ANALYSIS CODING INFORMATION
Macro-level Analysis Methods

Characterizing the initial student activity system followed the methods outlined by Jonassen and Rohrer-Murphy (1999) to systematically describe various components of the system. Steps one through five utilized a process of open and axial thematic coding (Strauss & Corbin, 1990) of student classroom actions and interview remarks during the first three weeks of the course to conceptually order the data components in each step. This process consisted of two key phases: data summary and synthesis. I first summarized relevant features of data from each class period. In this appendix, the ‘Sample Memos of Classroom Teacher and Student Actions’ provides an example of this process. For classroom recordings, I watched and outlined whole-class activity for the entire class. Subsequently, I listened to audio recordings at each student table to capture relevant small-group actions that occurred in between whole-class interactions.

Once I summarized the relevant features from each data source, I then synthesized the various data pieces to delineate the various components of the initial student activity system, as evidenced in this appendix (‘Summary of Classroom Actions and Interview Statements to Identify Initial Activity System’). Each data piece (classroom action, interview statement, etc.) was open coded and then organized axially into each system component. For example, student statements were coded to capture their different reasons for taking the course, such as ‘wanting to become a good elementary teacher’ or ‘to be marketable’. These emerging codes were then related to each other under the ‘Student Motives’ component of the initial system in the first step of analysis.

After describing the different components of the initial activity system, the final phase (step 6) of macro-level analysis involved stepping back from the system to
understand dynamics as catalyzed by tensions within system components, between system components, and between the student and teacher activity systems. These dynamics were identified by first looking across daily teacher and student actions over the course of the semester and then coordinating these actions with student and instructor interviews, as well as recorded planning sessions. The summary table used to organize the relevant semester-long data is provided in this appendix (‘Summary of Semester-Long Actions to Identify Catalysis for Change’) and includes abbreviated versions of the data pieces identified as part of the ‘data summary’ phase used to construct the initial student activity system (i.e., data pieces such as those in ‘Sample Memos of Classroom Teacher and Student Actions’ below).
Sample Memos of Classroom Teacher and Student Actions

Dr. Arkash wants them to establish relationships and get to know each other since they will take other courses together. Some of the ideas in the course will be seen in discrete math.

Discuss answers to #1 and #5 on page 2 of multiplication worksheet.

(Group 1) Done with these problems, so they work on modeling (a+b)(c+d). Kerri suggests using an example like 2x+2x5.

(Group 3) Work silently - Kim complains about how early it is.

(Group 4) Sharon explained her solution to #5. Alexis asked what the math was behind it and Sharon said it wasn’t really a problem and she didn’t know how to do it another way.

(Group 4) Dr. Arkash wants Sharon to present. Sharon says she gets nervous presenting, but Alexis and Lucy say they will support her from the table.

(Group 5) Jordan explains her solution to the group.

(Group 6) Ronnie asks Emilie to talk through her idea. Emilie asks Molly what she thinks. Molly says it’s the same question in different words.

(Group 7) Dr. Arkash asks them what they are thinking. Dr. Arkash asks Anna to explain her incorrect answer and then how they changed to the correct answer (what they were thinking). She comments this is a powerful thing for everyone to hear.

(Group 8) Done so move on to next page. Mary said she didn’t know how to draw a picture so she used algebra.

Anna presents: first she and Lina made a wrong tree diagram.

Dr. Arkash pauses to ask class what question this diagram is answering. Kerri and Ronnie share ideas. Dr. Arkash says she wants the whole class thinking about it and asks Isley to share her thought.

Lance shares the right answer. Dr. Arkash says that’s not what we are talking about now. Sometimes you claim something and you have to think about what you did and decide what that representation actually represents. Does this answer make sense? Students should ask
Summary of Classroom Actions and Interview Statements to Identify Initial Activity System (Data from First 3 Weeks of the Course)

**Step 1: Clarify the purpose of the activity system.** Describe the motives and conscious goals of the activity system. Specifies the broader motives (and related goals) within which activity and actions of the system reside, to aid in understanding components of the activity system and any potential contradictions within or between activity systems.

(Broader) Motives [taking this course because...]
- Become a (good) elementary teacher
  - Sharon wonders if Dr. Arickadong is asking questions because they are going to be elementary teachers [Day 01 class video]
  - Kerri liked the video they were assigned to watch and ended up watching more TED talks from teachers [Day 02 class video]
  - Likes working with kids/wants to help children
    - Amber, Amie, Sasha, Isley, Kerri, Taryn, Molly, Della, Ronnie, Courtney, Lucy, Lance, Naomi, Skylar, Sharon, Mary, Paula, Jordan [HW 0]
    - Lance, Alexia, Lima, Isley [Interview 1]
- Had good elementary math teachers/inspired by them and/or bad experiences with HS teachers
  - Alexia, Anna, Courtney, Mary [HW 0]
  - Lance [Interview 1] – teacher let him use different methods to solve problems, be creative (unlike middle/high school teachers where he could only do it one way)
  - Alexia [Interview 1] – had a bad experience with high school teachers and only one way to solve the problem – 185 is dividing because it’s not like that
  - Della [Interview 1]
- Had a good secondary teacher who inspired them [Taryn, Delia, Isley, Interview 1]
- Taking this course because it’s required
  - Amber, Alexia, Anna, Sasha, Lima, Ronnie, Lance, Skylar, Jordan [HW 0]
- Enjoy math
  - Group 5 (Isley, Naomi, Mary, Anna) discuss how much they love math and their roommates don’t understand this love for math [Day 02 class video]
  - Isley, Kerri, Kim, Anna, Della, Courtney, Skylar [HW 0]
  - Lance, Taryn, Kristy, Naomi, Courtney [Interview 1]
  - Mary [Interview 1] – likes it more than other subjects
- Good at math
  - Alexia, Molly, Sharon [HW 0]
  - Alexia, Amie, Kerri, Lima, Sharon, Delia [Interview 1]
- Told by advisor to take the course
  - Kim [HW 0]
  - Lance, Amie [Interview 1]
- Marketable [Lance, Amie, Ronnie [Interview 1]
- Teaching has a better schedule than nursing so she can be around for her son [Lina, Interview 1]

(Conscious) Goals
- Want to learn how to be a (good) math teacher [primary goal]
  - Learn to teach the math concepts they will need to teach
    - Alexia, Sasha, Kim, Taryn, Lima, Della, Ronnie, Lucy, Naomi, Sharon, Mary, Paula, Jordan [HW 0]
    - Molly, Day 04 class video – doesn’t know why they are learning different bases if they won’t be teaching it)
  - Lima, Taryn, Delia [Interview 1]
  - Learn multiple methods/procedures/representations to approach problems for needs of different learners
    - Amber, Alexia, Sasha, Isley, Taryn, Anna, Lima, Ronnie, Courtney, Lance, Sharon, Mary [HW 0]
    - Lance [Interview 1] – to give students a bigger database of tools
    - Mary [Interview 1] – remember this flexibility from earlier grades, since it wasn’t like this in calculus
    - Lima, Taryn, Courtney [Interview 1]
  - Learn how kids solve math problems [Alexia, Courtney, HW 0]
  - Want to get better at explaining context/thinking “showing steps”
  - Isley, Lima, Rennie, Lance, Naomi, Skylar, Sharon, Mary, Paula [Jordan [HW 0]
  - Alexia, Mary, Amie, Ronnie, Naomi, Isley [Interview 1]
  - Learn about common core (it’s mentioned in the text) [Delia, Interview 1]
- Develop as a “mathmatician” [secondary goal – students largely talk about their teaching goals first, and then personal mathematical growth in an afterthought]
  - Develop a growth mindset (mediated by Jo Boaler video)
    - “Grow” math brace” [Amber, HW 0]
    - Learn from mistakes, not afraid to make them [Amber, Lima, Lucy, Lance HW 0]
    - Resilience/Confidence [Kerr, Kim, Lucy, Mary HW 0]
    - Taryn, Mary [HW 0]
  - Learn multiple methods/ways to think about problems (especially visual) [Amber, Amie, Sasha, Della, Courtney, Mary, HW 0]
  - Improve problem-solving skills [Alexia, HW 0]
  - Get better at math (“fundamentals”)
    - Amie, Sasha, Kim, Taryn, Mary, Paula (“big ideas”) [HW 0]
    - Mary, Amie, Kristy [Interview 1]
  - Review math content (implies assumed proficiency) [Molly, Ronnie, Lucy, HW 0]
  - Learn math content beyond geometry, algebra, AP calculus [Molly, HW 0]
  - Be more creative [Amie, HW 0]
  - Better collaborator [Isley, Kerri, Della, Ronnie, Courtney HW 0]
  - Patients with others [Kerri, Ronnie HW 0]
  - Rekindle passion for math (previous bad experience with calculus) [Kerri, HW 0 / Int. 1]
  - Become aware of how they and others learn
    - Kerri [HW 0]
    - Mary [Interview 1]
Step 2: Analyze the activity system. Define the subject, object, community, division of labor, and rules. Outlines (1) how subjects perceive their roles in relation to the goals of the system, (2) how objects fulfill the goals of the system, (3) the nature of social interactions within the community and community beliefs and values impacting the system, (4) the implicit/explicit norms that influence the system, and (5) the division of labor that mediates the relationship between community and object.

Subject
- Students in a first-year math content course for pre-service elementary teachers with math emphasis.
- Perceived Role: Build an elementary math teaching toolkit

Object (There are many objects to choose, but I care about the outcomes of problem solving, the way that problem solving is transformed through mediation)

- Developing skills/knowledge toolkit of packaged products to find optimal/most efficient rules/procedures to get the correct answer for classes of elementary school problems (daring problem-solving situations)
  - Knowing about over knowing to act in the moment
  - Fulfills goal of becoming a better teacher (building a skill/knowledge toolkit), their primary goal in this course; Fulfills goal of getting better at math (secondary goal) by building content knowledge/products (as opposed to problem-solving habits of mind/process)
  - Supported by student beliefs based on past math experiences about what math and math teaching is (getting to the correct answer)
  - There is a best method to get to the answer – good math is efficient
  - Supported by student beliefs that the teacher is the authority of thinking (the process), and students are the recipients of the result, where the teacher needs to be able to explain the process in different ways so that all students get the process
  - Note: You can see evidence of this object transforming through the different actions/operations being taken as the days go on through time period 3
  - Realized through actions:
    - Rely on the teacher to do the explaining and metacognitive thinking (teacher metacognitive authority)
    - Supporting concretized operations:
      - Not part of their repertoire to explain their thinking:
        - Kim wonders what they are supposed to do when the activity says "explain" and asks Paula if that means they should just write down the answers. Paula says they are supposed to talk about their thought process, leading to a discussion of different methods. (Day 1 class video)
        - Sharon wonders if the reasons they need to explain the process (their thinking) is because they are going to be teachers (as if she would not do this otherwise) (Day 1 class video)
        - Students talk about having difficulty writing up PPS 1 because they got the answer, but had trouble explaining their thinking (Day 1 class video)
        - Mary asks later about one of the activity problems - are they supposed to discuss how they got to the answer on the paper? icy says they can discuss with their group and not write anything down (Day 2 class video)
        - Kerri, Courtney, and Alexis finish first problem. Kerri says to go to the next one. Alexis says her saying they need to explain their process. (Day 3 class video)
        - Naomi asks if they have to explain, and Mary says they need to explain to each other (Day 3 class video)
        - Naomi tries to explain her answer to Mary, who isn't listening. Anna asks if they are supposed to be explaining (Day 3 class video)
        - While modeling subtracting a negative, Sharon wonders if negatives canceling is enough of an explanation. (Day 4 class video)
        - While modeling adding 0, Amber comes up with adding 1Y/R. Naomi asks her how she would explain that (Day 4 class video)
        - Sharon explained her solution to #5. Alexis asked what the "math" was behind it and Sharon said it wasn't really a problem and she didn't know how to do it another way (Day 5 class video)
        - While working on a class problem, Sharon, Lucy, and Alexis say they know how to do it but can't explain why it works, but they know they need to. They try to explain why. They think that's why they are getting these types of problems (Day 6 class video)
        - Lina and Anna figure out why a particular multiplication method works. They wonder if figuring out the method is enough, but decide they have to explain why it works. They will work on this over the weekend. (Day 6 class video)

- Alexis says this has been hard [Interview 1]
  - "If I don't understand a problem, I need help. It's not like I can just kind of pull my way around. Does that make sense? Like if I don't get a problem right then and there, I need help to understand this before I move on. It's not something where I can just get myself around it and then move on." [Noemi, interview 1]
  - Shares gives a method and Alexis says it sounds like a rule and would be confusing for students.
  - Shares says that what a teacher needs to do - explain difficult concepts [Day 4 video]
  - Kerri and Lance work together on typing addition and subtraction with chips. They decide 2 negatives just make a positive. Paula says Dr. Arkadash will just explain it. (Day 4 class video)
  - Mary says she likes when Dr. Arkadash comes over because she can do it but not explain it - and that's the role of the teacher. (Day 1 class video)
  - "Teach problems in more than 1 way," "explain things in an understandable way" asks "force" students to explain their thinking (so they can practice for teaching), ask questions to help students think about what they are doing, be "to the point" [HW 0]
- Prioritize the most efficient/easy/fast method
  
  - Supporting concretized operations:
    - While working on the warmup problem, Lucy reflects on efficiency of method - want the most efficient method [Day 2 class video]
    - Keith proposes a method to solve a question from the warmup and Dr. Arkadash asks students to discuss this method. Amber, Naomi, and Jordan discuss that their method is easier and wonder if they can solve the problem their way. During a class discussion about the method led by Dr. Arkadash, a student proposes a different method instead of thinking about Keith’s method - other students comment that Keith’s method is “ridiculously long” [Day 2 class video]
    - While looking at student work, Ravi says it isn’t efficient, though Lance says it might be efficient for a little kid [Day 5 class video]
    - While modeling Pres/VP counting, Molly talks about which methods are easier (according to her) [Day 5 class video]
    - Naomi talks about trying not to have to show all her steps to be more “effective” [Interview 1]

- Value correct answers over correct thinking processes
  
  - Supporting concretized operations:
    - Meta-cognitive action 4 in PPS 2
      - After watching the Jo Boaler video in HW 0, students discuss goals aligned with a growth mindset, but this isn’t discussed in the student interviews. Growth and process are afterthoughts [HW 0]
      - Oscar dominates conversation (tells Jordan for answer is wrong and what the correct answer is, but doesn’t care to go through Jordan’s process) [Day 2 class video]
      - Students work mostly individually/silently, and then just check answers [Day 2, 3, 4 class video - this tapers off as the days go on]
      - Della and Isley discuss how the methods are impractical when looking at different multiplication method [Day 8 class video]
    - Meta-cognitive action 4 during PPS 2 [micro-level analysis]
    - Molly doesn’t know why they are learning different bases if they won’t be teaching it [Day 4, class video]

Community (“those who share the same object”)
- Perceived (elementary) teacher community (they aren’t actually in this community yet) and undergraduate math classroom community:
  - Community beliefs and values impacting the system:
    - Teachers need to be good at explaining how to do math problems to students (teacher delivers packaged products received by students)
      - Evidenced by expectations for Dr. Arkadash in the course (see reflection 1)
      - Dr. Arkadash must be asking students to explain because they will be future teachers [Day 91 - Sharon]
      - Students Goals for the course: Want to get better at explaining content/thinking “showing steps” [Day 91 - Sharon]
      - Alexia, Mary, Amie, Ronnie, Naomi, Isley [Interview 1]
      - The teacher explains problems the students don’t know how to solve [Lance, Interview 1]
      - Students can “just see it” while problem solving as long as the teacher has already taught them the method. If students can’t solve it, then there’s some trick students shouldn’t necessarily be expected to just see on their own [Alexia, Interview 1]
    - Teachers need to appeal to diverse needs of learners by explaining how to do math problems in different ways, meaning they have to build knowledge of multiple ways to solve a problem
      - Students Goals for the course: Learn multiple methods/procedures’ representations to approach problems for needs of different learners so that everyone understands the “rules” [Day 91 - Sharon]
      - Lance [Interview 1] - and to give students a bigger database of tools
      - Alexia, Mary, Amie, Ronnie [Interview 1]
      - Amie notes that while she’s not good at thinking about problems in multiple ways (she’s “one-track minded”), she has to be able to do that as a teacher to explain to others [Interview 1]
    - Teachers need to value the voices of all students
      - 185 students realize they need to understand how others think about problems

- Good teachers that inspired the 185 students include those who let them do problems in different ways
  - Lance, Mary [Interview 1]
- Teacher asks questions to elicit student thinking and help them get unstuck to solve problems (meta-cognitive authority)
  - Mary says she likes when Dr. Arkadash comes over because she can do it but not explain it - and that’s the role of the teacher [Day 1 class video]
  - Ask questions to help students think about what they are doing [HW 0]
  - Getting questions from other people can make you see it in a different way and maybe help to solve it. [Lance, Interview 1]
  - When stuck on a problem, wait until the next day to get help from the teacher, though this makes test taking difficult because you may not be able to do the problems on your own [Alexia, Interview 1]
- Teach the common core “new math” [Della, Interview 1]
  - Group work not a right or wrong answer, more about the process, not as much emphasis on the most efficient method [Della, Interview 1]
- Math is marketable [Lance, Amie, Ronnie [Interview 1]
High school math “Algebra/Calculus/Statistics” Brain/Mindset [what it means to “play school” in math class] – traditional math community
  - Only one way to do most problems [Lance, Alexis, Mary, Amie, Kristy, Interview 1]
    - Students note that IB is very different from this setting
    - Amie notes that there’s only one way to do things in high school because it is “higher” math [Interview 1]
  - No group work and emphasis on student thinking in previous courses – what to do and then practice it [Alexis, Keri, Shanna, Kristy, Romée, Delta, Isley, Interview 1]
  - “Plug and chug” in a formula [Taryn, Romée, Interview 1]
  - More about the answer than the process [Delta, Interview 1]
  - “Not have to think about what I was doing” like they have to now in the PPS write ups [Delta, Interview 1]
  - More choice to try strategies you want [Isley, Interview 1]

Division of Labor (mediates between object [building a toolkit in this case] and community)

- Group work
  - This is new to them. They didn’t have any group work or emphasis on student thinking in IB.
  - Start by working in silence unless instructor comes by to ask a question or if they want to check an answer [Day 01]
  - Slightly later in groups, as analyzed by directions in problem to explain/discuss with each other [Day 02 – Mary, Isley] – working in groups is not new to many students, and talking about their thinking is new to students, especially at the beginning of the problem-solving process
  - Individual think time followed by helping each other when they are stuck [Interview 1, Lance]

- Students want teacher to help, but teacher asks them to explain their own thinking [Day 01]
  - Tension between individual and group needs? [Type I]
  - Discuss norms to talk about what is needed to work together [Day 01]
  - Different people explain how they got to the answer to help each other and practice explaining as teachers [Day 01 – Krissy, Courney, Lima]
  - Check answers with each other [Day 01 – Taryn]
  - The teacher should ask students how they got their answers since they can do the problems but not explain (this is the role of the teacher) [Day 01 – Mary]
    - Dr. Arkada asks this back on students – they should ask each other questions
  - Work together [Day 01 – Naomi]
  - Individual Think Time [Day 01 – Keri, Lance, Lima]
  - Move through activities together [Day 01 – Lucy]

- Since they need to talk about why things are happening, students are taking more time to solve the problems (than if they were outside of this class), meaning they talk a lot more [Mary, Interview 1]
  - It seems like students are attributing the IBL components (like group work) where you have to share your thinking) to them preparing to be teachers, not because it is good for them as students
  - Most have to “create the steps” instead of being told what they are, experiment more instead of just applying a known formula [Shanna, Taryn, Interview 1]
  - Groups are assigned

Rules (mediates between subject and community)

- Explicit
  - Explicit rule – implicit: Students need to explain their thinking [Dr. Arkada asks them to do this repeatedly] [Day 01 – e.g., Isley]
  - Explicit rule – implicit: Need to discuss solutions [Dr. Arkada asks them to share ideas] [Day 01]
  - Explicit rule: participate in class, “claim your learning” [Day 02]
  - Grading mechanisms (PPS grading – emphasizes value on process)

- Implicit
  - Tell the teacher what they want to hear
    - (as evidenced by PPS write-ups and actions during PPS – think Amie saying she is going to write everything down)
  - Justify by means of lots of examples, looking for patterns [PPS write-ups, Interview 1]
  - Teacher is authority of correctness and thinking, though others can help triangulate answers
  - Product over process

Step 3: Analyze the activity structure. Delineate the hierarchy of activity, concrete actions, and automated operations. Describes the interrelationships of thinking and performances (focused on the object) while purposefully including an understanding of the intentionality of actions and operations.

- Operations that evidence the norm of process (why) over product (what) developing:
  - Kim wonders what they are supposed to do when the activity asks “explain” and asks Paula that means they should just write down the answers. Paula says they are supposed to talk about their thought process, leading to a discussion of their different methods. [Day 1 class video]
  - Sharon wonders if the reason they need to explain the process (their thinking) because they are going to be teachers (as if she would not do this otherwise) [Day 1 class video]
  - Students talk about having difficulty writing up PPS 1 because they got the answer, but had trouble explaining their thinking [Day 3 class video]
  - Mary asks Isley about one of the activity problems – are they supposed to discuss how they got to the answer on the paper? Isley says they can discuss with their group and not write anything down [Day 2 class video]
  - Keri, Courney, and Alexis finish first problem. Keri says to go to the next one. Alexis says her saying they need to explain their process, [Day 3 class video]
  - Naomi asks if they have to explain, and Mary says they need to explain to each other [Day 3 class video]
  - Naomi tries to explain her answer to Mary, who is not listening. Anna asks if they are supposed to be explaining [Day 3 class video]
  - While modeling subtracting a negative, Sharon wonders if negatives canceling is enough of an explanation. [Day 4 class video]
  - While modeling adding 0, Amber comes up with adding 1/1/0. Naomi asks her how she would explain that [Day 4 class video]
  - Sharon explained her solution to 45. Alexis asked what the “math” was behind it and Sharon said it wasn’t really a problem and she didn’t know how to do it another way [Day 3 class video]
  - While working on a class problem, Sharon, Lucy, and Alexis say they know how to do it but can’t explain why it works, but they know they need to. They try to explain why. They think that’s why they are getting these types of problems [Day 6 class video]
  - Lisa and Anna figure out why a particular multiplication method works. They wonder if figuring out the method is enough, but decide they have to explain why it works. They will work on this over the weekend. [Day 6 class video]
  - Mary says they have to be able to explain why they are doing what they are doing [Interview 1]
Step 4: Analyze tools and mediators. Describe the tools, rules, and roles of participants that mediate activity within the system. Explicate the instruments (physical and cognitive), implicit and explicit rules, and divisions of labor that mediate and constrain activity, actions, and operations.

- Jo's Boiler Video
  - Made Amie excited for class [Day 01]
  - Kerr liked it and watched more videos online [Day 02]
  - Influenced reflection on HW 0 with respect to goals
  - Emphasizing process (Della talks about this in Interview 1)

- Dr. Arkadash
  - elicits thinking from students rather than telling them what to do and having them work on problems after she tells them what to do
  - asking students to explain ‘why’, ‘how do you know’, ‘give the details’
  - Lima, Interview 1 values this approach
  - ‘I thought it was so funny cause after my meeting yesterday with Dr. [Arkadash]. I went back to my dorm room and all my sentences were like, “How did it go?” And I was like, “My life has changed. I understand math and I was just like freaking out and I basted out the three pages of the portfolio... We worked on stuff and it was weird that I understood because she didn’t even tell me what to do, she made me do it myself and figure it out myself. And I thought that was weird because usually like teachers in high school would be like, oh, yeah, this is how you do it; now go work on it yourself, but she was pushing me to think and try to find the process of how it works. And when I did it... I think that was why I was like so excited because I figured it out myself, with help obviously, but it was my own thinking’” [Krissy, Interview 1]
  - lots of other quotable quotes

- Activities
  - directions: need to explain
    - Kim wonders what they are supposed to do when the activity says “explain” and asks Paula if they should just write down Paula’s answers. Paula says they are supposed to talk about their thought process. Kim realizes they did the problem differently — they discuss their methods [Day 01 class video]
  - Problems instead or exercises
  - Multiple representations/methods for the same problem
  - Group members
    - Since this is new for most students, they don’t really know how to use their group members. Over the course of the semester they figure out how to use each other for more productive thinking
  - Different representations (mostly manipulatives in this time period)
    - Provide a way to explain thinking to others (teacher has) [Mary, Interview 1]
  - Helps her catch her own mistakes (student has) [Mary, Interview 1]

- HW
  - Amie talks about how the homework is problems, not just more practice with the same rules (like in statistics or trig), “you really have to get the manipulatives out” and “thinking through things more” [Interview 1]

- PPS 1/2
  - Kristy almost talks about there being one way of doing the problems, like there’s some sort of “trick” to doing them
  - “It’s not really about getting the answer, it’s about your process in getting to the answer, and being able to explain yourself, and being able to understand it in such a deep way that you can explain it to another person” [Roenie, Interview 1]
  - (About PPS 1) First explained her solution with mathematical justification, then filled up the rest of the pages with questions and reasoning. “It was so different than writing an English paper and writing about math is hard, so she didn’t think it was good [Alexis, Interview 1]
    - It makes sense, but they are writing what they think Dr. Arkadash wants to see. Because they are focused on the answer, they were focused more: on a complete solution, not a complete solution attempt.
  - Not ready for generalization tool in PPS 2 (based on write ups)

- Solving problems where the method is not obvious (or within a narrow number of possibilities)
  - This is new for most students — they don’t know how to deal with it. They just want to quickly find a rule and go with it. This is what happens with PPS 2, where they just find a rule and go with it.
  - Of course they aren’t good at process-focused meta-cognition. They haven’t really had to do it before.
  - “I really get excited going to class because it’s fun in my opinion. But I think it’s gone really well like the fact that we do have to think for ourselves and problem solve ourselves instead of having it written on the board and then copying answers” [Krissy, Interview 1]
  - Kristy talks about how they have to move back and do the “first step” to get to algebraic equations instead of having them just given to you [Kristy, Interview 1]
Step 5: Analyze the context. Characterize the internal, subject-driven and external, community driven contextual bounds. Defines the larger non-instructional context within which activity is bounded, including beliefs, assumptions, preferences, etc. of the subject, as well as formal and informal rules and roles imposed by the larger community. The goal is to describe how things get done within the broader context by making obvious the conversation and collaboration tools required to support the activity structure.

**Subject beliefs, assumptions, preferences/ Formal and informal rules and roles imposed by the larger community**

- Teacher is authority [Day 01]
  - Explains the answers [Day 01 – Naomi]
- Need to get the right answer (product over process)
- Delta says she has common core and doesn’t want to teach elementary school anymore [Day 3 class video]
- Beliefs about teacher role in the “community” step
- What is math?
  - Everything is “set” [Lance, Interview 1]
  - Problem solving [Lance, Alexis, Interview 1]
    - Figure out any problem based using what you know, even if it isn’t the most efficient method (builds on itself) [Lance, Interview 1]
    - Important to be able to solve problems in different ways since you don’t always know which method is “best” [Lance, Interview 1]
    - Engages creativity [Lance, Interview 1]
    - Lets you discover [Alexis, Interview 1]
    - Using numbers in solve problems (”get other places and find different things”). [Alexis] [Alexis, Amie, Interview 1]
    - “starting with one product and morphing it into something else” [Alexis, Interview 1]
  - Straightforward, “black and white”, “direct” [Day 01 – Isley, Naomi, Asna] [Naomi, Interview 1]
  - “Solving a problem with numbers where it can give you an answer” [Mary, Interview 1]
  - You are given rules/instructions to use and have to solve a problem with that rule [Mary, Kerr, Interview 1]
  - English is more “inferring” and math is more “doing” [Amie, Interview 1]
  - Rules that will work every time, whether or not you think deeply about them [Kerr, Interview 1]
    - This class is slightly different because math feels more like a “game” where you can think about problems in lots of different ways, though there are still rules [Kerr, Interview 1]
  - Can be emotional [Kerr, Interview 1]
  - Only one answer, “right or wrong”, even if you can think of it differently [Lina, Interview 1]
  - Clear “endpoints and goals” and you put pieces together to get there
  - Having an equation where you have to solve for a variable or number
  - Numbers, equations, getting to the answer (possibly different routes, but always a way to get to the answer) [Sharon, Interview 1]
  - Patterns with numbers, calculations, shares [Tanya, Courtsey Interview 1]
  - “How numbers are used in real life and the influences they have” [Kristy, Interview 1]
  - Not something you could just pick up on your own [Kristy, Interview 1]
  - Solving numeric puzzles [Ronnie, Interview 1]
  - Same things just have to be told to you by the teacher [Ronnie, Interview 1]
  - Builds on itself so much that you can get left behind [Naomi, Interview 1]
  - Applying what you already did before to solve problems (unlike in the real world where you don’t have something to look back to) [Kim, Interview 1]
  - Numbers and procedures that will help you with a future job (engineering, computer tech, etc.) [Delta, Courtney Interview 1]
  - Solving for x in an equation [Delta, Interview 1]
  - It’s everywhere [Isley, Courtney Interview 1]
  - Finding multiple solutions and different ways of looking at things [Isley, Interview 1]

- What can you do about math?
  - Persevering when what you tried something that didn’t work [Isley, Interview 1]
  - What do you like about math?
    - Solving problems I don’t know how to do [Lance, Interview 1]
    - Easier to follow the steps than in English where you have to be more creative [Amie, Interview 1]
    - It is a universal language [Sharon, Interview 1]
    - There is just one answer, a formula [Tanya, Interview 1]
    - Math is unique and creative; “you get to bring your own thinking into it” [Kristy, Interview 1]
    - Not as fun as other subjects [Ronnie, Interview 1]
- **What is problem solving?**
  - Observation, then one step at a time to break down the problem solution method into manageable pieces [Lance, Interview 1]
  - Science you start with an observation and go backwards to where it started, whereas with math you start with something and "move forward" [Alexis, Interview 1]
  - Check answer at the end
    - First “go with instinct” and if that doesn’t work (you get the wrong answer), try something else [Alexis, Interview 1]
    - “trial and error,” “guess and check” [Kerri, Taryn, Mary, Isley, Interview 1]
    - unless you know the shortcut
  - Showing your work to find the one “right” answer [Mary, Interview 1]
  - Read the question and underline important information, then try to find a rule to use (like the quadratic formula), then guess and check [Mary, Interview 1]
  - “Step-by-step” using rules/procedures [Mary, Kerri, Taryn, Interview 1]
  - Applying what you know [Kerri, Taryn, Interview 1]
  - Finding an answer using a sequence of steps [Sharon, Interview 1]
  - Identify what you’re looking for and then apply your knowledge to develop a method to solve the problem [Taryn, Interview 1]
  - Not just using a memorized procedure/conversion calculation, where you have to “know what’s actually going on” [Taryn, Interview 1]
  - Analyze a problem, then use different methods or ways of thinking to solve it [Kristy, Interview 1]
  - Less ways to solve problems than in other subjects, but still multiple methods [Kristy, Interview 1]
  - Before this class, “working through the problem and finding the answer”, but learning in 185 that “is more about understanding how you got to where you did. It’s not just doing the problem, it’s understanding each step, then being able to explain your process, and being able to explain why you understand it, and how it works the way it does” [Ronnie, Interview 1]
  - “Doing math to find a solution” [Naomi, Interview 1]
  - numbers and letters, mental math, not really visual [Ronnie, Naomi, Interview 1]
  - find the connections and the relations between where you are at and where you are going [Isley, Interview 1]

- **Preferences for group work**
  - Positive feelings about group work [Lance, Alexis, Amie, Kerri, Sharon, Ronnie, Interview 1]
    - Time to figure it out on your own, and then ask questions and get ideas [Lance, Alexis, Lima, Interview 1]
    - Help you to get the answer by filling in the pieces you don’t have. [Lance, Sharon Interview 1]
    - Helps you be more efficient/solve problems faster [Naomi, Interview 1]
      - If being good at math means you’re fast and efficient, then of course they want others to just tell them what to do and not break down their thinking at all
      - Wants to be careful working with others [Kerri, Interview 1]
    - So as not to be left behind [Kerri, Naomi, Interview 1]
    - Wants to make sure she has time to think about what she’s doing to make sure she understands the concepts behind the rules [Kerri, Naomi, Interview 1]
  - Prefers working individually, though values being able to hear other’s thoughts [Lina, Isley, Interview 1]

- **Molly (anomaly/ “negative case”)**
  - No need to show work on “easier” problems [Day 01 – Molly]
  - They aren’t “learning” in class [Day 02 – Molly]
  - This class doesn’t seem like “math” [Day 02 – Molly and Sasha]
  - Wants Dr. Arkadash to “teach” them first before they do the problems [Molly, Day 2 class video]
<table>
<thead>
<tr>
<th>Pre-Class</th>
<th>Teacher</th>
<th>Students</th>
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<tbody>
<tr>
<td></td>
<td>Sends growth mindset/inquiry video</td>
<td>Confusion about explaining</td>
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| Day 1           |                         |                                              |
| (Unit 1)        | Ask students: How do you know? Why? Explain more slowly. Slow your thinking. Use manipulatives |                                              |
|                 | You will need this as teachers | Group norms                                 |
|                 | Explain/discuss in activity directions | Share you got answer                         |
|                 | Make connections (between bases 5 and 10) | Check answers                               |
|                 | Reason with manipulatives | Teacher explains (students do)               |
|                 |                          | Individual think time                        |

| Planning Mtg (Goals) |                         |                                              |
|                      | Highlight                |                                              |
|                      | Compare/contrast solutions |                                              |
|                      | Use manipulatives and representations to reason |                                              |
|                      | Notice patterns           |                                              |
|                      | Content #1 goal (not PS for its own sake) |                                              |

| Day 2           |                         |                                              |
| (Unit 1)        | Tell groups to share process |                                              |
|                 | Draw attention to understanding what question is asking |                                              |
|                 | Connect process to math meaning |                                              |
|                 | Model with manipulatives (tell students to do this too) |                                              |
|                 | Define math terms           |                                              |
|                 | Students should claim learning |                                              |
|                 | Introduce PPS – Interested in thinking |                                              |
|                 |                          |                                              |

| PPS 1 (class)   |                         |                                              |
|                 | Praise students sharing reasoning and asking for examples/clarification |                                              |
|                 | Need to generalize        |                                              |
|                 | Explains write up directions (reasoning, decisions, why) |                                              |
|                 | "slow your thinking" |                                              |
|                 |                          |                                              |

| HW0 Reflection  |                         |                                              |
|                |                          |                                              |
|                |                          |                                              |

| Reflection 1   |                         |                                              |
|                |                          |                                              |
|                |                          |                                              |

| Day 3           |                         |                                              |
| (Unit 1)        | Meaning of +/-          |                                              |
|                 | Make connections between +/- in bases/different representations | Hard to write about PPS 1 b/c didn’t take that long to answer |
|                 | Adding right to left isn’t only method | Not sure if they are supposed to be explaining – many groups still work silently and check answers |
|                 | Manipulatives slow your thinking |                                              |
|                 | Multiple methods asked for in activities |                                              |
|                 | Continue telling students to explain |                                              |

| Planning Mtg (Goals) |                         |                                              |
|                      | More presentations |                                              |
|                      | Pick PPS’s to match content |                                              |
|                      | PPS 2 easier to solve with manipulatives |                                              |
|                      | Emphasize math justification/generalization beyond examples |                                              |

| Interview 1       |                         |                                              |
|                   | See TP1 summary for initial student activity system |                                              |

| Day 4           |                         |                                              |
| (Unit 2)        | "you are peculiar" |                                              |
|                 | the have different ways of learning and their students will too | Wait for teacher to tell them what to do when they get stuck |
|                 | practice explaining and understanding your and each other’s thinking | Present solution, not thinking |
|                 | Struggling good because forces making connections | Some think visualizing helps (but is different) |
|                 | Connect to meaning of +/- made explicit (put down, take away) | |
| PPS 2.1 (class) | After finding a pattern, need to make a conjecture and then try to explain why the conjecture is true  
- Why/when do you keep getting 1089?  
- Suggest using ‘abc’ and manipulatives to generalize (for any number)  
- Talk to each other to understand what question is asking | Try lots of examples to find patterns, but not connecting to math meaning and paying attention to superficial features (even/odd, big/small)  
- Make conjectures about what #’s you can get (1089, 198, 0) – run out of time to think about why |
| Day 5 (Unit 2) | Warm up to practice using manipulatives to show +/-  
- Student makes conjecture and she tells student to explain with ‘abc’  
- Points out that explaining helps you hear your thinking and catch your mistakes  
- Need to make connections to math (when looking at student work)  
- Continue telling students to explain and understand what the question is asking | Students say sample method is correct, but not efficient |
| PPS 2.2 (class) | Push students to generalize past a bunch of examples | Some students know they need to explain why, but others OK with more examples  
- Don’t know how to use ‘abc’ to generalize different cases/rules they find |
| Day 6 (Unit 2) | Showing examples not enough to generalize/need math-based justification  
- Has presenter explain thinking to get from incorrect to correct answer  
- Asks class what the wrong diagram represents mathematically  
- Highlight representation connections  
- Push presenter to understand why FOIL has no math meaning | Know they need to explain but don’t know how  
- Try different visuals, but not ones that have mathematical meaning  
- Think nonstandard multiplication methods are “impractical”  
- Focus on explaining the procedure, not really why it works (though some groups do) |
| Day 7 (Unit 2) | Connect pattern to meaning of × (be precise with language)  
- Push presenters to explain mathematical justification, not just the procedure  
- Connect multiplication methods to math meaning  
- Connect definition of + to × | Struggle to explain why negative time negative is positive using mathematical justification instead of rules |
| Planning Mtg (Goals) | Students take control of challenging/questioning each other (instead of her)  
- Explicit with language  
- Connections to definitions |
| Day 8 (Unit 2) | *** HW means explanations  
- Practice with meaning of + (connect to definitions) | Feels “weird” explaining, but needed for teaching |
| PPS 3.1 (class) | Gives examples from previous write-ups  
- Tried example (guess and check) to understand question (strategy and why strategy used)  
- Include questions they asked themselves  
- Other strategies used: counterexamples, cases, working backwards  
- Use representations to organize thinking and help communicate  
- Highlights an ‘abc’ explanation to say, again, students can’t just use examples as justification | Spend most of the time looking for patterns and understanding the question  
- Acknowledge they will need to know why their conjectures/observations work (but this is something they do after they find the answer) |
| Day 9 (Unit 2) | Push to use precise language, connect to definitions, math-based justification  
- “what do the numbers represent?”  
- “what does that operation represent?” | Practice compare/contrast methods  
- Students present/interview on their own  
- Pose question (students work on this for next class) |
| Day 10 (Unit 3) | Highlight 2M, 2N+1 representation to help with generalization (instead of writing lots of sentences) |
| Day 11 (Unit 3) | • Connect to definitions  
• Use mathematically-based justification | • Practice with definitions and meaning |
| Day 12 (Unit 4) | • Connection to definitions  
• Representations help this  
• Different representations | • Practice definitions/explanations  
• Struggle explaining the procedures |
| Day 13 (Unit 4) | • Use representation to see mathematical meaning  
• Precision with language and definitions | • Students practice via student presentations – catching themselves giving rules and changing to more math-based explanations |
| Day 14 (Test 1) | | |
| Day 15 (Unit 4) | • Support claim with visual or manipulatives  
• Pushes presenters to dig further into mathematical meaning (“what does that mean?”)  
• Precision with language  
• Looks for different methods  
• Use a picture to help think through things | • Discuss multiple methods – leads to comparing/contrasting |
| Day 16 (Unit 4) | • Ask for multiple methods shared in presentations | • Question if explanations are sufficient  
• Ask each other how they would explain their method to students  
• Pictures are helpful for these problems |
| Reflection (goals) | • Students seem like good “self-starters” now – now it is time to “shape” their PS strategies/skills  
• Made explicit connecting to previous knowledge  
• Make explicit the questions she (the teacher) is asking herself | |
| Day 17 (Unit 4) | • Pushes using pictures and definition of fractions instead of algebraic manipulation  
• Models asking herself questions to connect representation to definitions – reasoning through the representation  
• Says explicitly that making connections to previous knowledge is important for them as teachers and as students | • Students insist on using algebraic manipulation instead of reasoning in the pictures  
• They practice reasoning with different representations |
| Planning (Goals) | | |
| Day 18 (Unit 4) | • Remind students about “description and meaning of things”  
• Explicitly making connections between representations (presenting multiple solutions)  
• Students need to be precise with language  
• Makes students do fraction division using the meaning of division and pictures/manipulatives, not copy-dot-flip | • Struggle with explaining why copy-dot-flip works using precise language and connecting operations to their mathematical meaning |
| PPS 4.1 (class) | • Dr. Arkadash tells groups who used denominators 8 and 9 to be explicit about where those come from – why 72? Why 71? (connect to meaning of fraction) | • Try to use different representations (percent, decimals, picture) – but these are disconnected from mathematical meaning – not helping with reasoning
• Those who think about what “close” means and connect size of pieces to definition of fractions are successful quickly (still a problem for them though)
• Similar to PPS 1 |
| Planning Mtg (Goals) | • Worried about confidence affecting students’ questioning if they are correct, rather than their method (opposite of Schoenfeld) – wants to highlight that problem solving isn’t a function of time, so they need to think about what they can do instead of questioning their confidence (turn the emotional trigger into action) |  |
| HW 6 Reflection | • Learned
• Explaining math (being explicit)
• Multiple ways to solve a problem
• Meaning of math terms
• Why rules/operations work instead of memorizing
• Importance of visual representations/pictures/language
• Accomplishments
• Explaining better (more specific)
• Explaining in multiple ways
• Questioning methods
• Goals for end of semester
• More strategies/tools
• Communication, Explain better | • Struggle to explain copy-dot-flip connecting to mathematical meaning of operations |
| Day 19 (Unit 4) | • Models what she asks herself to connect to definition of division and understand what the question is asking
• Highlights importance of language | • Students down the rabbit hole last time, still down the rabbit hole |
| PPS 4.2 (class) | • Dr. Arkadash tells groups who got 8 and 9 to be explicit about where those come from – why 72? Why 71? (connect to meaning of fraction) |  |
| Planning Mtg (Goals) | • Wants to discuss how to deal with struggle |  |
| Day 20 (Unit 5) | • Highlights that manipulatives can help students reason through the problem, not just model the solution after the problem
• Highlights that moving a decimal point has mathematical meaning | • Students debate "best" way to explain their work for someone else to understand
• Practice with double number line |
| Day 21 (Unit 5) | • clarifies that HW is for exercising ideas, and PPS is for problem solving where they don’t know immediately how to solve it – this takes practice (articulating what they are thinking and asking themselves why they think certain things).
• Plays Andrew wiles video – frustration is part of the process – sometimes you need to take a break
• The whole discussion today is about connecting expressions and picture representations to word problems - all about connecting to mathematical meaning | • practice connecting to mathematical meaning |
<table>
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<th>Day 22 (Test 2)</th>
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| **Day 23** (Unit 6) | • purposefully group students with new group members to practice explaining to few people  
• wants them to practice listening and pushing each other to explain things differently (different representations, explaining what they mean, etc.)  
• they need to think about why they are doing certain operations  
• pushes for different representations, and connecting them  
• Decide which representations are applicable for ratios (initial discussion about representing ratios and coming up with multiple methods) |
| **Day 24** (Unit 6) | • explicitly talks to groups that are stuck. Tells them to think about the context, what things mean (connect to meaning), use a picture, make sure the picture represents what the context says  
• students are practicing, but struggling with, reasoning within the representation itself, rather than solving the problem and then using the visual to show how to get the answer, or not using a visual at all |
| **PPS 5.1 (class)** | • reminds students to make connections (to the section material)  
• pushes students to connect to meaning of 4:9 (possibly using the representation to see this)  
• spend much of the time trying to represent the problem in a meaningful way – solving the problem through the representation  
• consider different representations, or variations of a representation  
• attend to ratio, pick up the first 4 numbers, and go with 4:9 (not knowing what this means) |
| **Day 25** (Unit 6) | • continue telling students to connect to mathematical meaning with the representations, work within the representations to solve  
• continue practicing connecting representations to mathematical meaning – students still struggling to work within the representation itself |
| **Day 26** (Unit 6) | • Focus today is on connecting representations of functions, finding expressions to represent visual patterns  
• Practice connecting between representations and with mathematical meaning |
| **Day 27** (Unit 6) | • Practice connecting between representations and with mathematical meaning |
| **PPS 6.1 (class)** | • Reminds students they need to justify their patterns – beyond just checking with a few examples – connect to representation  
• Answer is an algebraic expression – think they are done because they found the answer  
• confused about how to justify this |
| **Day 28** (Unit 6) | • Discuss more than one correct answer for graphs – need to explain assumptions  
• Practice connecting between representations and with mathematical meaning |
| **Day 29** (Unit 6) | • Final exam review (part of take home)  
• Final exam review |
APPENDIX F

PORTFOLIO PROBLEMS
Portfolio Problem 1: (Dr. Steven Leth, personal communication) The last digit of a number is a 0 when it is represented in base 5 and a 1 when represented in base 2. What is the last digit when it is represented base 10?

Portfolio Problem 2: (Driscoll, 1999) *Take a three-digit number, reverse its digits, subtract the smaller from the larger. Reverse the digits of the result and add it to the original result. For example,*

123 becomes 321, and 321 − 123 = 198

198 becomes 891, and 198 + 891 = 1089

*Try this process with several numbers. What do you observe? Why?*

Portfolio Problem 3: (Liljedahl, Chernoff, & Zazkis, 2007) A pentomino is a shape that is created by joining five squares such that every square touches at least one other square along a full edge. There are 12 such shapes, named for the letters they most clearly resemble.

![Pentominoes](image)

Now consider a 100’s chart! *If a pentomino is placed somewhere on a 100’s chart, will the sum of the numbers be divisible by 5? If not, what will the remainder be? Explain how you can know “quickly”!*

Portfolio Problem 4: (Northern Colorado Math Circles, 2013) Find four different digits a, b, c, d so that the sum \( \frac{a}{b} + \frac{c}{d} < 1 \) and the sum is as close to 1 as possible. Justify why your answer is the largest such number less than 1. (When we say a, b, c, d digits, we mean that they can be any whole number between 0 and 9.)
**Portfolio Problem 5:** (Adapted from Dorichenko, 2011) *At sunrise, two old women started to walk towards each other. One started from point A and went towards point B while the other started at B and went towards A. They met at noon but did not stop; each one continued to walk maintaining her speed and direction. The first woman came to the point B at 4:00 pm, and the other one came to point A at 9:00 pm. At what time did the sun rise that day?*

**Portfolio Problem 6:** (Adapted from Mathematics Achievement Partnership, 2002)

Below is a triangle formed with numbers.

| Row 1: | 1 |
| Row 2: | 3 5 |
| Row 3: | 7 9 11 |
| Row 4: | 13 15 17 19 |
| Row 5: | 21 23 25 27 29 |
| Row 6: | ___ ___ ___ ___ ___ |
| Row 7: | ___ ___ ___ ___ ___ ___ |

- What are the first and last numbers in the n<sup>th</sup> row? (E.g., the first number in 3<sup>rd</sup> row is 7 and the last number in the 3<sup>rd</sup> row is 11). Justify your answer.

- What is the sum of the numbers in this n<sup>th</sup> row? (E.g., the sum of the numbers in the 3<sup>rd</sup> row is $7 + 9 + 11 = 27$.) Justify your answer.

- What is the sum of all the numbers up to and including the n<sup>th</sup> row? (E.g., the sum of the numbers up to and including the 3<sup>rd</sup> row is 36). Justify your answer.

- What other patterns do you notice in this triangle? Justify your answer.