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Time Series for the Boolean Random Sets

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TIME SERIES FOR THE BOOLEAN RANDOM SETS

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ABSTRACT

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Although Random closed sets (RACS) are rich in modeling complex objects, its model parameter inference is simple. The most important type of the RACS is Boolean random set (BRS). BRS is a parametric model that is formed by placing random closed sets (grains) at points of a Poisson process (germs), and taking union of these sets. The radius of these grains may be fixed and known, fixed but unknown, and random. Furthermore, the intensity parameter λ, which is one source of randomness in the BRS model, has been estimated by the method of Intensity, method of Minimum contrast, method of moments, and ordinary and generalized least squares regression for the independent BRS. A time series model was then developed for the intensity estimation of the correlated BRS using maximum-likelihood, and method of moments for the radius estimation. The model used past observations (n_t^+ or $\hat{n}_t$), and past intensity to estimate current intensity. In addition, twelve parameter schemes were employed to study the properties of these parameter estimates, including biasness, consistency, and asymptotic normality. Simulation results showed that the parameter estimates inherited the properties of maximum likelihood estimators. Thus, the future intensity of phenomena that are correlated random sets (Boolean) in nature can be predicted, using the model built in this study. This model was built for the Rocky Mountain Pine Beetle data from 2001 to 2010, which consisted of

seasonal attacks of trees by Pine Beetles in the Rocky Mountain region. This was then used to predict the intensity for the year 2010.

DEDICATION

To the memories of the shoulders I stand on: my mother-Comfort Wagya, grandpa—Agya Wagya, grandma—Mame Manu, uncles—Ayaim & Ndabiah. Rest well. Thank you for the foundation you laid for me.

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CHAPTER I

INTRODUCTION

The Boolean random set is formed by placing random closed sets (referred to as grain) at points of a Poisson process (germ) and taking unions of these sets. Two components of the BRS are the homogeneous Poisson point process, with intensity λ and a sequence of independent random closed sets independent of the Poisson point process.

For example, during the in–cloud seeding experiment, the clouds are bombarded with bits of magnesium oxide. Some craters appear in points, which hit these bits, and with some approximation, the shape of their projection on $\mathbb{R}^2$ are circles centered at the hit points with random radius (Khazaee, 2004). This can be taken as a Boolean random set realization. Another interesting example is fire destruction in the forest. The scar of an area burnt as a result of fire can also be modeled as a Boolean random set. If one takes the position where the fire first started burning as the point of a Poisson process, then the scar can be approximated as a closed set in $\mathbb{R}^2$. Taking unions of such scars results in a Boolean random set.

Given this, the intensity of these occurrences (RACS) is of interest to scientists, specifically, the intensity across time, which enables future intensity predictions of these sets. Hence, the subject of this study.

Boolean models have been used extensively throughout the years. From Armitage (1949) in the random clumping of dust or powder particles, to Widom & Rowlinson

(1970) in the system of water droplets in the study of liquid-vapor phase transitions, to Diggle (1981) in the distribution of heather in forests, to Garcia-Sevilla & Petrou (1999) in classifying binary and spatial textures to Mattfeldt, Gottfried, Schmidt, & Kestler (2000) in classifying spatial textures in benign and cancerous tissues, to Kärkkäinen, Jensen, & Jeulin (2002) in orientational characteristics of fibers in digital images, and to Khazaee (2004) in modeling the effect of some explanatory variables on a BRS. Even though interest in the Boolean model has grown over the years, more work is needed to expand its frontiers.

Count data appear in diverse phenomena, from monthly numbers of people with certain diseases to the daily number of customers at a particular branch of a bank. Time series for count data is simply defined as dependent count observations of a random phenomenon. For example, the weekly number of customers at a bank over the course of a year can be viewed as a count time series.

In ordinary linear regression, the problem is to relate the mean response of a variable to the explanatory variables by means of a linear equation under the assumption of independent and normal data. However, there are situations where non-normal data, such as count, leads to wrong conclusions due to failures in assumptions of normality. Generalized linear models bridge this gap. A lot of literature has been built on this. The work of Nelder & Wedderburn (1972) can be used as a point of reference. Furthermore, when observations are not independent, then models such as time series are used to capture the dependency that exist in the data. Specifically, for dependent count data, count time series is a solution to the non-independent assumption failure.

For example, let Y_t, $t = 1, \ldots, N$ denote a time series for count data, which consists of nonnegative integer values. Therefore, we can naturally assume that the conditional density of the Y_t given the past observation is Poisson with mean μ_t. With this choice, we know that the conditional expectation of the response is equal to the conditional variance. The problem is to relate the conditional expectation to some covariates. Kedem & Fokianos (2005) used partial likelihood and generalized linear model methodologies to solve this problem. Fokianos, Rahbek, & Tjøstheim (2009); Fokianos & Tjøstheim (2011) provided a framework that accounts for both positive and negative correlations. Additional review of this framework will be discussed in chapter two of this study.

Statement of the Problem

Every statistical model should take the dependency or the correlational structure of the data into consideration. Liang & Zeger (1993) stated that the impact of ignoring the correlation among observations produced incorrect variance. And as a result, the discrepancy between the correct and incorrect variances increased with the degree of correlation ρ. Another effect was the loss of efficiency, meaning that the uncertainty in biased estimates β was greater than the uncertainty in the best unbiased estimate. Any of the above discrepancies impacts scientific conclusions negatively. Thus, models that take the correlation of observations into account will be preferred in modeling the effects of explanatory variables along with past observations on the intensity of the Boolean model across time.

Purpose of the Study

The Boolean model has two sources of randomness. This includes the Poisson process D, which is responsible for the germ distribution, and a sequence of independent random closed sets independent of D, with their distributions identical to that of Z_0. Khazaee (2004) studied the effect of some explanatory variables observed along the BRS on the model's behavior and prediction. In doing so, Khazaee classified the explanatory variables into three categories, i.e.: propagation variables, which affect D, growth variables, affect Z_0, and propagation-growth variables, affect both D, and Z_0.

The purpose of this research was to build a count time series, specifically, the log-linear Poisson autoregression. It modeled the intensity of the Boolean random sets across time, the effect of past observations, past intensities, and time-dependent covariates (propagation variables) on the estimation of its parameters. In addition, the log-linear model catered to both negative and positive correlational structures across time (lags).

The maximum-likelihood based method was used to estimate the parameters of the models under a set of different conditions.

Research Questions

The following questions guided this study:

Q1 Can we build a model to estimate the intensity of a time-dependent BRS?

- Q1a How do both time-dependent covariates and past observations affect the estimation of the intensity?
- Q1b How do the past observations of BRS affect the estimation of the intensity?
- Q1c How do time-dependent covariates affect the estimation of the intensity?

Q2 Are the estimators of parameters of these models unbiased?

Q3 What are the characteristics of these estimates under different times?

- Q3a When the radius of the grains is known and fixed?

Q3b When the radius of the grains is unknown and fixed?

Q3c When the radius of the grains is random?

Q4 Are these estimators asymptotic normal and consistent?

To answer the above questions, simulations were conducted to study the properties of this model for the Boolean random set. The proposed model was then applied to a real world data set—the Rocky Mountain Pine Beetle data. The mountain pine beetles seasonally attack the host tree species (trees where the damage has taken place) on the Rocky Mountains. Thus the seasonal (yearly) nature of these attacks produce sets (data) that are correlated across time. Kaufeld (2014) applied the Generalized Method of Moments Approach for Spatial-Temporal Binary Data to the Pine Beetle data from 2001 to 2010. This data were collected by the United States Forest Service through aerial survey methods. The weather variable, which is the mean annual precipitation in inches, was obtained from the PRISM dataset, which is publicly available at the prism site of the Oregon State University website. Thresholding and smoothing were applied to the data and a stationary grid of dimension 42 x 55 was constructed at each site. These data were treated as Boolean random sets, with the location of the damaged trees as the point of a Poisson process and a grain distribution of radius of 0.02.

Definitions

Random closed set (RACS) is a measurable map $Y : \Omega \longmapsto \mathfrak{F}$ from a probability space $(\Omega, \mathfrak{U}, P)$ to a family of closed sets in a locally compact, Hausdorff, and separable space (LCHS) E.

Boolean random set (BRS) is a special type of RACS formed by placing a random closed sets at points of a Poisson process and taking their union.

Germ is a point from the Poisson process in the formulation of BRS.

Grain is a random closed set placed at the points of the Poisson process in the formulation of BRS.

The layout of this study is as follows: Chapter 2 presented a relevant review of existing literature on the subject of RACS, BRS, time series for count models, and estimation methods used in fitting them. We also reviewed some properties of the Boolean random model and some advancements made in this field. In Chapter 3, the model and its estimation methods that were used in answering the above research questions was presented. The results of the simulation study was presented and discussed in Chapter 4, along with the application of the model to the Rocky Mountain Pine Beetle data. Finally, in Chapter 5, conclusions for the Time Series for the Boolean random sets as well as future developments and limitations were presented.

CHAPTER II

REVIEW OF LITERATURE

In this section, we review the relevant literature and advancements made in the field of Random set theory, especially, the Boolean random set.

Review of Random Closed Sets

Molchanov (2005) posits that the study of random geometrical objects dates back to the Buffon needle problem. In fact, the concept of Geometric Probability traces back to the origins of probability. Together with the mathematical foundations of probability theory, the concept of random set was mentioned for the first time.

Not only did Kendall (1974) and Matheron (1975) introduce the mathematical theory of random sets, but the idea of random sets with different shapes as well. In regards to the development of relevant probabilistic and geometric techniques, the modern definition of random set is attributed to Georges Matheron.

Fundamentals of Topology

Fundamental definitions needed to define random closed sets are given as follows:

Topological space. Let E be a set, then a collection of subsets ϕ of E is called a topology if the following are satisfied:

i. $\emptyset$ and E belong to ϕ.

ii. Union of any subcollections of ϕ belongs to ϕ.

iii. Intersection of any finite subcollections of ϕ belongs to ϕ.

The set E along with the topology ϕ is called a topological space denoted by (E,ϕ), where elements of ϕ are called open sets.

Base of a topology. Let $\mathscr{B}$ be a collection of subsets of E, such that for every $e \in E$:

i. There is at least a member $B \in \mathscr{B}$, such that $e \in B$.

ii. If $e \in B_1, B_2$, then there is a $B_3 \in \mathscr{B}$ such that $e \in B_3$, and $B_3 \subset B_1 \cap B_2$.

Then, $\mathscr{B}$ is called *base of a topology* such that any open set is can be constructed as a union of elements of $\mathscr{B}$. A topology $\phi_{\mathscr{B}}$ generated by a base $\mathscr{B}$ is defined as " the subset U from $\mathscr{B}$ with respect to $\phi_{\mathscr{B}}$ is open, if for each $e \in U$, there is an element $B \in \mathscr{B}$, such that $e \in B$ and $B \subset U$.

Hausdorff topological space. The topological space (E,ϕ) is Hausdorff, if for every distinct pair of elements $x, y \in E$, there are open sets, U of x, and V of y such that $U \cap V = \emptyset$. Thus, distinct pairs of points have distinct neighborhoods.

Separable topological space. Let (E,ϕ) be a topological space, $C \subset E$ is closed (with regards to ϕ), if its complement (a subset of E) is an open set. The intersection of all closed sets containing C is termed "the closure of C" and denoted by $\bar{C}$. A subset C of E whose closure is E is called a dense subset. i.e. $\bar{C} = E$. If E has a countable dense subset, then (E,ϕ) is a separable topological space.

Locally compact topological space. If any open covering of E has a finite sub-covering, then (E,ϕ) is said to be *compact*. Stated differently, if $\{O_\alpha | \alpha \in I\}$ is a collection of open subsets, such that $E = \cup_{\alpha \in I} O_\alpha$, then there are $\alpha_1, \alpha_2, ..., \alpha_m \in I$ such that $E = \cup_{i=1}^{m} O_{\alpha_i}$. If any element $e \in E$ has a compact neighborhood, (E, ϕ) is called a locally compact topological space.

The Euclidean space $\mathbb{R}^d$ is the most common *locally compact Hausdorff separable* topological space (LCHS space). An example of a base is the collection of all open balls such that $B(v,r) = \{w \in \mathbb{R}^d |, |w-v|_d < r\}, \forall v \in \mathbb{R}^d$, with $|.|_d$ as the usual d-dimensional Euclidean norm. $\phi_{\mathbb{R}^d}$ is the topology generated by the base.

Algebra and σ-algebra. A family of sets is called an algebra if this family contains $\emptyset$ and is closed under taking complements and finite unions. An algebra $\mathfrak{U}$ is called a σ-algebra if it is closed under countable unions. If $\mathfrak{M}$ is any family of sets, then $\sigma(\mathfrak{M})$ denotes the smallest σ-algebra generated by $\mathfrak{M}$. The minimal σ-algebra which contains the family $\mathscr{G}$ of all open sets is called the Borel σ-algebra on E and denoted by $\mathfrak{B}(E)$.

Fell topology. Let $\mathfrak{F}$ be the space of closed subsets of a topological space. Topologies on $\mathfrak{F}$ are often introduced by describing their sub-bases. To define one of such sub-bases;

for $A \subset E$, define

$$\mathfrak{F}_A = \{F \in \mathfrak{F} : F \cap A \neq \emptyset\} \quad \mathfrak{F}^A = \{F \in \mathfrak{F} : F \cap A = \emptyset\},$$

as the family of closed sets, which have non-empty intersection with A (in other words, the family of closed sets that hits A), and as the family of closed sets, which miss A respectively. The Fell topology has a sub-base, which consists of $\mathfrak{F}_G \; \forall \; G \in \mathscr{G}$ and $\mathfrak{F}^K \; \forall \; K \in \mathscr{K}$, where $\mathscr{G}$ and $\mathscr{K}$ are collections of open subsets and compact subsets of a

Hausdorff, separable and locally compact topological space E. Let the collection of elements of $\mathfrak{F}$ which 'hit' the sets $G_1, G_2, ..., G_n \in \mathscr{G}$ and 'miss' $K \in \mathscr{K}$ be denoted by $\mathfrak{F}^K_{G_1,G_2,...,G_n}$, then $\mathfrak{F}^K_{G_1,G_2,...,G_n} = \mathfrak{F}^K \cap \mathfrak{F}_{G_1} \cap ... \cap \mathfrak{F}_{G_n}$ when $n \neq 0$ otherwise $\mathfrak{F}^K_{G_1,G_2,...,G_n} = \mathfrak{F}^K$ (Molchanov, 2005). Matheron (1975) used this class $\mathfrak{F}^K_{G_1,G_2,...,G_n}$ as a base of a topology on $\mathfrak{F}$. This topology is called the hit-or-miss topology. This hit-or-miss topology is crucial in generating a σ-field used in defining random closed sets.

Definition of Random Closed Set

Let Σ be the σ-field in $\mathfrak{F}$ generated by the hit-or-miss topology. This field can be generated separately by the family $\{\mathfrak{F}^K | K \in \mathscr{K}\}$ and likewise by $\{\mathfrak{F}_G | G \in \mathscr{G}\}$. Molchanov (2005) gives the following definitions:

Definition 1. Let E be a Locally compact, Hausdorff, and Separable space (LCHS), $\mathfrak{F}$, Σ be defined as above and $(\Omega, \mathfrak{U}, P)$ be a probability space. A random closed set (RACS) Y on E is defined as a measurable map $Y : \Omega \longmapsto \mathfrak{F}$ from a probability space $(\Omega, \mathfrak{U}, P)$ to a family of closed sets in a LCHS space E. For each $\omega \in \Omega$, $Y(\omega)$ is a closed subset of E and for all $V \in \Sigma$,

$$Y^{-1}(V) = \{\omega \in \Omega | Y(\omega) \in V\} \in \mathfrak{U}.$$

Definition 2. Let E be a LCHS. A measurable map $Y : \Omega \longmapsto \mathfrak{F}$ is called a random closed set if Y is measurable with respect to the Borel σ-algebra on $\mathfrak{F}$ with respect to the Fell topology, i.e. for each y $\in \mathfrak{B}(\mathfrak{F})$

$$Y^{-1}(y) = \{\omega \in \Omega | Y(\omega) \in \chi\} \in \mathfrak{U}.$$

Note that $Y^{-1}(V) = \{\omega : Y(\omega) \in V\} \in \mathfrak{U}$ can be restated as

$$Y^{-1}(\mathfrak{F}_K) = \{\omega : Y(\omega) \in \mathfrak{F}_K\} \in \mathfrak{U}.$$

The probability distribution P_Y on Σ defined by X is

$$P_Y = P(Y^{-1}(V)), \quad V \in \Sigma.$$

Example 1 (half line). If ξ is a real-valued random variable with distribution function $F_\xi(.)$, then $Y = (-\infty, \xi\ ]$ is a random closed set on the line $E = \mathbb{R}$. Let $K = [a,b]$, then observe that, $\{Y \cap K \neq \emptyset\} = \{\xi \geq \inf K\}$ is a measurable event for every $K \subset E$ with probability:

$$P_Y(Y \cap K \neq \emptyset) = P(\{\omega : Y(\omega) \cap K \neq \emptyset\}) = P_\xi(\xi \geq a) = 1 - F_\xi(a).$$

Along the same lines, $Y = (-\infty, \xi_1\ ] \times (-\infty, \xi_2\ ] \times ... \times (-\infty, \xi_d\ ]$ is a random closed subset of $\mathbb{R}_d$ if $(\xi_1, ..., \xi_d)$ is a d-dimensional random vector (Molchanov, 2005).

Example 2 (random triangle and random ball). Let (ξ_1, ξ_2, ξ_3) be random vectors in $\mathbb{R}^d$, then the triangle with vertices ξ_1, ξ_2, and ξ_3 is a random closed set. If ξ is a random vector in $\mathbb{R}^d$ and η is a non-negative random variable, then the random ball $B_\eta(\xi)$ of radius η centered at η is a random closed set (Molchanov, 2005). The above examples show that probabilities such as: $P_Y(Y \cap K \neq \emptyset)$ for $K \in \mathscr{K}$ entirely determines the probability distribution of RACS Y.

Hitting Functional and Choquet Theorem

In the above section, it was noted that the knowledge of $P_Y(Y \cap K \neq \emptyset)$ determines entirely the distribution of RACS Y. However, the most important tool for determining the probability properties of RACS is the hitting functional.

Definition 3. The functional $T_Y : \mathscr{K} \longmapsto [0,1]$ given by

$$T_Y(K) = P_X(\mathfrak{F}_K) = P_Y\{Y \cap K \neq \emptyset\}, \quad K \in \mathscr{K},$$

is said to be the capacity (hitting) functional of Y denoted by $T_Y(K)$.

The functional defined above for RACS is analogous to the distribution function of a random variable. Hence, it is straightforward to show that, if $T_Y(.)$ is defined on $\mathscr{K}$, then the probability distribution $P_Y(.)$ of RACS Y is entirely determined. In addition, Choquet (1954); Kendall (1974); Matheron (1975), state some properties of the hitting functional, that are easily verified:

i. $T_Y(\emptyset) = 0$, since no closed set Y hits the empty set. i.e. $Y \cap \emptyset = \emptyset$. Also, T_Y being probability satisfies $0 \leq T_Y(K) \leq 1$, for every $K \in \mathscr{K}$

ii. The functional is increasing on $\mathscr{K}$; i.e. $K_1, K_2 \in \mathscr{K}$ and $K_1 \subseteq K_2$ implies $T_Y(K_1) \leq T_Y(K_2)$.

iii. $T_Y(K)$ is upper semi-continuous on $\mathscr{K}$, which, is equivalent to $K_n \downarrow K$ in $\mathscr{K} \Rightarrow T_Y(K_n) \downarrow T_Y(K)$. Let $K, K_i \in \mathscr{K}, i = 1, 2, ...$, recursively define the functionals $Q_Y^{(0)}, Q_Y^{(1)}, \ldots$ by:

$$Q_Y^{(0)}(K) = Q_Y(K) = P_Y(Y \cap K = \emptyset) = 1 - T_Y(K),$$

and

$$Q_Y^{(1)}(K; K_1) = Q_Y^{(0)}(K) - Q_Y^{(0)}(K \cup K_1),$$

and for $n = 2, 3, \ldots$

$$Q_Y^{(n)}(K; K_1, K_2, \ldots, K_n) =$$

$$Q_Y^{(n-1)}(K, K; K_1, K_2, \ldots, K_{n-1}) - Q_Y^{(n-1)}(K \cup K_n; K; K_1, K_2, \ldots, K_{n-1}).$$

It can easily be shown that:

$$Q_Y^{(n)}(K; K_1, K_2, \ldots, K_n) = P_Y(Y \cap K = \emptyset, \ \& \ Y \cap K_i \neq \emptyset, i = 1, 2, \ldots, n).$$

Thus, $Q_Y^{(n)}(K; K_1, K_2, \ldots, K_n)$ is the probability that the RACS Y misses K and hits $K_i, i = 1, 2, \ldots, n$.

iv. $0 \leq Q_Y^{(n)}(K; K_1, K_2, \ldots, K_n) \leq 1$, for $K, K_i \in \mathscr{K}, \ i = 1, 2, \ldots, n$ and for every $n \geq 1$.

The functional $Q_Y(K)$ is known as the generating functional of RACS Y or the finite-dimensional distribution functional of the Y. Additionally, properties $(ii - iV)$ makes T_Y a Choquet capacity of infinite order (Matheron, 1975). Hence, T_Y is also a Choquet capacity, which satisfies $T_Y(\emptyset) = 0$ and $0 \leq T_Y(K) \leq 1$ for every $K \in \mathscr{K}$.

A groundbreaking theorem for random set theory is the Choquet theorem. This theorem was proven independently by Kendall (1974) and Matheron (1975). Hence, we state it without proof.

Theorem 1 (Choquet). Let T be a real-valued functional on $\mathscr{K}$. Then, $\exists$ a probability space $(\Omega, \mathfrak{U}, P)$ and RACS Y, $(Y : (\Omega, \mathfrak{U}) \rightarrow (\mathfrak{F}, \Sigma))$ satisfying:

$$P(\{\omega \in \Omega | Y(\omega) \cap K \neq \emptyset\}) = T(K) \quad K \in \mathscr{K},$$

if and only if T is a Choquet capacity of infinite order, such that $T(\emptyset) = 0$ and $0 \leq T_Y(K) \leq 1$ for every $K \in \mathscr{K}$.

Furthermore, any probability distribution P_Y on Σ such that $P_Y(\mathfrak{F}_K) = T(K)\ \forall\ K \in \mathscr{K}$, is necessarily unique.

Choquet's theorem implies that the probability distribution of a RACS Y is completely determined by the hitting (capacity) functional $T_Y(K),\ \ K \in \mathscr{K}$. This theorem gives a rather computationally simple way of deriving probabilistic properties of random sets.

Review of the Boolean Random Set

A very important class of random closed sets is the Boolean random set. It is a very simple structured parametric model with a lot of applications in the field of random set theory. We formally define Boolean random set (BRS) below:

Definition of the Boolean Random Set

Definition 4. Let $D = \{x_1, x_2, ...\}$ be a homogeneous Poisson point process in $\mathbb{R}^d$, with intensity of λ. Also, suppose $Z_1, Z_2, ..$ is a sequence of independent random closed sets independent of D, with their distributions identical to that of Z_0. Then, the Boolean model (BRS) is defined as:

$$Y = \bigcup_{x_i \in D} (Z_i \oplus x_i), \quad (1)$$

where $Z_i \oplus x_i = \{z + x_i | x \in Z_i\}$ is the Minkowski sum of Z_i and x_i. The points of the Poisson process are called germs and the corresponding RACS Z_i are called grains. Commonly used examples of grains are line segments of random length, balls of random radius and random finite clusters of points, where the Boolean model becomes a Neymann-Scott process (Dietrich & Helga, 1994).

The Hitting Functional

The Boolean model, like other random closed sets, has its hitting functional T_Y. Mattfeldt (1996) showed that

$$T_Y(K) = 1 - exp\{-\lambda E[\nu_d(\check{Z}_0 \oplus K)]\}, \quad K \in \mathscr{K}, \quad (2)$$

where $\nu_d(.)$ is the Lebesgue measure and $\check{Z}_0 = \{-z | z \in Z_0\}$. Analogously,

$$T_Y(K) = 1 - exp\{-\lambda E[\nu_d(\check{Z}_0 \oplus K)]\} = 1 - exp\{-\lambda E[\nu_d(Z_0 \oplus \check{K}]\},$$

using the relationship $(\check{Z}_0 \oplus K) = -(Z_0 \oplus \check{K})$ (Khazaee, 2004).

Some Characteristics of the Model

When the test set K is a compact set, in $\mathscr{K}$, then very "nice" descriptive characteristics can be obtained. Khazaee (2004) summarized some of the properties of the Boolean model. We state the relevant ones below :

Stationarity and isotropicity. If $E[\nu_d(Z_0 \oplus \check{K})] = E[\nu_d((Z_0 \oplus x) \oplus \check{K})], \;\; \forall x \in \mathbb{R}^d$, and $T_{Y \oplus x}(K) = T_Y(K)$, using (2) then the BRS Y is said to be stationary. Similarly, for an isotropic Z_0, $E[\nu_d(Z_0 \oplus \check{K})] = E[\nu_d(R_\alpha(Z_0) \oplus \check{K})]$ implies that Y is isotropic.

Porosity (q). If a point of the space falls in the complement of RACS Y, then porosity is the probability of falling in the pores, i.e.:

$$q = P(x \in Y^c) = 1 - T_Y(\{x\}),$$

using (2.2.2),

$$= exp\{-\lambda E[\nu_d(Z_0)]\}. \tag{3}$$

Volume fraction (p) . Let B be a Borel set with measure one, i.e.: $(\nu_d(B) = 1)$. For RACS Y, the expected value of the ratio of B covered by Y is defined as the volume fraction, p_B. When Y is stationary, then p_B is independent of the region B and can be interpreted as the mean proportion of space covered by Y. Note that $p = P(x \in Y)$, and

$$\begin{aligned} p = E[\nu_d(Y \cap B)] &= E \int_{\mathbb{R}^d} I_{Y \cap B}(Y) d\nu_d(x) \\ &= E \int_B I_Y(Y) d\nu_d(x) = \int_B P(x \in Y) d\nu_d(x) \\ &= P(x \in Y) \nu_d(x) = P(x \in Y). \end{aligned} \tag{4}$$

That of the Boolean model is of the form:

$$p = 1 - exp\{-\lambda E[\nu_d(Z_0)]\}. \tag{5}$$

Estimation of Model Parameters

Molchanov (1997) introduced two classes of parameters of the Boolean model. These include the macroscopic and microscopic parameters. Whilst the macroscopic parameters are determined by visible set, microscopic parameters are not directly observable. Moreover, macroscopic parameters deal with aggregate properties whilst microscopic deals with individual properties. Examples of macroscopic parameters include the volume fraction, covariance, hitting functional, and contact distribution functions. Microscopic includes intensity of the Poisson process, expected value of perimeter, and area of grains. Below, we discuss the methods of estimating some of these parameters:

Volume fraction (p). The volume fraction can be thought of as the mean proportion of space covered by the RACS Y, then a natural estimator of p is the proportion of W covered by Y, i.e.:

$$\hat{p} = \frac{\nu_d(Y \cap W)}{\nu_d(W)}. \tag{6}$$

Using a digitalized image as done in practice, Khazaee (2004) shows the estimator takes the form:

$$\begin{aligned} \hat{p} &= \frac{N(Y \cap W)}{N(W)} \\ &= \frac{\sum_{x \in W} I_{Y \cap W}(x)}{N(W)}. \end{aligned} \tag{7}$$

Where x is a pixel of a digitalized image and $N(W)$, the number of pixels inside W. With Y stationary, a similar proof of (4) can be used to show that (6) and (7) are unbiased. Thus, the variance is:

$$\begin{aligned} Var(\hat{p}) &= \frac{1}{N(W)^2} \sum_{x,y \in W} Cov(I_{Y \cap W}(x), I_{Y \cap W}(y)) \\ &= \frac{1}{N(W)^2} \sum_{x,y \in W} [P(x \in Y \cap W, y \in Y \cap W) - p^2] \\ &= \frac{1}{N(W)^2} \sum_{x,y \in W} C(x-y) - p^2, \end{aligned} \tag{8}$$

where $C(.)$ is the covariance of Y. In a similar manner to (6),

$$Var(\hat{p}) = \frac{1}{\nu_d^2(W)} \int_{W^2} (C(x-y) - p^2) d\nu_d(x) d\nu_d(y). \tag{9}$$

These estimators of p are strongly consistent, i.e.: $\hat{p} \to p$ a.s.,$W \to \mathbb{R}^d$ and $\nu_d(W)(\hat{p} - p) \xrightarrow{D} N(0, \sigma^2)$, where $\sigma^2 = \int_{\mathbb{R}^d} C(r)dr - p^2$ (Mase, 1982).

Covariance function C(r). The covariance function $C(r)$ of the Boolean model can be estimated as:

$$
\begin{aligned}
\hat{C}(r) &= \frac{\nu_d(\{x|x \in (Y \cap W), x+r \in (Y \cap W)\})}{\nu_d(\{x|x \in W, x+r \in W\})} \\
&= \frac{\nu_d((Y \cap W) \cap ((Y \cap W) \oplus \{-r\})}{\nu_d(W \cap (W \oplus \{-r\}))} \\
&= \frac{\nu_d((Y \cap W) \ominus \{o,-r\})}{\nu_d(W \ominus \{o,-r\})}.
\end{aligned} \tag{10}
$$

Observe that (10) is an estimator of the volume fraction of RACS $Y \ominus \{o,-r\}$ in the window $W \ominus \{o,-r\}$. Hence, $\hat{C}(r)$ is an unbiased estimator of $C(r)$. In pixel context, we have the form:

$$\hat{C}(r) = \frac{1}{N_r(W)} \sum_{x \in W \ominus \{o,-r\}} I_{Y \ominus \{o,-r\}}(x), \tag{11}$$

where $N_r(W) = N(W \ominus \{o,-r\})$ (Khazaee, 2004).

Coverage probability. It is obvious that $I_{K \subset Y}(.)$ is an unbiased estimator of $P(K \subset Y)$. However, when RACS Y is stationary, estimators of the form $I_{K_x \subset Y}(.)$ are still unbiased estimators of $P(K \subset Y)$ $\forall x \in \mathbb{R}^d$, where K_x is the dilation of K by x. In practice, the estimator used is the mean of estimators of $I_{K_x \subset W}$

$$\hat{P}(K \subset Y) = \frac{\sum_{x \in W} I_{(K_x \subset Y \cap W)}(x)}{\sum_{x \in W} I_{(K_x \subset W)}(x)} \tag{12}$$

$$= \frac{\nu_d(\{x|K_x \subset Y \cap W\})}{\nu_d(\{x|K_x \subset W\})}. \tag{13}$$

Since

$$\{x|K_x \subset W\} = \{x|x+y \in W, \ \ \forall y \in K\}$$
$$= \bigcap_{y \in K} \{x|x+y \in W\}$$
$$= \bigcap_{y \in K} \{z-y|z \in W\} = W \ominus \check{K},$$

then from (13),

$$\hat{P}(K \subset Y) = \frac{\nu_d((Y \cap W) \ominus \check{K})}{\nu_d(W \ominus \check{K})}. \tag{14}$$

Hitting probability. With the relation $P(K \subset Y^c) = P(Y \cap K = \emptyset) = Q_x(K) = 1 - T_Y(K)$, the hitting probability can be derived from $T_Y(K) = 1 - P(K \subset Y^c)$. Thus, an unbiased estimator of $T_Y(K)$ can be obtained using (13) as follows:

$$\hat{T}_Y(K) = 1 - \hat{P}(K \subset Y^c)$$
$$= 1 - \frac{\nu_d(\{x|K_x \subset Y^c \cap W\})}{\nu_d(\{x|K_x \subset W\})}$$
$$= \frac{\nu_d(\{x|K_x \subset W\}) - \nu_d(\{x|K_x \subset W, K_x \subset Y^c\})}{\nu_d(\{x|K_x \subset W\})}$$
$$= \frac{\nu_d(\{x|K_x \subset W, (K_x \subset Y^c)^c\})}{\nu_d(\{x|K_x \subset W\})}$$
$$= \frac{\nu_d(\{x|K_x \subset W, K_x \cap Y \neq \emptyset\})}{\nu_d(\{x|K_x \subset W\})},$$

and with the relation:

$$\{x|K_x \cap Y \neq \emptyset\} = \{x|\exists y \in K \ni x+y \in Y,\}$$
$$= \bigcup_{y\in K} \{x|x+y \in Y\}$$
$$= \bigcup_{y\in K} \{z-y|z \in YW\} = Y \ominus \check{K},$$

$$\hat{T}_Y(K) = \frac{\nu_d(W \ominus \check{K}) \cap (Y \oplus \check{K}))}{\nu_d(W \ominus \check{K})}. \tag{15}$$

And, the pixel equivalent:

$$\hat{T}_Y(K) = \frac{1}{N(W \ominus \check{K})} \sum_{x \in W \ominus \check{K}} I_{Y \oplus \check{K}}(x). \tag{16}$$

From $T_{Y\oplus\mathbb{K}}(\{o\}) = T_Y(L)$, it can be concluded that $T_Y(K)$ is the volume fraction of the BRS $Y \oplus \check{K}$. Therefore, $\hat{T}_Y(K)$ is an estimator for the volume fraction in the window $W \ominus Y$. And from previous results, $\hat{T}_Y(K)$ is an unbiased and strongly consistent estimator for $T_Y(K)$ with $(T_Y(K) - \hat{T}_Y(K)) \xrightarrow{D} N(0, \sigma^2)$ (Khazaee, 2004).

Next, we review some spatial means and their estimators besides the volume fraction, which is considered the simplest spatial mean.

Specific boundary length (L). Specific boundary length is defined as the length per unit area for planar random sets. Matheron (1975) showed that

$$L = (1-p)\lambda E[U(Z_0)], \tag{17}$$

where $U(Z_0)$ is the perimeter of Z_0. Suppose $l(A)$ denotes the perimeter of A, then:

$$\hat{L} = \frac{l(Y \cap W)}{\nu_2(W)}. \tag{18}$$

Positive tangent points process and special convexity number (N^+). Suppose each grain Z_0 is associated with its tangent point $n_u(Z_0)$, where u is a fixed direction in $\mathbb{R}^d$. If u is directed upwards, then the lower tangent point of Z_0 is defined as the lexicographical minimum among all points at which a hyperplane orthogonal to u first touches Z_0. Consider the set of such tangent points determined for all shifted grains. Although grains may cover some points, the exposed tangent points form a point process denoted by $N^+(u)$ with intensity $\lambda(1-p)$. This is called the special convexity number. If n^+ is the number of lower tangent points in window W, then

$$\hat{N}^+ = \frac{n^+}{\nu_2(W)}, \tag{19}$$

is an unbiased and strong consistent estimator of N^+ (Molchanov, 1995).

Specific Euler-Poincar*é* and special connectivity number. Euler-Poincar*é* characteristics are defined as:

i. $\chi(\emptyset) = 0$

ii. $\chi(K) = 1$ for any nonempty compact convex set K

iii. Suppose K admits the representation $K = \cup_{i=1}^{p} K_i$ for compact convex sets $K_i, i = 1, ..., p$, then

$$\chi(K) = \sum_i \chi(K_i) - \sum_{i_1 < i_2} \chi(K_{i1} \cap K_{i2}) + ... + (-1)^{p-1} \chi(K_1 \cap K_2 \cap ... \cap K_p). \tag{20}$$

It can be shown that the value $\chi(K)$ is not dependent on the representation of $K = \cup_{i=1}^{p} K_i$. In addition, let $\chi(K) =$ the number of connected components of K—the number of holes in K. If Y is a planar BRS with almost surely (a.s.) convex grain, the mean of the Euler-Poincaré characteristics in the unit area is called specific Euler-Poincaré characteristics or specific connectivity number of Y denoted by χ. Kellerer (1984) showed that

$$\chi = (1-p)\lambda(1 - \frac{\lambda}{4\pi}\{E[U(Z_0]\}^2). \tag{21}$$

Suppose $\chi(Y \cap W)$ is the Euler-Poincareé characteristics for the realization Y in window W, then an estimator of χ is :

$$\hat{\chi} = \frac{\chi(Y \cap W)}{\nu_2(W)}, \tag{22}$$

where Kellerer (1984) showed that $\chi(Y \cap W) = N(N^{+}(u) \cap W) - N(N^{c}(-u) \cap W)$. Note that $N^{c}(-u)$ is the point process of tangent points in the complement of Y in the direction of $-u$ and $N(N^{c}(-u) \cap W)$ is the number of points in window W. Below, we review some methods of estimation for the microscopic parameters of the Boolean model.

Method of intensities. Molchanov (1997) stated that similar to the method of moments in classical statistics, the intensity method chooses the estimators of the parameters to match the empirical values of the aggregate parameters. In the plane, equations (5) , (17) and (21) make it possible to express λ, $E[U(Z_0)]$, $E[\nu_2(Z_0)]$ through p, L, and χ. Equations (17) and (21) can be solved to get:

$$\lambda = \frac{\chi}{1-p} + \frac{1}{4\pi}\frac{L^2}{(1-p)^2}, \tag{23}$$

and

$$E[U(Z_0)] = \frac{4\pi L(1-p)}{4\pi(1-p)\chi + L^2}. \tag{24}$$

The final step is to replace the aggregate parameters with their empirical counterparts in (6), (18), and (22). The estimators resulting from the method of intensities are biased, but strongly consistent and asymptotic normality. However, it is difficult to express the variance of the limiting distribution through the parameters of the grain (Molchanov, 1997). Khazaee (2004) used $N^+ = \lambda(1-p)$ instead of (21) and arrived at

$$\hat{\lambda} = \frac{\hat{N^+}}{1-\hat{p}}, \tag{25}$$

which is also consistent, asymptotically normal and $[\nu_2(W)]^{\frac{1}{2}}(\hat{\lambda} - \lambda)$ converges weakly to a centered normal distribution with variance $\frac{\lambda}{1-p}$.

Method of minimum contrast. With this method, the contact distribution functional is written in polynomial terms using the Steiner formula, where the coefficients of the polynomial are functions of the unknown parameters. These parameters are estimated by statistical methods, such as method of moments, ordinary least squares or generalized least squares.

Let $d = 2$, Y be a Boolean model with almost surely convex grain Z_0, and $K \in \mathscr{K}$, then by the Steiner formula in (2),

$$\begin{aligned} Q_Y(K) &= 1 - T_{Y(K)} \\ &= exp\{-\lambda[E[\nu_2(Z_0)] + \frac{1}{2\pi}E[U(Z_0)]U(K) + \nu_2(K)]\}. \end{aligned} \tag{26}$$

In the planar case, the distribution of the Boolean model is completely determined by λ, $E[\nu_2(Z_0)]$, and $E[U(Z_0)]$. Below are some methods for estimating the parameters.

Method of moments. When the test sets are K_0—the origin, K_1—straight line segment of length l, and K_2—the closed square of side l, equation (26) becomes,

$$\begin{aligned} Q_Y(K_0) &= exp\{-\lambda E[\nu_2(Z_0)]\}, \\ Q_Y(K_1) &= exp\{-\lambda E[\nu_2(Z_0)] + \frac{l}{\pi}E[U(Z_0)]\}, \\ Q_Y(K_2) &= exp\{-\lambda[E[\nu_2(Z_0)] + \frac{2l}{2\pi}E[U(Z_0)]U(K) + l^2]\}. \end{aligned}$$

Equation (16) can be used to calculate $\hat{Q}_Y(K_i) = 1 - \hat{T}_Y(K_i),\ i = 0, 1, 2$, then substituted into the above equations and solved for the method of moment estimators. Khazaee (2004), solved the following estimators:

$$\begin{aligned} \hat{\lambda} = \frac{\hat{S}}{l^2},\ \ \hat{E}[\nu_2(Z_0)] &= -\frac{l^2}{\hat{S}} \ln \hat{Q}_Y(K_0), \\ \hat{E}[U(Z_0)] &= -\frac{\pi l}{S} \ln\left[\frac{\hat{Q}_Y(K_1)}{\hat{Q}_Y(K_0)}\right], \\ \hat{S} &= \ln\left[\frac{\hat{Q}_Y(K_1)}{\hat{Q}_Y(K_0)\hat{Q}_Y(K_2)}\right]. \end{aligned}$$

Ordinary least squares method (OLS). Suppose in (26) the test sets K are circles with variable radius (i.e. K_t could be a circle with radius $t = t_1, t_2, ..., t_n$), then for t, we can rewrite $Q_Y(K_t)$ as:

$$Q_Y(K_t) = exp\{-\lambda[E[\nu_2(Z_0)] + E[U(Z_0)]t + \pi t^2 \tag{27}$$

$$-\ln Q_Y(K_t) = -\lambda\{[E[\nu_2(Z_0)] + E[U(Z_0)]t + \pi t^2\} \tag{28}$$

$$= \beta_0 + \beta_1 t + \beta_2 t^2, \tag{29}$$

where, $\beta_0 = \lambda E[\nu_2(Z_0)$, $\beta_1 = \lambda E[U(Z_0)]$, $\beta_2 = \lambda\pi$. We can use (26) to compute and substitute $\hat{Q}_Y(K_t), t = t_1, t_2, ..., t_n$ for $Q_Y(K_t)$. Now, treating $-\ln\hat{Q}_Y(K_t)$ as the dependent variable, and $(1, t, t^2)$ as independent variables, we can fit the linear regression model (29).

Let $Y = (-\ln\hat{Q}_Y(K_{t_1}), ..., -\ln\hat{Q}_Y(K_{t_n}))^{'}$ and T be an $n \times 3$ matrix with i^{th} row $(1, t, t^2)$, then the OLS estimator of $\beta = (\beta_0, \beta_1, \beta_2)$ is

$$\hat{\beta} = (T^{'}T)^{-1}T^{'}Y.$$

With the estimates from the OLS, equation (**??**) will take the form:

$$\hat{\lambda} = \frac{\hat{\beta}_2}{\pi},\ \hat{E}[U(Z_0)] = \frac{\hat{\beta}_1}{\hat{\lambda}},\ \text{and}\ \hat{E}[\nu_2(Z_0)] = \frac{\hat{\beta}_0}{\hat{\lambda}},$$

(Khazaee, 2004).

Generalized least squares method (GLS). In the OLS, we assumed that $-\ln\hat{Q}_Y(K_t)'s$ for $t = t_1, t_2, ..., t_n$ had constant variances and were uncorrelated, but that will be an invalid assumption in the context of BRS. Khazaee (2004) showed that

$$C_{t,u} = Cov(\hat{Q}_Y(K_t), \hat{Q}_Y(K_u)) = \sum_{x_i \in W_t} \sum_{x_j \in W_u} c_{t,u}(x_i - x_j)/N_t N_u, \tag{30}$$

where $W_t = W \ominus \breve{K}_t,\ W_u = W \ominus \breve{K}_u,\ N_t = N(W_t),\ N_u = N(W_u)$, and

$c_{t,u}\{exp\{\lambda E[\nu_2((Z_0 \oplus \breve{K}_t) \cap (Z_0 \oplus \{-h\} \oplus \breve{K}_u)]\} - 1\} \times Q_Y(K_t)Q_Y(K_u)$.

$$Cov(\hat{Q}_Y(K_t), \hat{Q}_Y(K_u)) = \frac{1}{N_t N_u} \sum_{x_i \in W_t} \sum_{x_j \in W_u} [E(1 - I_{Y \ominus \breve{K}_t}(x_i)) \times (1 - I_{Y \ominus \breve{K}_u}(x_j))] - Q_Y(K_t)Q_Y(K_u). \tag{31}$$

The empirical estimator $\hat{C}_{t,u}$ for $C_{t,u}$ is obtained by substituting

$$\hat{c}_{t,u}(h) = [\sum_{x_i, x_j \in M(h)} (1 - I_{Y \ominus \breve{K}_t}(x_i))(1 - I_{Y \ominus \breve{K}_u}(x_j))/N(M(h))] - Q_Y(K_t)Q_Y(K_u), \tag{32}$$

where $M(h) = \{(x_i, x_j) : x_I \in W_t, x_j \in W_u, x_i - x_j = h\}$.

Let Σ be the variance-covariance matrix of $Y = (-\ln \hat{Q}_Y(K_{t_1}), ..., -\ln \hat{Q}_Y(K_{t_n}))'$, then the elements of Σ can be approximated by

$$Cov(-\ln \hat{Q}_Y(K_t), -\ln \hat{Q}_Y(K_u)) \approx \frac{C_{t,u}}{Q_Y(K_t), Q_Y(K_u)}. \tag{33}$$

(16) and (32) can then be used to estimate elements of $\hat{\Sigma}$. Thus, the GLS estimator of β becomes:

$$\hat{\beta}^{*} = (T^{'}\hat{\Sigma}T)^{-1}T^{'}\hat{\Sigma}Y. \tag{34}$$

The initial parameter estimators of the Boolean model can be used to calculate the intimal estimates of $\hat{\Sigma}'s$ element by (33). Denote the resulting positive definite matrix by Σ_0. Also, suppose the GLS estimators of β using Σ_0 is denoted by $\beta^{\hat{(1)}*}$, then

$$\beta^{\hat{(1)}*} = (T^{'}\hat{\Sigma_0}T)^{-1}T^{'}\hat{\Sigma_0}Y.$$

An iterative process can be continued using $\beta^{\hat{(i)}*}, i = 1, 2, ...$ to calculate the elements $\Sigma_{(i-1)}$ for Σ to get

$$\beta^{\hat{(i)}*} = (T^{'}\hat{\Sigma}_{i-1}T)^{-1}T^{'}\hat{\Sigma}_{i-1}Y,$$

which is a better approximation for β (Khazaee, 2004).

Advancement of the Model

The Boolean model has seen a lot of advancement and application since Solomon (1953) first considered it. We list a few below:

Marcus (1966, 1967) applied the Boolean model to the study of meteoroidal impact hypothesis for the origin of lunar craters. Dupač (1980) considered the etching of tracks formed by the fission of randomly located uranium atoms in a fission material as a Boolean random set. Also, Ohser & Stoyan (1980) modeled the form of geological deposits of potassium as a BRS. Whilst Serra (1981) used the BRS in the ore-sintering, Diggle (1981) applied it to the incidence of heather in a forest. Robbins (1945) and Neyman & Scott (1972) studied random closed sets with such application in view.

More recent developments include Molchanov & Chiu (2000) studying smoothing techniques and estimation methods for non-stationary Boolean models with applications to coverage processes. Khazaee & Shafie (2006) introduced regression models for Boolean random sets, in order to model the effects of explanatory variables on the distribution of random sets. In addition, Khazaee & Shafie (2010) worked on the propagation models and fitted them to set-valued observations. Gallego, Ibanez, & Simó (2015) studied parameter estimation in Non-Homogeneous Boolean models. And, Last & Gieringer (2017) investigated the concentration inequalities for measures of a Boolean model.

Review of Count Time Series

A time series is a collection of sequential observations made through time. When the observations are counts, we call it count time series. Count time series appear in diverse areas. Take for example, the monthly number of people with a certain disease and the number of trees attacked by a seasonal pest. Cox et al. (1981), classified time-dependent data models into two groups, namely, observation-driven and parameter-driven.

Observation-Driven Count Time Series

For this model, the dependence structure is introduced via the incorporation of lagged values of the observed count into the model. An example is the integer-valued autoregressive moving average (INARMA) class of models. These are some of the most important members of observation-driven time series, which employs the use of appropriate thinning operations to replace the scalar multiplications in the Gaussian ARMA framework. Other groups of observation-driven models are the generalized linear autoregressive moving average (GLARMA). They incorporate the serial correlation as well as binomial variation in the data into the generalized linear model framework by specifying the log of the conditional mean process as a linear function of previous observations. See Davis, Dunsmuir, & Streett (2003) for more on this model. The autoregressive conditional Poisson (ACP) is similar to the GLARMA. Both of these models, GLARMA and ACP, allow for easy incorporation of covariates—an advantage over the INARMA models (Heinen, 2003).

Parameter-Driven Count Time Series

The parameter-driven extends the generalized linear models by incorporating into the conditional mean function a latent dynamic process whose evolution is independent of past observations. Let $\theta_t = \log \mu_t$ be the canonical parameter for $\log \mu_t = \beta^T x_t$ (log-linear model). Then, θ_t is assumed to depend on a latent noise process ε_t, which introduces the autocorrelation and over-dispersion in the observations. The conditional distribution of $y_t|(x_t, \mu_t)$ is assumed to be Poisson with mean $\mu_t = \exp\{x_t^{/}\beta\}\varepsilon_t, \quad t : 1, ..., T$. i.e.

$$y_t|(x_t, \mu_t) \sim Pois(\exp\{x_t^{/}\beta\}\varepsilon_t), \;\; t : 1, ..., T. \tag{35}$$

Zeger (1988) first introduced the specifications for parameter-driven models. Below, we review some estimation methods used in analyzing these types of models.

Some Estimation Methods

Count time series like any other statistical models have their estimation. The relevant ones to this study are discussed below:

Iterative weighted and filtered least-squares algorithm (Zeger-Approach). This method is analogous to the quasi-likelihood proposed by McCullagh et al. (1983) for independent data. The $\hat{\beta}$, which is equal to the root of

$$U(\beta) = \sum_{t=1}^{n} \frac{\partial \mu_t^{'}}{\partial \beta} v_t^{-1}(y_t - \mu_t) = 0, \tag{36}$$

is consistent, asymptotically Gaussian, and optimal in the extended Gauss-Markov sense. For the time series' estimating equations, Zeger (1988) proposed the following: if we let

$$y = (y_1, .., y_n)^{'}, \;\; X = (x_1, .., x_n)^{'}, \;\; \mu = (\mu_1, .., \mu_n)^{'}, \;\; V = \text{var}(y),$$

then the estimating equation can be rewritten as,

$$\frac{\partial \mu^{'}}{\partial \beta} V^{-1}(y - \mu) = 0. \tag{37}$$

It is worth noting that with the time series, the off-diagonal terms of V has dependence on the nuisance parameters. In addition, if we let R_ε be an $n \times n$ autocorrelation matrix with j,k element $\rho_\varepsilon(|j-k|)$, then for the parameter-driven model,
$V = \text{var}(y) = A + \sigma^2 A R_\varepsilon A,$ where $A = \text{diag}(\mu_1, .., \mu_n)$. Hence, we estimate β by solving

the $p \times 1$ system of equations,

$$\frac{\partial \mu'}{\partial \beta} V^{-1}(\beta, \theta)(y - \mu) = 0. \tag{38}$$

Equation (37) depends on both the nuisance parameter θ and β. Suppose $\hat{\theta}$ is an $\sqrt{\text{n}}$-consistent estimate of θ that depends on the observations and β, then let $\hat{\beta}$ be a solution of

$$U(\beta) = \frac{\partial \mu'}{\partial \beta} V^{-1}\{\beta, \hat{\theta}(\beta)\}(y - \mu) = 0. \tag{39}$$

Zeger (1988) proposed the following proposition:

Proposition 1. Suppose ε_t is a stationary process. Under mild regularity conditions and given $\sqrt{n}(\hat{\theta} - \theta) = o_p(1)$ for some fixed θ, $\sqrt{n}(\hat{\beta} - \beta)$ is asymptotically multivariate Gaussian with zero mean and covariance matrix $V_{\hat{\beta}} = \lim_{n \to \infty} \left(\frac{\partial \mu'}{\partial \beta} V^{-1} \frac{\partial \mu}{\partial \beta} / n \right)^{-1}$. An analogous proof is found in Liang & Zeger (1986). To compute $\hat{\beta}$ for a given β, an iterative weighted least-squares is used. The parameter estimates at the $j + 1^{th}$ iteration $\hat{\beta}^{(j+1)}$, are given by

$$\hat{\beta}^{(j+1)} = \left(\frac{\partial \mu'}{\partial \beta} V^{-1} \frac{\partial \mu}{\partial \beta} \right)^{-1} \left(\frac{\partial \mu'}{\delta \beta} V^{-1} Z \right), \tag{40}$$

where $Z = (\frac{\partial \mu'}{\partial \beta})\beta + (y - u)$. $\hat{\beta}$ is found by alternately solving (39) for $\hat{\beta}^{(j+1)}$ given $\hat{\theta}^{(j)}$, then updating $\hat{\beta}^{(j+1)}$ to find $\hat{\theta}^{(j+1)}$ until convergence.

In the solution of (39), there is difficulty in the inversion of V due to the absence of stationary autocorrelation function for the parameter driven process. An approximation of R_ε in (37) leads to an approximate $V_R^{-1} \approxeq D^{\frac{-1}{2}} L' L D^{\frac{-1}{2}}$ where $D = \text{diag}(\mu_t + \sigma^2 \mu_t^2)$ and

LL applies the autoregressive filter i.e. Ly has elements $y_t - \alpha_1 y_{t-1} - ... - \alpha_p y_{t-p} (t > p)$. This leads to

$$\hat{\beta}_R^{(j+1)} = \left\{ \left(LD^{\frac{-1}{2}} \frac{\delta\mu}{\delta\beta} \right)' \left(LD^{\frac{-1}{2}} \frac{\delta\mu}{\delta\beta} \right) \right\}^{-1} \left(LD^{\frac{-1}{2}} \frac{\delta\mu}{\delta\beta} \right)' (LD^{\frac{-1}{2}} Z), \tag{41}$$

which is referred to as Iterative weighted and filtered least-squares algorithm. The nuisance parameters $(\sigma^2, \rho_\varepsilon)$ can be estimated by a method of moments i.e:

$$\hat{\sigma}^2 = \sum_{t=1}^{n} \{(y_t - \hat{\mu}_t)^2 - \hat{\mu}_t\} / \sum_{t=1}^{n} \hat{\mu_t}^2 \tag{42}$$

$$\hat{\rho}_\varepsilon = \hat{\sigma}^{-2} \sum_{t=\tau+1}^{n} \{(y_t - \hat{\mu}_t)(y_{t-\tau} - \hat{\mu_{t-\tau}})\} / \sum_{t=\tau+1}^{n} \hat{\mu}_t \hat{\mu_{t-\tau}}. \tag{43}$$

For more see Zeger (1988).

Quasi maximum likelihood method. Denote a count time series by $\{n_t : t \in \mathbb{N}\}$ and a time-varying r-dimensional covariate vector $\{X_t : t \in \mathbb{N}\}$, say $X_t = (X_{t,1}, ...X_{t,r})^T$. Also, denote by $\mathscr{F}_{t-1}$, the history of the joint process $\{n_{t-1}, \lambda_{t-1}, X_t : t \in \mathbb{N}\}$ up to time $t-1$, including the covariate information at time t. Then, observe that the conditional $n_t | \mathscr{F}_{t-1} \sim Poisson(\lambda_t)$. Hence, we model the conditional mean $E(n_t|\mathscr{F}_{t-1})$ of the count time series by a process, $\{\lambda_t : t \in \mathbb{N}\}$, such that $E(n_t|\mathscr{F}_{t-1}) = \lambda_t$. The general form of the model is,

$$g(\lambda_t) = \beta_0 + \sum_{k=1}^{p} \beta_k \tilde{g}(n_{t-i_k}) + \sum_{l=1}^{q} \alpha_l g(\lambda_{t-j_l}) + \eta^T X_t, \tag{44}$$

where $g : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a link function and $\tilde{g} : \mathbb{N}_0 \longrightarrow \mathbb{R}$ is a transformation function. The parameter vector $\eta = (\eta_1, ...\eta_r)^T$ corresponds to the effects of covariates. For example, Liboschik, Fokianos, & Fried (2015) assumed $g(m) = \log\ m$, $\tilde{g}(x) = \log\ (x+1)$, $p = q = 1$, and $X = 0$ (for simplicity), so that (44) reduces to

$$log\ \lambda_t = \beta_0 + \beta_1 log\ (n_{t-1}+1) + \alpha_1 log \lambda_{t-1}, \tag{45}$$

where λ_t is the intensity of the Poisson process $n_t|\mathscr{F}_{t-1}$. They studied the likelihood inference for (45), with the three dimensional parameter space of $\theta = (\beta_0, \beta_1, \alpha_1)$, and the initial value of λ_0 in terms of observations $n_1, ..., n_T$. Thus, the conditional likelihood function for θ was given by:

$$L(\theta) = \prod_{t=1}^{T} \frac{exp(\lambda_t(\theta))\lambda_t^{n_t(\theta)}}{n_t!},$$
$$\text{where } \lambda_t(\theta) = \exp(\beta_0 + \beta_1 log(n_{t-1}+1) + \alpha_1 \log \lambda_{t-1}).$$

If we let $\log \lambda_t = \gamma_t$. Then, the log-likelihood function has the form,

$$l(\theta) = \sum_{t=1}^{T} (n_t \gamma_t(\theta) - \exp(\gamma_t(\theta)) - \log n_t),$$

where $\gamma_t(\theta) = \beta_0 + \beta_1 log(n_{t-1}+1) + \alpha_1 \gamma_{t-1}$. Furthermore, the score function wass given by

$$S_T(\theta) = \frac{\partial l(\theta)}{\partial \theta} = \sum_{t=1}^{T} (n_t - \exp(\gamma_t(\theta))) \frac{\partial \gamma_t(\theta)}{\partial \theta},$$

where $\frac{\partial \gamma_t(\theta)}{\partial \theta}$ is a vector with components:

$$\frac{\partial \gamma_t}{\partial \beta_0} = 1 + \alpha_1 \frac{\partial \gamma_{t-1}}{\partial \beta_0},\ \frac{\partial \gamma_t}{\partial \beta_1} = \log(1+n_{t-1}) + \alpha_1 \frac{\partial \gamma_{t-1}}{\partial \beta_1},\ \frac{\partial \gamma_t}{\partial \alpha_1} = \gamma_{t-1} + \alpha_1 \frac{\partial \gamma_{t-1}}{\partial \alpha_1}.$$

The solution of $S_T(\theta) = 0$, i.e $\hat{\theta}$, yields the conditional maximum likelihood estimator of θ, if it exists. The Hessian matrix of (45) was obtained from

$$H_T(\theta) = \frac{\partial^2 l(\theta)}{\partial \theta^2} = \sum_{t=1}^{T} (\exp(\gamma_t(\theta)) \left(\frac{\partial \gamma_t(\theta)}{\partial \theta}\right) \left(\frac{\partial \gamma_t(\theta)}{\partial \theta}\right)' \\ - \sum_{t=1}^{T} (n_t - \exp(\gamma_t(\theta))) \frac{\partial^2 \gamma_t(\theta)}{\partial \theta \partial \theta'}.$$

Fokianos et al. (2009) and Fokianos & Tjøstheim (2011) showed the asymptotic normality of $\hat{\theta}$. For more, see Liboschik et al. (2015); Fokianos et al. (2009); Fokianos & Tjøstheim (2011).

CHAPTER III

METHODOLOGY

In this chapter, we proposed models for dependent Boolean random sets and their estimation method. We began with a review of the work of Khazaee (2004) on regression for the Boolean model. It laid the foundation for time series for the Boolean random sets and its fitting methods, which was the subject of this study.

Regression Model for the Boolean Random Sets

If we let Y be a Boolean random set, and X a vector of explanatory variable that affects Y, then the distribution of Y depends on X, i.e.:

$$Y_X = \bigcup_{d_i \in D_X} (Z_{iX} \oplus d_i). \tag{46}$$

$Z'_{iX}s$ are independent copies of Z_{0X}, $\forall\, i$, and D_X is a Poisson point process with intensity λ_X.

Khazaee (2004) developed regression models for the intensity for independent Boolean random sets, with propagation explanatory variables. Also, the dependence of X on Y in (46) takes the following form:

$$Y_X = \bigcup_{d_i \in D_X} (Z_i \oplus d_i), \tag{47}$$

where D_X is a Poisson point process with intensity $\lambda_X = f(X,\beta)$ —f, a positive function. Furthermore, $\beta = (\beta_1,...,\beta_p)^{'}$ is a vector of unknown parameters of the model. Finally, the observations are $(Y_i, X_i),\ \ i = 1,2,...,n$, where, Y_i are realizations of the Boolean model Y_{X_I} in a window W_i with Lebesgue measure 1. $X_i = (x_{i1},...,x_{ik})$ is an observation of X. Then, the number of points in D_{X_i}, n_i in a window W_i with Lebesgue measure 1 has a Poisson distribution with mean $\lambda_i = \lambda_{X_i}$. If we let $\lambda_i = f(X_i,\beta) = h(Y_i^{'}\beta)$, where h is a differentiable monotonic function, the estimation of β then becomes a parameter estimation of a generalized linear model (glm) in the Poisson family, with link function $g(.) = h^{-1}(.)$ problem.

Khazaee (2004) used iterative reweighted least squares algorithm to obtain the maximum likelihood estimate of β from the linear regression model $z_i = X_i^{'}\beta + \varepsilon_i$, where

$$z_i = g(\lambda_i) + (n_i - \lambda_i)g^{'}(\lambda_i), \tag{48}$$

with weights $w_i = [Var(z_i)]^{-1} = [Var(n_i)]^{-1}[g^{'}(\lambda_i)]^{-2}$ and adjusted weights $w_i^* \simeq \left[[g^{'}(\lambda_i)]^(2)\nu_2(W_i)\frac{\lambda_i}{1-p_i}\right]^{-1}$. Since the germs n_i cannot be observed for overlapping grains, Zhazaee used

$$\hat{n}_i = \left[\frac{n_i^+}{1-\hat{p}_i}\right] \tag{49}$$

as an estimate for n_i in (48). The number of observable lower tangent points, n_i^+ in W_i, and $\hat{p}_i$ is the estimated volume fraction for the i^{th} Boolean model. An unbiased estimator of p

when Y is observed in a window W is given by,

$$\hat{p} = \frac{|Y \cap W|}{|W|}. \tag{50}$$

A second estimate used by Khazaee (2004), was n_i^+, the observable lower tangent points in W_i. These tangent points, according to Molchanov (1995), has an approximate Poisson distribution, i.e.:

$$n^+ \sim Poisson\left(\nu_2(W_i)\lambda_i \exp\{-E[\nu_2(Z_0)]\lambda_i\}\right). \tag{51}$$

Then, with the log-likelihood function of β given by

$$l(\beta) = \sum_{i=1}^{n}\{\lambda_i^* + (n_i^+)\ln\lambda_i^*\} - \sum_{i=1}^{n}\ln n_i^+,$$

the parameter estimates $\hat{\beta}$ are the solution to the above equation.

Clearly, these Boolean random sets in the above estimations were assumed to be independent. However, in practice, these observations are often correlated, and the above models may produce inaccurate results. Therefore, this dissertation sought to propose a new model for dependent Boolean random sets. As stated earlier in chapter one, the questions that guided this study are as follows:

Q1 Can we build a model to estimate the intensity of a time-dependent BRS?

- Q1a How do both time-dependent covariates and past observations affect the estimation of the intensity?
- Q1b How do the past observations of BRS affect the estimation of the intensity?
- Q1c How do time-dependent covariates affect the estimation of the intensity?

Q2 Are the estimators of parameters of these models unbiased?

Q3 What are the characteristics of these estimates under different times?

- Q3a When the radius of the grains is known and fixed?
- Q3b When the radius of the grains is unknown and fixed?
- Q3c When the radius of the grains is random?

Q4 Are these estimators asymptotic normal and consistent?

In order to answer the above questions, we propose time series models for the Boolean random set and their parameter estimation methods in the next section.

Time Series Model for the Boolean Random Set

As stated before, our aim in this study was to introduce time series models for random sets (RACS) i.e. $Y = (Y_1, ..., Y_T)'$, estimate the parameters, and study the properties (behavior) of these estimators. Suppose we let

$$Y_{X_t} = \bigcup_{d_t \in D_{X_t}} (Z_t \oplus d_t),$$

where Z_t, $t = 1, 2, ...$ are independent copies of the random closed set Z_0. Also, $Z_t \oplus d_t$ is a realization of the a.s bounded RACS translated to point d_t of a homogeneous Poisson process D_{X_t}. Furthermore, the intensity parameter λ_t, of D_{X_t}, and the probability law of the bounded random grain Z_0, are independent sources of randomness in the Boolean model. Then, X_t only affects D_t and not Z_t. Hence, the intensity parameter of Y_{X_t} is equivalent to the number of points in the Poisson process D_t (n_t) , in the window W_t of Lebesgue measure 1.

We denote a count time series by $\{n_t : t \in \mathbb{N}\}$, and a time-varying r-dimensional covariate vector as $X_t = (X_{t,1}, ...X_{t,r})^T$. Also, denote by $\mathscr{F}_{t-1}$, the history of the joint process $\{n_{t-1}, \lambda_{t-1}, X_t\}$ up to time $t-1$, including the covariate information at time t. Then, the conditional distribution of n_t giving the history $\mathscr{F}_{t-1}$, is distributed as

$$n_t|\mathscr{F}_{t-1} \sim Poisson(\lambda_t). \tag{52}$$

We propose the following methodologies to answer the research questions stated earlier.

Research Question 1

In order to answer the first research question, we denote the conditional mean $E(n_t|\mathscr{F}_{t-1})$ of the count time series by $\{\lambda_t : t \in \mathbb{N}\}$. Then, $E(n_t|\mathscr{F}_{t-1}) = \lambda_t$ is modeled. We propose the following count time series model:

$$g(\lambda_t) = \beta_0 + \sum_{k=1}^{p} \beta_k \tilde{g}(n_{t-i_k}) + \sum_{l=1}^{q} \alpha_l g(\lambda_{t-j_l}) + \eta^T X_t, \tag{53}$$

where $g : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a link function and $\tilde{g} : \mathbb{N}_0 \longrightarrow \mathbb{R}$ is a transformation function. The parameter vector $\eta = (\eta_1, ...\eta_r)^T$ corresponds to the effects of covariates. When $g(m) = \log\ m$, and $\tilde{g}(x) = \log\ (x+1)$, then, the above model takes the form oulined byLiboschik et al. (2015). The advantage to this form lies in the model's ability to cater to both negative and positive correlations in any data. This is what we adopt for our study. Also, for simplicity, we let $p = q = 1$, and η—be the effect from one-dimensional time-dependent covariate. Then, (53) reduces to an $AR(1)$ model of the form

$$\log\ \lambda_t = \beta_0 + \beta_1 log\ (n_{t-1}+1) + \alpha_1 \log \lambda_{t-1} + \eta X_t, \tag{54}$$

where λ_t is the intensity of the Poisson point process of the Boolean RACS. i.e.: $n_t|\mathscr{F}_{t-1}$.

We assume all realizations of $n_t|\mathscr{F}_{t-1}$ are observed in window W_t of Lebesgue measure of 1. Also, we assume Z_t in the Boolean RACS Y_{X_t} are independent of D_{X_t}. For our study, the grains are circles of radius R_t. Hence, we can study the intensity of the BRS Y through $n_t|\mathscr{F}_{t-1}$, by studying the relation in (54), since the λ_t controls the point process. To incorporate the information of n_{t-1} in (54), we use the suggested bijective transformation by Liboschik et al. (2015) i.e., $\tilde{g}(n_{t-1}) = log\ (n_{t-1}+1)$, instead of n_{t-1}. This ensures that the n_{t-1} is transformed onto a similar scale as the rates $\lambda_t' s$ and also deal with zero values of n_{t-1}.

The special cases of (54) that address research question 1, subparts (i) and (ii) respectively, are as follows:

$$log\ \lambda_t = \beta_0 + \beta_1 log\ (n_{t-1}+1) + \alpha_1 log \lambda_{t-1}, \tag{55}$$

$$log\ \lambda_t = \beta_0 + \alpha_1 log \lambda_{t-1} + \eta^T X_t. \tag{56}$$

In the ensuing sections, we will propose two methods for fitting (54). For brevity, we treat only (54), with the result extending naturally to (55) and (56), since they are a reduced form of (54).

Fitting method I. In practice, we cannot observe n_t for overlapping grains. We use an estimate proposed by Khazaee & Shafie (2006) instead i.e.:

$$\hat{n}_t = [|W_t|\hat{\lambda}_t] = \left[\frac{n_t^+}{1-\hat{p}_t}\right], \tag{57}$$

where, n_t^+ and $\hat{p}_t$ are the number of lower tangent points and estimated volume fraction obtained from (1) in window W_t. Then, we can learn about $n_t|\mathscr{F}_{t-1} \sim Poisson(\lambda_t)$, instead, by studying

$$\hat{n}_t|\mathscr{F}_{t-1} \sim Poisson(\hat{\lambda}_t).$$

We can now model the conditional mean $E(\hat{n}_t|\mathscr{F}_{t-1})$ of the count time series by a process say, $\{\hat{\lambda}_t : t \in \mathbb{N}\}$, such that $E(\hat{n}_t|\mathscr{F}_{t-1}) = \hat{\lambda}_t$. Let $log\ \hat{\lambda}_t = \nu_t$, then

$$\nu_t = \hat{\beta}_0 + \hat{\beta}_1 \log(\hat{n}_{t-1}+1) + \hat{\alpha}_1 \nu_{t-1} + \hat{\eta} X_t. \tag{58}$$

We will call this Method I.

Fitting method II. In Method I, we employed the use of exposed lower tangent points n_t^+ in the estimation of $\hat{n}_t$. However, n_t^+ has an approximate Poisson distribution (see Molchanov (1995)) i.e.:

$$n_t^+ \sim Poisson(\lambda_t^+),\ where\ \lambda_t^+ = |W_t|\lambda_t exp[-E|Z_0|\lambda_t]$$

Hence, we can use n_t^+ in (54), then use the relationship between λ_t^+ and $\hat{\lambda}_t$ to derive the effect of the parameters on the intensity. The estimate n_t^+ was obtained by counting the lower tangent point of the set-valued observation in window W_t. This can be achieved by Laslett's transformation implemented in the Spatstat package by Baddeley, Rubak, & Turner (2015) in R Core Team (2019). The Laslett function returns the number of exposed lower tangent points. Then, we use that as an estimate to model the conditional mean $E(n_t^+|\mathscr{F}_{t-1}) = \lambda_t^+$. Again, let $log\lambda_t^+ = \mu_t$, then,

$$\mu_t = \beta_0^+ + \beta_1^+ \, log(n_{t-1}^+ + 1) + \alpha_1^+ \mu_{t-1} + \eta^+ X_t. \tag{59}$$

However, to get the estimates of (54) from μ_t, we use the relationship $\lambda_t^+ = |W_t|\lambda_t exp[-E|Z_0|\lambda_t]$. By the first order Taylor expansion of the relationship, approximation of parameters in (54) through μ_t are as follows:

$$\beta_0 \approx \frac{\beta_0^+ + C}{1-C},\ \beta_1 \approx \frac{\beta_1^+}{1-C},\ \alpha_1 \approx \frac{\alpha_1^+}{1-C},\ \eta^T \approx \frac{\eta^{+T}}{1-C},\ \text{where } C = E|Z_0| = \pi r^2 \tag{60}$$

This, we call Method II.

Thus, to answer research question 1, we can use the proposed models in (58) and (59) to study the effects of propagation explanatory variables on the evolution of Y_t. With the same models, we can predict the future rate of the $n_{t+1}|F_T$, and by extension, of the Boolean random set Y.

Estimation and likelihood inference. The likelihood inference for (54), with parameter space of $\theta = (\beta_0, \beta_1, \alpha_1, \eta)$, is presented here. With initial value of λ_0, the conditional likelihood function for θ is given in terms of observations $n_1,...,n_T$, by

$$L(\theta)=\prod_{t=1}^{T}\frac{exp(-\lambda_t(\theta))\lambda_t^{n_t(\theta)}}{n_t!}, \text{where } \lambda_t(\theta)=\exp(\beta_0+\beta_1\log(n_{t-1}+1)+\alpha_1\log\lambda_{t-1}+\eta X_t).$$

We let $\log\lambda_t=\gamma_t$. Then, the log-likelihood function has the form up to a constant,

$$l(\theta)\approx\sum_{t=1}^{T}(n_t\gamma_t(\theta)-\exp(\gamma_t(\theta))), \text{ where } \gamma_t(\theta)=\beta_0+\beta_1\log(n_{t-1}+1)+\alpha_1\gamma_{t-1}+\eta X_t. \tag{61}$$

The score function is given by,

$$S_T(\theta)=\frac{\partial l(\theta)}{\partial\theta}=\sum_{t=1}^{T}(n_t-\exp(\gamma_t(\theta)))\frac{\partial\gamma_t(\theta)}{\partial\theta}. \tag{62}$$

The $\frac{\partial\gamma_t(\theta)}{\partial\theta}$ is a vector with components,

$$\frac{\partial\gamma_t}{\partial\beta_0}=1+\alpha_1\frac{\partial\gamma_{t-1}}{\partial\beta_0},\ \frac{\partial\gamma_t}{\partial\beta_1}=\log(1+n_{t-1})+\alpha_1\frac{\partial\gamma_{t-1}}{\partial\beta_1},\ \frac{\partial\gamma_t}{\partial\alpha_1}=\gamma_{t-1}+\alpha_1\frac{\partial\gamma_{t-1}}{\partial\alpha_1}$$

$$\frac{\partial\gamma_t}{\partial\eta_s}=\sum_{l=1}^{q}\alpha_1\frac{\partial\gamma_{t-1}}{\partial\eta_s}+X_{t,s},\ \ s=1,...,r.$$

The solution of $S_T(\theta)=0$ i.e. $\hat{\theta}$ yields the conditional maximum likelihood estimator of θ, if it exists.

The Hessian matrix of (54) is obtained from

$$H_T(\theta)=\frac{\partial^2 l(\theta)}{\partial\theta^2}=\sum_{t=1}^{T}(\exp(\gamma_t(\theta))\left(\frac{\partial\gamma_t(\theta)}{\partial\theta}\right)\left(\frac{\partial\gamma_t(\theta)}{\partial\theta}\right)'-\sum_{t=1}^{T}(n_t-\exp(\gamma_t(\theta)))\frac{\partial^2\gamma_t(\theta)}{\partial\theta\partial\theta'}.$$

With the estimates from the solution of $S_T(\theta) = 0$, we will get a fitted model, which will be able to estimate the intensity of a time-dependent BRS.

Research Questions 2, 3, and 4

Fokianos et al. (2009) and Fokianos & Tjøstheim (2011) proved the following theorem for $\hat{\theta}$:

Theorem 2. Consider model (54) and that at the true value $\theta_0, |\alpha_{1_0} + \beta_{1_0}| < 1$, if both $\alpha_{1_0}, \beta_{1_0}$ have the same sign, and $\alpha_{1_0}^2 + \beta_{1_0}^2 < 1$, if both $\alpha_{1_0}, \beta_{1_0}$ have the different signs. Then, there exists a fixed open neighborhood $O = O(\theta_0)$ of θ_0- such that with probability tending to 1, as T $\rightarrow \infty$, the log-likelihood has a unique maximum point $\hat{\theta}$. Furthermore, $\hat{\theta}$. is consistent and asymptotically normal;

$$\sqrt{T}\left(\hat{\theta} - \theta_0\right) \xrightarrow{D} \mathcal{N}(0, G^{-1}).$$

A consistent estimator of **G** is given by $\mathbf{G_T}(\hat{\theta})$, where

$$\mathbf{G_T}(\theta) = \sum_{t=1}^{T}(\exp(\gamma_t(\theta))\left(\frac{\partial \gamma_t(\theta)}{\partial \theta}\right)\left(\frac{\partial \gamma_t(\theta)}{\partial \theta}\right)'.$$

The above theorem shows the existence of a unique solution of (61), which is actually consistent and asymptotically normal. Therefore, we will study the characteristics of the estimators under different times, and confirm the unbiasedness and asymptotic behavior through simulation.

Estimation of radius. The grain of z_0 in this study is a circle with radius R. Also, note that z_t are independent copies of z_0. Hence, R may be known and fixed, or unknown but fixed, or random. The estimation of R under each case is presented below:

i. Known radius: Suppose the radius R of grain is known and fixed, then the fitted model of the Boolean random sets realizations has estimates:

$$\hat{\theta} = (\hat{\beta}_0, \hat{\beta}_1, \hat{\alpha}_1, \hat{\eta}_1) \text{ with known radius } R.$$

$\hat{\theta}$ are the solutions from $S_T(\theta) = 0$ using the BRS Y_{X_t} of grains with radius R.

ii. Unknown but fixed radius : Suppose, the radius R is unknown but fixed. Then, R is estimated along with θ. Hence, we estimate R, by using the relationship between the volume fraction of a Boolean model, p, and the hitting functional, $p = 1 - exp\left\{-\lambda E[|Z_0|]\right\}$. If we let $c = E[|Z_0|] = \pi R^2$, and since the grains are assumed to be circles with radius R, then

$$\begin{aligned} p &= 1 - \exp\{-\lambda_t \pi R^2\}, \\ 1 - p &= \exp\{-\lambda_t \pi R^2\}, \\ \ln(1-p) &= -\lambda_t \pi R^2, \\ R &= \sqrt{\frac{-\ln(1-p)}{\pi\lambda}}. \end{aligned}$$

Hence, the method of moment estimator of R_t for the t^{th} realization is

$$\hat{R}_t = \sqrt{\frac{-\ln(1-\hat{p}_t)}{\pi\hat{\lambda}_t}}. \tag{63}$$

Where $\hat{p}_t$ can be obtained (50) and $\hat{\lambda}_t$ from (49). The mean of these $\hat{R}_t, t = 1, ..., T$ can be used as the estimator of

$$\hat{R} = \frac{1}{T}\sum_{t=1}^{T}\sqrt{\frac{-\ln(1-\hat{p}_t)}{\pi\hat{\lambda}_t}}. \tag{64}$$

iii. Grains with random radius: In estimating the random radius for the Boolean random sets regression model, Khazaee (2004) assumed the radius was uniformly distributed on (a,b), i.e.

$$R_t \sim Uniform(a,b).$$

Then, one can solve for a and b using the first and second moment equations below;

$$E[R_t] = \frac{a+b}{2}, \tag{65}$$

$$E[R_t^2] = \frac{a^2+ab+b^2}{3}. \tag{66}$$

These parameters a and b, have the form,

$$\begin{cases} a &= E[R_t] - \sqrt{3}(E[R_t^2] - E[R_t]^2)^{1/2}, \\ b &= E[R_t] + \sqrt{3}(E[R_t^2] - E[R_t]^2)^{1/2}. \end{cases}$$

With suitable estimators $\hat{E}[R_t^2]$ and $\hat{E}[R]_t$ of $E[R_t^2]$ and $E[R_t]$, we get as moment methods estimators:

$$\begin{cases} \hat{a} &= \hat{E}(R_t) - \sqrt{3}(\hat{E}(R_t^2) - \hat{E}(R_t)^2)^{1/2}, \\ \hat{b} &= \hat{E}(R_t) + \sqrt{3}(\hat{E}(R_t^2) - \hat{E}(R_t)^2)^{1/2}. \end{cases}$$

We observe from (63) that,

$$E(R_t^2) = -\frac{\ln(1-p_t)}{\pi\lambda_t}. \tag{67}$$

Also, $\hat{p}_t$ can be obtained from(50) and $\hat{\lambda}_t$ from (49). Hence, it can be shown that:

$$\begin{aligned} \hat{E}(R^2) &= \frac{1}{T}\sum_{t=1}^{T} \hat{E}_t(R_t^2) \\ &= \frac{1}{T}\sum_{t=1}^{T} \frac{-\ln(1-\hat{p}_t)}{\pi\hat{\lambda}_t}. \end{aligned} \tag{68}$$

With the method of minimum contrast formula in (26), along with random grains of circles, (26) takes the form:

$$Q_X(K_h) = \exp\{-\lambda[\pi E(R_t^2) + 2\pi h E(R_t) + \pi h^2]\}.$$

Then, method of moment estimator of $E_(R_t)$is given by,

$$\hat{E}(R_t) = \frac{1}{2\pi h}\left[-\frac{\ln \hat{Q}_{X_t}}{\hat{\lambda}_t} - \pi \hat{E}_t(R_t^2) - \pi h^2\right]. \quad (69)$$

The mean of these $E(R_t), t = 1, ..., T$, gives,

$$\hat{E}(R) = \frac{1}{2\pi h T}\sum_{t=1}^{T}\left[-\frac{\ln \hat{Q}_{X_t}}{\hat{\lambda}_t} - \pi \hat{E}_t(R_t^2) - \pi h^2\right]. \quad (70)$$

Hence, along with these estimators and the intensity parameter estimators from above, we will be able to answer the research questions stated.

Simulation Setup

We present the setup for the simulation in order to answer the research questions. To simulate the Boolean model, we first simulated the germs from a homogeneous Poisson point process. The grains were then simulated from circles with given radii. Secondly, the *discs* function in R Core Team (2019) package Spatstats by Baddeley & Turner (2005) was used to generate the Boolean realization in a window W_t of Lebesgue measure 1. Then, the Laslett's transform function implemented in the Spatstats package was used to compute the number of exposed lower tangent points n^+. Finally, the number of exposed tangent points, along with (50) and (49) were used to compute $\hat{p}_t$ and $\hat{n}_t$.

Moreover, we took λ_0 to be a known initial intensity. This λ_0 can be obtained from prior information or an educated guess. If we let $\nu_0 = \mu_0 = 1$, then, the estimated $\hat{\lambda}_t$ can be obtained from the model equation (58) or (59). Also, for simplicity, we assumed that $X = (x_1, .., x_T)^{'}$ is a $Tx1$ vector. This vector was simulated from a Normal distribution with mean $(\mu = 0.5)$ and variance $(\sigma^2 = 0.04)$ as suggested by Liboschik et al. (2015), so that their effect sizes were comparable. We then set the time T to take different values, i.e., T=10, 50, 100, 200, 500, 1000, & 2500. Finally, with these conditions $\alpha_1^2 + \beta_1^2 < 1$ and $|\alpha_1 + \beta_1| < 1$, and one thousand, we solved our research problems.

Schemes of Parameters

Below are the schemes of parameters setup for the simulations. The parameters' value combinations for this limited simulation are given below.

Table 1
Parameter Schemes for Simulation

Scheme	β_0	β_1	α_1	η_1	*Radius*	a	b	Condition
1	-0.5	0.65	-0.5	0.5		0	0.1	$\alpha_1^2 + \beta_1^2 < 1$
2	0.5	-0.35	-0.5	0.5		0	0.1	$\|\alpha_1 + \beta_1\| < 1$
3	1.7	0.65	-0.5	0.5	0.01			$\alpha_1^2 + \beta_1^2 < 1$
4	5.5	-0.35	-0.5	0.5	0.01			$\|\alpha_1 + \beta_1\| < 1$
5	-0.5	0.65	-0.5	0		0	0.1	$\alpha_1^2 + \beta_1^2 < 1$
6	0.5	-0.35	-0.5	0		0	0.1	$\|\alpha_1 + \beta_1\| < 1$
7	1.7	0.65	-0.5	0	0.01			$\alpha_1^2 + \beta_1^2 < 1$
8	5.5	-0.35	-0.5	0	0.01			$\|\alpha_1 + \beta_1\| < 1$
9	-0.5	0	-0.5	0.5		0	0.1	$\alpha_1^2 + \beta_1^2 < 1$
10	0.5	0	0.5	0.5		0	0.1	$\|\alpha_1 + \beta_1\| < 1$
11	5.5	0	-0.5	0.5	0.01			$\alpha_1^2 + \beta_1^2 < 1$
12	1.7	0	0.5	0.5	0.01			$\|\alpha_1 + \beta_1\| < 1$

Table 1 contains the different schemes that are applied to models (58) and (59). The schemes from 1 to 4 correspond to models with time-dependent covariate, where schemes 1 and 2 use random radius, whilst 3 and 4 have fixed radius. Also, schemes 5 to 8 correspond to models without time-dependent covariate, with random radius (5, and 6) and fixed radius (7 and 8). And finally, schemes 9 to 12 corresponds to models without past observation $\hat{n}$ or n^{+}, with random radius (9 and 10) and fixed radius (11 and 12). Also, the conditions $\alpha_1^2 + \beta_1^2 < 1$ and $|\alpha_1 + \beta_1| < 1$ are used to ensure stationarity of the intensity (Liboschik et al., 2015). Then, each scheme will be applied to $T = 10,\ 50,\ 100,\ 200,\ 500,\ \&\ 1000,\ 2500$. Lastly, to study the asymptotic behavior of these estimators, we used $T = 1000,$ and 2500.

Application to the Mountain Pine Beetle Data

As stated earlier in chapter one, we applied the models built in this research to the mountain pine beetle data from the Rocky Mountain region. We treated this data as Boolean random set realizations with a fixed unknown radius of 0.02. In addition, the annual average precipitation was used as a time-dependent covariate. Then, we applied thresholding and smoothing to the data. The parameter estimates were obtained using data from 2001 to 2009. Then, with the estimated parameters, we predicted the intensity λ_{2010} of Y_{2010}. Below are the data from 2001 to 2002, with the remaining 2003 to 2010 in Appendix A

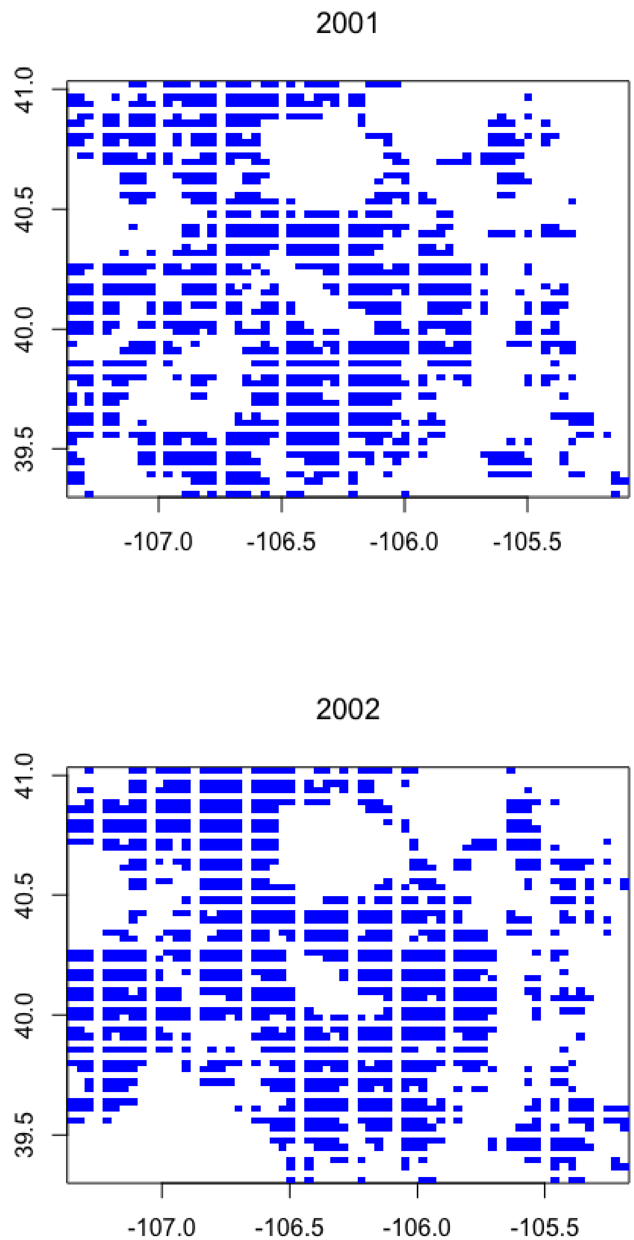

Figure 1. The Rocky Mountain Pine Beetle Data from 2001 to 2002.

CHAPTER IV

RESULTS

In this chapter, we present and discuss the results of this study. In the previous chapter, two methods were introduced. That is, the estimated number of points $\hat{n}$, and the number of exposed lower tangents points n^+. Both were used in modeling the log intensity of the Boolean random sets. In addition, $\log\hat{\lambda}_t = \nu_t$ and $\log\lambda^+ = \mu_t$. Using twelve different schemes, which were discussed in chapter 3, the Boolean RACS realizations were simulated. These schemes ensured that the Boolean realizations were generated from grain processes with unknown radii, and random radii from a uniform distribution. Additionally, the schemes controlled β_1 and α_1, which ensured stationarity of the realizations. The results consist of maximum likelihood estimates for the model parameter and method of moments estimates of the unknown and random radii. Also, the biases, and standard errors for the estimators for different schemes are presented and discussed. Thus, answering the research questions raised. This chapter consists of two sections, which discuss the results of each method.

Results for Method I

The conditional mean $E(\hat{n}_t|\mathscr{F}_{t-1})$ of the count time series for method I is given by:

$$\nu_t = \hat{\beta}_0 + \hat{\beta}_1 log(\hat{n}_{t-1} + 1) + \hat{\alpha}_1 \nu_{t-1} + \hat{\eta}_1 x_t.$$

The conditions, $\beta_1^2 + \alpha_1^2 < 1$, and $|\beta_1 + \alpha_1| < 1$ ensure stationarity. Also, the log intensity allows for both positive and negative lag 1 correlations. Below in Figure 2 and Table 2, are examples of a Boolean realization and statistics extracted from the same.

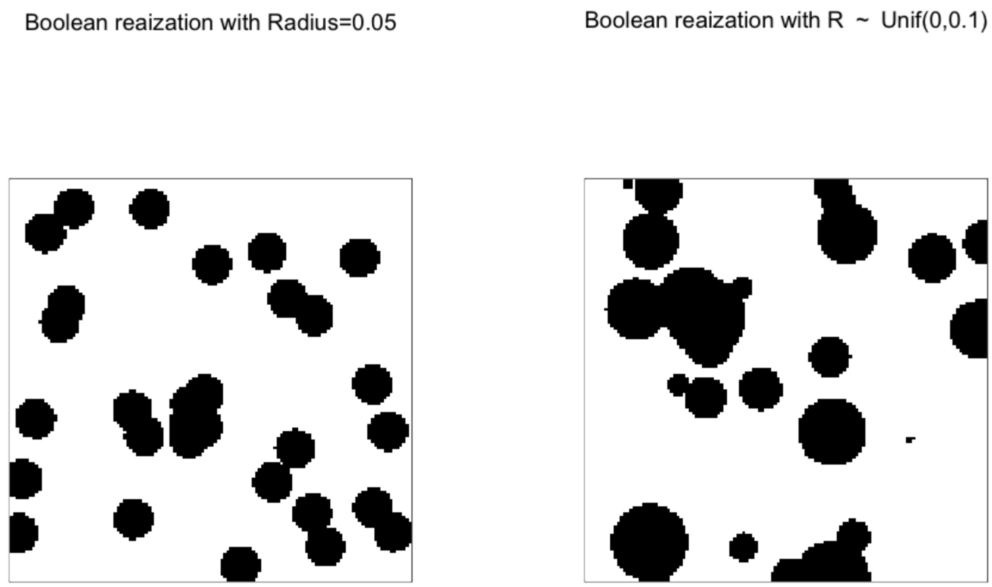

Figure 2. Boolean Realization With Fixed $r_t = 0.05$ and Random Radius, $r_t \sim Unif(0, 0.1)$.

Table 2

Statistics from Ten (10) Boolean Realizations Using Scheme 1

$Time = t$	n_t	n_t^+	p_t^*	$\hat{n}_t$	$\hat{r}_t$	x_t	Q_t	$E(R_t^2)$	$E(R_t)$	$\hat{\lambda}_t$
1	0.00	0.00	0.00	0.00	-0.00	0.69	0.00	0.00	0.00	0.52
2	2.00	2.00	0.01	2.02	0.05	0.19	0.01	0.00	0.06	0.93
3	0.00	0.00	0.00	0.00	-0.00	0.23	0.00	0.00	0.00	1.44
4	1.00	1.00	0.01	1.01	0.08	0.59	0.02	0.01	0.08	0.68
5	2.00	2.00	0.03	2.07	0.09	0.24	0.04	0.01	0.10	1.31
6	5.00	4.00	0.06	4.24	0.13	0.14	0.07	0.02	0.21	1.16
7	1.00	0.00	0.00	0.00	-0.00	0.54	0.00	0.00	0.00	2.36
8	0.00	0.00	0.00	0.00	-0.00	0.38	0.00	0.00	0.00	0.75
9	2.00	2.00	0.02	2.05	0.09	0.52	0.03	0.01	0.11	0.91
10	3.00	2.00	0.01	2.01	0.03	0.36	0.01	0.00	0.03	1.56

The n_t are points generated from a random Poisson process. Where n_t^+ is the number of exposed tangent points recovered from the t^{th} Boolean realization. Whilst, $\hat{n}_t$ is the estimates for n_t using (49). In addition, p_t^* is the volume fraction and Q_t the hitting functional value for each image. Simulated from Normal distribution with $\mu = 0.50$ and $\sigma = 0.04$, we can take x_t as a time-dependent covariate. Then, $\hat{\lambda}_t$ is the estimated intensity from (58). $E(R_t^2)$ and $E(R_t)$ are the first and second moment estimates, which are used in solving for a and b for the unknown parameters of the Uniform distribution. In order to answer research questions $1i$, and $3iii$ corresponding to (58), we ran the schemes in Table 1. i.e.: schemes 1 to 2. The results are displayed in the tables below:

Table 3

Results from Scheme 1

Time	θ	Parameters	$\hat{\theta}$	Std.Error	Bias	Statistic	P value
10	β_0	-0.50	-0.4756049	1.7474151	-0.0243951	0.1240	<0.001
	β_1	0.65	0.1867123	0.7572443	0.4632877	0.1496	<0.001
	α_1	-0.50	-0.1244810	0.7445394	-0.3755190	0.1221	<0.001
	η_1	0.50	0.3540084	3.0381681	0.1459916	0.1170	<0.001
	a	0.00	-0.0370609	0.0168511	0.0370609	0.0998	<0.001
	b	0.10	0.1041074	0.0258972	-0.0041074	0.0451	1e-04
50	β_0	-0.50	-0.5843659	0.5113693	0.0843659	0.0168	0.7053
	β_1	0.65	0.4825539	0.3569580	0.1674461	0.0831	<0.001
	α_1	-0.50	-0.3169092	0.5237645	-0.1830908	0.0961	<0.001
	η_1	0.50	0.4990660	0.8142450	0.0009340	0.0201	0.4189
	a	0.00	-0.0428164	0.0060557	0.0428164	0.0422	2e-04
	b	0.10	0.1096714	0.0117068	-0.0096714	0.0312	0.0228
100	β_0	-0.50	-0.5716119	0.3587157	0.0716119	0.0205	0.3896
	β_1	0.65	0.5370042	0.2165289	0.1129958	0.0170	0.6901
	α_1	-0.50	-0.5176675	0.3370081	0.0176675	0.0762	<0.001
	η_1	0.50	0.4556020	0.5143164	0.0443980	0.0190	0.5108
	a	0.00	-0.0430681	0.0043712	0.0430681	0.0432	2e-04
	b	0.10	0.1108408	0.0080706	-0.0108408	0.0169	0.700
200	β_0	-0.50	-0.5918654	0.2587009	0.0918654	0.0239	0.1794
	β_1	0.65	0.5486800	0.1461364	0.1013200	0.0143	0.8867
	α_1	-0.50	-0.4311021	0.2410231	-0.0688979	0.0678	<0.001
	η_1	0.50	0.4941714	0.3702917	0.0058286	0.0208	0.3711
	a	0.00	-0.0434383	0.0030744	0.0434383	0.0321	0.0169
	b	0.10	0.1110467	0.0057893	-0.0110467	0.0174	0.6528
500	β_0	-0.50	-0.5899845	0.1538225	0.0899845	0.0319	0.0182
	β_1	0.65	0.5546322	0.0957894	0.0953678	0.0196	0.4595
	α_1	-0.50	-0.4482066	0.1427250	-0.0517934	0.0294	0.0411
	η_1	0.50	0.4952096	0.2121559	0.0047904	0.0257	0.1121
	a	0.00	-0.0437036	0.0018408	0.0437036	0.0278	0.0663
	b	0.10	0.1113458	0.0036703	-0.0113458	0.0274	0.0742
1000	β_0	-0.50	-0.5834755	0.1052596	0.0834755	0.0204	0.3951
	β_1	0.65	0.5575079	0.0641257	0.0924921	0.0198	0.4445
	α_1	-0.50	-0.4361159	0.0940835	-0.0638841	0.0438	0.0001
	η_1	0.50	0.4870386	0.1501619	0.0129614	0.0138	0.9152
	a	0.00	-0.0437323	0.0013123	0.0437323	0.0178	0.6180
	b	0.10	0.1114453	0.0025959	-0.0114453	0.0174	0.6537
2500	β_0	-0.50	-0.5825164	0.0649762	0.0825164	0.0216	0.3087
	β_1	0.65	0.5565001	0.0402819	0.0934999	0.0143	0.8873
	α_1	-0.50	-0.4371118	0.0584869	-0.0628882	0.0307	0.0271
	η_1	0.50	0.4896683	0.0922974	0.0103317	0.0124	0.9685
	a	0.00	-0.0437374	0.0008245	0.0437374	0.0218	0.2991
	b	0.10	0.1114021	0.0016339	-0.0114021	0.0198	0.4425

Table 3 shows the results of simulations from scheme 1 for various times T. The grains Z_t of these realizations were circles with random radius R from $Uniform\ (0,0.1)$. Then, using a test set—a circle with center at the origin with radius of 0.01—the method of moment estimates of a and b, stabilizes across T. Also we note that the method of moment estimates equations in $\hat{a} = \hat{E}(R) - \sqrt{3}(\hat{E}(R^2) - \hat{E}(R)^2)^{1/2}$ and $\hat{b} = \hat{E}(R) + \sqrt{3}(\hat{E}(R^2) - \hat{E}(R)^2)^{1/2}$ are not range preserving. Hence, the underestimation of a. However, the standard errors of these estimates decrease with increasing sample size (T), with the distribution of estimates passing the normality test.

Furthermore, the maximum likelihood estimates $\hat{\theta}$ improves to parameter values with increasing sample size t. Especially at $T = 2500$, the biases and the standard errors become significantly small, thus confirming the characteristics of maximum likelihood estimates. In addition, the distribution of the estimates also approach normal, which was seen in the p-values of the Kolmogorov-Smirnov normality tests. Thus, with both past observations $\hat{n}_t$ and time dependent covariates x_t in the model (58), we can accurately estimate the intensity parameter λ_t of the Boolean model.

Similar behaviors and patterns are seen in Table 4 for scheme 2, for both method of moments, and maximum likelihood estimates. Thus, for both conditions i.e.: $\beta_1^2 + \alpha_1^2 < 1$, and $|\beta_1 + \alpha_1| < 1$, the Boolean model Y_t with grains Z_t of random radius from $Unif\ (0,0.1)$, with intensity $\hat{\lambda}_t$ can be estimated using model (58) accurately.

Table 4
Results from Scheme 2

Time	θ	Parameters	$\hat{\theta}$	Std.Error	Bias	Statistic	P value
10	β_0	0.50	0.0581898	1.2843532	0.4418102	0.0622	<0.001
	β_1	-0.35	-0.2910588	0.6933754	-0.0589412	0.1533	<0.001
	α_1	-0.50	-0.0572888	0.7447230	-0.4427112	0.1085	<0.001
	η_1	0.50	0.3480330	2.2473791	0.1519670	0.0570	<0.001
	a	0.00	-0.0385806	0.0148499	0.0385806	0.0843	<0.001
	b	0.10	0.1062335	0.0226632	-0.0062335	0.0296	0.0392
50	β_0	0.50	0.1737841	0.4519590	0.3262159	0.0269	0.0845
	β_1	-0.35	-0.3338667	0.3020805	-0.0161333	0.0155	0.8132
	α_1	-0.50	-0.2803542	0.5414197	-0.2196458	0.0998	<0.001
	η_1	0.50	0.4999262	0.7164263	0.0000738	0.0455	<0.001
	a	0.00	-0.0425677	0.0057413	0.0425677	0.0528	<0.001
	b	0.10	0.1106224	0.0106214	-0.0106224	0.0235	0.1986
100	β_0	0.50	0.1954056	0.2909666	0.3045944	0.0344	0.0073
	β_0	-0.35	-0.3081971	0.2002832	-0.0418029	0.0367	0.003
	α_0	-0.50	-0.4217077	0.4119127	-0.0782923	0.1172	<0.001
	η_1	0.50	0.4960932	0.4574629	0.0039068	0.0199	0.4391
	a	0.00	-0.0432584	0.0037808	0.0432584	0.0318	0.0184
	b	0.10	0.1109860	0.0072991	-0.0109860	0.0285	0.0533
200	β_0	0.50	0.2284459	0.2016789	0.2715541	0.0209	0.3638
	β_1	-0.35	-0.3141885	0.1283050	-0.0358115	0.0201	0.4228
	α_1	-0.50	-0.4490368	0.2920806	-0.0509632	0.1145	<0.001
	η_1	0.50	0.4961666	0.3038345	0.0038334	0.0260	0.1048
	a	0.00	-0.0433141	0.0027568	0.0433141	0.0463	<0.001
	b	0.10	0.1110261	0.0053745	-0.0110261	0.0130	0.9515
500	β_0	0.50	0.2496623	0.1196900	0.2503377	0.0287	0.0504
	β_1	-0.35	-0.3248081	0.0763713	-0.0251919	0.0257	0.1118
	α_1	0.50	0.4895231	0.1846565	0.0104769	0.0204	0.398
	a	0.00	-0.0436906	0.0016713	0.0436906	0.0391	0.001
	b	0.10	0.1112172	0.0034137	-0.0112172	0.0158	0.7856
1000	β_0	0.50	0.2531837	0.0851960	0.2468163	0.0218	0.2951
	β_1	-0.35	-0.3188453	0.0551241	-0.0311547	0.0161	0.7665
	α_1	-0.50	-0.5241991	0.0908810	0.0241991	0.0457	<0.001
	η_1	0.50	0.4901651	0.1261732	0.0098349	0.0231	0.2214
	a	0.00	-0.0437230	0.0012354	0.0437230	0.0221	0.276
	b	0.10	0.1115092	0.0023900	-0.0115092	0.0115	0.9874
2500	β_0	0.50	0.2579814	0.0517617	0.2420186	0.0203	0.4094
	β_1	-0.35	-0.3226764	0.0327150	-0.0273236	0.0189	0.5211
	α_1	-0.50	-0.5262106	0.0531311	0.0262106	0.0293	0.0425
	η_1	0.50	0.4869960	0.0780131	0.0130040	0.0207	0.3720
	a	0.00	-0.0437757	0.0007407	0.0437757	0.0127	0.9590
	b	0.10	0.1113730	0.0014925	-0.0113730	0.0175	0.6417

Research questions 1i, and 3ii correspond to (58). Table 5 below is an example of descriptive statistics from the BRS generated from schemes 3, and 4 in Table 1. Where the grain Z_t have fixed and unknown radius R.

Table 5
Statistics from Ten (10) Boolean Realizations from Scheme 3

$Time = t$	n_t	n_t^+	p_t^*	$\hat{n}_t$	$\hat{R}_t$	x_t	$\hat{\lambda}_t$	R
1	20.0000	20.0000	0.0058	20.1166	0.0119	0.4166	13.1040	0.0100
2	16.0000	16.0000	0.0052	16.0844	0.0107	0.5709	14.5549	0.0100
3	10.0000	11.0000	0.0032	11.0357	0.0098	0.3543	10.8024	0.0100
4	9.0000	9.0000	0.0029	9.0259	0.0093	0.5791	10.5731	0.0100
5	11.0000	11.0000	0.0034	11.0371	0.0100	0.7202	10.7791	0.0100
6	9.0000	9.0000	0.0029	9.0264	0.0090	0.6351	11.5183	0.0100
7	8.0000	9.0000	0.0024	9.0220	0.0097	0.2610	8.2088	0.0100
8	7.0000	6.0000	0.0021	6.0125	0.0080	0.5282	10.3783	0.0100
9	6.0000	6.0000	0.0017	6.0103	0.0078	0.6182	8.9429	0.0100
10	9.0000	8.0000	0.0028	8.0225	0.0100	0.6326	8.8970	0.0100

From Table 5, we observe that when the radius was sufficiently small, the Laslett function was able to recover a lot of the exposed lower tangent points. This is because, with a sufficiently small grain radius, the grains shrink towards the germ from the point process. Also, $\hat{n}_t$ is the estimates for n_t using (49). p_t^* is 1 minus the hitting functional value Q_t for each image. x_t is the time-dependent covariate simulated from normal distribution with $\mu = 0.50$ and $\sigma = 0.04$. $\hat{\lambda}_t$ is the estimated intensity from (58) and $\hat{R}_t$ is the estimate for R_t.

Table 6
Results from Scheme 3

Time	θ	Parameters	$\hat{\theta}$	Std.Error	Bias	Statistic	P value
10	β_0	1.70	1.9942575	0.8457011	-0.2942575	0.0524	<0.001
	β_1	0.65	0.2993197	0.6207261	0.3506803	0.1318	<0.001
	α_1	-0.50	-0.2690104	0.7307089	-0.2309896	0.1586	<0.001
	η_1	0.50	0.4436066	0.7493935	0.0563934	0.0595	<0.001
	Radius	0.01	0.0098352	0.0005410	0.0001648	0.0336	0.0098
50	β_0	1.70	1.7871003	0.6059542	-0.0871003	0.0172	0.669
	β_1	0.65	0.5769652	0.1582886	0.0730348	0.0285	0.0539
	α_1	-0.50	-0.4683012	0.2589570	-0.0316988	0.0593	<0.001
	η_1	0.50	0.4846358	0.2246510	0.0153642	0.0211	0.3466
	Radius	0.01	0.0098288	0.0002331	0.0001712	0.0244	0.1578
100	β_0	1.70	1.7571994	0.4203642	-0.0571994	0.0313	0.0223
	β_1	0.65	0.5864859	0.1073966	0.0635141	0.0279	0.0641
	α_1	-0.50	-0.4675856	0.1614900	-0.0324144	0.0360	0.0038
	η_1	0.50	0.4946568	0.1597091	0.0053432	0.0166	0.7197
	Radius	0.01	0.0098226	0.0001641	0.0001774	0.0149	0.8540
200	β_0	1.70	1.7359856	0.3038111	-0.0359856	0.0258	0.1082
	β_1	0.65	0.5878822	0.0770359	0.0621178	0.0210	0.3557
	α_1	-0.50	-0.4586346	0.1117631	-0.0413654	0.0252	0.1279
	η_1	0.50	0.4921712	0.1048772	0.0078288	0.0171	0.6811
	Radius	0.01	0.0098225	0.0001223	0.0001775	0.0133	0.9387
500	β_0	1.70	1.7104920	0.1914353	-0.0104920	0.0184	0.5665
	β_1	0.65	0.5927031	0.0468776	0.0572969	0.0176	0.6385
	α_1	-0.50	-0.4527485	0.0686246	-0.0472515	0.0205	0.3893
	β_1	0.50	0.4932953	0.0708219	0.0067047	0.0166	0.7204
	Radius	0.01	0.0098234	0.0000761	0.0001766	0.0251	0.1311
1000	β_0	1.70	1.7040678	0.1325939	1.5674061	0.0190	0.5125
	β_1	0.65	0.5906505	0.0334899	0.0593495	0.0230	0.2251
	α_1	-0.50	-0.4473722	0.0471905	-0.0526278	0.0224	0.2594
	η_1	0.50	0.4929810	0.0467460	0.0070190	0.0198	0.4463
	Radius	0.01	0.0098257	0.0000528	0.0001743	0.0190	0.5123
2500	β_0	1.70	1.7012990	0.0816827	-0.0012990	0.0197	0.4517
	β_1	0.65	0.5940552	0.0220534	0.0559448	0.0109	0.9947
	α_1	-0.50	-0.4501487	0.0296920	-0.0498513	0.0228	0.2353
	η_1	0.50	0.4944287	0.0300216	0.0055713	0.0262	0.0984
	Radius	0.01	0.0098243	0.0000327	0.0001757	0.0266	0.0928

Table 6 shows the results of simulations from scheme 3 for various times T. The grains Z_t of these realizations were fixed, but unknown radius, ie.: $R = 0.01$. Similarly, using the same test set, the method of moment estimates of $\hat{R}$ across T improves to the parameter value. Also, the standard errors and biases of the estimates decrease with increasing sample size (T). Thus, both approach approximately zero with increasing T. This scheme's parameter estimates also achieve asymptotic normality, which was evident in the table.

Likewise, the maximum likelihood estimates $\hat{\theta}$ approached parameter values with increasing sample size t. In addition, at $T = 2500$, the biases and the standard errors were approximately zero, confirming the characteristics of maximum likelihood estimates. Moreover, the distribution of the estimates also approached normal, seen in the p-values of the Kolmogorov-Smirnov normality tests. Thus, with both past observations $\hat{n}_t$, time dependent covariates x_t in the model (58) and unknown but fixed radius $R = 0.01$, we can accurately estimate the intensity parameter λ_t of the Boolean model.

Similar behavior and pattern are seen Table 7 below for scheme 4 for both the method of moments and maximum likelihood estimates. For both conditions, $\beta_1^2 + \alpha_1^2 < 1$ and $|\beta_1 + \alpha_1| < 1$, the Boolean model Y_t with a grains Z_t with fixed but unknown radius $R = 0.01$, the intensity of the $\hat{n}_t|\mathscr{F}_{t-1} \sim Poisson(\hat{\lambda}_t)$ can be estimated using model (58) accurately.

Given this, observe that for research questions $1i$ and $3i$, corresponding to (58), the result was similar to that of Tables 6 and 7. Also, we do not estimate R. Thus, the maximum likelihood estimation of Schemes 3 and 4, without the method of moment estimation of radius is the solution for research questions $1i$, $3i$.

Table 7
Results from Scheme 4

Time	θ	Parameters	$\hat{\theta}$	Std.Error	Bias	Statistic	P value
10	β_0	5.50	4.2376823	1.1167344	1.2623177	0.0332	0.0115
	β_1	-0.35	-0.4892052	0.3364832	0.1392052	0.0869	<0.001
	α_1	-0.50	0.1495335	0.6575786	-0.6495335	0.0979	<0.001
	η_1	0.50	0.1798061	2.1549719	0.3201939	0.0603	<0.001
	Radius	0.01	0.0098649	0.0004275	0.0001351	0.0208	0.3692
50	β_0	5.50	5.2761800	0.6048663	0.2238200	0.2060	<0.001
	β_1	-0.35	-0.5234261	0.1334463	0.1734261	0.0731	<0.001
	α_1	-0.50	-0.2769451	0.2399359	-0.2230549	0.1857	<0.001
	η_1	0.50	0.6724066	0.4573186	-0.1724066	0.0376	0.002
	Radius	0.01	0.0098953	0.0001641	0.0001047	0.0178	0.6225
100	β_0	5.50	5.4430438	0.2306754	0.0569562	0.0893	<0.001
	β_1	-0.35	-0.4876263	0.0863276	0.1376263	0.0607	<0.001
	α_1	-0.50	-0.3655698	0.1164948	-0.1344302	0.0724	<0.001
	η_1	0.50	0.5988273	0.2250217	-0.0988273	0.0282	0.0594
	Radius	0.01	0.0099039	0.0001071	0.0000961	0.0226	0.2449
200	β_0	5.50	5.4523947	0.1299340	0.0476053	0.0318	0.0187
	β_1	-0.35	-0.4342370	0.0614069	0.0842370	0.0257	0.1115
	α_1	-0.50	-0.4211310	0.0763940	-0.0788690	0.0411	0.0004
	η_1	0.50	0.5456752	0.1259152	-0.0456752	0.0282	0.0585
	Radius	0.01	0.0098978	0.0000780	0.0001022	0.0162	0.7536
500	β_0	5.50	5.4401678	0.0773314	0.0598322	0.0245	0.1542
	β_1	-0.35	-0.3863618	0.0398661	0.0363618	0.0188	0.5310
	α_1	-0.50	-0.4641789	0.0464301	-0.0358211	0.0237	0.1887
	η_1	0.50	0.5138528	0.0595666	-0.0138528	0.0327	0.0136
	Radius	0.01	0.0099010	0.0000485	0.0000990	0.0264	0.0977
1000	β_0	5.50	5.4409038	0.0564279	0.0590962	0.0336	0.0097
	β_1	-0.35	-0.3617414	0.0281792	0.0117414	0.0133	0.9372
	α_1	-0.50	-0.4890881	0.0336776	-0.0109119	0.0174	0.6533
	η_1	0.50	0.5023116	0.0374019	-0.0023116	0.0285	0.0534
	Radius	0.01	0.0098972	0.0000345	0.0001028	0.0275	0.0723
2500	β_0	5.50	5.4417141	0.0347971	0.0582859	0.0142	0.8969
	β_1	-0.35	-0.3424395	0.0178288	-0.0075605	0.0114	0.9887
	α_1	-0.50	-0.5077921	0.0216477	0.0077921	0.0278	0.0664
	η_1	0.50	0.4923831	0.0203014	0.0076169	0.0221	0.2747
	Radius	0.01	0.0099005	0.0000212	0.0000995	0.0241	0.1710

We noted that without the time-dependent covariate, (58) reduced to (55), thus leaving only the effect of past observations in the model. Also, with a similar setup and $\eta_1 = 0$, schemes 5 to 8 repeats similar analyses to the one above, where the investigation of the estimation of λ_t without the covariate effect can be seen. Tables 8 to 11 below presents the results of the simulation study.

Table 8
Results from Scheme 5

Time	θ	Parameters	$\hat{\theta}$	Std.Error	Bias	Statistic	P value
10	β_0	-0.50	-0.3347657	0.7069925	-0.1652343	0.0780	<0.001
	β_1	0.65	0.1117826	0.7637328	0.5382174	0.1572	<0.001
	α_1	-0.50	-0.0428181	0.6940062	-0.4571819	0.0906	<0.001
	a	0.00	-0.0361794	0.0178552	0.0361794	0.0992	<0.001
	b	0.10	0.1036912	0.0277260	-0.0036912	0.0466	<0.001
50	β_0	-0.50	-0.5058920	0.3970688	0.0058920	0.0286	0.0518
	β_1	0.65	0.4543833	0.3924313	0.1956167	0.0965	<0.001
	α_1	-0.50	-0.2203802	0.5737438	-0.2796198	0.1154	<0.001
	a	0.00	-0.0420547	0.0071099	0.0420547	0.0565	<0.001
	b	0.10	0.1099430	0.0119235	-0.0099430	0.0236	0.1947
100	β_0	-0.50	-0.5993220	0.2431623	0.0993220	0.0360	0.0038
	β_1	0.65	0.5519261	0.2109249	0.0980739	0.0195	0.4752
	α_1	-0.50	-0.4612838	0.3387734	-0.0387162	0.0675	<0.001
	a	0.00	-0.0428222	0.0047946	0.0428222	0.0349	0.006
	b	0.10	0.1113204	0.0087453	-0.0113204	0.0267	0.0906
200	β_0	-0.50	-0.5876557	0.1861741	0.0876557	0.0246	0.1492
	β_1	0.65	0.5535943	0.1779451	0.0964057	0.0224	0.2619
	α_1	-0.50	-0.4021627	0.3183430	-0.0978373	0.0816	<0.001
	a	0.00	-0.0432908	0.0033083	0.0432908	0.0230	0.2243
	b	0.10	0.1110780	0.0063313	-0.0110780	0.0296	0.0382
500	β_0	-0.50	-0.5996980	0.1029191	0.0996980	0.0254	0.1230
	β_1	0.65	0.5689125	0.1058278	0.0810875	0.0170	0.6878
	α_1	-0.50	-0.4522304	0.1577205	-0.0477696	0.0397	0.0008
	a	0.00	-0.0436061	0.0020011	0.0436061	0.0162	0.7534
	b	0.10	0.1114748	0.0038976	-0.0114748	0.0259	0.1059
1000	β_0	-0.50	-0.5972694	0.0739679	0.0972694	0.0136	0.9261
	β_1	0.65	0.5670747	0.0744655	0.0829253	0.0146	0.8697
	α_1	-0.50	-0.4565785	0.1097052	-0.0434215	0.0449	0.0001
	a	0.00	-0.0436799	0.0014368	0.0436799	0.0182	0.5851
	b	0.10	0.1113546	0.0027209	-0.0113546	0.0119	0.9801
2500	β_0	-0.50	-0.5988478	0.0486692	0.0988478	0.0329	0.0129
	β_1	0.65	0.5727167	0.0481620	0.0772833	0.0368	0.0028
	α_1	-0.50	-0.4495354	0.0690978	-0.0504646	0.0325	0.0146
	a	0.00	-0.0437907	0.0009416	0.0437907	0.0242	0.1647
	b	0.10	0.1114041	0.0017694	-0.0114041	0.0205	0.3943

Table 8 shows the results of simulations from scheme 5 for various times T. Here, the grains Z_t of these realizations are circles with random radius R from $Uniform\ (0, 0.1)$. The results are very similar to Scheme 1 in Table 3. This was due to the fact that the serial dependence of the covariate does not interact with the dependency in the observations.

The standard errors of these estimates decreased with increasing sample size (T), with the distribution of estimates approaching normality over time.

The maximum likelihood estimates $\hat{\theta}$ improved with increasing sample size t, similar to results in Table 3. At $T = 2500$, the biases and the standard errors became significantly small, thus confirming the characteristics of maximum likelihood estimates. The distribution of the estimates also approach normal distribution, which can be seen in the p-values of the Kolmogorov-Smirnov normality tests. As a result, with past observations $\hat{n}_t$ in the model (55), we can accurately estimate the intensity parameter λ_t of the Boolean model.

Similar behavior and pattern are seen in Table 9 below for scheme 6 for both the method of moments and maximum likelihood estimates. Thus, for both conditions, $\beta_1^2 + \alpha_1^2 < 1$ and $|\beta_1 + \alpha_1| < 1$, the Boolean model Y_t with grains Z_t of random radius from $Unif\ (0, 0.1)$, the intensity of the $\hat{n}_t|\mathscr{F}_{t-1} \sim Poisson(\hat{\lambda}_t)$ can be estimated using model (55), accurately.

Table 9
Results from Scheme 6

Time	θ	Parameters	$\hat{\theta}$	Std.Error	Bias	Statistic	P value
10	β_0	0.50	0.2044597	0.6632083	0.2955403	0.1142	<0.001
	β_1	-0.35	-0.4139489	0.6505817	0.0639489	0.1838	<0.001
	α_1	-0.50	0.0550340	0.6320331	-0.5550340	0.0802	<0.001
	a	0.00	-0.0368994	0.0173446	0.0368994	0.1171	<0.001
	b	0.10	0.1048638	0.0250135	-0.0048638	0.0396	9e-04
50	β_0	0.50	0.2663501	0.2337756	0.2336499	0.0485	<0.001
	β_1	-0.35	-0.3971586	0.3036582	0.0471586	0.0517	<0.001
	α_1	-0.50	-0.0672672	0.5550388	-0.4327328	0.0956	<0.001
	a	0.00	-0.0423578	0.0058369	0.0423578	0.0462	<0.001
	b	0.10	0.1098913	0.0117255	-0.0098913	0.0270	0.0817
100	β_0	0.50	0.2515484	0.1783202	0.2484516	0.0431	2e-04
	β_1	-0.35	-0.3346904	0.2095204	-0.0153096	0.0173	0.66
	α_1	-0.50	-0.4196551	0.4225002	-0.0803449	0.1115	<0.001
	a	0.00	-0.0432270	0.0040237	0.0432270	0.0368	0.0028
	b	0.10	0.1107726	0.0075212	-0.0107726	0.0179	0.6131
200	β_0	0.50	0.2531008	0.1170876	0.2468992	0.0265	0.0961
	β_1	-0.35	-0.3300965	0.1383019	-0.0199035	0.0301	0.0329
	α_1	-0.50	-0.3774909	0.3876433	-0.1225091	0.1351	<0.001
	a	0.00	-0.0434270	0.0028671	0.0434270	0.0293	0.0416
	b	0.10	0.1110546	0.0054744	-0.0110546	0.0301	0.0324
500	β_0	0.50	0.2575159	0.0751384	0.2424841	0.0191	0.5048
	β_1	-0.35	-0.3312770	0.0889700	-0.0187230	0.0169	0.6939
	α_1	-0.50	-0.4970832	0.1823180	-0.0029168	0.0845	<0.001
	a	0.00	-0.0437265	0.0018438	0.0437265	0.0406	5e-04
	b	0.10	0.1113912	0.0035799	-0.0113912	0.0243	0.1621
1000	β_0	0.50	0.2587403	0.0523489	0.2412597	0.0213	0.3321
	β_1	-0.35	-0.3255645	0.0605186	-0.0244355	0.0220	0.2822
	α_1	-0.50	-0.5123642	0.1203780	0.0123642	0.0549	<0.001
	a	0.00	-0.0437147	0.0012951	0.0437147	0.0260	0.103
	b	0.10	0.1114175	0.0025302	-0.0114175	0.0148	0.8607
2500	β_0	0.50	0.2570308	0.0334176	0.2429692	0.0167	0.7148
	β_1	-0.35	-0.3235941	0.0388293	-0.0264059	0.0238	0.1859
	α_1	-0.50	-0.5222265	0.0701911	0.0222265	0.0280	0.0615
	a	0.00	-0.0437945	0.0007711	0.0437945	0.0139	0.9125
	b	0.10	0.1113376	0.0016393	-0.0113376	0.0263	0.0957

The ensuing Tables 10 and 11 are for Boolean random sets with grains of fixed but unknown radius $R = 0.01$.

Table 10
Results from Scheme 7

Time	θ	Parameters	$\hat{\theta}$	Std.Error	Bias	Statistic	P value
10	β_0	1.70	1.8723692	0.7090535	-0.1723692	0.0549	<0.001
	β_1	0.65	0.3936916	0.5106610	0.2563084	0.1294	<0.001
	α_1	-0.50	-0.3304898	0.6007120	-0.1695102	0.1325	<0.001
	Radius	0.01	0.0098009	0.0006341	0.0001991	0.0240	0.1766
50	β_0	1.70	1.8014570	0.5920243	-0.1014570	0.0371	0.0025
	β_1	0.65	0.5754635	0.1557404	0.0745365	0.0268	0.087
	α_1	-0.50	-0.4805663	0.2566208	-0.0194337	0.0573	<0.001
	Radius	0.01	0.0097794	0.0002797	0.0002206	0.0219	0.2911
100	β_0	1.70	1.7342172	0.4184706	-0.0342172	0.0310	0.0241
	β_1	0.65	0.5848675	0.1115658	0.0651325	0.0147	0.8630
	α_1	-0.50	-0.4594400	0.1835352	-0.0405600	0.0342	0.0079
	Radius	0.01	0.0097677	0.0001941	0.0002323	0.0167	0.7143
200	β_0	1.70	1.7157622	0.3021399	-0.0157622	0.0202	0.4177
	β_1	0.65	0.5942236	0.0794240	0.0557764	0.0216	0.3088
	α_1	-0.50	-0.4581225	0.1226270	-0.0418775	0.0429	0.0002
	Radius	0.01	0.0097749	0.0001354	0.0002251	0.0171	0.6828
500	β_0	1.70	1.7104079	0.1743871	-0.0104079	0.0168	0.7083
	β_1	0.65	0.5933565	0.0505177	0.0566435	0.0213	0.3310
	α_1	-0.50	-0.4544700	0.0690476	-0.0455300	0.0263	0.0960
	Radius	0.01	0.0097740	0.0000900	0.0002260	0.0198	0.4497
1000	β_0	1.70	1.7122275	0.1236519	-0.0122275	0.0189	0.5257
	β_1	0.65	0.5942075	0.0350559	0.0557925	0.0217	0.3012
	α_1	-0.50	-0.4565386	0.0509232	-0.0434614	0.0234	0.2024
	Radius	0.01	0.0097715	0.0000614	0.0002285	0.0163	0.7462
2500	β_0	1.70	1.7042273	0.0827892	-0.0042273	0.0269	0.0842
	β_1	0.65	0.5947220	0.0224310	0.0552780	0.0216	0.3124
	α_1	-0.50	-0.4528117	0.0317134	-0.0471883	0.0164	0.7411
	Radius	0.01	0.0097739	0.0000386	0.0002261	0.0185	0.5550

Similar to Tables 6 and 7, the standard errors and biases of the estimates in table 10 decrease with increasing sample size (T), approaching approximately zero at $T = 2500$. Here also, the distribution of the estimates pass the normality test, thus establishing asymptotic normality.

Whilst the maximum likelihood estimates $\hat{\theta}$ improved with increasing sample size t, at $T = 2500$, the biases and standard errors decreased significantly, which confirmed the characteristics of maximum likelihood estimates. The distribution of the estimates also approach normal, as seen in the p-values of the Kolmogorov-Smirnov normality tests. Hence, with only past observations $\hat{n}_t$ in the model (55) and unknown but, fixed radius $R = 0.01$, we can accurately estimate the intensity parameter λ_t of the Boolean model.

Additionally, the results in Table 7 below for scheme 4 is comparable to that of scheme 8. Which is true for both the method of moments and maximum likelihood estimates. For both conditions, $\beta_1^2 + \alpha_1^2 < 1$ and $|\beta_1 + \alpha_1| < 1$, the Boolean model Y_t with grains Z_t with fixed, but unknown radius $R = 0.01$, the intensity of the $\hat{n}_t|\mathscr{F}_{t-1} \sim Poisson(\hat{\lambda}_t)$ can be estimated using model (55) accurately.

We observed that for research questions $1ii$, and $3i$ corresponding to (55), the result were similar to that of the Tables 10 and 11. Because we do not estimate R, the maximum likelihood estimation of Schemes 7 and 8 without the method of moment estimation of radius is the solution for research questions $1ii$, $3i$.

Table 11
Results from Scheme 8

Time	θ	Parameters	$\hat{\theta}$	Std.Error	Bias	Statistic	P value
10	β_0	5.50	5.4902651	0.4662198	0.0097349	0.1005	<0.001
	β_1	-0.35	-0.4754661	0.1565197	0.1254661	0.0946	<0.001
	α_1	-0.50	-0.2446484	0.2510843	-0.2553516	0.1095	<0.001
	Radius	0.01	0.0098270	0.0004538	0.0001730	0.0268	0.0867
50	β_0	5.50	5.4110923	0.3307174	0.0889077	0.1872	<0.001
	β_1	-0.35	-0.5134992	0.1001124	0.1634992	0.0789	<0.001
	α_1	-0.50	-0.2958641	0.1608498	-0.2041359	0.1076	<0.001
	Radius	0.01	0.0098785	0.0001742	0.0001215	0.0283	0.0574
100	β_0	5.50	5.4093050	0.3487685	0.0906950	0.2149	<0.001
	β_1	-0.35	-0.4637240	0.0710972	0.1137240	0.0265	0.0956
	α_1	-0.50	-0.3579586	0.1341100	-0.1420414	0.1100	<0.001
	Radius	0.01	0.0098904	0.0001193	0.0001096	0.0173	0.6599
200	β_0	5.50	5.4496695	0.1845474	0.0503305	0.1131	<0.001
	β_1	-0.35	-0.4226373	0.0573960	0.0726373	0.0232	0.2133
	α_1	-0.50	-0.4216338	0.0887348	-0.0783662	0.0361	0.0038
	Radius	0.01	0.0098885	0.0000850	0.0001115	0.0190	0.5149
500	β_0	5.50	5.4504262	0.1070550	0.0495738	0.0609	<0.001
	β_1	-0.35	-0.3804772	0.0406186	0.0304772	0.0207	0.3719
	α_1	-0.50	-0.4691927	0.0620927	-0.0308073	0.0313	0.0224
	Radius	0.01	0.0098906	0.0000519	0.0001094	0.0319	0.018
1000	β_0	5.50	5.4415728	0.0756987	0.0584272	0.0433	0.0001
	β_1	-0.35	-0.3584342	0.0283397	0.0084342	0.0169	0.6954
	α_1	-0.50	-0.4902872	0.0427330	-0.0097128	0.0228	0.2377
	Radius	0.01	0.0098897	0.0000352	0.0001103	0.0244	0.1587
2500	β_0	5.50	5.4418969	0.0516802	0.0581031	0.0284	0.0558
	β_1	-0.35	-0.3416311	0.0180800	-0.0083689	0.0301	0.0332
	α_1	-0.50	-0.5082133	0.0284640	0.0082133	0.0219	0.2934
	Radius	0.01	0.0098919	0.0000236	0.0001081	0.0238	0.1828

In the following tables, the result of how only time-dependent covariate affect the estimation of the intensity presented from schemes 9 and 12. Both random radius and, unknown but fixed radius results are presented below.

Table 12

Results from Scheme 9

Time	θ	Parameters	$\hat{\theta}$	Std.Error	Bias	Statistic	p value
10	β_0	-0.5	-1.2121520	9.0640439	0.7121520	0.3443	<0.001
	α_1	-0.5	-0.5096134	0.5488904	0.0096134	0.1858	<0.001
	η_1	0.5	-1.1136221	30.8403348	1.6136221	0.4077	<0.001
	a	0.0	-0.0366682	0.0179839	0.0366682	0.0951	<0.001
	b	0.1	0.1022382	0.0281244	-0.0022382	0.0479	<0.001
50	β_0	-0.5	-0.7395628	0.4905854	0.2395628	0.0518	<0.001
	α_1	-0.5	-0.6800055	0.4517623	0.1800055	0.2394	<0.001
	η_1	0.5	0.3055890	0.7791583	0.1944110	0.0685	<0.001
	a	0.0	-0.0423787	0.0068114	0.0423787	0.0599	<0.001
	b	0.1	0.1101738	0.0123652	-0.0101738	0.0326	0.014
100	β_0	-0.5	-0.6582016	0.3875375	0.1582016	0.0278	0.0656
	α_1	-0.5	-0.2359352	0.5242940	-0.2640648	0.0866	<0.001
	η_1	0.5	0.5475800	0.6149299	-0.0475800	0.0731	<0.001
	a	0.0	-0.0429870	0.0047452	0.0429870	0.0486	<0.001
	b	0.1	0.1103703	0.0087170	-0.0103703	0.0218	0.2977
200	β_0	-0.5	-0.7009021	0.2427061	0.2009021	0.0260	0.1027
	α_1	-0.5	-0.7234316	0.3703806	0.2234316	0.2276	<0.001
	η_1	0.5	0.3487078	0.4194353	0.1512922	0.1039	<0.001
	a	0.0	-0.0434029	0.0033630	0.0434029	0.0359	0.004
	b	0.1	0.1113126	0.0062622	-0.0113126	0.0172	0.6756
500	β_0	-0.5	-0.6739123	0.1833469	0.1739123	0.0529	<0.001
	α_1	-0.5	-0.3880041	0.4011278	-0.1119959	0.1052	<0.001
	η_1	0.5	0.5118461	0.2433074	-0.0118461	0.0256	0.1143
	a	0.0	-0.0435910	0.0020490	0.0435910	0.0263	0.0966
	b	0.1	0.1113053	0.0039265	-0.0113053	0.0252	0.1296
1000	β_0	-0.5	-0.6794796	0.1201047	0.1794796	0.0201	0.4255
	α_1	-0.5	-0.6028536	0.3280836	0.1028536	0.1130	<0.001
	η_1	0.5	0.3987543	0.2686625	0.1012457	0.0999	<0.001
	a	0.0	-0.0437452	0.0014621	0.0437452	0.0263	0.0957
	b	0.1	0.1114228	0.0028400	-0.0114228	0.0193	0.4906
2500	β_0	-0.5	-0.6887554	0.0790816	0.1887554	0.0437	1e-04
	α_1	-0.5	-0.4749012	0.1889841	-0.0250988	0.0767	<0.001
	η_1	0.5	0.4948373	0.1156905	0.0051627	0.0265	0.0961
	a	0.0	-0.0437410	0.0009323	0.0437410	0.0230	0.2242
	b	0.1	0.1114504	0.0018170	-0.0114504	0.0168	0.7038

Table 12 shows the results of simulations from scheme 9 for various times T. Again, the grains Z_t of these realizations are circles with random radius R from $Uniform\ (0, 0.1)$. The results were not different from Scheme 1 and 5 in Tables 3 and 8.

The standard errors of these estimates decreased with increasing sample size (T), with the distribution of estimates passing the normality test.

The maximum likelihood estimates $\hat{\theta}$ increased towards parameter values with increasing sample size t, similar to results in Table 3. Also, at $T = 2500$, the biases and standard errors decreased significantly, thus confirming the characteristics of maximum likelihood estimates. Finally, the distribution of the estimates also approached normal distribution, as seen in the p-values of the Kolmogorov-Smirnov normality tests. Thus, time-dependent covariate x_t in the model (56), does not change the estimation of the intensity parameter λ_t of the Boolean model.

Similar behavior and trends are seen in Table 13 below for scheme 10, for both the method of moments and maximum likelihood estimates. Thus, for both conditions, i.e: $\beta_1^2 + \alpha_1^2 < 1$ and $|\beta_1 + \alpha_1| < 1$, the Boolean model Y_t with grains Z_t of random radius from $Unif(0,0.1)$, with intensity $\hat{\lambda}_t$ can accurately be estimated using model (56).

Table 13

Results from Scheme 10

Time	θ	Parameters	$\hat{\theta}$	Std.Error	Bias	Statistic	P value
10	β_0	0.5	1.0727646	1.0477631	-0.5727646	0.0869	<0.001
	α_1	0.5	-0.0537654	0.5663579	0.5537654	0.0685	<0.001
	η_1	0.5	0.4809130	1.2800679	0.0190870	0.0663	<0.001
	a	0.0	-0.0405973	0.0085884	0.0405973	0.0568	<0.001
	b	0.1	0.1084605	0.0182022	-0.0084605	0.0306	0.0281
50	β_0	0.5	0.8350797	0.7808077	-0.3350797	0.1562	<0.001
	α_1	0.5	0.1817967	0.5227505	0.3182033	0.1241	<0.001
	η_1	0.5	0.4841498	0.4271517	0.0158502	0.0591	<0.001
	a	0.0	-0.0428560	0.0034706	0.0428560	0.0213	0.3293
	b	0.1	0.1110672	0.0079636	-0.0110672	0.0173	0.6594
100	β_0	0.5	0.9216591	0.8038451	-0.4216591	0.1739	<0.001
	α_1	0.5	0.1224047	0.5646708	0.3775953	0.1428	<0.001
	η_1	0.5	0.4538953	0.3076223	0.0461047	0.0506	<0.001
	a	0.0	-0.0434254	0.0023878	0.0434254	0.0258	0.1103
	b	0.1	0.1114786	0.0056105	-0.0114786	0.0273	0.0756
200	β_0	0.5	0.5255728	0.4040953	-0.0255728	0.1498	<0.001
	α_1	0.5	0.4166399	0.3084371	0.0833601	0.1287	<0.001
	η_1	0.5	0.4907473	0.1834603	0.0092527	0.0237	0.1885
	a	0.0	-0.0433873	0.0017090	0.0433873	0.0155	0.8137
	b	0.1	0.1111841	0.0041660	-0.0111841	0.0182	0.582
500	β_0	0.5	0.5181749	0.4128508	-0.0181749	0.1997	<0.001
	α_1	0.5	0.4249701	0.2949630	0.0750299	0.1643	<0.001
	η_1	0.5	0.4821159	0.1387734	0.0178841	0.0433	1e-04
	a	0.0	-0.0435789	0.0010611	0.0435789	0.0193	0.4908
	b	0.1	0.1112946	0.0025204	-0.0112946	0.0200	0.4313
1000	β_0	0.5	0.4370888	0.1501523	0.0629112	0.0604	<0.001
	α_1	0.5	0.4857096	0.1230046	0.0142904	0.0412	4e-04
	η_1	0.5	0.4845004	0.0821328	0.0154996	0.0246	0.152
	a	0.0	-0.0435825	0.0007169	0.0435825	0.0176	0.6387
	b	0.1	0.1113224	0.0017804	-0.0113224	0.0294	0.0409
2500	β_0	0.5	0.4250141	0.0914333	0.0749859	0.0587	<0.001
	α_1	0.5	0.4970918	0.0752941	0.0029082	0.0518	<0.001
	η_1	0.5	0.4797896	0.0512439	0.0202104	0.0199	0.4412
	a	0.0	-0.0435531	0.0004652	0.0435531	0.0244	0.1576
	b	0.1	0.1114625	0.0011671	-0.0114625	0.0248	0.1442

Table 14
Results from Scheme 11

Time	θ	Parameters	$\hat{\theta}$	Std.Error	Bias	Statistic	P value
10	β_0	5.50	4.0577566	1.9541283	1.4422434	0.1320	<0.001
	$\alpha 1$	-0.50	-0.1306595	0.6613829	-0.3693405	0.2054	<0.001
	$\eta 1$	0.50	0.6238600	1.6459018	-0.1238600	0.1565	<0.001
	Radius	0.01	0.0099352	0.0002419	0.0000648	0.0248	0.1426
50	β_0	5.50	5.5770392	1.0494701	-0.0770392	0.2374	<0.001
	$\alpha 1$	-0.50	-0.5398369	0.2916777	0.0398369	0.2959	<0.001
	$\eta 1$	0.50	0.6108057	0.3067851	-0.1108057	0.0820	<0.001
	Radius	0.01	0.0099347	0.0001054	0.0000653	0.0199	0.4419
100	β_0	5.50	5.6814662	0.3417724	-0.1814662	0.0258	0.1090
	$\alpha 1$	-0.50	-0.5682357	0.0763343	0.0682357	0.0194	0.4840
	$\eta 1$	0.50	0.5572583	0.1514100	-0.0572583	0.0373	0.0023
	Radius	0.01	0.0099354	0.0000732	0.0000646	0.0160	0.7698
200	β_0	5.50	5.5842694	0.2376955	-0.0842694	0.0170	0.6923
	$\alpha 1$	-0.50	-0.5416309	0.0569045	0.0416309	0.0204	0.4023
	$\eta 1$	0.50	0.5155057	0.0840732	-0.0155057	0.0264	0.0989
	Radius	0.01	0.0099338	0.0000542	0.0000662	0.0227	0.2417
500	β_0	5.50	5.4982054	0.1605428	0.0017946	0.0237	0.1897
	$\alpha 1$	-0.50	-0.5197265	0.0398378	0.0197265	0.0268	0.0874
	$\eta 1$	0.50	0.4989020	0.0419911	0.0010980	0.0186	0.5445
	Radius	0.01	0.0099307	0.0000346	0.0000693	0.0161	0.7618
1000	β_0	5.50	5.4613960	0.1166063	0.0386040	0.0188	0.5316
	$\alpha 1$	-0.50	-0.5099119	0.0293461	0.0099119	0.0168	0.7039
	$\eta 1$	0.50	0.4929193	0.0260312	0.0070807	0.0207	0.3743
	Radius	0.01	0.0099324	0.0000232	0.0000676	0.0143	0.8886
2500	β_0	5.50	5.4363288	0.0769723	0.0636712	0.0238	0.1855
	$\alpha 1$	-0.50	-0.5035972	0.0193122	0.0035972	0.0232	0.2147
	$\eta 1$	0.50	0.4902750	0.0157619	0.0097250	0.0350	0.0058
	Radius	0.01	0.0099312	0.0000145	0.0000688	0.0280	0.0614

Similar to Tables 6, 7, 10, and 11 the standard errors and biases of the estimates in table 10 decreased with increasing sample size (T), approaching approximately zero at $T = 2500$ with the distribution of the estimates passing the normality test. With increasing sample size t, the maximum likelihood estimates $\hat{\theta}$ improved. Also, at $T = 2500$, the biases and standard errors significantly decreased, which prove the characteristics of maximum likelihood estimates. In addition, the distribution of the estimates attained asymptotic normality. Which was evident in the p-values of the Kolmogorov-Smirnov normality tests. Hence, with only time-dependent covariate x_t in the model (56) and unknown but fixed radius $R = 0.01$, we can accurately estimate the intensity parameter λ_t of the Boolean model.

Similar results are seen in Table 15 below for scheme 12 comparable to scheme 11. That applies to both method of moments and maximum likelihood estimates. For both conditions, $\beta_1^2 + \alpha_1^2 < 1$ and $|\beta_1 + \alpha_1| < 1$, the Boolean model Y_t with a grains Z_t with fixed but unknown radius $R = 0.01$, the intensity of the $\hat{n}_t|\mathscr{F}_{t-1} \sim Poisson(\hat{\lambda}_t)$ can be estimated using model (55) accurately.

Observe that for research questions $1iii$, and $3i$ corresponding to (56), the result is same as that of the Tables 14 and 15—especially since we do not have to estimate R. Hence the maximum likelihood estimation of Schemes 11 and 12 without the method of moment estimation of radius is the solution for research questions $1iii$, $3i$.

Table 15

Results from Scheme 12

Time	θ	Parameters	$\hat{\theta}$	Std.Error	Bias	Statistic	P value
10	β_0	1.70	1.8582177	2.1010080	-0.1582177	0.3693	<0.001
	α_1	0.50	0.4241947	0.5124538	0.0758053	0.3220	<0.001
	η_1	0.50	0.5833665	0.6903803	-0.0833665	0.1814	<0.001
	Radius	0.01	0.0099085	0.0002788	0.0000915	0.0148	0.8594
50	β_0	1.70	0.8250173	0.2923081	0.8749827	0.1676	<0.001
	α_1	0.50	0.7185515	0.0788668	-0.2185515	0.1410	<0.001
	η_1	0.50	0.5119432	0.1069982	-0.0119432	0.0420	3e-04
	Radius	0.01	0.0099291	0.0001088	0.0000709	0.0152	0.8341
100	β_0	1.70	0.8955234	0.1684820	0.8044766	0.0463	<0.001
	α_1	0.50	0.7061393	0.0499808	-0.2061393	0.0456	<0.001
	η_1	0.50	0.4654575	0.0724253	0.0345425	0.0321	0.017
	Radius	0.01	0.0099316	0.0000760	0.0000684	0.0244	0.1592
200	β_0	1.70	1.0817690	0.1967864	0.6182310	0.0352	0.0054
	α_1	0.50	0.6579855	0.0561720	-0.1579855	0.0295	0.0392
	η_1	0.50	0.4616129	0.0567894	0.0383871	0.0167	0.7194
	Radius	0.01	0.0099306	0.0000535	0.0000694	0.0171	0.6785
500	β_0	1.70	1.3941794	0.1591340	0.3058206	0.0240	0.1737
	α_1	0.50	0.5745655	0.0439522	-0.0745655	0.0234	0.2047
	η_1	0.50	0.4788495	0.0338068	0.0211505	0.0158	0.7904
	Radius	0.01	0.0099325	0.0000330	0.0000675	0.0154	0.8181
1000	β_0	1.70	1.5345716	0.1177747	0.1654284	0.0235	0.1995
	α_1	0.50	0.5373571	0.0322013	-0.0373571	0.0206	0.3797
	η_1	0.50	0.4850831	0.0249818	0.0149169	0.0149	0.8497
	Radius	0.01	0.0099330	0.0000240	0.0000670	0.0219	0.2923
2500	β_0	1.70	1.6221029	0.0714763	0.0778971	0.0177	0.6310
	α_1	0.50	0.5142504	0.0194545	-0.0142504	0.0137	0.9211
	η_1	0.50	0.4886399	0.0147932	0.0113601	0.0204	0.3991
	Radius	0.01	0.0099341	0.0000144	0.0000659	0.0191	0.5021

The findings in the above results and analyses show that, indeed, we can build various models to estimate the intensity of time-dependent or correlated Boolean random sets Y_t. Moreover, the estimators for the models parameters, exhibit maximum likelihood characteristics. Finally, method of moments estimators for the radius estimations, are unbiased and asymptotically normal.

Theorem 2 states that given the conditions of the above simulations, the $\hat{\theta}$ is consistent and asymptotically normal. Tables 16, 17, and 18, show the asymptotic behavior of $\hat{\theta}_{MLEs}$ for all schemes. It is evident that these estimates are consistent. Thus, as $T \to \infty$, the biases approach zero., i.e.: $\hat{\theta}_{MLE} \to \theta$.

Table 16

Asymptotic Behavior of Estimates

Scheme	Time	θ	Parameters	$\hat{\theta}$	Std.Error	Bias	Statistic	P value
1	2500	β_0	-0.50	-0.5825164	0.0649762	0.0825164	0.0216	0.3087
		β_1	0.65	0.5565001	0.0402819	0.0934999	0.0143	0.8873
		α_1	-0.50	-0.4371118	0.0584869	-0.0628882	0.0307	0.0271
		η_1	0.50	0.4896683	0.0922974	0.0103317	0.0124	0.9685
		a	0.00	-0.0437374	0.0008245	0.0437374	0.0218	0.2991
		b	0.10	0.1114021	0.0016339	-0.0114021	0.0198	0.4425
2	2500	β_0	0.50	0.2579814	0.0517617	0.2420186	0.0203	0.4094
		β_1	-0.35	-0.3226764	0.0327150	-0.0273236	0.0189	0.5211
		α_1	-0.50	-0.5262106	0.0531311	0.0262106	0.0293	0.0425
		η_1	0.50	0.4869960	0.0780131	0.0130040	0.0207	0.3720
		a	0.00	-0.0437757	0.0007407	0.0437757	0.0127	0.9590
		b	0.10	0.1113730	0.0014925	-0.0113730	0.0175	0.6417
3	2500	β_0	1.70	1.7012990	0.0816827	-0.0012990	0.0197	0.4517
		β_1	0.65	0.5940552	0.0220534	0.0559448	0.0109	0.9947
		α_1	-0.50	-0.4501487	0.0296920	-0.0498513	0.0228	0.2353
		η_1	0.50	0.4944287	0.0300216	0.0055713	0.0262	0.0984
		Radius	0.01	0.0098243	0.0000327	0.0001757	0.0266	0.0928
4	2500	β_0	5.50	5.4417141	0.0347971	0.0582859	0.0142	0.8969
		β_1	-0.35	-0.3424395	0.0178288	-0.0075605	0.0114	0.9887
		α_1	-0.50	-0.5077921	0.0216477	0.0077921	0.0278	0.0664
		η_1	0.50	0.4923831	0.0203014	0.0076169	0.0221	0.2747
		Radius	0.01	0.0099005	0.0000212	0.0000995	0.0241	0.1710

Figures 3 and 4 show graphically, the approximate normality of the estimates for conditions found in schemes 1 and 3. When both past observations $\hat{n}$ and X_t are present in the model, the estimates are both consistent and asymptotically normal.

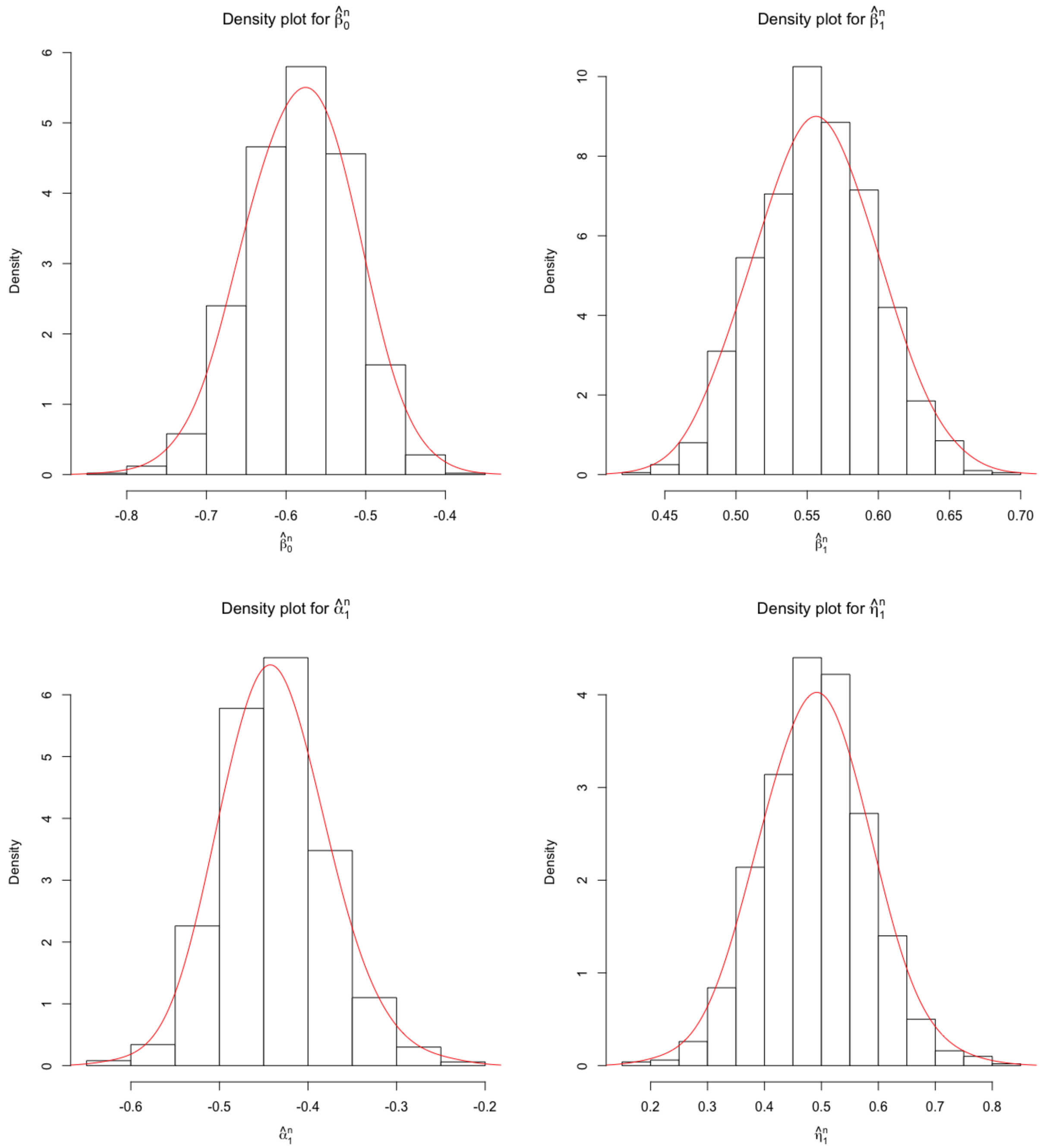

Figure 3. Normal Density Plots for Scheme 1, $T = 2500$.

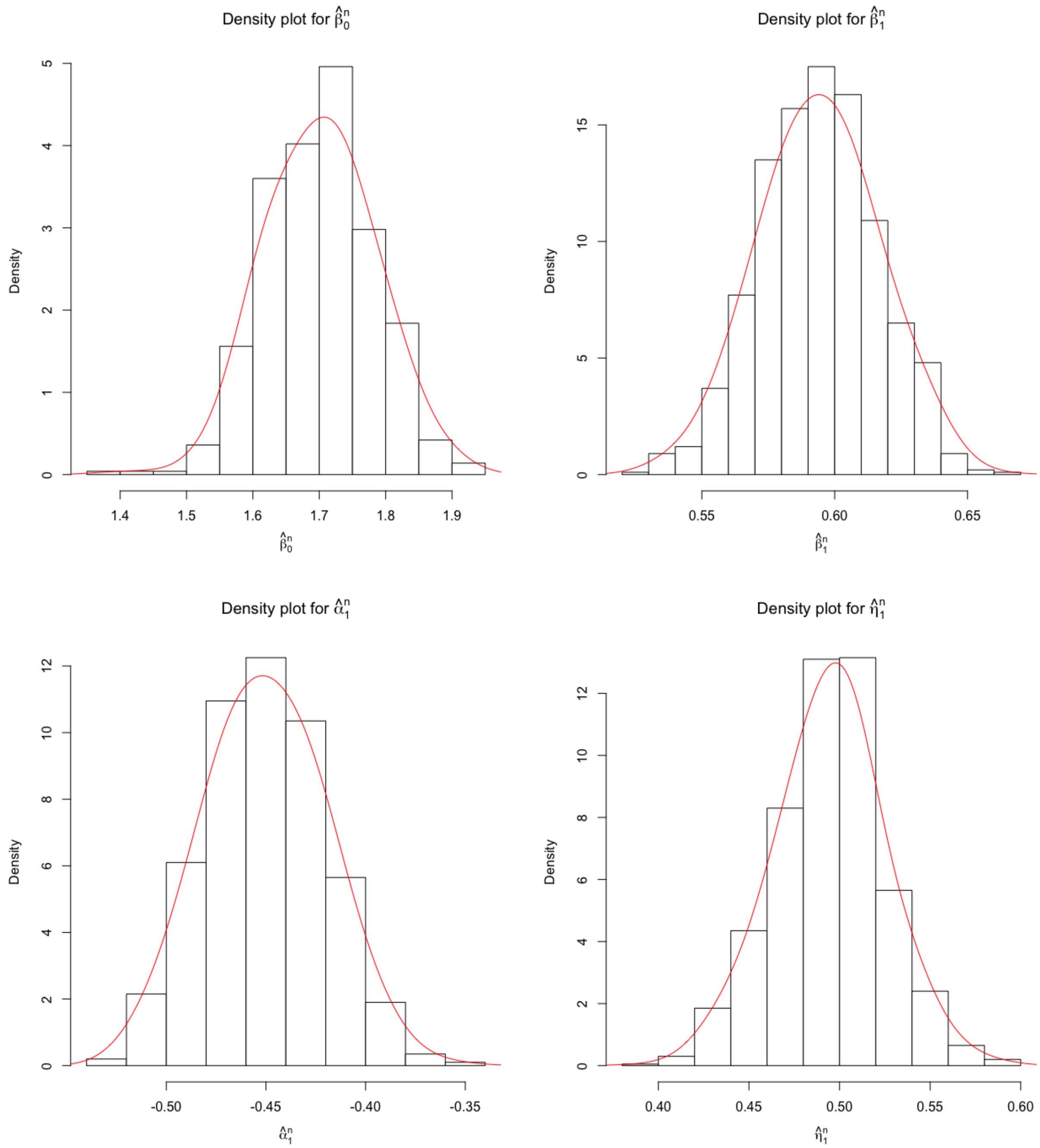

Figure 4. Normal Density Plots for Scheme 3, $T = 2500$.

Tables 17 and 18 show the asymptotic behavior of $\hat{\theta}_{MLEs}$ for schemes 6 to 8 and 9 to 12 respectively. The estimates are consistent in that as $T \to \infty$, the biases approach zero., i.e $\hat{\theta}_{MLE} \to \theta$.

Table 17

Asymptotic Behavior of Estimates

Scheme	Time	θ	Parameters	$\hat{\theta}$	Std.Error	Bias	Statistic	P value
5	2500	β_0	-0.50	-0.5988478	0.0486692	0.0988478	0.0329	0.0129
		β_1	0.65	0.5727167	0.0481620	0.0772833	0.0368	0.0028
		α_1	-0.50	-0.4495354	0.0690978	-0.0504646	0.0325	0.0146
		a	0.00	-0.0437907	0.0009416	0.0437907	0.0242	0.1647
		b	0.10	0.1114041	0.0017694	-0.0114041	0.0205	0.3943
6	2500	β_0	0.50	0.2570308	0.0334176	0.2429692	0.0167	0.7148
		β_1	-0.35	-0.3235941	0.0388293	-0.0264059	0.0238	0.1859
		α_1	-0.50	-0.5222265	0.0701911	0.0222265	0.0280	0.0615
		a	0.00	-0.0437945	0.0007711	0.0437945	0.0139	0.9125
		b	0.10	0.1113376	0.0016393	-0.0113376	0.0263	0.0957
7	2500	β_0	1.70	1.7042273	0.0827892	-0.0042273	0.0269	0.0842
		β_1	0.65	0.5947220	0.0224310	0.0552780	0.0216	0.3124
		α_1	-0.50	-0.4528117	0.0317134	-0.0471883	0.0164	0.7411
		Radius	0.01	0.0097739	0.0000386	0.0002261	0.0185	0.5550
8	2500	β_0	5.50	5.4418969	0.0516802	0.0581031	0.0284	0.0558
		β_1	-0.35	-0.3416311	0.0180800	-0.0083689	0.0301	0.0332
		α_1	-0.50	-0.5082133	0.0284640	0.0082133	0.0219	0.2934
		Radius	0.01	0.0098919	0.0000236	0.0001081	0.0238	0.1828

Figures 5 and 6 graphically show the approximate normality of the estimates, for conditions found in schemes 1 and 3. When past observations $\hat{n}$ are present in the model alone, the estimates are both consistent and asymptotically normal.

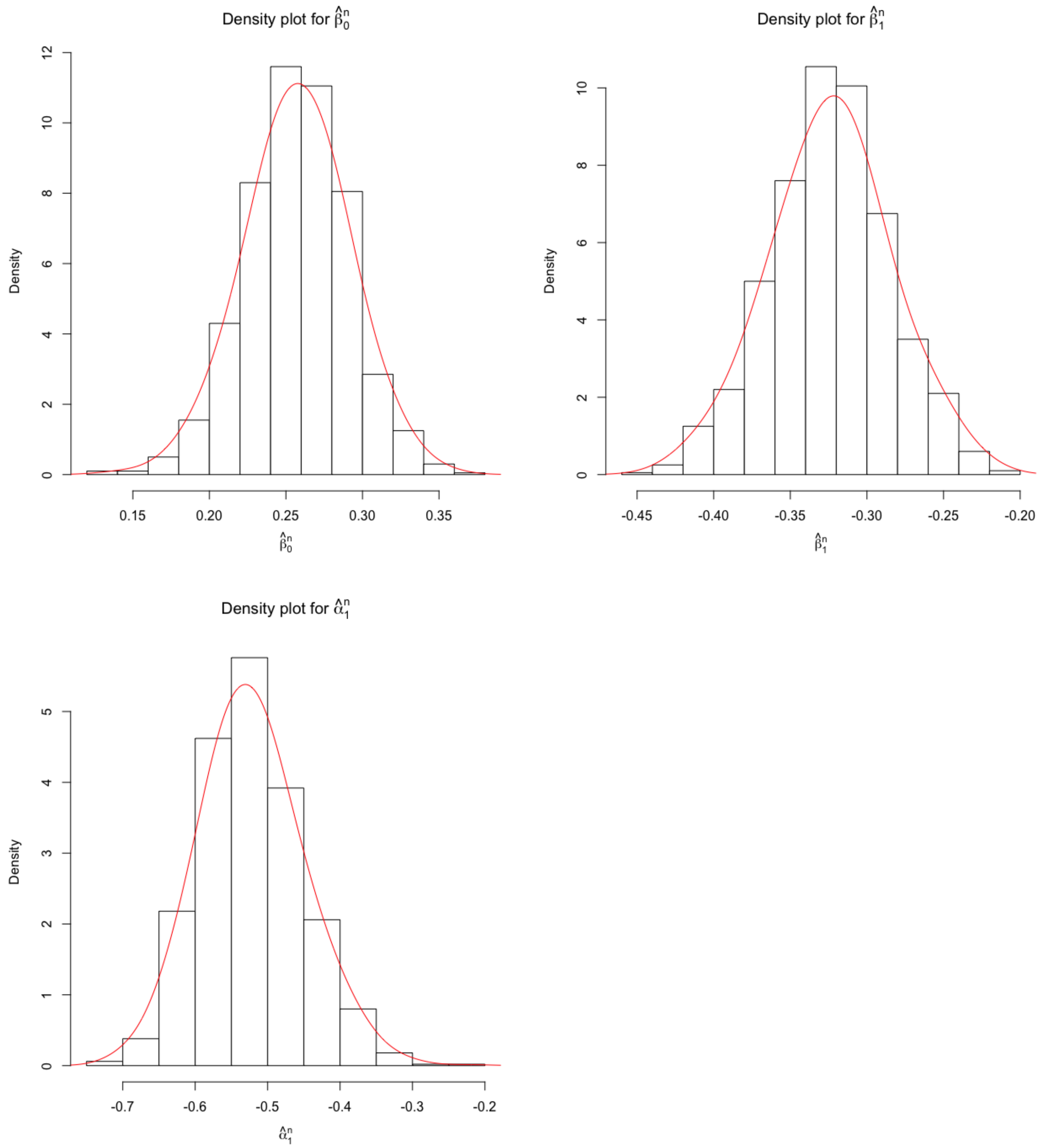

Figure 5. Normal Density Plots for Scheme 6, $T = 2500$.

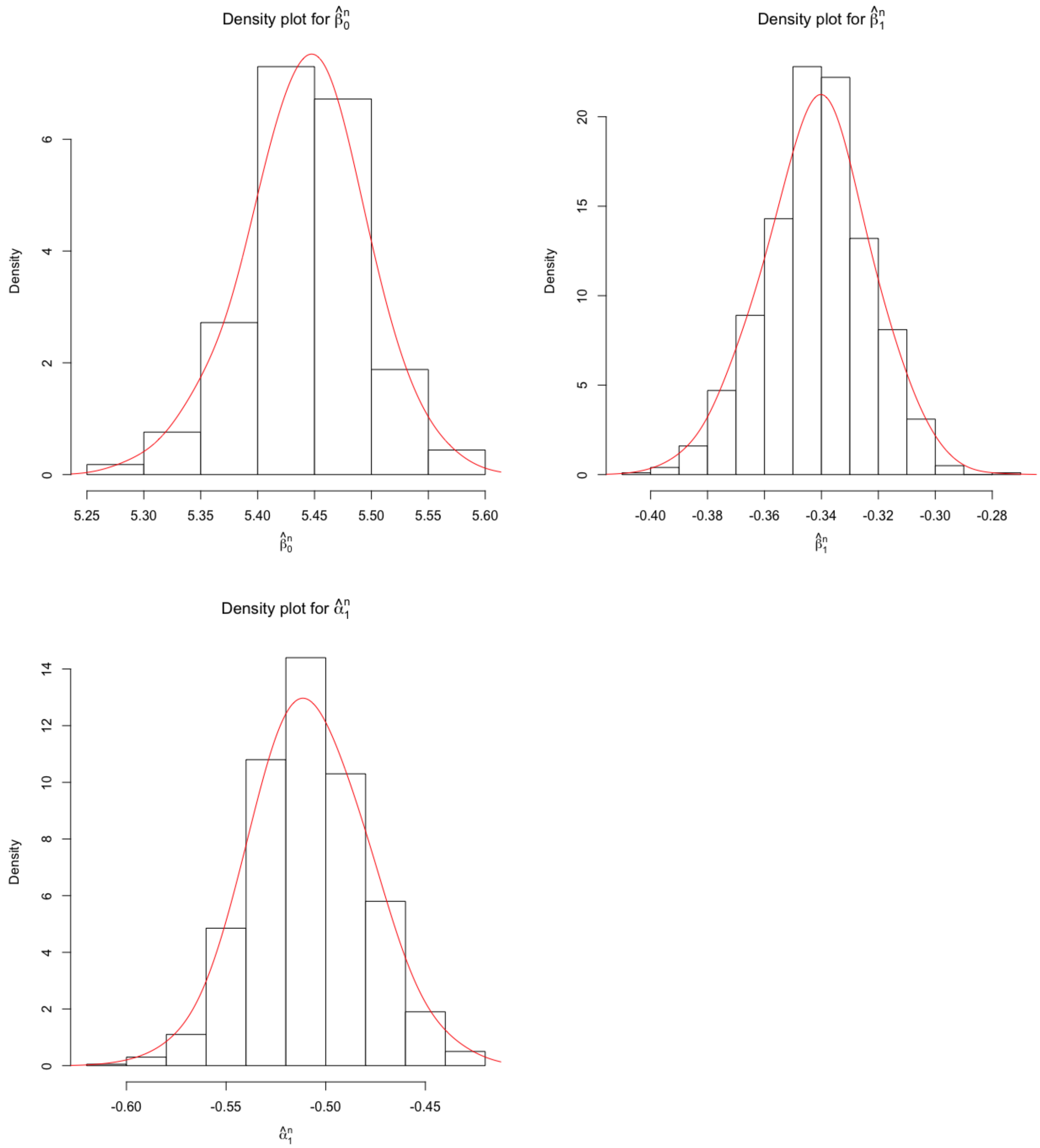

Figure 6. Normal Density Plots for Scheme 8, $T = 2500$.

Table 18 shows the asymptotic behavior of $\hat{\theta}_{MLEs}$ for schemes 9 to 12 respectively. We observe that estimates are consistent. Thus, as $T \to \infty$, the biases approach zero., i.e.: $\hat{\theta}_{MLE} \to \theta$.

Table 18

Asymptotic Behavior of Estimates.

Scheme	Time	θ	Parameters	$\hat{\theta}$	Std.Error	Bias	Statistic	P value
9	2500	β_0	-0.5	-0.6887554	0.0790816	0.1887554	0.0437	1e-04
		α_1	-0.5	-0.4749012	0.1889841	-0.0250988	0.0767	<0.001
		η_1	0.5	0.4948373	0.1156905	0.0051627	0.0265	0.0961
		a	0.0	-0.0437410	0.0009323	0.0437410	0.0230	0.2242
		b	0.1	0.1114504	0.0018170	-0.0114504	0.0168	0.7038
10	2500	β_0	0.5	0.4250141	0.0914333	0.0749859	0.0587	<0.001
		α_1	0.5	0.4970918	0.0752941	0.0029082	0.0518	<0.001
		η_1	0.5	0.4797896	0.0512439	0.0202104	0.0199	0.4412
		a	0.0	-0.0435531	0.0004652	0.0435531	0.0244	0.1576
		b	0.1	0.1114625	0.0011671	-0.0114625	0.0248	0.1442
11	2500	β_0	5.50	5.4363288	0.0769723	0.0636712	0.0238	0.1855
		$\alpha 1$	-0.50	-0.5035972	0.0193122	0.0035972	0.0232	0.2147
		$\eta 1$	0.50	0.4902750	0.0157619	0.0097250	0.0350	0.0058
		Radius	0.01	0.0099312	0.0000145	0.0000688	0.0280	0.0614
12	2500	β_0	1.70	1.6221029	0.0714763	0.0778971	0.0177	0.6310
		α_1	0.50	0.5142504	0.0194545	-0.0142504	0.0137	0.9211
		η_1	0.50	0.4886399	0.0147932	0.0113601	0.0204	0.3991
		Radius	0.01	0.0099341	0.0000144	0.0000659	0.0191	0.5021

Theorem 2 states the $\hat{\theta}_{MLE}$ attains multivariate normality. In the above tables, we showed that univariate normality tests were attained as $T \to \infty$ for individual estimates. Even though multivariate normality is difficult to show, due to the sensitivity to outliers, Table 19 shows that some multivariate normality are attained for schemes 1 and 3. Thus, with both past observations and time-dependent covariate in the model, estimates attain multivariate normality. And confirm the theorem.

Table 19

Asymptotic Multivariate Normality Test Using T=2500.

(a) Scheme 1				(b) Scheme 3			
Test	Statistic	P value	MVN	Test	Statistic	P value	MVN
E-statistic	1.15	0.10	YES	E-statistic	1.13	0.10	YES
Royston	9.60	0.02	NO	Royston	8.71	0.04	NO
Henze-Zirkler	0.88	0.68	YES	Henze-Zirkler	1.01	0.13	YES
Mardia Skewness	42.823	0.002	NO	Mardia Skewness	20.657	0.418	YES
Mardia Kurtosis	1.298	0.194	YES	Mardia Kurtosis	-0.843	0.400	YES
Mardia-MVN			NO	Mardia-MVN			YES

When only past observations were present in the model, estimates attained multivariate normality for fixed radius (scheme 8). Table 20 shows that multivariate normality are attained in some of the tests. Additionally, when the grains have random radius, the presence of outliers hinder the proof of multivariate normality. However, for the most part, univariate normality are attained.

Table 20
Asymptotic Multivariate Normality Test Using T=2500

(a) Scheme 6				(b) Scheme 8			
Test	Statistic	P value	MVN	Test	Statistic	P value	MVN
E-statistic	1.56	0.00	NO	E-statistic	1.10	0.05	NO
Royston	11.59	0.01	NO	Royston	4.96	0.05	YES
Henze-Zirkler	1.10	0.04	NO	Henze-Zirkler	1.14	0.02	NO
Mardia Skewness	69.060	0.00	NO	Mardia Skewness	21.283	0.019	NO
Mardia Kurtosis	3.191	0.001	NO	Mardia Kurtosis	-0.117	0.907	YES
Mardia-MVN			NO	Mardia-MVN			NO

Finally, with only time-dependent covariates in the model, estimates attained multivariate normality for fixed radius (scheme 12). Table 21 shows that multivariate normality are attained in some of the tests.

Table 21
Asymptotic Multivariate Normality Test Using T=2500

(a) Scheme 10				(b) Scheme 12			
Test	Statistic	P value	MVN	Test	Statistic	P value	MVN
E-statistic	3.62	0.00	NO	E-statistic	0.96	0.16	YES
Royston	29.41	0.00	NO	Royston	0.20	0.71	YES
Henze-Zirkler	2.00	0.00	NO	Henze-Zirkler	0.83	0.59	YES
Mardia Skewness	150.072	0.00	NO	Mardia Skewness	17.063	0.073	YES
Mardia Kurtosis	5.327	0.00	NO	Mardia Kurtosis	-0.273	0.785	YES
MVN			NO	MVN			YES

Results for Method II

According to Molchanov (1995), n_t^+ has an approximate Poisson distribution, i.e.: $n_t^+ \sim Poisson(\lambda_t^+), where\ \lambda_t^+ = |W_t|\lambda_t exp[-E|Z_0|\lambda_t]$. In this method, we modeled the conditional mean $E(n_t^+|\mathscr{F}_{t-1}) = \lambda_t^+$. Again, let $log\lambda_t^+ = \mu_t$, then

$$\mu_t = \beta_0^+ + \beta_1^+ log(n_{t-1}^+ + 1) + \alpha_1^+ \mu_{t-1} + \eta^+ X_t$$

However, to get the estimates of (54) from μ_t, we used the relationship $\lambda_t^+ = |W_t|\lambda_t exp[-E|Z_0|\lambda_t]$. Thus, by the first order Taylor expansion, approximate estimates of parameters in (54) through μ_t are as follows:

$$\beta_0 \approx \frac{\beta_0^+ + C}{1-C},\ \beta_1 \approx \frac{\beta_1^+}{1-C},\ \alpha_1 \approx \frac{\alpha_1^+}{1-C},\ \eta \approx \frac{\eta^+}{1-C},\ \text{where } C = E|Z_0| = \pi R^2.$$

Observe that, the parameter estimate of $\log\lambda_t$ is just an approximate constant multiple of $\log\lambda_t^+$. Thus, we can analyze the result for μ_t and extend the results naturally to $\log\lambda_t$. Hence, we present and discuss the result of method II below. Similar to method I, to answer research questions $1i$,and $3iii$ corresponding to (59), we again ran the schemes in Table 1 i.e.: schemes 1 to 2. The results are displayed in the tables below:

Table 22

Results from Scheme 1

Time	θ	Parameters	θ^+	Std.Error	Bias	Statistic	P value
10	β_0	-0.50	-0.4759523	1.7480788	-0.0240477	0.1247	<0.001
	β_1	0.65	0.1865244	0.7575065	0.4634756	0.1495	<0.001
	α_1	-0.50	-0.1240170	0.7449906	-0.3759830	0.1223	<0.001
	η_1	0.50	0.3520870	3.0398495	0.1479130	0.1173	<0.001
	a	0.00	-0.0370609	0.0168511	0.0370609	0.0998	<0.001
	b	0.10	0.1041074	0.0258972	-0.0041074	0.0451	1e-04
50	β_0	-0.50	-0.5844218	0.5114097	0.0844218	0.0168	0.706
	β_1	0.65	0.4824701	0.3569630	0.1675299	0.0832	<0.001
	α_1	-0.50	-0.3169425	0.5237804	-0.1830575	0.0961	<0.001
	η_1	0.50	0.4988620	0.8139187	0.0011380	0.0200	0.4296
	a	0.00	-0.0428164	0.0060557	0.0428164	0.0422	2e-04
	b	0.10	0.1096714	0.0117068	-0.0096714	0.0312	0.0228
100	β_0	-0.50	-0.5709625	0.3588177	0.0709625	0.0209	0.3607
	β_1	0.65	0.5366315	0.2168096	0.1133685	0.0181	0.5962
	α_1	-0.50	-0.5181557	0.3374225	0.0181557	0.0766	<0.001
	η_1	0.50	0.4545392	0.5142931	0.0454608	0.0194	0.4821
	a	0.00	-0.0430681	0.0043712	0.0430681	0.0432	2e-04
	b	0.10	0.1108408	0.0080706	-0.0108408	0.0169	0.700
200	β_0	-0.50	-0.5918920	0.2586015	0.0918920	0.0239	0.1794
	β_1	0.65	0.5486244	0.1460701	0.1013756	0.0141	0.8988
	α_1	-0.50	-0.4308836	0.2410083	-0.0691164	0.0671	<0.001
	η_1	0.50	0.4940830	0.3702445	0.0059170	0.0210	0.3545
	a	0.00	-0.0434383	0.0030744	0.0434383	0.0321	0.0169
	b	0.10	0.1110467	0.0057893	-0.0110467	0.0174	0.6528
500	β_0	-0.50	-0.5904171	0.1522223	0.0904171	0.0304	0.0300
	β_1	0.65	0.5549402	0.0943657	0.0950598	0.0168	0.7083
	α_1	-0.50	-0.4475017	0.1417100	-0.0524983	0.0276	0.0705
	η_1	0.50	0.4953259	0.2118498	0.0046741	0.0265	0.0958
	a	0.00	-0.0437036	0.0018408	0.0437036	0.0278	0.0663
	b	0.10	0.1113458	0.0036703	-0.0113458	0.0274	0.0742
1000	β_0	-0.50	-0.5832961	0.1051667	0.0832961	0.0198	0.4469
	β_1	0.65	0.5573317	0.0640943	0.0926683	0.0199	0.4381
	α_1	-0.50	-0.4360885	0.0940858	-0.0639115	0.0439	0.0001
	η_1	0.50	0.4866967	0.1501133	0.0133033	0.0133	0.9375
	a	0.00	-0.0437323	0.0013123	0.0437323	0.0178	0.6180
	b	0.10	0.1114453	0.0025959	-0.0114453	0.0174	0.6537
2500	β_0	-0.50	-0.5824615	0.0649542	0.0824615	0.0205	0.3884
	β_1	0.65	0.5563652	0.0402725	0.0936348	0.0134	0.9347
	α_1	-0.50	-0.4370555	0.0585051	-0.0629445	0.0300	0.0342
	η_1	0.50	0.4895029	0.0922204	0.0104971	0.0123	0.9714
	a	0.00	-0.0437374	0.0008245	0.0437374	0.0218	0.2991
	b	0.10	0.1114021	0.0016339	-0.0114021	0.0198	0.4425

Table 22 shows the result of simulations from scheme 1 for various times T. With the same setup as Method I, the results are similar to using n_t^+. The standard errors of these estimates decreased with increasing sample size (T), with the distribution of estimates passing the normality test. The maximum likelihood estimates $\hat{\theta}$ improved to parameter values with increasing sample size t. At $T = 2500$, the biases and standard errors became significantly small and confirmed the characteristics of maximum likelihood estimates. The distribution of the estimates also approach normal, as seen in the p-values of the Kolmogorov-Smirnov normality tests. Thus, with both past observations n_t^+ and time-dependent covariates x_t in the model (59), we can accurately estimate the intensity parameter λ_t of the Boolean model. Similar results are seen in Table 23 below for scheme 2 for both the method of moments and maximum likelihood estimates. Now, in order to find the estimates $\hat{\Theta}$ of Θ from Θ^+, we find $C = \pi R^2$ using the expected value of $R \sim U(a,b)$ for each T.

Let's take $T = 2500$ as an example, then C will be the expected value of $R \sim Unif\ (-0.04, 0.11)$ from Table 22, i.e. $C = \pi(0.0338)^2 = 0.00360$. Thus the

$$\hat{\beta}_0 \approx \frac{-0.5825 + 0.00360}{1 - 0.00360} = -0.581,\ \hat{\beta}_1 \approx \frac{0.5564}{1 - 0.00360} = 0.558,$$

$$\hat{\alpha}_1 \approx \frac{-0.4371}{1 - 0.00360} = -0.439,\ \hat{\eta} \approx \frac{0.4895}{1 - 0.00360} = 0.491.$$

Table 23

Results from Scheme 2

Time	θ	Parameters	θ^+	Std.Error	Bias	Statistic	P value
10	β_0	0.50	0.06	1.28	0.44	0.0622	<0.001
	β_1	-0.35	-0.29	0.69	-0.06	0.1533	<0.001
	α_1	-0.50	-0.06	0.74	-0.44	0.1085	<0.001
	η_1	0.50	0.35	2.25	0.15	0.0570	<0.001
	a	0.00	-0.04	0.01	0.04	0.0843	<0.001
	b	0.10	0.11	0.02	-0.01	0.0296	0.0392
50	β_0	0.50	0.17	0.45	0.33	0.0269	0.0856
	β_1	-0.35	-0.33	0.30	-0.02	0.0155	0.8106
	α_1	-0.50	-0.28	0.54	-0.22	0.0997	<0.001
	η_1	0.50	0.50	0.72	0.00	0.0458	<0.001
	a	0.00	-0.04	0.01	0.04	0.0528	<0.001
	b	0.10	0.11	0.01	-0.01	0.0235	0.1986
100	β_0	0.50	0.20	0.29	0.30	0.0347	0.0064
	β_1	-0.35	-0.31	0.20	-0.04	0.0367	0.0029
	α_1	-0.50	-0.42	0.41	-0.08	0.1178	<0.001
	η_1	0.50	0.50	0.46	0.00	0.0197	0.4567
	a	0.00	-0.04	0.00	0.04	0.0318	0.0184
	b	0.10	0.11	0.01	-0.01	0.0285	0.0533
200	β_0	0.50	0.23	0.20	0.27	0.0208	0.3663
	β_1	-0.35	-0.31	0.13	-0.04	0.0194	0.483
	α_1	-0.50	-0.45	0.29	-0.05	0.1136	<0.001
	η_1	0.50	0.50	0.30	0.00	0.0245	0.1541
	a	0.00	-0.04	0.00	0.04	0.0463	<0.001
	b	0.10	0.11	0.01	-0.01	0.0130	0.9515
500	β_0	0.50	0.25	0.12	0.25	0.0281	0.0601
	β_1	-0.35	-0.32	0.08	-0.03	0.0259	0.1058
	α_1	-0.50	-0.50	0.15	0.00	0.0753	<0.001
	η_1	0.50	0.49	0.18	0.01	0.0206	0.3795
	a	0.00	-0.04	0.00	0.04	0.0391	0.001
	b	0.10	0.11	0.00	-0.01	0.0158	0.7856
1000	β_0	0.50	0.25	0.09	0.25	0.0224	0.2599
	β_1	-0.35	-0.32	0.06	-0.03	0.0158	0.7858
	α_1	-0.50	-0.52	0.09	0.02	0.0457	<0.001
	η_1	0.50	0.49	0.13	0.01	0.0228	0.2364
	a	0.00	-0.04	0.00	0.04	0.0221	0.276
	b	0.10	0.11	0.00	-0.01	0.0115	0.9874
2500	β_0	0.50	0.26	0.05	0.24	0.0193	0.4881
	β_1	-0.35	-0.32	0.03	-0.03	0.0183	0.5714
	α_1	-0.50	-0.53	0.05	0.03	0.0282	0.0591
	η_1	0.50	0.49	0.08	0.01	0.0183	0.5724
	a	0.00	-0.04	0.00	0.04	0.0127	0.9590
	b	0.10	0.11	0.00	-0.01	0.0175	0.6417

For research questions 1i, 3ii corresponding to (59), the schemes in Table 1, i.e.: schemes 3 and 4, where the grain Z_t have a fixed but, unknown radius R were ran for each T.

Table 24
Results from Scheme 3

Time	θ	Parameters	θ^+	Std.Error	Bias	Statistic	P value
10	β_0	1.70	1.9942575	0.8457011	-0.2942575	0.0524	<0.001
	β_1	0.65	0.2993197	0.6207261	0.3506803	0.1318	<0.001
	α_1	-0.50	-0.2690104	0.7307089	-0.2309896	0.1586	<0.001
	η_1	0.50	0.4436066	0.7493935	0.0563934	0.0595	<0.001
	Radius	0.01	0.0098352	0.0005410	0.0001648	0.0336	0.0098
50	β_0	1.70	1.7871003	0.6059542	-0.0871003	0.0172	0.669
	β_1	0.65	0.5769652	0.1582886	0.0730348	0.0285	0.0539
	α_1	-0.50	-0.4683012	0.2589570	-0.0316988	0.0593	<0.001
	η_1	0.50	0.4846358	0.2246510	0.0153642	0.0211	0.3466
	Radius	0.01	0.0098288	0.0002331	0.0001712	0.0244	0.1578
100	β_0	1.70	1.7571994	0.4203642	-0.0571994	0.0313	0.0223
	β_1	0.65	0.5864859	0.1073966	0.0635141	0.0279	0.0641
	α_1	-0.50	-0.4675856	0.1614900	-0.0324144	0.0360	0.0038
	η_1	0.50	0.4946568	0.1597091	0.0053432	0.0166	0.7197
	Radius	0.01	0.0098226	0.0001641	0.0001774	0.0149	0.8540
200	β_0	1.70	1.7359856	0.3038111	-0.0359856	0.0258	0.1082
	β_1	0.65	0.5878822	0.0770359	0.0621178	0.0210	0.3557
	α_1	-0.50	-0.4586346	0.1117631	-0.0413654	0.0252	0.1279
	η_1	0.50	0.4921712	0.1048772	0.0078288	0.0171	0.6811
	Radius	0.01	0.0098225	0.0001223	0.0001775	0.0133	0.9387
500	β_0	1.70	1.7104920	0.1914353	-0.0104920	0.0184	0.5665
	β_1	0.65	0.5927031	0.0468776	0.0572969	0.0176	0.6385
	α_1	-0.50	-0.4527485	0.0686246	-0.0472515	0.0205	0.3893
	η_1	0.50	0.4932953	0.0708219	0.0067047	0.0166	0.7204
	Radius	0.01	0.0098234	0.0000761	0.0001766	0.0251	0.1311
1000	β_0	1.70	1.7040678	0.1325939	-0.0040678	0.0190	0.5125
	β_1	0.65	0.5906505	0.0334899	0.0593495	0.0230	0.2251
	α_1	-0.50	-0.4473722	0.0471905	-0.0526278	0.0224	0.2594
	η_1	0.50	0.4929810	0.0467460	0.0070190	0.0198	0.4463
	Radius	0.01	0.0098257	0.0000528	0.0001743	0.0190	0.5123
2500	β_0	1.70	1.7012990	0.0816827	-0.0012990	0.0197	0.4517
	β_1	0.65	0.5940552	0.0220534	0.0559448	0.0109	0.9947
	α_1	-0.50	-0.4501487	0.0296920	-0.0498513	0.0228	0.2353
	η_1	0.50	0.4944287	0.0300216	0.0055713	0.0262	0.0984
	Radius	0.01	0.0098243	0.0000327	0.0001757	0.0266	0.0928

Table 24 shows the results of simulations from scheme 3 for various times T. The grains Z_t of these realizations have fixed but unknown radius $R = 0.01$. Similar to method I, the method of moments estimates of $\hat{R}$ were stable across T. The standard errors and biases of the estimates decreased with increasing sample size (T). They approached approximately zero at $Time = 2500$ with the distribution of the estimates exhibiting asymptotic normality.

The maximum likelihood estimates $\hat{\theta}$ improved towards parameter values with increasing sample size t. The biases and standard errors decreased significantly with increasing T, thus confirming the characteristics of maximum likelihood estimates. The distribution of the estimates also approached normal, which was evident in the p-values of the Kolmogorov-Smirnov normality tests. Hence, with both past observations $\hat{n}_t^+$ and time-dependent covariates x_t in the model (59) and unknown but fixed radius $R = 0.01$, we can accurately estimate the intensity parameter λ_t of the Boolean model. Similar results are seen in Table 25 below for scheme 4 for both the method of moments and maximum likelihood estimates.

Note that for the research questions $1i$, $3i$ corresponding to (59), the results were the similar to that of the Tables 24 and 25. Since we do not estimate R, the maximum likelihood estimation of schemes 3 and 4 without the method of moments estimation of radius, is the solution for research questions $1i$, $3i$.

Table 25

Results from Scheme 4

Time	θ	Parameters	$\hat{\theta}$	Std.Error	Bias	Statistic	P value
10	β_0	5.50	4.2523439	1.1205878	1.2476561	0.0267	0.0906
	β_1	-0.35	-0.4986284	0.3370451	0.1486284	0.0838	<0.001
	α_1	-0.50	0.1484715	0.6508758	-0.6484715	0.0954	<0.001
	η_1	0.50	0.2002153	2.0924682	0.2997847	0.0621	<0.001
	Radius	0.01	0.0098649	0.0004275	0.0001351	0.0208	0.3692
50	β_0	5.50	5.2786373	0.5942747	0.2213627	0.2025	<0.001
	β_1	-0.35	-0.5260311	0.1328037	0.1760311	0.0743	<0.001
	α_1	-0.50	-0.2754984	0.2401429	-0.2245016	0.1990	<0.001
	η_1	0.50	0.6652643	0.4448693	-0.1652643	0.0302	0.0319
	Radius	0.01	0.0098953	0.0001641	0.0001047	0.0178	0.6225
100	β_0	5.50	5.4439079	0.2267121	0.0560921	0.0868	<0.001
	β_1	-0.35	-0.4816842	0.0831125	0.1316842	0.0567	<0.001
	α_1	-0.50	-0.3711715	0.1125614	-0.1288285	0.0657	<0.001
	η_1	0.50	0.5907560	0.2157583	-0.0907560	0.0224	0.2597
	Radius	0.01	0.0099039	0.0001071	0.0000961	0.0226	0.2449
200	β_0	5.50	5.4536964	0.1318726	0.0463036	0.0333	0.0111
	β_1	-0.35	-0.4313700	0.0601840	0.0813700	0.0361	0.0038
	α_1	-0.50	-0.4243167	0.0772416	-0.0756833	0.0626	<0.001
	η_1	0.50	0.5421889	0.1212027	-0.0421889	0.0297	0.0373
	Radius	0.01	0.0098978	0.0000780	0.0001022	0.0162	0.7536
500	β_0	5.50	5.4390388	0.0766778	0.0609612	0.0229	0.2280
	β_1	-0.35	-0.3847900	0.0391853	0.0347900	0.0187	0.5440
	α_1	-0.50	-0.4653583	0.0463613	-0.0346417	0.0262	0.0982
	η_1	0.50	0.5125823	0.0578612	-0.0125823	0.0351	0.0056
	Radius	0.01	0.0099010	0.0000485	0.0000990	0.0264	0.0977
1000	β_0	5.50	5.4403944	0.0558459	0.0596056	0.0277	0.0684
	β_1	-0.35	-0.3606201	0.0277894	0.0106201	0.0135	0.9284
	α_1	-0.50	-0.4900243	0.0334542	-0.0099757	0.0178	0.6199
	η_1	0.50	0.5015649	0.0368154	-0.0015649	0.0200	0.4327
	Radius	0.01	0.0098972	0.0000345	0.0001028	0.0275	0.0723
2500	β_0	5.50	5.4417443	0.0350822	0.0582557	0.0234	0.2060
	β_1	-0.35	-0.3418183	0.0175787	-0.0081817	0.0133	0.9385
	α_1	-0.50	-0.5084093	0.0214671	0.0084093	0.0228	0.2334
	η_1	0.50	0.4920419	0.0201854	0.0079581	0.0215	0.3152
	Radius	0.01	0.0099005	0.0000212	0.0000995	0.0241	0.1710

Again, to find the estimates $\hat{\Theta}$ of Θ from Θ^+, we find $C = \pi R^2$ using the expected value of $\hat{R}$ as the estimate for R.

When $T = 1000$ is used an example, the expected value of $\hat{R} = 0.0098972$ from Table 25, and $C = \pi(0.0098972)^2 = 0.0003077334$. Thus, the

$$\hat{\beta}_0 \approx \frac{5.440 + 0.00031}{1 - 0.00031} = 5.4424, \; \hat{\beta}_1 \approx \frac{-0.361}{1 - 0.00031} = -0.3607,$$

$$\hat{\alpha}_1 \approx \frac{-0.4900}{1 - 0.00031} = -0.4902, \; \hat{\eta} \approx \frac{0.5016}{1 - 0.00031} = 0.5017.$$

For all T's, $\hat{\Theta}$ can be found using the procedure in the above example.

The results of the simulations from schemes 5 to 12 for method II have similar discussions as that of method I for various times T. The standard errors of these estimates decreased with increasing sample size (T), with the distribution of estimates passing the normality test.

The maximum likelihood estimates θ^+ improved to parameter value with increasing sample size t, which is similar to results from method I in Table 8 to 15. At $T = 2500$, the biases and standard errors decreased significantly to zero, thus confirming the characteristics of maximum likelihood estimates. The distributions of these estimates also approached normal, which is seen in the p-values of the Kolmogorov-Smirnov normality tests across all schemes. Thus, with past observations n_t^+ in the model (55), we can accurately estimate the intensity parameter λ_t of the Boolean model. And with only time-dependent covariate x_t in the model (56), it does not change the estimation of the intensity parameter λ_t of the Boolean model.

We do observe that both methods I and II are functions of n^+. Hence, similar behavior, trends, and results were expected under the same schemes. To avoid repetition, Tables 31 to 38 are included in Appendix *A*.

Figures 7 and 8 show the approximate normality of the estimates, for conditions found in scheme 2 and 4. When both past observations n_t^+ and X_t were present in the model, the estimates were both consistent and asymptotically normal.

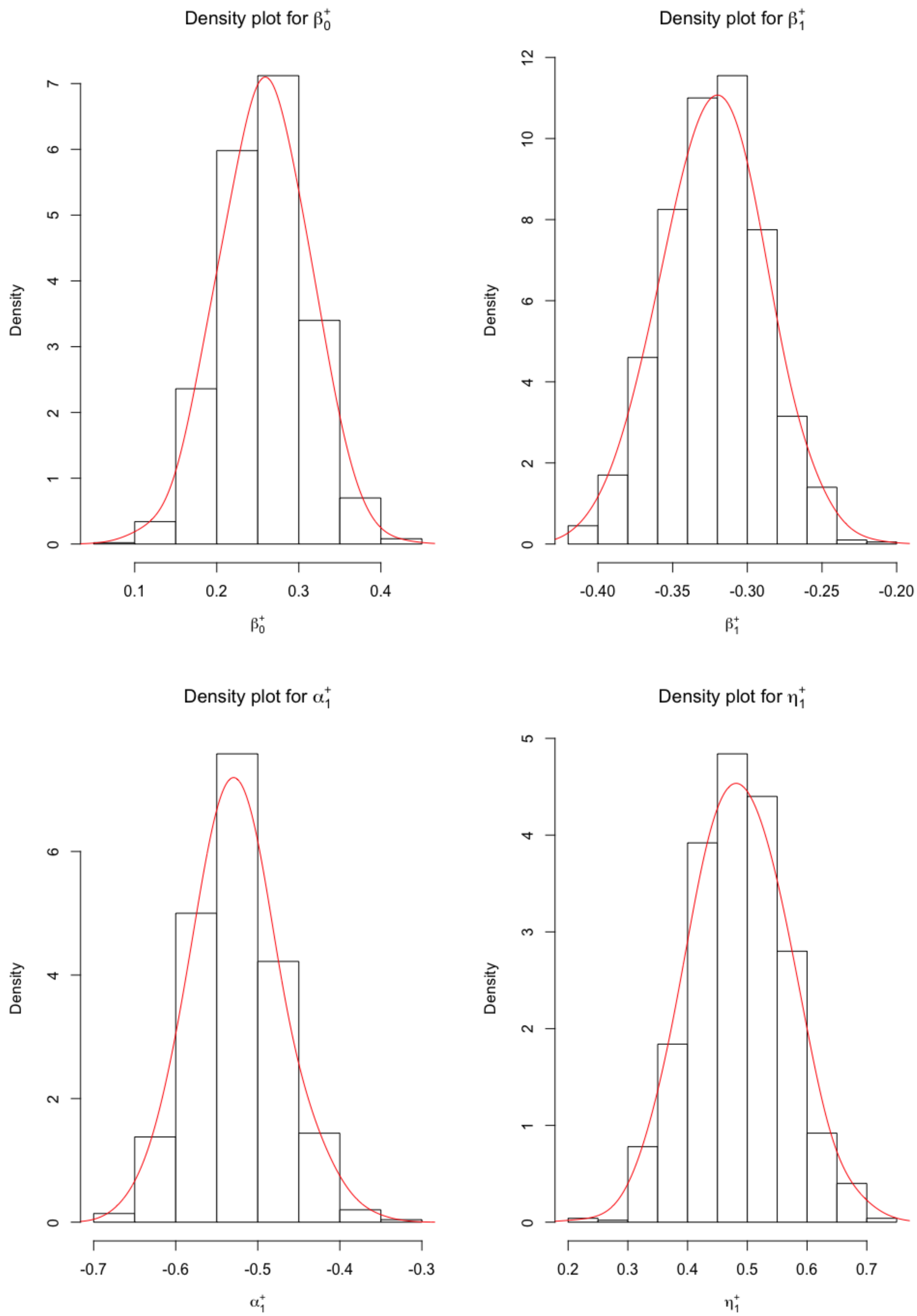

Figure 7. Normal Density Plots for Scheme 2, $T = 2500$.

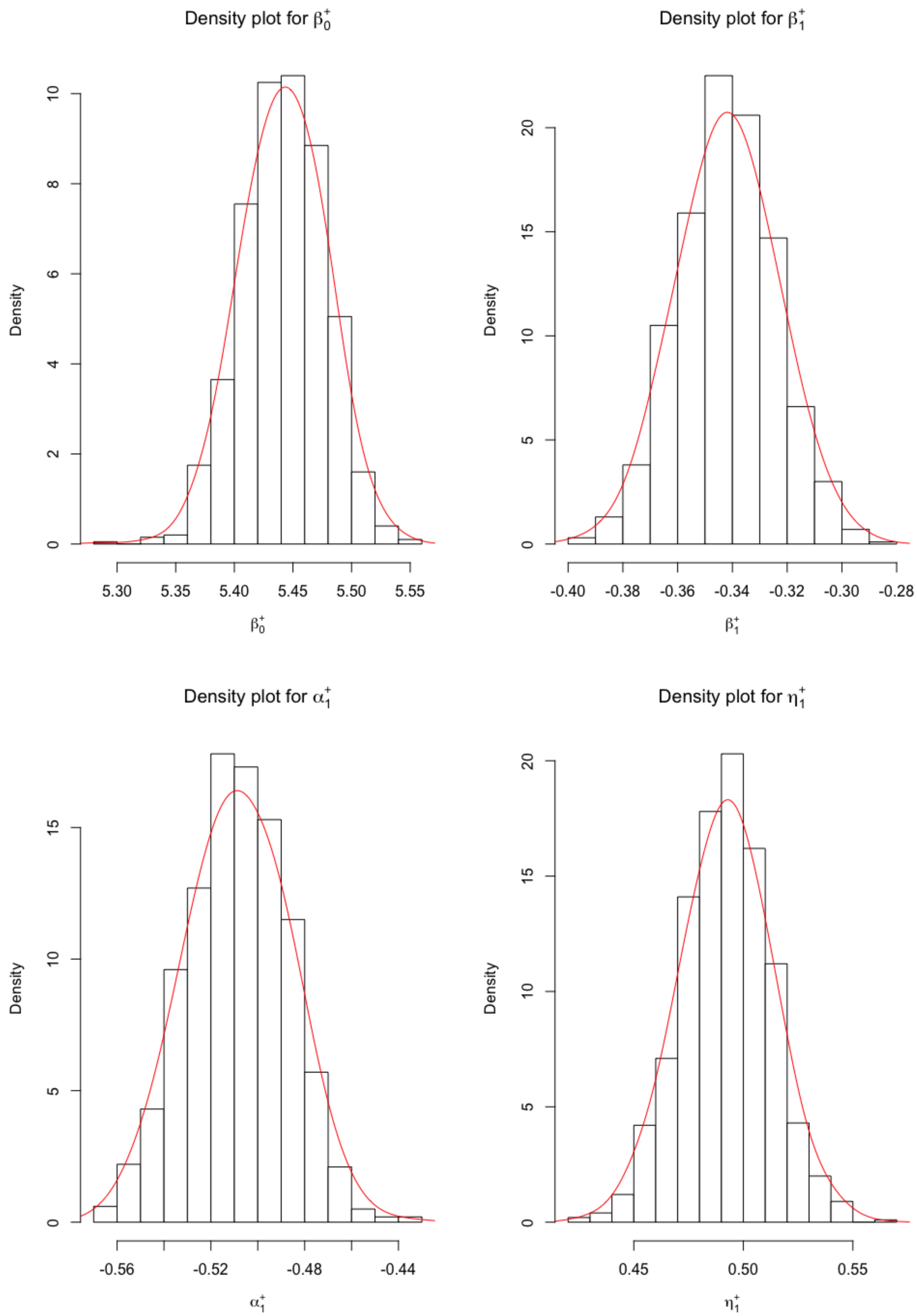

Figure 8. Normal Density Plots for Scheme 4, $T = 2500$.

Even though multivariate normality is difficult to show due to the presence of outliers, Table 26 shows that some multivariate normality tests are attained for both when the Boolean model is formed by grains Z_t with random radius (scheme 2) and fixed and unknown radius (scheme 4). Thus, with both past observations and time-dependent covariate in the model, estimates attain multivariate normality.

Table 26

Asymptotic Multivariate Normality Test Using T=2500

(a) Scheme 2				(b) Scheme 4			
Test	Statistic	P value	MVN	Test	Statistic	P value	MVN
E-statistic	1.575	0	NO	E-statistic	1.098	0.165	YES
Royston	4.882	0.154	YES	Royston	2.766	0.314	YES
Henze-Zirkler	1.131	0.006	NO	Henze-Zirkler	0.919	0.4814571	YES
Mardia Skewness	52.483	9.680e-05	NO	Mardia Skewness	23.240	0.277	YES
Mardia Kurtosis	0.303	0.762	YES	Mardia Kurtosis	-0.197	0.843	YES
MVN			NO	MVN			YES

Figures 9 and 10 shows the approximate normality of the estimates for conditions found in scheme 5 and 7. When both past observations n_t^+ and X_t were present in the model, the estimates were both consistent and asymptotically normal.

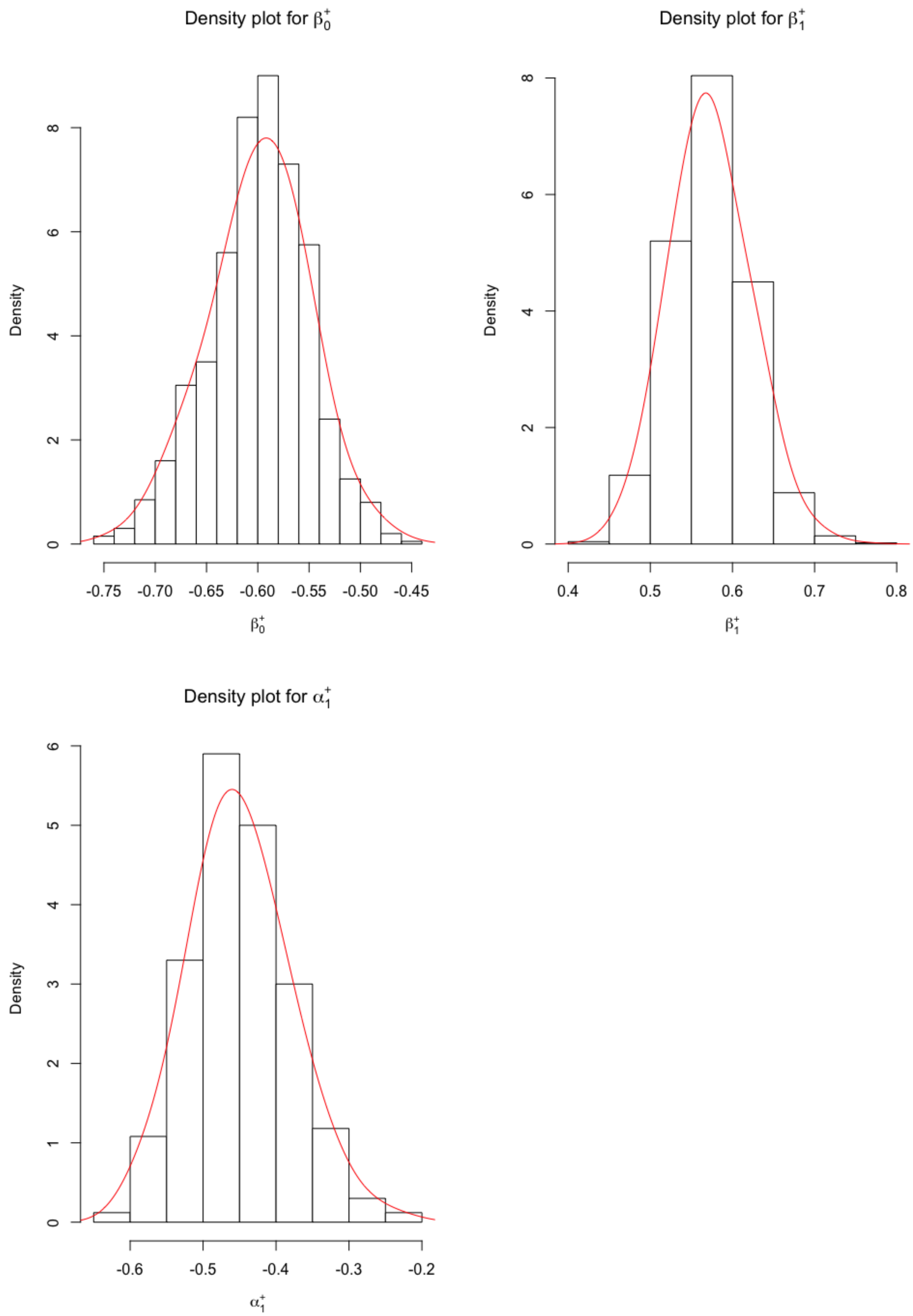

Figure 9. Normal Density Plots for Scheme 5, $T = 2500$.

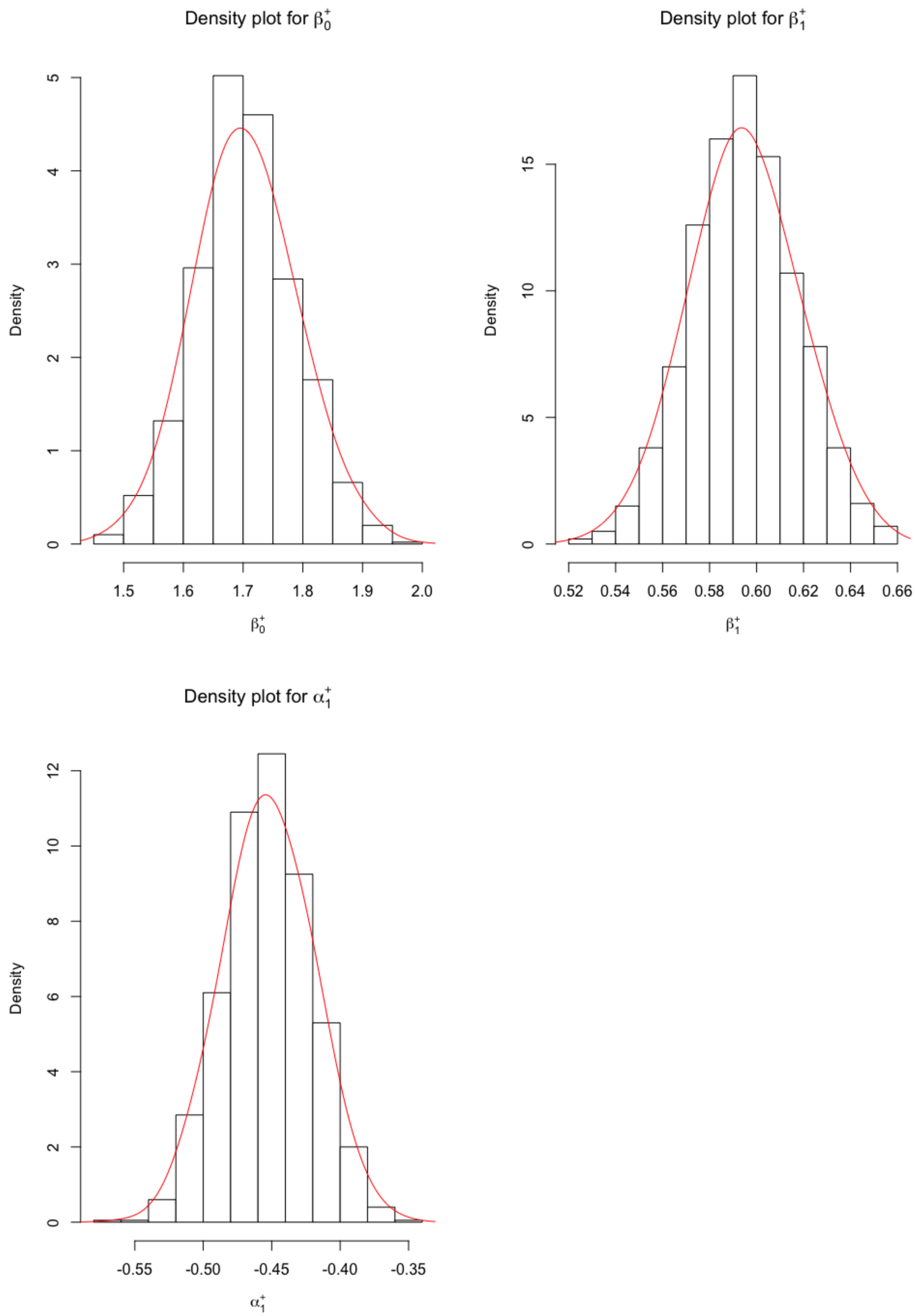

Figure 10. Normal Density Plots for Scheme 7, $T = 2500$.

Table 27 shows that some multivariate normality tests are attained for both when the Boolean model is formed by grains Z_t with random radius (Scheme 5) and fixed but unknown radius (Scheme 7).

Table 27
Asymptotic Multivariate Normality Test Using T=2500

(a) Scheme 5				(b) Scheme 7			
Test	Statistic	P value	MVN	Test	Statistic	p value	MVN
E-statistic	1.774	0	NO	E-statistic	0.803	0.582	YES
Royston	24.035	1.642e-05	NO	Royston	24.035	1.642e-05	NO
Henze-Zirkler	1.337	0.001	NO	Henze-Zirkler	0.787	0.729	YES
Mardia Skewness	57.383	1.129e-08	NO	Mardia Skewness	10.081	0.433	YES
Mardia Kurtosis	-0.134	0.893	YES	Mardia Kurtosis	0.483	0.629	YES
MVN			NO	MVN			YES

Figures 11 and 12 shows the approximate normality of the estimates, for conditions found in scheme 1 and 3. When only past observations n_t^+ was present in the model, the estimates were both consistent and asymptotically normal.

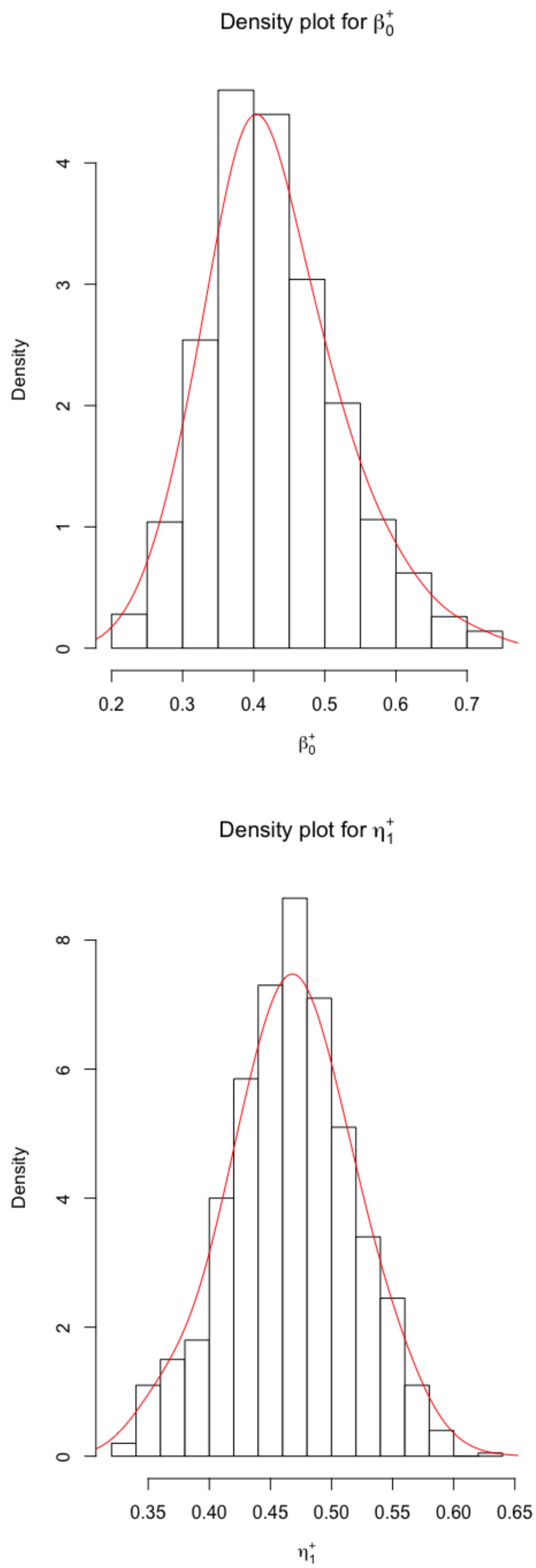

Figure 11. Normal Density Plots for Scheme 10, $T = 2500$.

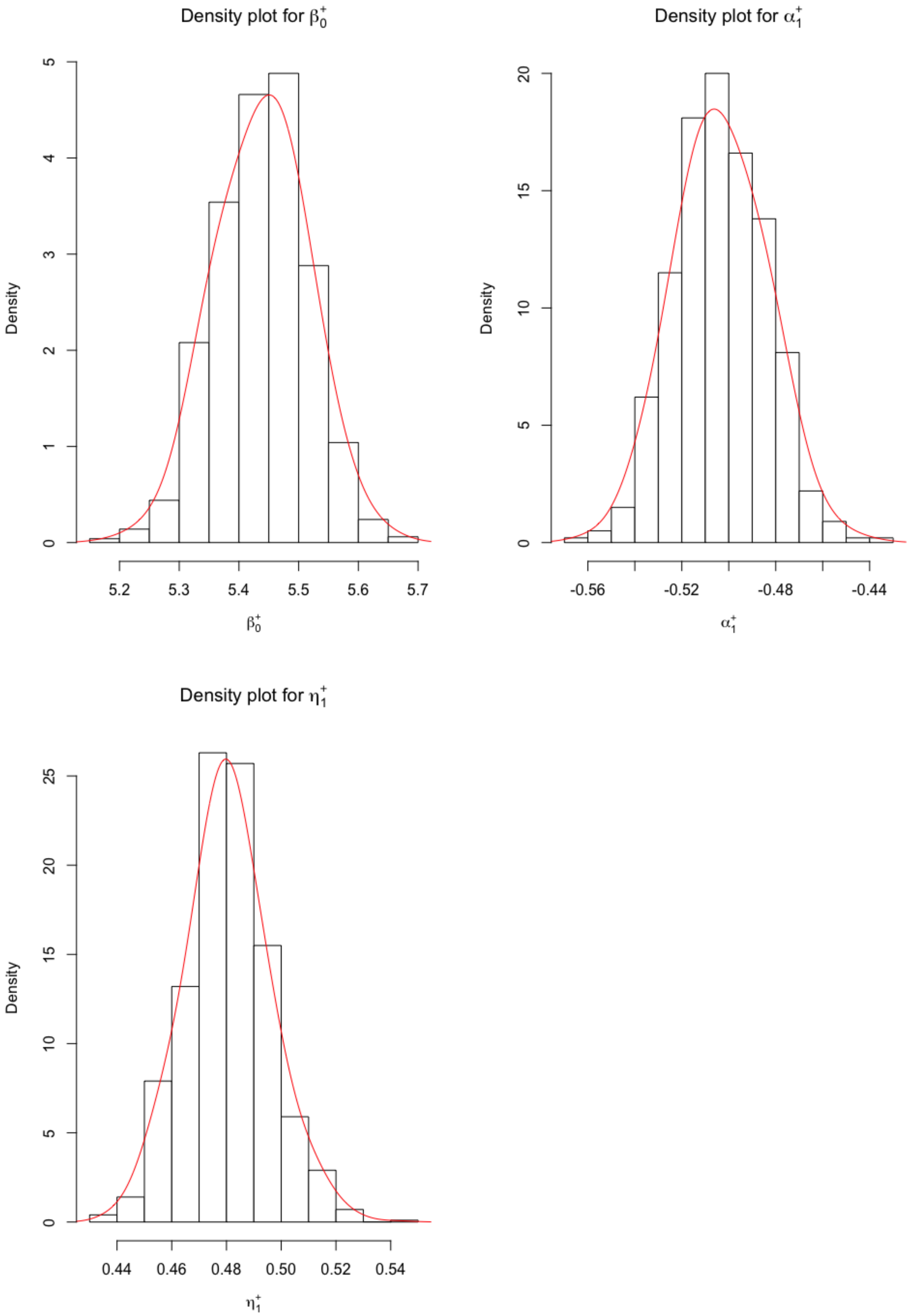

Figure 12. Normal Density Plots for Scheme 11, $T = 2500$.

Table 28 shows that some multivariate normality tests are attained for both when the Boolean model is formed by grains Z_t with random radius (scheme 10) and fixed but unknown radius (scheme 11).

Table 28
Asymptotic Multivariate Normality Test Using T=2500

(a) Scheme 10				(b) Scheme 11			
Test	Statistic	P value	MVN	Test	Statistic	P value	MVN
E-statistic	3.676	0	NO	E-statistic	0.980	0.139	YES
Royston	31.087	4.526e-08	NO	Royston	0.179	0.727	YES
Henze-Zirkler	2.010	1.339e-09	NO	Henze-Zirkler	0.845	0.546	YES
Mardia Skewness	154.193	5.107e-28	NO	Mardia Skewness	18.013	0.05475	YES
Mardia Kurtosis	5.473	4.416e-08	NO	Mardia Kurtosis	-0.191	0.849	YES
MVN			NO	MVN			YES

Thus, the above results of method I and II give similar answers to the research questions for this study.

Rocky mountain pine beetle data application

In this section, we applied the $AR(1)$ times series models built to the mountain pine beetles data. The data consist of correlated images taken over time, with time-dependent covariate mean annual precipitation (inches). Below, we discuss the results.

Table 29
Descriptive Statistics from the Beetle Data.

Year	n_t^+	p_t^*	$\hat{n}_t$	Precip(in)
2001	357	0.35	546.00	644
2002	293	0.38	476.00	622
2003	311	0.47	585.00	522
2004	355	0.44	639.00	457
2005	248	0.40	416.00	668
2006	269	0.46	495.00	616
2007	240	0.46	445.00	679
2008	260	0.46	482.00	608
2009	313	0.39	513.00	618
2010	316	0.36	491.00	645
Mean	296.2	0.416	508.8	620.3
Median	302.0	0.424	493.0	608
Std. Dev	41.62	0.047	65.95	21.58
Cor	n^+	p^*	$\hat{n}$	Precip
n^+	1.00	-0.46	0.82	-0.56
p^*	-0.46	1.00	0.13	-0.40
$\hat{n}$	0.82	0.13	1.00	-0.89
Precip	-0.56	-0.40	-0.89	1.00

Table 29 shows the average number of tangent points n^+ across the years is 296 and $\hat{n}$ is 508 with an average area of damage of 0.41 units. The correlation between n^+ and $\hat{n}$ is 0.82, which was expected. p^* and $\hat{n}$ have a correlation of 0.13.

Table 30
Results from the Mountain Pine Beetle Data.

Method I		(a)		
Parameter	$\hat{\theta}$	Std.Error	CI(lower)	CI(upper)
β_0	6.3169012	0.6909231	4.9627167	7.6710856
β_1	-0.8655854	0.6175579	-2.0759766	0.3448059
α_1	0.9999997	0.5892992	-0.1550054	2.1550048
η_1	-0.0015651	0.0002898	-0.0021331	-0.0009972
		(b)		
β_0	5.2175867	0.3515736	4.5285151	5.9066584
β_1	0.0719525	0.7263160	-1.3516007	1.4955057
α_1	0.1657881	0.7411295	-1.2867990	1.6183752
η_1	-0.0015281	0.0002879	-0.0020923	-0.0009639
		(c)		
β_0	2.5938431	1.6590583	-0.6578513	5.8455375
β_1	-0.3177490	0.1227373	-0.5583096	-0.0771883
α_1	0.9018683	0.2833472	0.3465181	1.4572186
Method II		(d)		
Parameter	$\hat{\theta}$	Std.Error	CI(lower)	CI(upper)
β_0	4.5293637	5.8116154	-6.8611932	15.9199205
β_1	0.1463701	0.1470922	-0.1419253	0.4346655
α_1	0.0570623	1.0548821	-2.0104686	2.1245932
		(e)		
β_0	6.5499817	0.26777400	6.0251543	7.0748091
α_1	0.1147018	0.03445781	0.0471658	0.1822379
η_1	-0.0017403	0.00021100	-0.0021538	-0.0013267
		(f)		
β_0	5.1505975	0.3845394	4.3969142	5.9042808
α_1	0.2460171	0.0451076	0.1576078	0.3344265
η_1	-0.0014981	0.0002830	-0.0020527	-0.0009435

The models with (and without) covariates in Tables 30 indicated that, when the average annual precipitation (inches) increased, the rate of mountain pine beetle damage decreased. However, the dependence on past observations has negative effect for method I compared to positive effect from method II. The estimated unknown fixed radius is 0.0185 and 0.024 from (c) and (d).

2010 Prediction of Mountain Pine Beetle Damage

Using model (c), $\nu_t = 2.594 - 0.318\log(\hat{n}_{t-1}+1) + 0.902\nu_{t-1}$ and (d), $\mu_t = 4.529 + 0.146\log(n^{+}_{t-1}+1) + 0.057\mu_{t-1}$ from Table 30, the predicted rate for 2010 are $\hat{\lambda} = 529.742$ and $\hat{\lambda}^{+} = 297.154$ respectively.

Below are the images from the original datum from 2010, the predicted images from model *c* and *d* respectively.

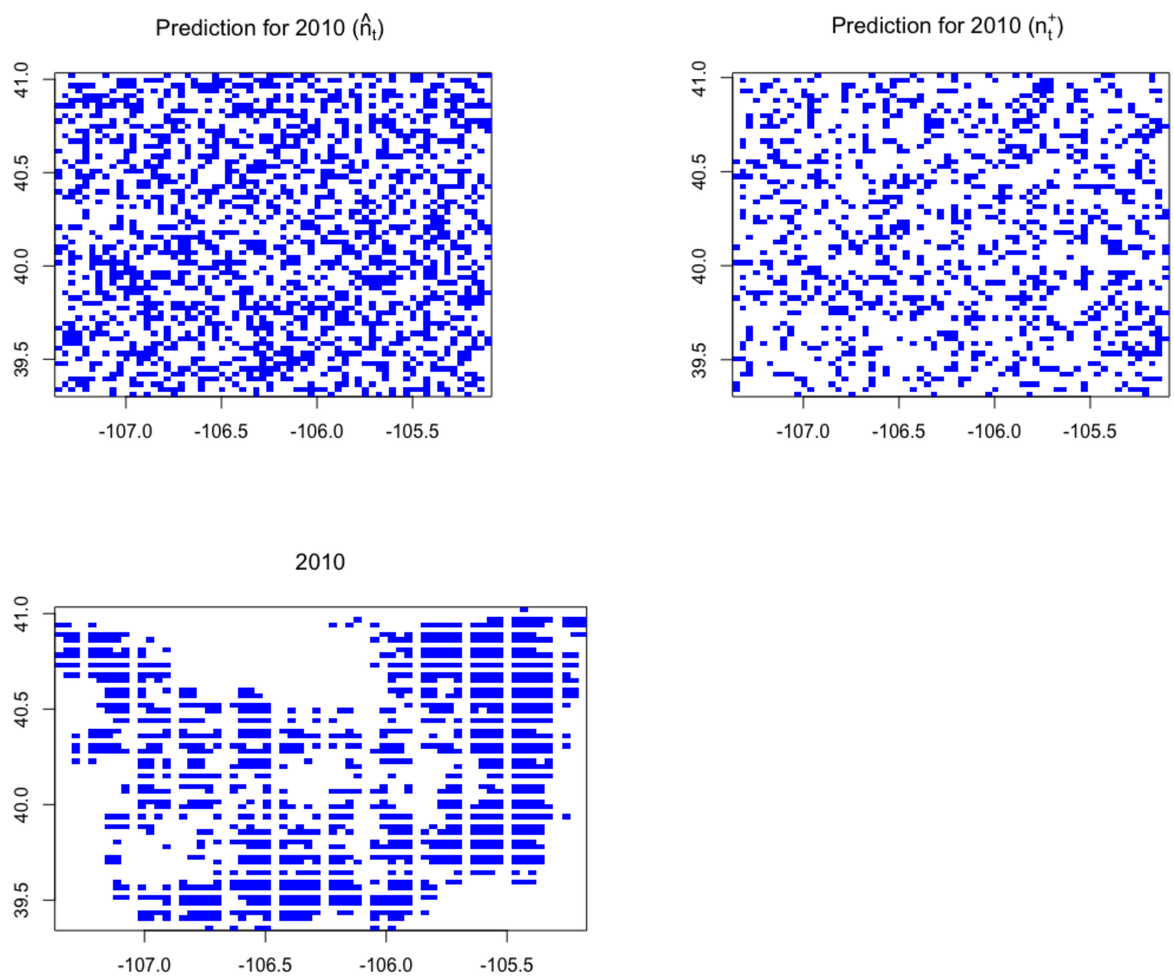

Figure 13. 2010 Datum and Predicted Images.

Regarding the predicted intensity for 2010, method II predicts the number of damaged trees comparable to the original data. However, the predicted images are not comparable. An explanation of this anomaly is that the model proposed in this study does not include the location of the damages in its formulation. Therefore, the time series model for the BRS will predict to some accuracy, the future intensity, but will fail to predict exact location of damaged trees.

CHAPTER V

CONCLUSIONS

The Boolean Random Set (BRS) is formed as a result of a union of two processes. These include, the Poisson point process, which is responsible for the random location of germs in the window, and an independent grain-process which places grains around the germs present.

A review of the literature for count time series and the Boolean model showed that there was a need for a time series model to predict the intensity of the Boolean model. Hence, the purpose of this study. This study proposed a model for estimating and predicting the intensity of Boolean random sets that are correlated over time. Additionally, the lagged observable points $\hat{n}_t$ and exposed lower tangent points n_t^+ from the BRS were used as observations in a log-linear Poisson autoregressive model. Then, maximum likelihood estimation was used to estimate the parameters of the models, whilst the method of moments were used for the parameters of the radius estimation. Moreover, simulations based on twelve (12) schemes in Table 1 for each model proposed were used to study the properties of these estimates, and also to answer the research questions raised in this study. Finally, these two models were applied to the mountain pine beetle data from 2001 to 2009. The resulting model was then used to predict the rate for the BRS for the year 2010.

Findings

Based on the methodologies proposed and the simulations carried, the results showed that when both time-dependent covariate x_t and past observations $\hat{n}$ or n_t^+ are included in the model, both biasedness and standard errors of the parameter estimates approached zero as time T was increased from 10 to 2500 for the maximum likelihood estimator of the model. The estimates improved towards parameter values, thus proving the large sample properties of maximum likelihood estimators. Also, the method of moments estimators for the uniform distribution's parameters a and b also showed similar results in the biasedness and standard errors.

When the radius was unknown but fixed, which was found in schemes 3 and 4, both maximum likelihood estimators and method of moments estimates had similar results. Also, the results were similar for fixed and known radius. We conclude that the estimators are unbiased for estimating the intensity of the Boolean random sets. This applies to models with and without covariates. However, when there is significant serial dependency from the covariate, the estimation results must be accepted with caution. Since the covariate's dependency could interact with that of the time series. The above estimators showed asymptotic normality and consistency, which was observed for sufficiently large T. Thus, we can estimate the intensity of the Boolean model using the time series built in this study.

In schemes 5 to 8, we showed that a model can be built to estimate the intensity of a time-dependent BRS that depends only on past observations i.e.: $\hat{n}_{t-1}$ or n_{t-1}^+. The results were similar to the model with the covariates, i.e.: estimators were unbiased, consistent and asymptotically normal. Additionally, from schemes 9 to 12, it was shown that a model based only on time-dependent covariate x_t had similar results as the models with the covariates. However, without dependence on past observations, the dependence on past values of the linear predictor may have no effect. Therefore, such models need to be considered with great care.

In all models, when the radii were non-random, the biasness and standard errors approached zero relatively faster than when radii were random. Also, under all conditions in all the models, when the radius was sufficiently large, the recovery rate for the lower tangent points was lower than when radii were small. This is due to the big grains absorbing smaller neighboring grains in the Boolean model.

The proposed models were applied to the Rocky Mountain pine beetle data, which captured the damages of trees in the region from 2001 to 2010. We used the annual average precipitation as a time-dependent covariate, covering the area of $[-107, -105.5]$ longitudes by $[39.5, 41]$ latitudes. Also, the smoothing parameter for these images was assumed to be constant across time. Finally, to test the predictive strength of the models built, the estimated models were used to predict the intensity for 2010, which was comparable to the original intensity of 2010 datum. However, predicting the location was not successful, due to the absence of a location parameter in the proposed model.

Limitations and Recommendations

Some limitations encountered in this study reveal suggestions for future research. These suggestions are stated below. In this study, the location was not modeled. Therefore, as seen in the application, the intensity of the correlated images were predicted accurately. However, the future image's prediction of the locations of the germ was not quite accurate. Thus, a future study should include a location parameter. This will ensure an accurate prediction of location is achieved in addition to the intensity prediction. Also, different time-dependent covariates should be included in the model to study the effect of different covariates' dependencies on the estimation and prediction of the intensity of the Boolean model. In application of the time series model to data, different nonrandom radii should be studied to see the effect of large and small fixed radii on the estimation process. Finally, in method II of this study, the first order Taylor series approximation was used to study the relationship between Θ^+ and $\hat{\Theta}$. Future studies should investigate the nonlinear relationship rather than an approximation.

Closing Remarks

In conclusion, this study has expanded the field of Boolean random sets by creating a time series for correlated Boolean random sets. Both options of past observations $\hat{n}$ and the exposed lower tangent points n^+ can be used accurately in estimating and predicting the future intensity of Boolean random sets.

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APPENDIX A

ROCKY MOUNTAIN PINE DATA

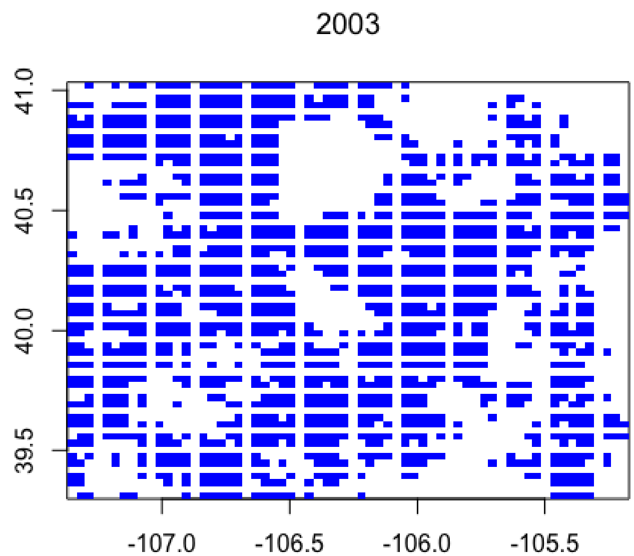

Figure 14. The Rocky Mountain Pine Beetle Data for 2003.

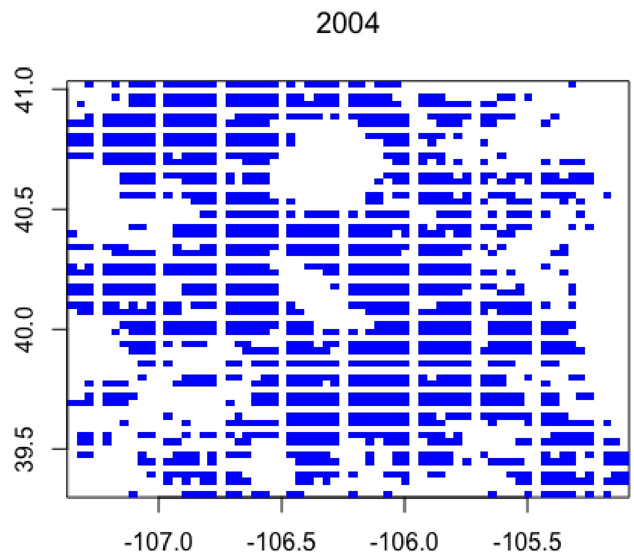

Figure 15. The Rocky Mountain Pine Beetle Data for 2004.

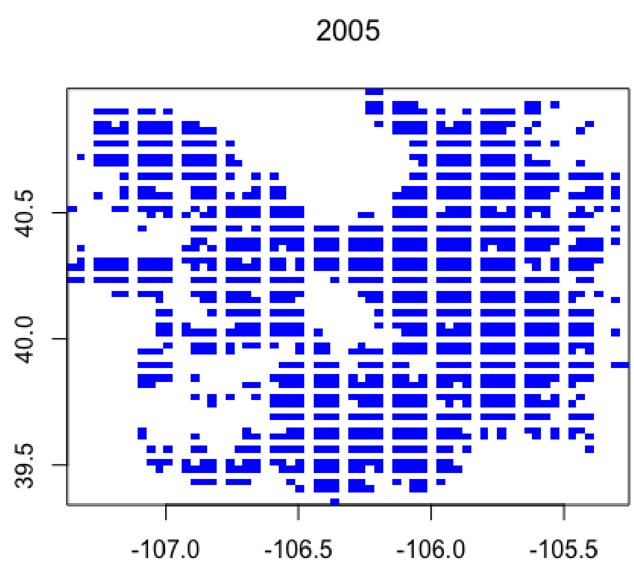

Figure 16. The Rocky Mountain Pine Beetle Data for 2005.

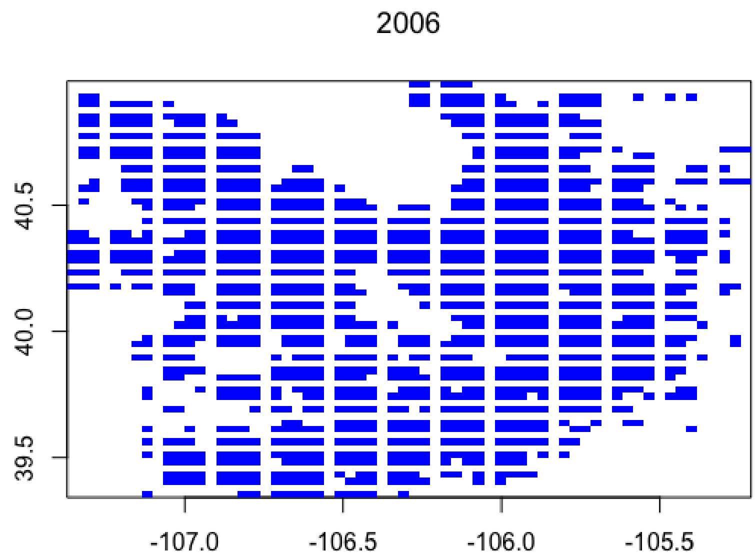

Figure 17. The Rocky Mountain Pine Beetle Data for 2006.

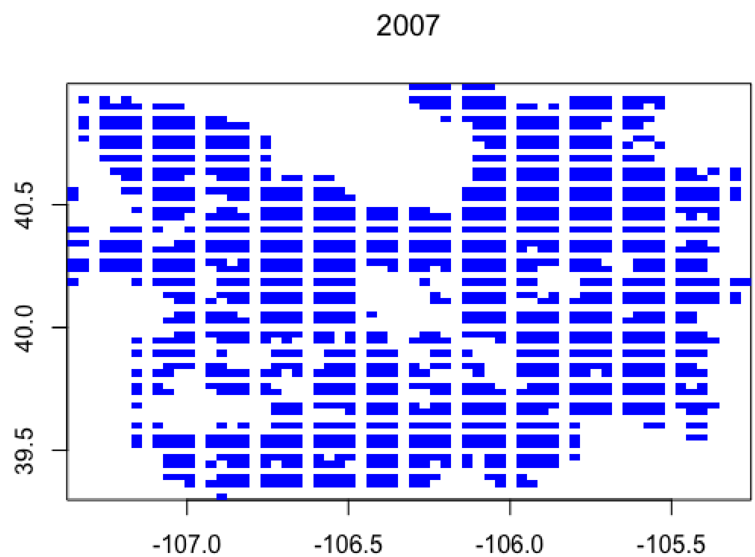

Figure 18. The Rocky Mountain Pine Beetle Data for 2007.

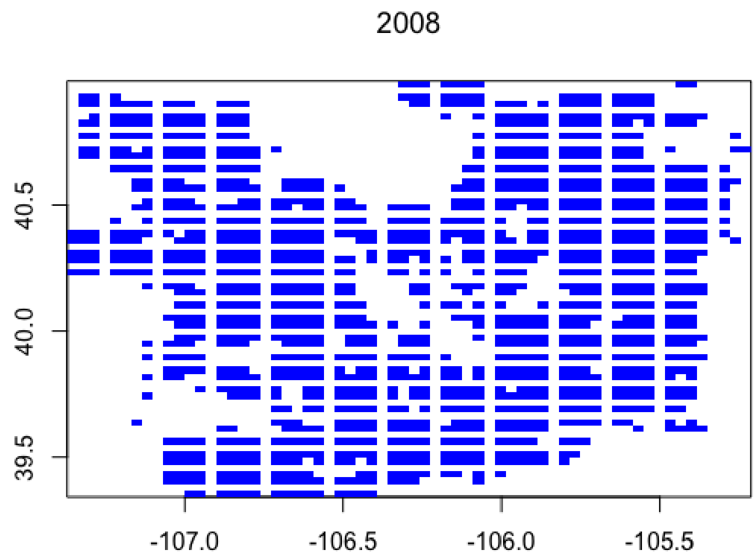

Figure 19. The Rocky Mountain Pine Beetle Data for 2008.

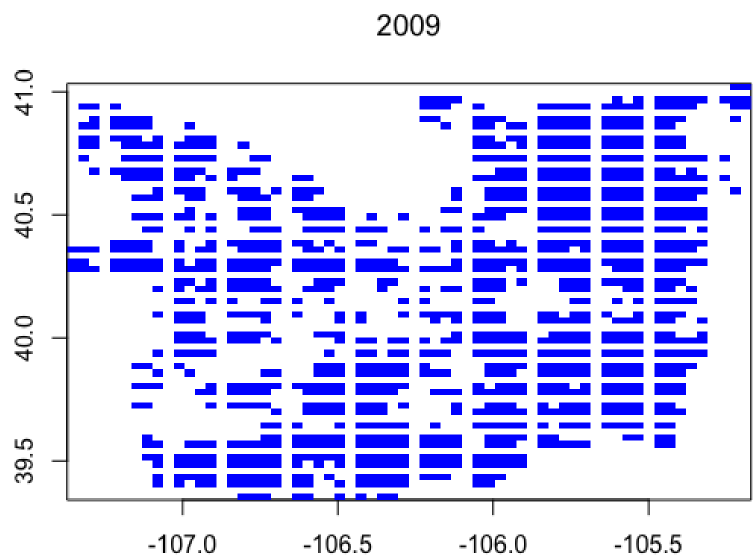

Figure 20. The Rocky Mountain Pine Beetle Data for 2009.

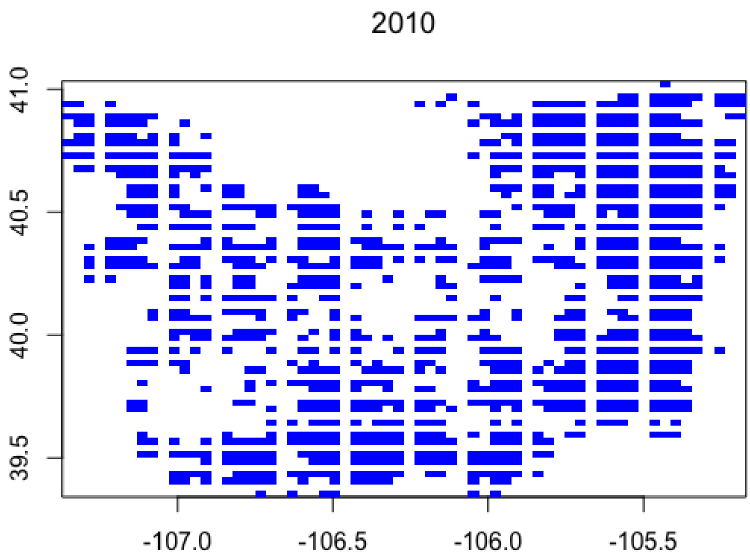

Figure 21. The Rocky Mountain Pine Beetle Data for 2010.

APPENDIX B

ADDITIONAL TABLES

Table 31
Results from Scheme 5, Method II

Time	θ	Parameters	θ^+	Std.Error	Bias	Statistic	P value
10	β_0	-0.50	-0.3348769	0.7069161	-0.1651231	0.0781	<0.001
	β_1	0.65	0.1118793	0.7638203	0.5381207	0.1572	<0.001
	α_1	-0.50	-0.0428348	0.6940198	-0.4571652	0.0906	<0.001
	a	0.00	-0.0361794	0.0178552	0.0361794	0.0992	<0.001
	b	0.10	0.1036912	0.0277260	-0.0036912	0.0466	<0.001
50	β_0	-0.50	-0.5059213	0.3970740	0.0059213	0.0287	0.0514
	β_1	0.65	0.4543636	0.3924046	0.1956364	0.0965	<0.001
	α_1	-0.50	-0.2204758	0.5736681	-0.2795242	0.1154	<0.001
	a	0.00	-0.0420547	0.0071099	0.0420547	0.0565	<0.001
	b	0.10	0.1099430	0.0119235	-0.0099430	0.0236	0.1947
100	β_0	-0.50	-0.5993548	0.2431495	0.0993548	0.0371	0.0025
	β_1	0.65	0.5520103	0.2108221	0.0979897	0.0187	0.5413
	α_1	-0.50	-0.4610817	0.3387216	-0.0389183	0.0678	<0.001
	a	0.00	-0.0428222	0.0047946	0.0428222	0.0349	0.006
	b	0.10	0.1113204	0.0087453	-0.0113204	0.0267	0.0906
200	β_0	-0.50	-0.5875698	0.1860492	0.0875698	0.0252	0.1297
	β_1	0.65	0.5533976	0.1778275	0.0966024	0.0226	0.2485
	α_1	-0.50	-0.4021167	0.3184057	-0.0978833	0.0807	<0.001
	a	0.00	-0.0432908	0.0033083	0.0432908	0.0230	0.2243
	b	0.10	0.1110780	0.0063313	-0.0110780	0.0296	0.0382
500	β_0	-0.50	-0.5996991	0.1028733	0.0996991	0.0258	0.1087
	β_1	0.65	0.5688350	0.1058154	0.0811650	0.0168	0.7069
	α_1	-0.50	-0.4521468	0.1578867	-0.0478532	0.0401	0.0007
	a	0.00	-0.0436061	0.0020011	0.0436061	0.0162	0.7534
	b	0.10	0.1114748	0.0038976	-0.0114748	0.0259	0.1059
1000	β_0	-0.50	-0.5972771	0.0739390	0.0972771	0.0135	0.9309
	β_1	0.65	0.5670460	0.0744259	0.0829540	0.0146	0.8699
	α_1	-0.50	-0.4565769	0.1096523	-0.0434231	0.0448	0.0001
	a	0.00	-0.0436799	0.0014368	0.0436799	0.0182	0.5851
	b	0.10	0.1113546	0.0027209	-0.0113546	0.0119	0.9801
2500	β_0	-0.50	-0.5988528	0.0486666	0.0988528	0.0327	0.0134
	β_1	0.65	0.5726614	0.0481791	0.0773386	0.0384	0.0014
	α_1	-0.50	-0.4495280	0.0691031	-0.0504720	0.0325	0.0144
	a	0.00	-0.0437907	0.0009416	0.0437907	0.0242	0.1647
	b	0.10	0.1114041	0.0017694	-0.0114041	0.0205	0.3943

Table 32
Results from Scheme 6, Method II

Time	θ	Parameters	θ^+	Std.Error	Bias	Statistic	P value
10	β_0	0.50	0.2042058	0.6629356	0.2957942	0.1143	<0.001
	β_1	-0.35	-0.4138851	0.6505584	0.0638851	0.1838	<0.001
	α_1	-0.50	0.0549702	0.6320740	-0.5549702	0.0802	<0.001
	a	0.00	-0.0368994	0.0173446	0.0368994	0.1171	<0.001
	b	0.10	0.1048638	0.0250135	-0.0048638	0.0396	9e-04
50	β_0	0.50	0.2662686	0.2337222	0.2337314	0.0486	<0.001
	β_1	-0.35	-0.3971686	0.3036561	0.0471686	0.0516	<0.001
	α_1	-0.50	-0.0672230	0.5550209	-0.4327770	0.0956	<0.001
	a	0.00	-0.0423578	0.0058369	0.0423578	0.0462	<0.001
	b	0.10	0.1098913	0.0117255	-0.0098913	0.0270	0.0817
100	β_0	0.50	0.2514346	0.1782821	0.2485654	0.0434	1e-04
	β_1	-0.35	-0.3346234	0.2094961	-0.0153766	0.0174	0.654
	α_1	-0.50	-0.4197436	0.4223442	-0.0802564	0.1114	<0.001
	a	0.00	-0.0432270	0.0040237	0.0432270	0.0368	0.0028
	b	0.10	0.1107726	0.0075212	-0.0107726	0.0179	0.6131
200	β_0	0.50	0.2529204	0.1169736	0.2470796	0.0267	0.0908
	β_1	-0.35	-0.3300212	0.1382172	-0.0199788	0.0292	0.0435
	α_1	-0.50	-0.3774756	0.3876345	-0.1225244	0.1351	<0.001
	a	0.00	-0.0434270	0.0028671	0.0434270	0.0293	0.0416
	b	0.10	0.1110546	0.0054744	-0.0110546	0.0301	0.0324
500	β_0	0.50	0.2573642	0.0751298	0.2426358	0.0195	0.4674
	β_1	-0.35	-0.3311777	0.0889402	-0.0188223	0.0167	0.7185
	α_1	-0.50	-0.4971447	0.1823686	-0.0028553	0.0855	<0.001
	a	0.00	-0.0437265	0.0018438	0.0437265	0.0406	5e-04
	b	0.10	0.1113912	0.0035799	-0.0113912	0.0243	0.1621
1000	β_0	0.50	0.2586189	0.0523329	0.2413811	0.0209	0.3603
	β_1	-0.35	-0.3255176	0.0605023	-0.0244824	0.0215	0.3187
	α_1	-0.50	-0.5122990	0.1203847	0.0122990	0.0541	<0.001
	a	0.00	-0.0437147	0.0012951	0.0437147	0.0260	0.103
	b	0.10	0.1114175	0.0025302	-0.0114175	0.0148	0.8607
2500	β_0	0.50	0.2568855	0.0334128	0.2431145	0.0164	0.7396
	β_1	-0.35	-0.3235158	0.0388118	-0.0264842	0.0229	0.2278
	α_1	-0.50	-0.5222168	0.0702031	0.0222168	0.0275	0.0716
	a	0.00	-0.0437945	0.0007711	0.0437945	0.0139	0.9125
	b	0.10	0.1113376	0.0016393	-0.0113376	0.0263	0.0957

Table 33

Results from Scheme 7, Method II

Time	θ	Parameters	θ_+	Std.Error	Bias	Statistic	P value
10	β_0	1.70	1.8723692	0.7090535	-0.1723692	0.0549	<0.001
	β_1	0.65	0.3936916	0.5106610	0.2563084	0.1294	<0.001
	α_1	-0.50	-0.3304898	0.6007120	-0.1695102	0.1325	<0.001
	Radius	0.01	0.0098009	0.0006341	0.0001991	0.0240	0.1766
50	β_0	1.70	1.8014570	0.5920243	-0.1014570	0.0371	0.0025
	β_1	0.65	0.5754635	0.1557404	0.0745365	0.0268	0.087
	α_1	-0.50	-0.4805663	0.2566208	-0.0194337	0.0573	<0.001
	Radius	0.01	0.0097794	0.0002797	0.0002206	0.0219	0.2911
100	β_0	1.70	1.7342172	0.4184706	-0.0342172	0.0310	0.0241
	β_1	0.65	0.5848675	0.1115658	0.0651325	0.0147	0.8630
	α_1	-0.50	-0.4594400	0.1835352	-0.0405600	0.0342	0.0079
Radius	0.01	0.0097677	0.0001941	0.0002323	0.0167	0.7143	
200	β_0	1.70	1.7157622	0.3021399	-0.0157622	0.0202	0.4177
	β_1	0.65	0.5942236	0.0794240	0.0557764	0.0216	0.3088
	α_1	-0.50	-0.4581225	0.1226270	-0.0418775	0.0429	0.0002
	Radius	0.01	0.0097749	0.0001354	0.0002251	0.0171	0.6828
500	β_0	1.70	1.7104079	0.1743871	-0.0104079	0.0168	0.7083
	β_1	0.65	0.5933565	0.0505177	0.0566435	0.0213	0.3310
	α_1	-0.50	-0.4544700	0.0690476	-0.0455300	0.0263	0.0960
	Radius	0.01	0.0097740	0.0000900	0.0002260	0.0198	0.4497
1000	β_0	1.70	1.7122275	0.1236519	-0.0122275	0.0189	0.5257
	β_1	0.65	0.5942075	0.0350559	0.0557925	0.0217	0.3012
	α_1	-0.50	-0.4565386	0.0509232	-0.0434614	0.0234	0.2024
	Radius	0.01	0.0097715	0.0000614	0.0002285	0.0163	0.7462
2500	β_0	1.70	1.7042273	0.0827892	-0.0042273	0.0269	0.0842
	β_1	0.65	0.5947220	0.0224310	0.0552780	0.0216	0.3124
	α_1	-0.50	-0.4528117	0.0317134	-0.0471883	0.0164	0.7411
	Radius	0.01	0.0097739	0.0000386	0.0002261	0.0185	0.5550

Table 34
Results from Scheme 8, Method II

Time	θ	Parameters	θ^+	Std.Error	Bias	Statistic	P value
10	β_0	5.50	5.4675300	0.4651444	0.0324700	0.1121	<0.001
	β_1	-0.35	-0.4855162	0.1541317	0.1355162	0.1028	<0.001
	α_1	-0.50	-0.2334213	0.2530244	-0.2665787	0.1188	<0.001
	Radius	0.01	0.0098270	0.0004538	0.0001730	0.0268	0.0867
50	β_0	5.50	5.4090830	0.3384554	0.0909170	0.1972	<0.001
	β_1	-0.35	-0.5154164	0.1048428	0.1654164	0.0919	<0.001
	α_1	-0.50	-0.2945414	0.1687878	-0.2054586	0.1206	<0.001
	Radius	0.01	0.0098785	0.0001742	0.0001215	0.0283	0.0574
100	β_0	5.50	5.4186821	0.3187415	0.0813179	0.1895	<0.001
	β_1	-0.35	-0.4637671	0.0704384	0.1137671	0.0367	0.003
	α_1	-0.50	-0.3617153	0.1310982	-0.1382847	0.1117	<0.001
	Radius	0.01	0.0098904	0.0001193	0.0001096	0.0173	0.6599
200	β_0	5.50	5.4441611	0.1883983	0.0558389	0.1197	<0.001
	β_1	-0.35	-0.4219371	0.0563366	0.0719371	0.0230	0.2221
	α_1	-0.50	-0.4208008	0.0898216	-0.0791992	0.0427	2e-04
	Radius	0.01	0.0098885	0.0000850	0.0001115	0.0190	0.5149
500	β_0	5.50	5.4491965	0.1036190	0.0508035	0.0601	<0.001
	β_1	-0.35	-0.3797037	0.0402148	0.0297037	0.0233	0.2094
	α_1	-0.50	-0.4696825	0.0606426	-0.0303175	0.0314	0.0217
	Radius	0.01	0.0098906	0.0000519	0.0001094	0.0319	0.018
1000	β_0	5.50	5.4408511	0.0754698	0.0591489	0.0456	<0.001
	β_1	-0.35	-0.3580998	0.0280942	0.0080998	0.0155	0.8107
	α_1	-0.50	-0.4904535	0.0425579	-0.0095465	0.0225	0.2531
	Radius	0.01	0.0098897	0.0000352	0.0001103	0.0244	0.1587
2500	β_0	5.50	5.4417913	0.0516558	0.0582087	0.0292	0.0437
	β_1	-0.35	-0.3413752	0.0179233	-0.0086248	0.0294	0.0410
	α_1	-0.50	-0.5084564	0.0283505	0.0084564	0.0186	0.5463
	Radius	0.01	0.0098919	0.0000236	0.0001081	0.0238	0.1828

Table 35

Results from Scheme 9, Method II

Time	θ	Parameters	θ^{+}	Std.Error	Bias	Statistic	P value
10	β_0	0.5	-1.2121520	9.0640439	0.7121520	0.3443	<0.001
	α_1	0.5	-0.5096134	0.5488904	0.0096134	0.1858	<0.001
	η_1	0.5	-1.1136221	30.8403348	1.6136221	0.4077	<0.001
	a	0.0	-0.0366682	0.0179839	0.0366682	0.0951	<0.001
	b	0.1	0.1022382	0.0281244	-0.0022382	0.0479	<0.001
50	β_0	-0.5	-0.7395628	0.4905854	0.2395628	0.0518	<0.001
	α_1	-0.5	-0.6800055	0.4517623	0.1800055	0.2394	<0.001
	η_1	0.5	0.3055890	0.7791583	0.1944110	0.0685	<0.001
	a	0.0	-0.0423787	0.0068114	0.0423787	0.0599	<0.001
	b	0.1	0.1101738	0.0123652	-0.0101738	0.0326	0.014
100	β_0	-0.5	-0.6581751	0.3874993	0.1581751	0.0279	0.065
	α_1	-0.5	-0.2359442	0.5242861	-0.2640558	0.0866	<0.001
	η_1	0.5	0.5475080	0.6147918	-0.0475080	0.0731	<0.001
	a	0.0	-0.0429870	0.0047452	0.0429870	0.0486	<0.001
	b	0.1	0.1103703	0.0087170	-0.0103703	0.0218	0.2977
200	$\beta_0 3$	-0.5	-0.7003852	0.2426265	0.2003852	0.0262	0.0976
	$\alpha_1 3$	-0.5	-0.7240535	0.3702756	0.2240535	0.2281	<0.001
	$\eta_1 3$	0.5	0.3473088	0.4187347	0.1526912	0.1041	<0.001
	a	0.0	-0.0434029	0.0033630	0.0434029	0.0359	0.004
	b	0.1	0.1113126	0.0062622	-0.0113126	0.0172	0.6756
500	β_0	-0.5	-0.6739350	0.1833481	0.1739350	0.0529	<0.001
	α_1	-0.5	-0.3879736	0.4011155	-0.1120264	0.1053	<0.001
	η_1	0.5	0.5118753	0.2432981	-0.0118753	0.0257	0.1135
	a	0.0	-0.0435910	0.0020490	0.0435910	0.0263	0.0966
	b	0.1	0.1113053	0.0039265	-0.0113053	0.0252	0.1296
1000	β_0	-0.5	-0.6793971	0.1201818	0.1793971	0.0204	0.3987
	α_1	-0.5	-0.6036172	0.3281188	0.1036172	0.1135	<0.001
	η_1	0.5	0.3981635	0.2689254	0.1018365	0.1003	<0.001
	a	0.0	-0.0437452	0.0014621	0.0437452	0.0263	0.0957
	b	0.1	0.1114228	0.0028400	-0.0114228	0.0193	0.4906
2500	β_0	-0.5	-0.6887593	0.0790802	0.1887593	0.0436	1e-04
	α_1	-0.5	-0.4748855	0.1889638	-0.0251145	0.0767	<0.001
	η_1	0.5	0.4948276	0.1156884	0.0051724	0.0265	0.0953
	a	0.0	-0.0437410	0.0009323	0.0437410	0.0230	0.2242
	b	0.1	0.1114504	0.0018170	-0.0114504	0.0168	0.7038

Table 36
Results from Scheme 10, Method II

Time	θ	Parameters	θ^{+}	Std.Error	Bias	Statistic	P value
10	β_0	0.5	1.0720303	1.0431434	-0.5720303	0.0821	<0.001
	α_1	0.5	-0.0548839	0.5657680	0.5548839	0.0704	<0.001
	η_1	0.5	0.4719820	1.2706779	0.0280180	0.0693	<0.001
	a	0.0	-0.0405973	0.0085884	0.0405973	0.0568	<0.001
	b	0.1	0.1084605	0.0182022	-0.0084605	0.0306	0.0281
50	β_0	0.5	0.8395556	0.7738290	-0.3395556	0.1585	<0.001
	α_1	0.5	0.1780761	0.5231464	0.3219239	0.1263	<0.001
	η_1	0.5	0.4715788	0.4202374	0.0284212	0.0598	<0.001
	a	0.0	-0.0428560	0.0034706	0.0428560	0.0213	0.3293
	b	0.1	0.1110672	0.0079636	-0.0110672	0.0173	0.6594
100	β_0	0.5	0.9584053	0.8195898	-0.4584053	0.1720	<0.001
	α_1	0.5	0.0957730	0.5762771	0.4042270	0.1389	<0.001
	η_1	0.5	0.4374326	0.3059018	0.0625674	0.0496	<0.001
	a	0.0	-0.0434254	0.0023878	0.0434254	0.0258	0.1103
	b	0.1	0.1114786	0.0056105	-0.0114786	0.0273	0.0756
200	β_0	0.5	0.5294143	0.4054557	-0.0294143	0.1513	<0.001
	α_1	0.5	0.4150226	0.3120527	0.0849774	0.1266	<0.001
	η_1	0.5	0.4782288	0.1783724	0.0217712	0.0191	0.503
	a	0.0	-0.0433873	0.0017090	0.0433873	0.0155	0.8137
	b	0.1	0.1111841	0.0041660	-0.0111841	0.0182	0.582
500	β_0	0.5	0.5244617	0.4208695	-0.0244617	0.2037	<0.001
	α_1	0.5	0.4216998	0.3034599	0.0783002	0.1679	<0.001
	η_1	0.5	0.4690289	0.1366033	0.0309711	0.0423	2e-04
	a	0.0	-0.0435789	0.0010611	0.0435789	0.0193	0.4908
	b	0.1	0.1112946	0.0025204	-0.0112946	0.0200	0.4313
1000	β_0	0.5	0.4397909	0.1513448	0.0602091	0.0571	<0.001
	α_1	0.5	0.4854633	0.1243818	0.0145367	0.0409	5e-04
	η_1	0.5	0.4716539	0.0809193	0.0283461	0.0276	0.0708
	a	0.0	-0.0435825	0.0007169	0.0435825	0.0176	0.6387
	b	0.1	0.1113224	0.0017804	-0.0113224	0.0294	0.0409
2500	β_0	0.5	0.4275567	0.0920702	0.0724433	0.0657	<0.001
	α_1	0.5	0.4970404	0.0762144	0.0029596	0.0530	<0.001
	η_1	0.5	0.4668783	0.0506389	0.0331217	0.0244	0.1571
	a	0.0	-0.0435531	0.0004652	0.0435531	0.0244	0.1576
	b	0.1	0.1114625	0.0011671	-0.0114625	0.0248	0.1442

Table 37
Results from Scheme 11, Method II

Time	θ	Parameters	θ^{+}	Std.Error	Bias	Statistic	P value
10	β_0	5.50	4.1025967	1.9743443	1.3974033	0.1341	<0.001
	η_1	-0.50	-0.1422244	0.6612251	-0.3577756	0.2091	<0.001
	η_1	0.50	0.6064356	1.5648207	-0.1064356	0.1516	<0.001
	Radius	0.01	0.0099352	0.0002419	0.0000648	0.0248	0.1426
50	β_0	5.50	5.6080685	0.9897076	-0.1080685	0.2272	<0.001
	η_1	-0.50	-0.5478898	0.2739805	0.0478898	0.2824	<0.001
	η_1	0.50	0.5938495	0.2840626	-0.0938495	0.0778	<0.001
	Radius	0.01	0.0099347	0.0001054	0.0000653	0.0199	0.4419
100	β_0	5.50	5.6806234	0.3368184	-0.1806234	0.0240	0.1769
	η_1	-0.50	-0.5674997	0.0762205	0.0674997	0.0175	0.6448
	η_1	0.50	0.5416910	0.1425714	-0.0416910	0.0360	0.0039
	Radius	0.01	0.0099354	0.0000732	0.0000646	0.0160	0.7698
200	β_0	5.50	5.5829587	0.2365030	-0.0829587	0.0225	0.2519
	η_1	-0.50	-0.5408385	0.0571011	0.0408385	0.0177	0.6312
	η_1	0.50	0.5033355	0.0795408	-0.0033355	0.0261	0.1016
	Radius	0.01	0.0099338	0.0000542	0.0000662	0.0227	0.2417
500	β_0	5.50	5.4981918	0.1605679	0.0018082	0.0254	0.1228
	η_1	-0.50	-0.5192521	0.0400266	0.0192521	0.0296	0.0388
	η_1	0.50	0.4883774	0.0401201	0.0116226	0.0175	0.6422
	Radius	0.01	0.0099307	0.0000346	0.0000693	0.0161	0.7618
1000	β_0	5.50	5.4623418	0.1166546	0.0376582	0.0169	0.6985
	η_1	-0.50	-0.5096782	0.0294426	0.0096782	0.0170	0.6935
	η_1	0.50	0.4828624	0.0250903	0.0171376	0.0186	0.5489
	Radius	0.01	0.0099324	0.0000232	0.0000676	0.0143	0.8886
2500	β_0	5.50	5.4379128	0.0768518	0.0620872	0.0233	0.2072
	η_1	-0.50	-0.5035102	0.0193260	0.0035102	0.0261	0.1005
	η_1	0.50	0.4804499	0.0153021	0.0195501	0.0375	0.0021
	Radius	0.01	0.0099312	0.0000145	0.0000688	0.0280	0.0614

Table 38

Results from Scheme 12, Method II

Time	θ	Parameters	θ^+	Std.Error	Bias	Statistic	p value
10	β_0	1.70	1.8752622	2.1071622	-0.1752622	0.3685	<0.001
	α_1	0.50	0.4202676	0.5143893	0.0797324	0.3216	<0.001
	η_1	0.50	0.5750847	0.6881455	-0.0750847	0.1808	<0.001
	Radius	0.01	0.0099085	0.0002788	0.0000915	0.0148	0.8594
50	β_0	1.70	0.8179122	0.2767546	0.8820878	0.1609	<0.001
	α_1	0.50	0.7216200	0.0750452	-0.2216200	0.1299	<0.001
	η_1	0.50	0.5010218	0.1048514	-0.0010218	0.0438	1e-04
	Radius	0.01	0.0099291	0.0001088	0.0000709	0.0152	0.8341
100	β_0	1.70	0.8880910	0.1655180	0.8119090	0.0471	<0.001
	α_1	0.50	0.7094483	0.0490928	-0.2094483	0.0484	<0.001
	η_1	0.50	0.4531031	0.0705764	0.0468969	0.0301	0.0324
	Radius	0.01	0.0099316	0.0000760	0.0000684	0.0244	0.1592
200	β_0	1.70	1.0718956	0.1954907	0.6281044	0.0361	0.0037
	α_1	0.50	0.6619998	0.0558025	-0.1619998	0.0286	0.0519
	η_1	0.50	0.4483626	0.0555596	0.0516374	0.0160	0.7717
	Radius	0.01	0.0099306	0.0000535	0.0000694	0.0171	0.6785
500	β_0	1.70	1.3897121	0.1594671	0.3102879	0.0199	0.4386
	α_1	0.50	0.5771351	0.0440648	-0.0771351	0.0255	0.1180
	η_1	0.50	0.4652310	0.0331490	0.0347690	0.0150	0.8454
	Radius	0.01	0.0099325	0.0000330	0.0000675	0.0154	0.8181
1000	β_0	1.70	1.5349361	0.1182146	0.1650639	0.0215	0.3187
	α_1	0.50	0.5386175	0.0323246	-0.0386175	0.0211	0.3488
	η_1	0.50	0.4715798	0.0243838	0.0284202	0.0151	0.8424
	Radius	0.01	0.0099330	0.0000240	0.0000670	0.0219	0.2923
2500	β_0	1.70	1.6254592	0.0715460	0.0745408	0.0179	0.6079
	α_1	0.50	0.5147115	0.0194706	-0.0147115	0.0135	0.9298
	η_1	0.50	0.4751259	0.0144324	0.0248741	0.0212	0.3401
	Radius	0.01	0.0099341	0.0000144	0.0000659	0.0191	0.5021