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Online Instructors’ Gestures For Euclidean Transformations

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ABSTRACT

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The purpose of this case study was to explore the nature of instructors' gestures as they teach Euclidean transformations in a synchronous online setting, and to investigate how, if at all, the synchronous online setting impacted the instructors' intentionality and usage of gestures. The participants in this case study were two collegiate instructors teaching Euclidean transformations to pre-service elementary teachers. The synchronous online instructors' gestures were captured in detail via two video cameras; one through the screen-capture software built into the online conference platform used to conduct the class and another separate auxiliary camera to capture the gestures that the instructors made outside the view of the screen-capture software. The perceived intentionality of the instructors' gestures was documented via an hour-long video-recorded interview after teaching the Euclidean transformation unit.

The findings indicated that synchronous online instructors make representational gestures and pointing gestures while teaching Euclidean transformations. Specifically, that representational gestures served as a second form of communication for the students while pointing gestures grounded synchronous online instructors' responses to student contributions within classroom materials. The findings further indicated the combination of the synchronous online instructors' gestures and language provided a more cohesive picture of the Euclidean transformation as opposed to the gestures alone. Additionally, the findings specified that synchronous online instructors believe the purpose of their gestures was for the benefit of their

students as well as for themselves. Finally, the findings highlighted a connection between instructors who previously thought about the potential impact of gestures in the mathematics classroom and intentionally producing gestures. Specifically, critically thinking about gestures within the mathematics classroom before teaching appeared to correspond with more intentional gestures while teaching.

Based on these findings, there were three recommendations. The first recommendation was for continued education on gesture as an avenue to communicate mathematical ideas. A professional development workshop may assist collegiate instructors to produce more intentional and mathematically precise gestures. The last two recommendations were for synchronous online instructors to utilize technology that affords students the opportunity to view all of their gestures and for the instructors to explicitly instruct their students to pay attention to their gestures. Knowing that the students can view all of their movements and are specifically looking for gestures might prompt the instructors to gesture with more intentionality and precision.

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CHAPTER I

INTRODUCTION

At first, the notion of using gestures to communicate with others may seem helpful only when playing a game of charades, traveling to a new country where you do not know the language, or talking to young children as they learn to speak. Certainly, using gestures as a way to learn mathematics feels ineffective and unnecessary with the accessibility and precision of mathematical notation. However, upon deeper reflection you may begin to realize, we use gestures to help understand and interact with mathematics throughout our schooling experience. When learning to count, add, or subtract in elementary school, young students use their fingers to understand the operations. In middle school, as students begin to interact with pre-algebra and algebra concepts, describing the steepness and direction of the slope of a linear equation is easily modeled with gestures. Finally, in high school a gesture for the right-hand limit and left-hand limit may be used when determining if a limit of a function exists at a point. When describing the concept of Euclidean transformations, it may even feel natural to describe sliding, flipping, or turning a shape with gestures to accompany speech. In this study, I investigated the gestures made by collegiate instructors while they taught Euclidean transformations in an online learning environment.

The world of higher education is rapidly changing, and there are many inherent challenges researchers must address. In 2017, Bettinger and Loeb reported that one out of three college students will elect to enroll in at least one online course during their college career.

However, with the COVID pandemic, Bustamante (2020) estimated that 3,278 higher education institutions in the United States transitioned to online learning by early April of 2020, displacing around 22.3 million on-campus students. The concerns for student and faculty health, and much needed potential for cost savings, fueled ongoing investments in online education by both public and private institutions (Bettinger & Loeb; Bustamante, 2020; Marcus, 2020). Marcus (2020) wrote that the COVID pandemic accelerated the necessity for innovation. Bustamante added that 43% of institutions invested in new online learning resources and that faculty or technological readiness for online learning became an immediate concern for college and university presidents. One technology that many higher education institutions heavily relied on was video conferencing software (Bustamante, 2020). This technology allowed faculty and students to interact in real time and provided both parties a sense of schedule and normalcy. Courses conducted in this manner were referred to as *synchronous online* courses. Public kindergarten through 12th grade (K-12) schools in the United States were forced to rapidly adapt to the online teaching environment as well. In a similar manner to higher education, K-12 schools in the United States quickly transitioned to a modified combination of homeschooling and synchronous online classes in the spring of 2020 (Black et al., 2020; Weir, 2020). Teachers used electronic technologies, such as video conferencing and emailing, to deliver content to students. The K-12 classrooms where the teacher sets up a learning path for students to finish at their own pace was referred to as an *asynchronous online* classroom. Black et al. (2020) stated that teachers "were unprepared and untrained to handle the complexities inherent to educating" (p. 119) in the foreign online learning environment. Weir (2020) commented on the importance of using video and audio technology during this trying time because "feeling connected to a teacher can make a big difference in educational outcomes" (p. 54).

With academia as a whole utilizing technology to host synchronous online learning opportunities, investigating the practices of teachers, instructors, and faculty members seemed to be a worthwhile endeavor. Even before the COVID-19 pandemic, Bustamante (2020) commented that the most important class activities for a majority of online students were videos and PowerPoint presentations implemented by their course instructors. To further investigate the importance of the online classroom activities, several researchers explored which features promoted student academic achievement and course satisfaction (Choi & Walters, 2018; Erixon, 2016; Gedeborg, 2016; Golding & Bretscher, 2018; Hadjinicolaou, 2014; Mayer et al., 2017). Their findings included the recommendation to utilize activities that promote inquiry-based learning with an increased amount of scaffolding (Choi & Walters, 2018; Gedeborg, 2016), to create spaces for social interactions amongst students (Choi & Walters, 2018; Gedeborg, 2016; Mayer et al., 2017; Hadjinicolaou, 2014), to include video and audio technologies for both the students and the instructors (Erixon, 2016; Gedeborg, 2016; Mayer et al., 2017), and to maintain clear and straightforward communication between the teacher and the students (Gedeborg, 2016; Golding & Bretscher, 2018). In Chapter II, I summarize some of the studies examining synchronous online classrooms as well as the recommendations for creating an impactful classroom in greater detail.

A feature of synchronous online learning that, until this point, has not been rigorously studied was the role of gestures in the synchronous online mathematics classroom. In several subject areas, not confined to mathematics, researchers examined the benefits of gesture on learning (Congdon et al., 2017; Cook et al., 2008, 2013, 2017; Fiorella & Mayer, 2016; Goldin-Meadow et al., 2009; Novack et al., 2014; Pi et al., 2017). All together, these studies provided evidence that students learned new material in a more efficient manner, both as a means of retention as well as transferability, when a human purposefully connected their verbal explanations to their gestures. In these studies, the learning environment was either face-to-face or asynchronous, not synchronous online. Within Chapter II, I define McNeill's (1992) categories of gestures which are widely accepted and used in the research community. I describe the two purposes for gestures, gestures produced for oneself (Alibali et al., 2001; Cohen & Harrison, 1973; Yang et al., 2020; Yoon et al., 2011; Zurina & Williams, 2011) and gestures produced for the benefit of others (Alibali & Nathan, 2012; Alibali et al., 2013, 2019; Weinberg et al., 2015). Lastly, I further explain the literature on gesturing for student learning.

As suggested earlier, Euclidean transformations may naturally invoke gestures. Far before the abrupt shift to online learning, Euclidean transformations received revived attention and policy makers expressed a desire to emphasize it in the K-12 curriculum (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010). The Common Core State Standards for Mathematics (CCSSM) included 43 high school geometry standards and about one-quarter of these standards specifically mentioned transformations. Usiskin (2014) wrote that these particular standards "caused consternation amongst many teachers because some teachers never encountered transformations in their study of geometry as high school students and many others would not be familiar with all the content mentioned in these standards" (p. 472). Teachers' unfamiliarity with Euclidean transformations as well as the inconsistency with the new geometry curriculum potentially contributed to students' incorrect conceptions surrounding Euclidean transformations (Ada & Kurtuluş, 2010; Hollebrands, 2003; Özerem, 2012; Qi et al., 2014; Seago et al., 2014; Yanik, 2014). Moreover, students appeared to struggle with procedural and conceptual understandings about Euclidean transformations (Ada & Kurtuluş, 2010; Aktaş & Ünlü, 2017; Özerem, 2012; Yanik, 2014). For example, students in Yanik's (2014) study viewed circles as untranslatable due to their procedural understanding when performing a translation. These students believed that to perform a translation on a geometric figure, one first translates the corners or vertices of the figure, then connects the translated vertices. However, this left the interpretation that "corner-less" figures, such as circles, could not be translated. I discuss a complete review of students' conceptions surrounding the Euclidean transformations, translation, reflection, rotation, and glide reflection in Chapter II.

When studying the learning of Euclidean transformations, scholars utilized the inclusion of dynamic geometric environments (DGEs) in the classroom as well as enriched activities or revitalized curriculum in their face-to-face classes (Bansilal & Naidoo, 2012; Chu & Kita, 2008; Guven, 2012; Idris, 2007; Price & Duffy, 2018; Valenzeno et al., 2003; Yao & Manouchehri, 2019). From both a quantitative and qualitative perspective, these face-to-face investigations resulted in positive academic outcomes or experiences for the students while learning Euclidean transformations. Chu and Kita (2008) as well as Valenzeno et al. (2003) specifically examined the impact of gesture on their participants' ability to recognize and create rotations and reflections respectively. Both studies suggested that students who viewed and created gestures outperformed their peers who did not. Each of the aforementioned studies investigated the teaching and learning of Euclidean transformations in a face-to-face classroom. This study followed a different population, specifically, synchronous online instructors. The purpose of this study was to examine synchronous online instructors' gestures in detail and to document their perceived intentionality behind their gestures. In particular this dissertation attempted to answer the following research questions:

Q1 What is the nature of instructors' gestures as they teach Euclidean transformations in a synchronous online setting?

Q2 How, if at all, does a synchronous online setting impact the instructors' intentionality and usage of gestures?

To answer these questions, I observed two synchronous online instructors teach Euclidean transformations. I asked the instructors to record their classes with two cameras, one with the screen-capture software built into the online conference platform and one from a separate, auxiliary camera. After the instructors finished teaching their sessions on Euclidean transformations, I began analysis of data captured on the video recordings to describe the nature of the two instructors' gestures. Next, I interviewed each instructor individually to validate my descriptions and perceptions of their gestures on Euclidean transformations and to gather information on the intentionality of their gestures while teaching Euclidean transformations. Finally, I qualitatively analyzed the interview recordings to produce my findings.

From the recordings of the instructors' synchronous online sessions, I identified and organized the instructors' gestures for Euclidean transformations in a way that aligned with the pre-existing literature for gesture production in the mathematics classroom. These descriptions extended the research on gesture production to a new population, synchronous online instructors. From the interviews, I not only verified my descriptions of the instructors' gestures, but also gathered data on the perceived impact of the synchronous online setting on the instructors' intentionality and usage of their gestures for Euclidean transformations.

From this study, I found that while teaching Euclidean transformations the synchronous online instructors produced a combination of representational and pointing gestures. Specifically, their representational gestures communicated a Euclidean transformation as a fluid, rigid motion and served as a secondary avenue for explaining the Euclidean transformations, while their pointing gestures grounded their verbal responses to student contributions and identified pre- and post-images. I discovered that many of the synchronous online instructors' gestures, both representational and pointing, failed to communicate all the distinctive qualities from the definitions of each Euclidean transformation. Instead, the instructors utilized a familiar motion and their verbiage to communicate a more complete notion of each Euclidean transformation. I found that the synchronous online instructors believe the purpose of their gestures were both for the benefit of the students and for themselves. Lastly, I uncovered a pattern between the synchronous online instructors' reported intentionality, their prior knowledge surrounding gesturing, and the need to adapt their described face-to-face gestures. To aid the reader throughout the remainder of the chapters, I discuss terms and mathematical definitions relevant to my research study below.

Definitions

My study painted a picture of synchronous online mathematics instructors' gestures when they teach Euclidean transformations to pre-service elementary teachers. According to McNeill (1992) *gestures* are the "movements of the hands and arms that we see when people talk"; they are the "creations of individual speakers unique and personal" (p. 1). These movements could be large or small, eccentric or minimal, and refer to physical objects as well as metaphorical ideas. As defined above, *synchronous online* classrooms are live online courses conducted in real-time through an online conference platform. The online conference platform supported both the instructors and students' use of audio and visual technology. The class regularly met in the same online space at the same time for class. The real-time aspect of a synchronous online classroom allowed for dynamic learning, peer collaboration, and immediate feedback. In *asynchronous online* classrooms, teachers usually created a learning path, which students engaged with individually. The teachers prepared videos, student workbook activities, or online modules for students to complete on their own time and pace. The interactions between the teacher and students then became optional and only necessary to assist a struggling student.

With the impact of the CCSSM (2010) on curriculum around the United States, aligning the definitions of the key mathematical terms, transformation, translation, rotation, reflection, and glide reflection of this dissertation study to the accepted definitions from the CCSSM was of the utmost importance. The CCSSM described a transformation in a holistic manner: a transformation was a "change in position, size, or shape of a geometric figure. The given figure is called the preimage and the resulting figure is called the image. A transformation maps a figure onto its image" (Hall et al., 2015, p. 891). This dissertation study focused on a subset of all transformations called *Euclidean transformations*, defined as a bijective mapping $M\colon\ R^2 \to R^2$ where for any $p, q \in R^2$, the distance between p and q, $d(p, q)$, was the same as the distance between the mapped points $M(p)$ and $M(q)$, $d\big(M(p), M(q)\big)$, where distance referred to the traditional metric of length.

There are four isometric Euclidean transformations: translation, rotation, reflection, and glide reflections. A *translation* in the direction of (h, k) of a pre-image point $P = (p, q)$ is the image point $P' = (p', q') = (p + h, q + k)$. In other words, a translation identified each point in the Cartesian plane and slid them all in the same direction. A *rotation* in the counterclockwise angle of rotation θ of a pre-image point (p, q) about the origin is $(p', q') = (p\cos(\theta) - q\sin(\theta), p\sin(\theta) + q\cos(\theta))$. Or similarly that a rotation of a pre-image point P is the image P' obtained by rotating point P about a point O (the center of rotation) either clockwise or counterclockwise, such that the measure of the angle POP' was constant for every point in the preimage and its image point and $PO = P'O$. A *reflection* of the point $P = (p, q)$ about the line l is the point $P' = (p', q')$ such that the segment PP' is perpendicular to the line l, and that a point O of line l, $\overline{PO} \equiv \overline{P'O}$. Hence, the line l became the perpendicular bisector of the segment PP'. Lastly, a *glide reflection* is a composition of a translation and a reflection across a line parallel to the direction of translation. The order of the translation and reflection does not alter the final location of the image.

Organization of the Dissertation

In Chapter II, I summarize the literature related to learning mathematics in a synchronous online platform and the role of mathematics instructors' gestures on communication for student learning, including their misconceptions about Euclidean transformations. Within Chapter III, I detail my researcher's stance, theoretical perspective, methodological choice, a description of my participants, and the nature of my data collection and analysis. In Chapter IV, I describe the instructor's actual gestures when teaching Euclidean transformations in detail as well as depict the post interview conversations on the intentionality of these gestures to address my research questions. Lastly in Chapter V, I interpret my findings from Chapter IV. These findings indicate that the synchronous online instructors made representational gestures and pointing gestures while teaching Euclidean transformations. Specifically, that representational gestures could serve as a second form of communication for students while pointing gestures grounded synchronous online instructors' responses to student contributions within classroom materials. My findings further indicate that the mathematics conveyed by synchronous online instructors' gestures alone did not always communicate all of the mathematical criteria for each Euclidean transformation. Additionally, the findings specify that synchronous online instructors believe the purpose of their gestures was for the benefit of their students as well as for themselves. Finally, my findings indicate a connection between a synchronous online instructor's reported prior knowledge of gesturing, their desire to adapt their gestures to the online setting, and their intentionality surrounding gesturing.

Based on the findings, I make three recommendations. First, I recommend continued education on gesture as an avenue to communicate mathematical ideas. A professional development workshop may assist collegiate instructors to produce more intentional and mathematically precise gestures. The second and third recommendations are for synchronous online instructors to utilize technology which affords students the opportunity to view all of their gestures and to explicitly instruct their students to pay attention to their gestures. Knowing that the students can see them at all times and are specifically looking for gestures may prompt the instructor to gesture with more intentionality and precision. I conclude Chapter V with limitations of the study and possible directions for future research.

CHAPTER II

LITERATURE REVIEW

The purpose of my study was to contribute to the literature on gesture, specifically the gestures of mathematics instructors as they teach Euclidean transformations in a synchronous online setting. In particular, my study sought to answer the following research questions:

Q1 What is the nature of instructors' gestures as they teach Euclidean transformations in a synchronous online setting?

Q2 How, if at all, does a synchronous online setting impact the instructors' intentionality and usage of gestures?

The instructors in my study taught Euclidean transformations to pre-service elementary teachers in a synchronous online setting. Therefore, this chapter begins with the literature surrounding synchronous online mathematics teaching. Next, I transition into the literature surrounding the impact and use of gesturing in the mathematics classroom. The literature on gesturing for communication and learning was vast, thus I drew a story line between communicating with gestures for one's own benefit and gesturing to advance others' progress while learning mathematics. Lastly, I share the literature related to students' conceptions of Euclidean transformations and summarized how several instructional interventions impacted students' understanding of Euclidean transformations. Each of the described instructional interventions occurred in a face-to-face classroom. However, this dissertation study investigated synchronous online instructors' gestures for Euclidean transformations. Hence, I merge the rich findings of Euclidean transformation teaching recommendations, gesturing in the mathematics classroom, and the synchronous online setting.

Synchronous Online Mathematics Learning

The notion of an online mathematics class is not new, however, in recent years there was an increase in attention toward including a space in these classes for synchronous learning (Choi & Walters, 2018; Erixon, 2016; Gedeborg, 2016; Golding & Bretscher, 2018; Hadjinicolaou, 2014; Mayer et al., 2017). Gedeborg (2016) outlined best practices for synchronous online mathematics learning, in which he described key pedagogical choices that instructors should implement to provide their students with an effective learning environment. His recommendations included using structured group investigations, teaching with both audio and video technologies, and maintaining clear and upfront expectations with students.

Regarding his first recommendation, Gedeborg (2016) further described structured group investigations as activities which promoted "freedom of discovery, but [were] organized enough to take advantage of the time students have dedicated for the activity" (p. 276). In his own classes, Gedeborg reported that a majority of his students expressed enjoyment in the group activities and opportunities to work with others on rich mathematical tasks. Several researchers specifically studied the impact of group investigations that Gedeborg shared about his own teaching experience (Choi & Walters, 2018; Hadjinicolaou, 2014). Choi and Walters (2018) investigated voluntary, group problem solving sessions where students explained and justified their mathematical ideas, listened carefully to their peers, asked thoughtful questions, and compared different approaches to the same mathematics problem. After analysis, Choi and Walters reported that students who participated in more group problem solving sessions had "both higher final course scores and higher odds of scoring at or above Proficient on the state assessments" (p. 61). Similarly, in a study on an online undergraduate calculus course, Hadjinicolaou (2014) placed the students into small online rooms where they were encouraged to freely express their ideas and to hold a dialogue among the group members using a shared virtual whiteboard. Hadjinicolaou reported that the use of the virtual class platform provided the necessary environment for the students to learn the conceptual backings of the integral. Hence, Choi and Walters and Hadjinicolaou's results supported Gedeborg's recommendation for using small group investigations when teaching mathematics in a synchronous online setting.

Accompanying the positive results, some drawbacks in the online technology for online synchronous teaching emerged. For example, the high school students in Mayer et al.'s (2017) study identified challenges related to access to microphones, web cameras, and stable internet. Gedeborg (2016) discussed the un-comfortability or unfamiliarity with the use of audio and video technology in synchronous learning. However, after acknowledging the challenge, his second recommendation was for online instructors to find a way to mediate this discomfort and use both audio and video technology in the synchronous online classroom. Erixon's (2016) study on a synchronous online course for secondary mathematics teachers in Sweden provided compelling evidence that both audio and video technologies were necessary for optimal learning. The purpose of the course in Erixon's study was to promote peer lesson planning and to strengthen the teachers' mathematics teaching abilities. A large online multi-party phone call, with no visual component, hosted the teachers during the synchronous online class. As a part of the course, the teachers video recorded themselves teaching a lesson and reflected on the lesson implementation. During the synchronous class time, the teachers each explained and justified their choices on the video and opened the floor to critiques or suggestions on how to improve their mathematics teaching. However, Erixon noted, this conversation "did not lead to a process in which meaningful solutions and explanations were created in the reflections" (p. 279). After watching a recorded lesson and listening to the teacher reminisce on the lesson, the class discord did not progress. Instead, the class moved on without offering many suggestions for more effective teaching practices. In reflecting on this established norm, Erixon determined that creating an online learning environment where the content delivery was through participation and communication required a willingness from the class members to speak often, freely, and without concern for time. This was not the online synchronous learning environment of the mathematics teachers' course. She added that "because of the absence of glances and body language, it is difficult for the participants to get an idea of how the other participants respond to messages" (p. 280). It appeared that the actual shape of the online learning environment mattered. For a course to produce rich, deep, and non-superficial learning or conversations, Erixon claimed that the online platform should not be auditory only.

Mayer et al. (2017) seconded this conclusion; the addition of a visual aspect in the classroom appeared helpful for some students. Mayer et al. studied the inclusion of structured group investigations in the remote teleconference option of their advanced calculus and linear algebra courses. Mayer et al. created a special synchronous section, during which the students worked in small groups on activities. The students communicated with a web conferencing software through a variety of ways including instant messaging, microphones, and the shared virtual whiteboard. Mayer et al. reported a statistically significant increase in social cohesion among the students in the special synchronous section as the semester progressed as well as an overall sense of course satisfaction at the end of the semester. The students in the special synchronous section proclaimed that seeing their classmates made attending recitations more enjoyable. These results provided evidence that the inclusion of visual synchronous collaboration spaces increased the feeling of social interconnection. Therefore, Mayer et al. suggested that when deciding whether or not to require students to have audio and video capabilities for recitations sessions, one must consider the tradeoffs between the social benefits these channels afford and the technical requirements that their students must meet in order to utilize them.

The last recommendation for best practices in synchronous online teaching that Gedeborg (2016) presented was to be clear and upfront with class expectations. He acknowledged that a large reason why students enroll in online mathematics courses was for the flexibility and that tension may arise if teachers required attendance at a synchronous session. However, to accommodate some flexibility while keeping the social activities in the synchronous session, Gedeborg advocated for instructors to hold several sessions throughout the day. Multiple sessions empowered the students to believe they had autonomy in their schedule. Golding and Bretscher (2018) investigated mathematics teachers' opinions of a professional development workshop when they had the option to choose the delivery method: face-to-face or synchronously online. Golding and Bretscher analyzed both the face-to-face and synchronous online professional development sessions and conducted focus group interviews with some of the teachers who opted to participate in the synchronous online sessions. Golding and Bretscher recounted that these teachers described "having the confidence to make contributions in a relatively strange group was a challenge; however, the possibility of remaining anonymous opened up opportunities" (p.110). Additionally, the teachers who chose to participate synchronously online enjoyed the location flexibility and time to think critically about the mathematics content. In a way, Golding and Bretscher's study supported Gedeborg's recommendation to provide students with options to participate synchronously online.

As more institutions introduce options for student enrollment in online classes, there must be careful time and consideration for the inclusion of a synchronous version. As described above, there was evidence that synchronous components with audio and visual requirements increased both student achievement and overall course satisfaction (Choi & Walters, 2018; Gedeborg, 2016; Hadjinicolaou, 2014; Mayer et al., 2017). In particular, the ability to contribute anonymously and to build relationships amongst peers appeared powerful. However, if the material and structure of the synchronous components were not well thought out, students found learning challenging and socially uncomfortable (Erixon, 2016; Golding & Bretscher, 2018). In this dissertation study, instructors hosted synchronous online sessions in a platform that supported audio and visual communication and allowed for breakout groups where students could collaborate on Euclidean transformation problems. Although this study did not focus on the effectiveness of the synchronous online setting, in Chapters IV and V, I described the gestures the instructors enacted and hypothesized how these gestures provided more information to the students. My implications supported and expanded the list of Gedeborg's (2016) suggestions for best practices of synchronous online teaching.

Gesture

In this section, I begin by defining the categories of gestures widely accepted and used in the gesture research community. I then describe two purposes for gestures, gestures produced for oneself and gestures produced for the benefit of others. I connect these two purposes specifically to the mathematics classroom. Lastly, I synthesize research results supporting the claim that gesturing in a mathematics classroom promoted student learning and achievement.

Categories of Gestures

McNeill (1992) defined gestures as the "movements of the hands and arms that we see when people talk… [they are] the creations of individual speakers, unique and personal" (p. 1). These movements could be large or small, eccentric or minimal, and referred to physical objects or metaphorical ideas. Researchers investigating gestures frequently cited McNeill's categorization of gestures as the origin point of analysis (Alibali et al., 2001, 2013; Alibali &

Nathan, 2007, 2012; Chu & Kita, 2008; Hostetter, 2011; Weinberg et al., 2015). McNeill defined four types of gestures: iconic, metaphoric, beat, and deictic. Iconic gestures "bear a close formal relationship to the semantic content of speech" (p. 12). These gestures revealed not only the speaker's mental image of the memory but also the particular point of view of the stored the memory. The speech and gesture referred to the same event and partially overlapped, but only looking at the gesture or only listening to the speech provided an incomplete understanding of the described event. It is only through a joint consideration of both gesture and speech that all the elements of the memory became clear. In a mathematics class, an example of an iconic gesture could be a student tracing a graph from the textbook in the air with their finger while they described the shape of the graph. The graph was a real image and the student's gesture mirrored their words.

McNeill (1992) defined metaphoric gestures like iconic gestures because they were graphic or pictorial, but they did not represent actual events or objects. These gestures represented an abstract imaginary concept. "The gesture presents an image of the invisible - an image of an abstraction. The gesture depicts a concrete metaphor for a concept of visual and kinesthetic image that we feel is in some fashion similar to the concept" (McNeill, 1992, p. 14). In a mathematics class, an example of a metaphoric gesture could be a teacher comparing the greater than symbol to an alligator that always ate the bigger number while they made a chomping gesture with their hand. McNeill (1992) defined beat gestures as simple flicks of the hand or fingers that appeared to follow the speaker's vocal rhythm. These gestures indicated "the word or phrase it accompanies as being significant not for its own semantic content but for the discourse-pragmatic content" (McNeill, 1992, p. 15). In a mathematics class, an example of a beat gesture could be a teacher exclaiming that a conclusion holds true for all real numbers. The teacher could raise and lower their hands on the words “all,” “real,” and “numbers,” to draw emphasis and importance to the words. Lastly, McNeill (1992) defined deictic or pointing gestures as those which indicated or located objects and events in the real world. These gestures occasionally pointed to void space, however, “although the space may seem empty it is full to the speaker” (McNeill, 1992, p. 18). In a mathematics class, an example of a pointing gesture could be a teacher pointing to an empty board and saying remember what we wrote yesterday. Although nothing was currently on the board, the gesture with the verbiage allowed the students to visualize what was on the board in the previous class.

Many researchers combined McNeill’s (1992) iconic and metaphoric gestures into one category called a representational gesture. This representational gesture signified a spatial or motor referent by demonstrating a spatial property, or by creating such a referent for an abstract idea (Alibali et al., 2001, 2013; Alibali & Nathan, 2007, 2012; Chu & Kita, 2008; Hostetter, 2011; Weinberg et al., 2015). In this dissertation study, I too adapted and adopted McNeill’s iconic and metaphoric gestures into representational gestures.

For Whom the Gesture is Made

There are two entities for whom a gesture could be made: oneself or others. Several researchers studied the type and frequency of a gesture an individual enacted when the speaker could not see their listener (Alibali et al., 2001; Cohen & Harrison, 1973). Researchers also investigated whether a teacher’s gestures for themselves improved their ability to explain a mathematical idea (Yang et al., 2020) or if a student’s gestures for themselves improved their level of understanding of a mathematical concept (Yoon et al., 2011; Zurina & Williams, 2011).

Cohen and Harrison (1973) tested the notion that people used gestures to better communicate. Specifically, Cohen and Harrison investigated whether the frequency of a speaker's gestures changed from speaking face-to-face to speaking over an intercom. To test their hypothesis, undergraduate students volunteered to give directions to a new staff member walking on campus. Half of the students spoke to a listener face-to-face, while the other half gave directions over the intercom. Cohen and Harrison found that the students used more gestures when speaking in a face-to-face situation as opposed to talking to the other person over an intercom. However, the existence of gestures when the students could not see their listener suggested that some gestures were purely for the speaker; the gestures were an effort to help focus and dictate the speaker's thoughts.

Alibali et al. (2001) also examined whether speakers used gestures differently when their gestures were visible to the listener and when they were not. Undergraduate students watched a short cartoon and retold the cartoon's plot to a listener. Parallel to Cohen and Harrison's (1973) study, the interaction between the undergraduate and listener was face-to-face in one group and blocked by a screen in the other. Alibali et al. found that the addition of the barrier between the student and listener did not lead to the absence of representational gestures. The students continued to make frequent representational gestures when they could not see their listener. The finding suggested that "representational gestures play a role in speech production as well as in communication" (p. 183). Overall, Alibali et al. agreed with Cohen and Harrison's results that some gestures created by speakers were for their use only.

Specifically looking at the mathematics classroom, Yang et al. (2020) examined whether mathematics instructors' gestures made without a live student audience enhanced their teaching performance. The instructors created a short video lecture on finding the x-intercept of a function using the same set of PowerPoint slides. They were not explicitly told to gesture and because no one supervised the instructors creating their video lecture, the gestures they created were for themselves. Three mathematics educator experts evaluated and scored the instructor's video lectures. Yang et al. noticed, based on the expert scores, the instructors who gestured for themselves received higher ratings than those who did not. The evaluators noted that the instructors who gestured not only integrated the visual information into their oral explanations but also linked their oral explanations with gestures. Yang et al. suggested that "instructors' gestures might have helped them retrieve stored knowledge and organize their oral explanations while recording video lectures, thus facilitating their teaching performance" (p. 193). Yang et al.'s results implied that when the mathematics instructors gestured for themselves their lecture was more connected and focused.

Zurina and Williams (2011) studied gestures middle school students made for themselves while learning mathematics; specifically, fractions. These gestures appeared to be miniature with an inward gaze, directed at no one else in the room, and enacted while avoiding eye contact. Zurina and Williams argued that these were "likely to be features characteristic of gestures for oneself as they help withdraw and intensify attention inwardly when reflecting" (p.185) and appeared to occur when a small group discussion reached a disagreement. For researchers and teachers, watching students gesture for themselves provided a window into the student's understanding of the mathematical concept. However, Zurina and Williams argued that gestures for oneself were more important for the learner themselves. Zurina and Williams closed by claiming that "such gestures bridge interpersonal interactions with intrapersonal reflection" (p. 186). This statement suggested that gestures made for oneself helped the students collect the shared classroom knowledge and rationalize how it fit into their mental schema of the mathematical concept.

Lastly, Yoon et al. (2011) also studied gestures made for oneself. The participants were secondary teachers enrolled in a calculus refresher course. The teachers, Ava and Noa, were not specifically instructed to use gestures while working on the Calculus activities, however, they created mathematical gestures while working on anti-derivative tasks with and without a physical context. Yoon et al. claimed that Ava and Noa's gestures were not inherently mathematical, rather, their gestures could be interpreted mathematically. Ava and Noa may have viewed each other's gestures in terms of the mathematical properties that blended the gesture with their own mental image of the mathematical concept. This noticing mirrored Zurina and Williams's claim that gestures for oneself connected interpersonal interactions with intrapersonal reflection. Yoon et al. further hypothesized that "students may likewise use gestures to create, reason with, and communicate through other mathematical constructs to help develop novel mathematical understandings" (p. 390). Yoon et al. suggested that gestures for oneself provided the teachers a space to experiment and test ideas without fear of repercussions if their gesture was incorrect.

In summary, research on gesture for oneself suggested that people gestured even knowing that their listener cannot see or use their movements. Some of these unseen gestures were representational gestures, or gestures which physically portrayed the speaker's words. Therefore, gestures for oneself enhanced the speaker's verbal descriptions and more clearly portrayed their thoughts. Gestures for oneself in the mathematics classroom focused, connected, and clarified a teacher's lectures and allowed students space to try ideas and internalize the shared classroom knowledge.

Alternatively, a person can gesture with the goal of benefitting someone else. Numerous researchers studied the type and frequency of gestures made while a mathematics teacher explained a concept to their students (Alibali & Nathan, 2012; Weinberg et al., 2015). Some researchers focused on a teacher's gesture production during specific classroom interactions such as trouble spots (Alibali et al., 2013) or student contributions (Alibali et al., 2019).

Alibali and Nathan (2012) focused on the gestures that teachers made for their students during class time. Their first claim was that deictic or pointing gestures ground or anchor abstract mathematical ideas in the physical classroom. In the mathematics classroom, Alibali and Nathan reported that mathematics teachers' pointing gestures referred to classroom objects, instructional manipulatives, and symbols or inscriptions such as equations, graphs, and diagrams. As the teachers pointed while speaking, their pointing gestures linked the verbiage to the physical referents. Alibali and Nathan commented that teachers frequently used "sets of pointing gestures to highlight corresponding aspects of related representations" (p. 258). When teachers used a series of pointing gestures, potentially, their students' became more focused on the lesson because the teacher communicated the mathematics concepts in a verbal and spatial manner. Alibali and Nathan's second claim was that representational gestures revealed the teacher's mental simulations of action and perception of action. In the classroom, Alibali and Nathan suggested that representational gestures often revealed characteristics of mathematical inscriptions, most frequently the mathematical inscriptions were visual images. For example, teachers' gestures could simulate or mimic the shape of an inscription within a textbook. Overall, Alibali and Nathan claimed that while explaining mathematics to students, teachers used pointing gestures to ground or anchor a mathematical idea in the classroom and used representational gestures to outwardly reproduce their mental images for a mathematics concept to their students.

Weinberg et al. (2015) detailed the opportunities to communicate mathematical ideas in an undergraduate abstract algebra mathematics lecture through the instructor's gestures. Much like Alibali and Nathan (2012), Weinberg et al. began with McNeill's (1992) categorization of gestures. Their resulting analysis produced an extension of McNeill's framework for characterizing individual gestures, specifically of the pointing gesture. According to Weinberg et al., nestled within the pointing gestures were six distinct features of individual pointing gestures.

> Whether the gesture is concrete or abstract, what the instructor is pointing to, whether the instructor is comparing multiple ideas, objects, or inscriptions, the position and orientation of the arms, hands, and fingers, what the instructor says while pointing, and the level of inference required on the part of the observer to link the pointing gesture to each reference point. (p. 240)

By expanding McNeill's pointing gestures, Weinberg et al. articulated the specific features which may play important roles in interpreting the gestures in advanced mathematic lectures. Each feature communicated more information about the mathematics than the gesture itself symbolized. According to Weinberg et al. when the instructor artfully combined their speech, gestures, and inscriptions, the students had the best opportunity to meaningfully interpret the mathematics.

Rather than analyze all gestures made by teachers in the classroom for students, Alibali et al. (2013) as well as Alibali et al. (2019) narrowed the scope of their studies to focus on particular classroom interactions. Alibali et al. (2013) focused their analysis on middle school mathematics teachers' gestures for students in trouble spots. A trouble spot was defined as a situation where a lack of common ground or shared understanding emerged among the teacher and students. After analysis, they found "that teachers systematically increase their use of gestures, both in absolute number and in rate, following trouble spots" (Alibali et al., 2013, p. 429). More specifically, the teachers increased their use of pointing and representational gestures following trouble spots suggested that the teachers used gestures to communicate pertinent mathematical ideas. For example, when the teachers became aware that they did not have a shared understanding with the students, the teachers increased their use of gestures, presumably in an effort to aid students' understanding. Alibali et al. (2013) argued that "gestures promote comprehension and learning because they contribute to establishing and maintaining common ground" (p. 436). By representing an abstract mathematical concept with a familiar physical action, the teacher's gesture prompted common ground and assisted the students in learning the mathematical concepts. These repercussions of the teacher's gestures helped establish and maintain common ground with their students throughout the lesson.

The other classroom scenario researchers investigated was when teachers used their own gestures to support and highlight a student's contribution. Alibali et al. (2019) focused on how mathematics teachers used their own gestures to support students' contributions to the classroom discourse. Again, their analysis suggested that teachers produced gestures for others to promote common ground within the classroom. To establish and maintain this common ground, teachers often used their own gestures to showcase and clarify the students' utterances when the content was abstract but highly connected to other class discussions. By pinpointing specific referents, the teachers' gestures connected student's ideas to the mathematical content, making the idea more readily accessible and accurate for the rest of the class. Another result suggested that there were "spatial, sociocultural, and semiotic reasons for teachers to use their own gestures to support students' turns at talk" (Alibali et al., 2019, p. 356). To assist students using ambiguous referents, teachers produced gestures that more clearly indicated referents out of the student's reach. Hence, the reasoning for the teachers' gesture was spatial. A teacher gestured for socio-cultural reasons when they chose to revoice a student's ideas as a way to provide that student a voice in a space where they may feel as though they are underrepresented. Lastly, teachers seemed to strategically use their gestures to interweave students' informal thinking with the formal mathematical knowledge or the focus of the lesson. Overall, Alibali et al. suggest that "teachers use their own gestures, not only to support individual students' contributions to the classroom discourse, but also to make those contributions prominent for other students" (p. 357). The teachers' gestures for their students had spatial, sociocultural, and semiotic reasons which helped establish and maintain common ground between all members of a classroom.

In summary, research on teachers' gestures for their students suggested that the most fruitful classroom interactions occurred when the teachers seamlessly connected speech, writing, and gesture. When explaining mathematics to their students, teachers used pointing gestures to ground a spoken mathematical idea in the classroom and used representational gestures to outwardly reproduce their mental images for the mathematics concept. When teachers encountered trouble spots or opportunities to highlight a student's idea, they used their own gestures to ascertain a common ground between themselves and the students. In Chapter IV, I identify the types of gestures that my synchronous online instructors enacted during their classes and document whether the instructors believe their gestures served themselves or their students.

Evidence for Gesture Promoting Learning

In this section, I synthesize research supporting the claim that instruction including gesture promoted learning at a higher rate than instruction without gesture. The following studies investigated the inclusion of gestures when learning a variety of topics including solving mathematics equations with the equalizer and add-subtract strategy, photography editing, and the Doppler effect. In an attempt to control for human error and biases, several studies specifically investigated body visibility and non-human gestures on student learning.

A series of studies focused on investigating the impact of gesturing on student's ability to find the correct number to solve equations of the form $a + b + c = \_\_ + c$ called ABC problems (Congdon et al., 2017; Cook et al., 2008, 2013; Goldin-Meadow et al., 2009; Novack et al., 2014). These studies highlighted two solution strategies, the equalizer (EQ) strategy and the add-subtract (AS) strategy. The EQ strategy explained the conceptual principle that the two sides of an equation must be equal while the AS strategy described a procedural algorithm for adding up numbers on the left side and subtracting the number on the right side. By comparing students' pre- and post-assessment scores when providing different lesson interventions, the research studies produced supporting results; student achievement when solving ABC problems increased with the presence of gesturing.

In 2008, Cook et al. examined third and fourth grade students solving ABC problems when using the EQ strategy. All students took a pre-assessment, received instruction, and then completed a post-assessment both immediately after the instruction and four-weeks later. Cook et al. divided the students into three instruction groups, a speech only group, a gesture only group, and a speech + gesture group. In the speech only group, Cook et al. verbally explained the EQ strategy to the students by stating "I want to make one side equal to the other side" and asked the students to repeat the phrase when solving the equation. In the gesture only group, Cook et al. did not speak and instead moved their hand under the equation's left side, paused, then moved their hand under the equation's right side, and asked the student to repeat their hand movements when solving the equation. In the speech + gesture group, Cook et al. combined the previous two instructions and spoke while gesturing. When comparing the post-assessment results across the instruction groups, Cook et al. found that the students from the gesture and speech + gesture groups retained what they learned from the instruction better than their peers: "These findings suggest that using the body to represent ideas may be especially helpful in constructing and retaining new knowledge" (p. 1054). Cook et al. claimed that gestures, in particular gestures identifying objects or locations, may aid in the connection between mental representations and the physical environment. Their results indicated that encouraging gesture may offer educators a technique for improving student learning.

In 2009, Goldin-Meadow et al. changed the instruction provided to the third and fourth grade students when learning how to solve ABC problems using the EQ solution strategy. Goldin-Meadow et al. partitioned the students into three instruction groups a no-gesture group, a correct-gesture group, and a partially correct gesture group. In the no-gesture group, the students only spoke the words "I want to make one side equal to the other side" (p. 268). In the correct-gesture group, in addition to speaking the same words, the students pointed with a V-hand, or peace sign, to the a and b numbers on the left-hand side and simultaneously pointed with their index finger to the blank on the right-hand side. In the partially-correct-gesture, the students spoke the same phrase and made similar gestures except the V-hand pointed to numbers whose sum were not the correct answer on the right-hand side. From the analysis of the post-assessments, Goldin-Meadow et al. found that students in the correct-gestures group outperformed the students in the partially correct gesture group, who, in turn, outperformed the students in the no-gestures group. Again, there was evidence suggesting that gesturing can facilitate learning by helping students extract information from their own hand movements.

Cook et al. (2013) took the idea of solving ABC addition problems using the EQ solution strategy with second and fourth graders and expanded their operation of focus to multiplication, $a \times b \times c = \_\_ \times c$. Cook et al. split their students into two groups, a gesture group and a speech only group. In the gesture group, the students watched a pre-recorded video on how to find a number for the blanks using simultaneous speech and gestures. In the speech only group, the students watched a pre-recorded video with only verbal instructions on how to find a number for the blanks. When comparing the analysis of the post-assessments Cook et al. found “a robust effect of observing gesture on both initial learning and maintenance of learning” and proposed that “gesture is changing something about the knowledge that children acquire from the instruction and this leads to improved performance across time” (p. 1867). Like the two previous studies, Cook et al. proclaimed that gestures, concurrent with speech, may clue students into the underlying structure of a mathematics problem, which may not only facilitate understanding in the moment, but also may impact how students represent knowledge over time. A new aspect to the study added by Cook et al. was the inclusion of transfer questions to the post-assessments. Transfer questions were mathematics problems related but not identical to the problems that the students watched in the videos. After analyzing these transfer problems, Cook et al. found that students from the gesture only group significantly outperformed their peers. This finding suggested that gesturing could be a way to promote learning for conceptual and procedural understanding.

Novack et al.'s (2014) novel addition to the study involved varying the level of concreteness in student's gestures while solving ABC addition problems using the EQ solution strategy. In their study, Novack et al. partitioned their third-grade students into three different instruction groups an action group, a concrete-gesture group, and an abstract-gesture group. Novack et al. taught the action group to physically pick-up number tiles lying over the numbers in a mathematics problem and then to hold the number tiles in their hand over the blank. They taught the concrete-gesture group to mime the action of picking up and moving the number tiles, without ever actually touching the tiles. Finally, they taught the abstract-gesture group to produce a V-point gesture towards the left most numbers and then to point at the blank on the opposite side with their finger. In all groups, students said the phrase "I want to make one side equal to the other side" in accordance with the equalizer strategy for solving addition ABC problems (p. 905). After analyzing the post-assessment, Novack et al. found that students in the action group demonstrated a relatively shallow understanding of novel mathematics concepts in the transfer questions, whereas students in the other groups appeared to portray a deeper and more flexible understanding. In particular, the abstract gesture group outperformed the concrete-gesture group on the transfer questions. Novack et al. noted that the concrete-gestures and actions appeared to tie the students' knowledge to the training context, suggesting that the beneficial effects of gesture on learning "may stem not only from gesture's base in action, but also from its ability to abstract away from action" (p. 909). Their findings provided additional evidence that students' gestures not only supported immediate learning but led to generalization beyond the task.

The last study highlighted here included a new solution strategy to their investigation, the AS strategy, and tested if the synchronous production of speech and gesture was necessary to achieve the best learning outcomes. Congdon et al. (2017) divided third-grade students into three groups. The first group of students listened to verbal explanations of both the EQ and AS strategies in training. The second group of students listened to a verbal explanation of the EQ strategy and afterwards watched a gestural explanation of the AS strategy with no accompanying speech. The third group of students received a concurrent explanation of a verbal EQ strategy and simultaneous AS strategy in gesture. When comparing the students' scores across the assessments, Congdon et al. reported that students in the third group retained more information and generalized the material more successfully than the other two groups. Presenting students with speech and gesture concurrently "appeared to encourage learners to simultaneously attend to and integrate ideas conveyed in the two modalities and thus create long-lasting and flexible new concepts" (p. 72). Lastly, Congdon et al. found no significant difference between the post-assessment scores of the first and second groups, which suggested that "the embodied nature of gesture, on its own, does not account for gesture's powerful role on this learning task" (p. 73). Therefore, to facilitate deep learning, generalization, and retention over time, the teacher should present gestures simultaneously with speech. The five studies together provided strong evidence for the claim that student achievement when solving ABC problems increased with the presence of gesturing.

With the results suggesting gesturing as an instructional tactic to improve student achievement, like the series of studies described above, some researchers investigated specific features of the gestures in an attempt to control for any human error and biases. These studies specifically investigated body visibility and non-human gestures on student learning. Fiorella and Mayer (2016) investigated whether observing the teacher's entire body while watching a video lesson on the Doppler effect impacted student achievement compared to only watching the instructor's hand. Undergraduate students volunteered to watch a video lesson and take a post-assessment over the Doppler effect to test their retention and transfer skills. Fiorella and Mayer split the students into two groups, a group whose video showed the teacher's whole body and a group who could only see the teacher's hand. From the post-assessments, Fiorella and Mayer found evidence that observing only the teacher's hand during the lesson led to a stronger understanding than observing the instructor's entire body. This result suggested that the visibility of the instructor's body could be distracting and could serve as a superfluous social cue. Fiorella and Mayer claimed that there might be a unique benefit linked with the presence of only the teacher's hand during the lesson.

In order to control both verbal and non-verbal behavior such as eye gaze, face, lip, and body movements while studying the impacts of gesture on student achievement, Cook et al. (2017) used a computer-generated character to teach a mathematics lesson. Cook et al. split their undergraduate students into two groups: a gesture group and a non-gesture group. In the gesture group, the students watched a mathematics lesson taught by an avatar using hand gestures to explain a mathematical equivalence problem. In the non-gesture group, students watched the identical mathematics lesson video except the avatar did not make any gestures while explaining the equivalence problem. Cook et al. reported that students who viewed the mathematics lesson from a computer-generated avatar using gestures learned more than their peers. Additionally, Cook et al. found that students watching the gesturing avatar were more likely to transfer their knowledge when compared with student watching the non-gesturing avatar. Their results suggested that "gesture and other deictic representations may provide a powerful cue for bridging internal and external representations in support of learning and transfer" (p. 529).

Further investigating Alibali and Nathan's (2012) claim that, while explaining an idea to students, teachers used pointing gestures to ground or anchor an idea, Pi et al. (2017) investigated whether the pointing gesture necessarily needed to come from a human teacher. Pi et al. created three short mathematics lectures for three groups of undergraduate students to watch. The first video lecture contained pointing gestures from a human, the second video lecture contained pointing gestures from a non-human source, and the third video did not contain any pointing gestures. In the human pointing group, the instructor stood next to a screen and produced eight pointing gestures. In the non-human pointing group, the video lecture included eight pointing arrows in the slides. In the no pointing group, students heard a vocal recording over the slides. After watching their assigned video, the undergraduate students took a post-assessment testing their retention of key ideas and ability to transfer the knowledge to new situations. Pi et al. found that students who viewed the human pointing video scored higher on their post-assessment than their counter parts. A consequence of this result was that the teacher's pointing gestures were beneficial to learning by serving a unique social function which non-human cues did not accomplish.

All together, these studies provided evidence that students learned new material in a more efficient manner, both as a means of retention as well as with transferability, when a human purposefully connected their verbal explanations to their gestures. Additionally, there was evidence that suggested viewing the entire body of the human instructor could be distracting, hence the proposed ideal situation was to only see the human's hand. In this study, I investigated the gestures of instructors when teaching Euclidean transformations and identified the gestures which potentially provided pre-service elementary teachers additional opportunities for learning in a synchronous online course. The studies above provided evidence that gesturing during instruction impacted students' academic achievement. In Chapter IV, I document the gestures providing students a second opportunity to engage with the Euclidean transformations as well as identify restrictions of the synchronous online setting on the students' opportunities for learning.

Euclidean Transformations

As noted in Chapter I, many researchers studied students' misconceptions of Euclidean transformations (Ada & Kurtuluş, 2010; Aktaş & Ünlü, 2017; Özerem, 2012; Seago et al., 2010, 2013; Yanik, 2014) as well as provided potential reasons for the misconceptions of Euclidean transformations (Hollebrands, 2003; Seago et al., 2010, 2013). Recently, investigations into these student misconceptions allowed researchers to suggest ways of more effectively teaching Euclidean transformations face-to-face (Guven, 2012; Hollebrands, 2003; Idris, 2007; Price & Duffy, 2018). In this section, I begin by focusing on students' misconceptions of the four Euclidean transformations: translation, rotation, reflection, and glide reflection. Next, I describe the potential explanations for these misconceptions and provide a brief review of the interventions used to bolster student achievement of Euclidean transformations. I conclude this section with two studies that explored ways to improve Euclidean transformational understanding using gesture as an intervention (Chu & Kita, 2008; Valenzeno et al., 2003).

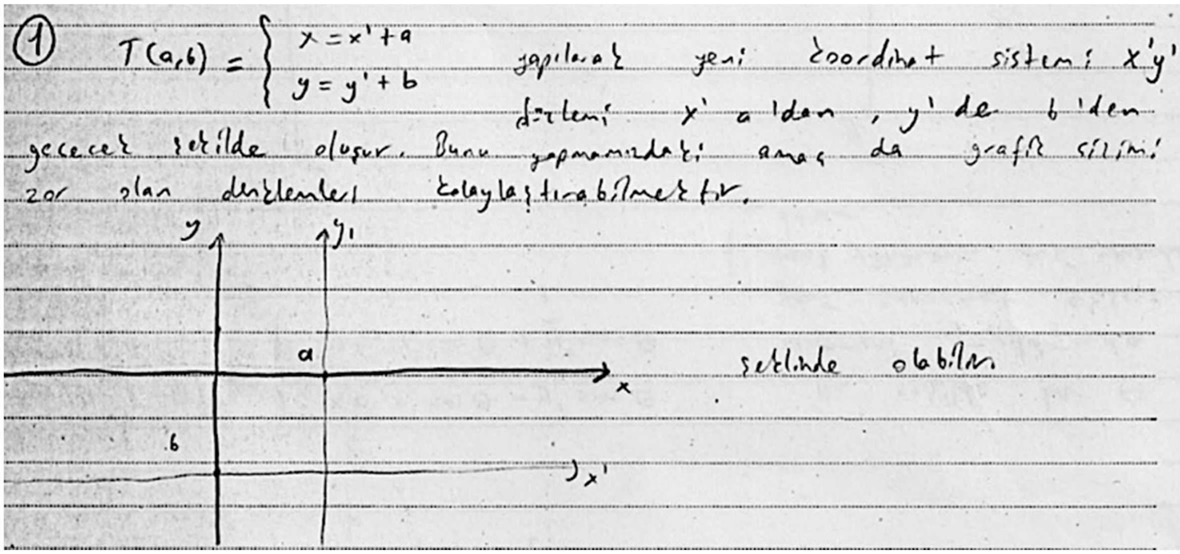
Misconceptions Surrounding Euclidean Transformations

Researchers found that students, elementary through post-secondary, incorrectly solve procedural translation, rotation, reflection, and glide reflection tasks and that students hold underdeveloped or incorrect conceptual understanding of these Euclidean transformations (Ada & Kurtuluş, 2010; Aktaş & Ünlü, 2017; Hollebrands, 2003; Özerem, 2012; Yanik, 2014). Seago et al. (2013) reported that U.S. middle school students correctly answered one-third of open-ended tasks related to Euclidean transformations on a nationwide assessment. This report suggested that students in the United States may procedurally comprehend a Euclidean transformation, but may struggle to effectively convey their knowledge of the concept of a Euclidean transformation to others.

In terms of a translation, some students made errors when procedurally translating an image. For example, Aktaş and Ünlü (2017) asked eighth-grade students to give the new coordinates when the point $M\ (-3,0)$ was translated five units along the x-axis. They found only 12.8% of their students answered the question completely correct. Students appeared to add five units to the y-variable or incorrectly solve the addition problem. For example, some students claimed $-3 + 5 = 8$. Other students demonstrated a lack of a strong conceptual understanding for a translation. For example, Ada and Kurtuluş (2010) found that only 16% of pre-service teachers correctly communicated an answer for a conceptual translation task. In response to the conceptual question, what is the geometric meaning of translation, a student wrote a translation $T(a, b)$ was "a new coordinate system, $x'y'$-plane, occurs such that a on x'-axis and b on y'-axis . . . equations that are difficult to draw a graph can be facilitated" (p. 908). This student proceeded to draw Figure 1 which appears to connect the movement of the x'-axis with b and the movement of the y'-axis with a which disagreed with their algebraic representation and written sentence.

Figure 1

Student's Explanation of the Geometric Meaning of Translation

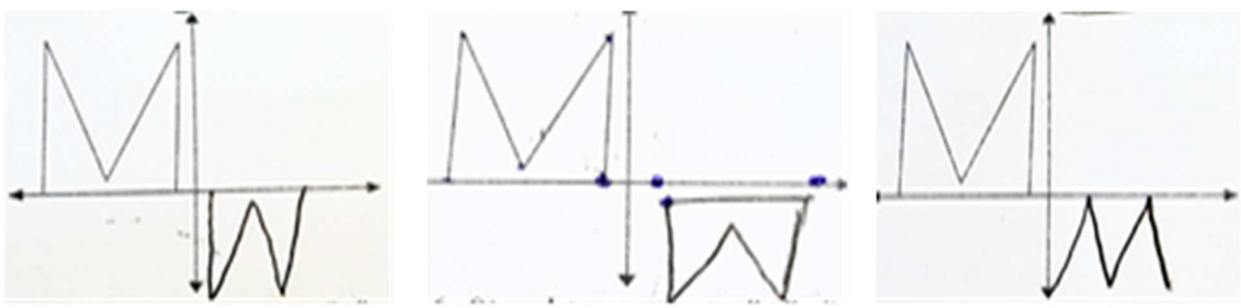


Yanik (2014) provided another example of students struggling with the concept of a translation. He discovered most of the middle school students in his study believed that to complete a translation, they must first move all the corners of a figure, then connect the new points. Although their procedural conception of a translation held for shapes with corners, this particular belief of a translation forced them to believe a circle was un-translatable because a circle has no corners or straight edges to move. Other students reported a circle could be translated only if the circle rolled to the new location. These students also held the conception that a translation can only occur in horizontal and vertical segments therefore, some students viewed an elevator as a translation and an escalator as a non-translation.

For rotations, Ada and Kurtuluş (2010) found that only 35% of pre-service teachers demonstrated mastery with procedural rotational tasks and only 10% correctly communicated an answer for a conceptual rotational task. On their conceptual task, what is the geometric meaning of rotation, a student wrote "when a rotation, $\theta,$ is made in this way, the new coordinate axis rotates counterclockwise and the point *P* rotates in the new coordinate system on graphic" (Ada & Kurtulus, 2010, p. 902). The student neglected to mention any size or shape-preserving properties of the rotation and resulted in a disfigured rotated image. Aktaş and Ünlü (2017) asked middle school students to rotate a capital M, about the origin 180° counterclockwise (Figure 2).

Figure 2

Student's Work When Asked to Rotate a Capital M, About the Origin 180° Counterclockwise (Aktaş & Ünlü, p. 108, 2017).



Less than half of the students correctly rotated the M from the fourth quadrant to the second quadrant (Aktaş & Ünlü, 2017). They reported that many students translated the M rather than rotating the M into the second quadrant. Also, in rotating the M, students did not preserve the shape and size of the M, this suggested that the students did not view a rotation as a rigid motion. Aktaş and Ünlü (2017) noted that their students solved rotational tasks correctly only when they physically lifted the paper off their desk, rotated the paper, and placed the paper back onto their desk with the new orientation. Similarly, Özerem (2012), who studied a year-long, seventh-grade course devoted to learning Euclidean geometry with a transformational lens, found that students incorrectly rotated images on their final exam. Even when provided with tracing paper, the students incorrectly rotated objects around non-origin points. Özerem speculated that small handheld manipulatives may not be enough to enforce the concept of rotation.

Aktaş and Ünlü (2017) asked their middle school students several questions about reflecting various shapes. The middle school students reflected a right triangle over a given line, but again encountered problems with maintaining the shape and size of the right triangle. Aktaş and Ünlü speculated the students' actions resulted from not viewing a reflection as a rigid motion. When the shapes increased in complexity, the students simply translated shapes across the line of reflection without changing the direction and orientation of the object; they only changed the position. Aktaş and Ünlü conjectured these students conceptualized a reflection as a special type of translation. Özerem (2012) found his seventh-grade students could accurately reflect objects but struggled to describe the Euclidean transformation in detail. Özerem speculated the students either did not know the information or simply forgot the mathematical terminology surrounding Euclidean transformations.

Lastly, in terms of glide reflections, while studying a course for pre-service teachers specifically designed to increase their content knowledge for teaching geometry, Mbusi (2019) found that the students struggled to identify a glide reflection. Only 23% of the students in his study correctly identified a glide reflection from the list of choices (see Figure 3). Mbusi's finding was less than the expected 25% chance of selecting the correct pattern if the students simply chose a pattern at random.

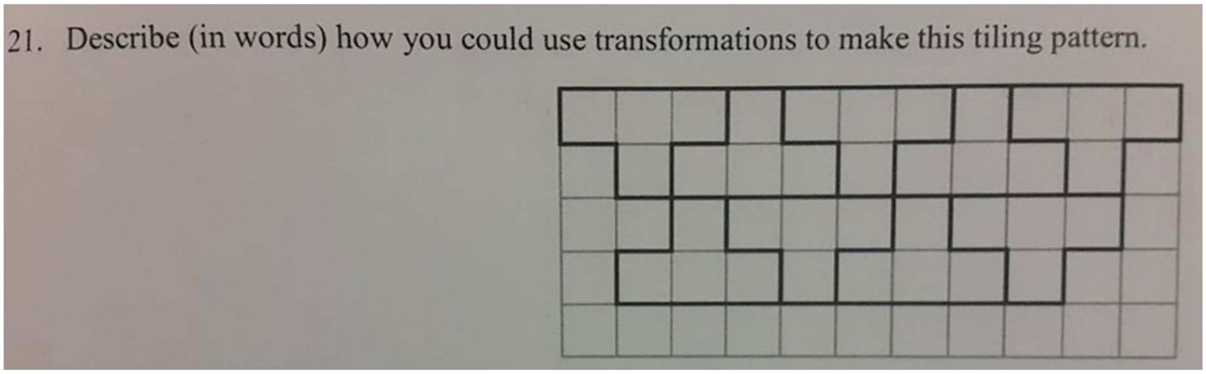
Figure 3

Mbusi's (2019) Question to Identify the Glide Reflection

In a different question, Mbusi (2019) asked the students to describe the transformation that created the pattern (see Figure 4). None of the students correctly answered the question. For example, one student said, "Moving the tile from its origin and not change anything" (p. 255). The student's description lacked all of the unique characteristics of a glide reflection. Another student in Mbusi's study wrote that "the upward tile [went] one unit downward and one unit to the left" (p. 256). This student realized that two transformations occurred to move the shape, they but they did not identify the second transformation as a reflection.

Figure 4

Mbusi's (2019) Question to Describe the Transformations Creating the Pattern or Tessellation



The last question from Mbusi's (2019) study asked the students to investigate a special type of glide reflection, one that created a tessellation. Dello Iacono and Ferrara Dentice (2020) specifically asked elementary school students to identify the shapes in a tessellation and to then describe how the shapes were transformed to create the tessellation. A majority of students in their study successfully described the shapes in the tessellations and that there was something symmetrical about the tessellation, but they were unable to identify the exact type of transformation. Aydin-Güç and Hacisalihoglu-Karadeniz (2020) also investigated students' ability to identify and complete tessellations. In their study, eighth grade students attempted to pick which motif was indeed a tessellation and to expand a given tessellation. Aydin-Güç and Hacisalihoglu-Karadeniz found that only 45% of the students correctly expanded the tessellation when the tessellation consisted of only glide reflections with familiar shapes. In a follow-up interview, some of the students explained they could not determine an individual image in the tessellation pattern, or they did not know the relationship between the image and the pattern.

Students, elementary through post-secondary, struggled with procedural and conceptual aspects of translations, rotations, reflections and glide reflections. Some students viewed circles as un-translatable; others thought a reflection was merely a translation across a specific line; and many students failed to recognize transformations as rigid motions that preserved distance and measure. Earlier in this chapter, I presented literature supporting the notion that gesture in the mathematics classroom provided students additional avenues to learn the material. Hence, in Chapter IV, where I craft a robust description of the synchronous online instructors' gestures for Euclidean transformations, I provide an account of non-verbal avenues that could alleviate some of the misconceptions held by the instructors' pre-service teachers.

Hypotheses for Roots of Misconceptions

The performances on Euclidean transformational tasks prompted researchers to study why these misconceptions occurred and how to potentially remedy them. The research documented formal definitions, specifically the domain and range of a Euclidean transformation (Hollebrands, 2003; Yanik, 2014) and teacher preparedness (Ada & Kurtuluş, 2010; Chinnappan & Lawson, 2005; Mbusi, 2019; Özerem, 2012; Qi et al., 2014; Seago et al., 2013, 2014) as potential explanations for these misconceptions. Hollebrands (2003) claimed students often struggled with the notions of the domain and range of Euclidean transformations. The domain and range of a Euclidean transformation potentially created pitfalls in students' understanding because the assortment of inputs for a Euclidean transformation vastly differs from an algebraic function. Algebraically, students transformed or mapped a single input value, while under geometric Euclidean transformations the inputs included single coordinate points, lines, shapes, and all other points in the plane. Similarly, Yanik (2014) reported some of the middle school students in his study perceived translation as the motion of a single figure or object. The students used their algebraic experiences and simply translated points on the plane one at a time. The students did not consider all points on the plane as the domain of the Euclidean transformation. These researchers suggested that if students obtained a firmer grasp on the notion of the domain and range of geometric Euclidean transformations, then students might gain procedural fluency and conceptual understanding of Euclidean transformations.

Additionally, teacher's lack of preparation in using a transformation lens to teach geometry contributed to students' incorrect conceptions of Euclidean transformations (Ada & Kurtuluş, 2010; Chinnappan & Lawson, 2005; Mbusi, 2019; Özerem, 2012; Seago et al., 2013, 2014). In studying how teachers in the United States taught geometry, Seago et al. (2014) found most teachers did not follow the CCSSM (2010) recommendation to integrate a transformation-based lens into the classroom. This could be because the CCSM geometry section emphasized an approach that differd from how most teachers learned geometry. Integrating this new lens required teachers to gain appropriate skills, content knowledge, and pedagogical content knowledge necessary to effectively implement activities expanding from the students' natural intuition (Ada & Kurtuluş, 2010; Özerem, 2012; Seago et al., 2013, 2014). Seago et al. reported the clarification of the new content, instructional design options, and example student responses in transformation-based curricular materials were inconsistent in terms of availability and robustness. Some materials encompassed copious amounts of detail, and in other materials, details were nonexistent. In a study on how Chinese teachers utilize their geometry resources when teaching transformations, Qi et al. (2014) discovered less than one-third of their teachers utilized their provided resources. Furthermore, the teachers felt as though part of their responsibilities included creating their visuals and manipulatives. This may suggest the materials provided to the Chinese teachers did not contain enough details for teaching. Mbusi (2019) wrote about South African teachers voicing their concern with their inability to use visualization when solving transformation-based geometry problems. Some of the teachers blamed the absence of in-class instruction when working with these problems, while other teachers claimed the transformation-based geometry curriculum felt like a mismatched addition to the course. Mbusi wrote, “these challenges suggest that these teachers, by the time they finished their teacher training, were not adequately prepared with the knowledge and skills involved in geometry” (p. 98). The combination of teacher's low exposure to transformation-based geometry and the inconsistency of transformation-based materials created steep challenges for teachers. Unsurprisingly, many teachers felt unprepared and unsupported to gain the recommended transformation-based geometry knowledge.

In an attempt to provide middle school teachers with support for teaching with a transformation-based lens, Seago et al. (2010) created a geometry professional development program. The professional development program encouraged teachers to explore the Euclidean transformations, to view, analyze and discuss video case studies, and to make links to their classroom practices. In 2014, Seago et al. implemented this professional development program and reported evidence of teacher gains in geometry content knowledge as well as in applications of understanding instructional practices surrounding Euclidean transformations. Even with the success of their small professional development program, Seago et al. (2014) stated there exist fewer professional development resources designed to foster mathematical knowledge for teaching geometry compared to teaching algebra. Therefore, continued research into in-service, as well as pre-service, teachers’ knowledge surrounding Euclidean transformations is required.

The literature suggested students struggle with concepts surrounding Euclidean transformations due to an incomplete conception of the domain and range of transformations and due to a potentially insufficient knowledge base from their teacher. By investigating the types of gestures made by the synchronous online instructors, I identified the gestures which may provide pre-service elementary teachers an additional avenue to learn Euclidean transformations.

Additionally, in Chapter V, I explain the continued need for professional development opportunities surrounding Euclidean transformations. Specifically, how the inclusion of gestures in the mathematics classroom may enhance the teachers' specialized content knowledge and pedagogical knowledge for teaching Euclidean transformations.

Progress Towards Attending to Misconceptions

To address students' erroneous conceptions about Euclidean transformations and to provide teachers with suggestions for transformation-based teaching materials, researchers turned towards three face-to-face intervention methods: DGEs (Guven, 2012; Hollebrands, 2003; Idris, 2007; Yao & Manouchehri, 2019), enrichment activities or curriculum revitalization (Bansilal & Naidoo, 2012; Price & Duffy, 2018), and the purposeful inclusion of gestures (Chu & Kita, 2008; Valenzeno et al., 2003). A technological approach to learning Euclidean transformations through interactions with DGEs appears promising from both a quantitative and qualitative standpoint. Idris (2007) and Guven (2012) found that students who explored, investigated, and discovered Euclidean transformations through interactions with a DGE significantly outperformed their peers. The students' academic achievement improved both in terms of percentage correct on the post-assessment as well as the student's rank within the Van Hiele (1986) levels of understanding. Using qualitative methods, Hollebrands (2003) reported that the DGE afforded her high school students a new collection of possible interpretations regarding Euclidean transformations. For example, with the addition of the DGE, students verbally described a Euclidean transformation as a one-to-one, onto function and viewed Euclidean transformations as both actions and objects. González and Herbst (2009) as well as Yao and Manouchehri (2019) reported the use of the DGE extracted new conceptions of mathematical ideas and realizations of mathematical structures. By incorporating new actions, such as dragging, the students developed new knowledge. For example, the measuring tool within the DGE allowed some students to shift their perspective about congruence from a purely visual understanding to a measure-preserving conception of congruency.

Another approach to improving the teaching of Euclidean transformations was through the addition of enriched activities or curriculum revitalization (Bansilal & Naidoo, 2012; Morgan & Twickenham, 2013; Price & Duffy, 2018). Bansilal and Naidoo (2012) focused on the connection between algebraic and geometric thinking as a way to enrich the learning of Euclidean transformations. They concluded that a student's ability to move between visual and analytic representations corresponded to a deepened understanding of the concepts surrounding Euclidean transformations and their tasks supported this movement between representations. Morgan and Twickenham (2013) described a hands-on workshop where high school students rolled out an ordinary untwisted band to produce repeated translations and a Möbius band to produce repeated glide reflections. The rolling out of the Möbius band gave a physical action that, according to Morgan and Twickenham, facilitated the conceptualization of a glide reflection. Price and Duffy (2018) studied the ways in which elementary school students thought about and interacted with angles and shapes in a whole-body geometry activity. In the activity, students used their bodies to represent different angle measures, to create different types of triangles, and to reflect convex shapes across lines. Price and Duffy then described the different ways in which the students' own bodies impacted their experience and opportunities to learn Euclidean transformations. Each of these studies demonstrated more effective face-to-face teaching strategies for Euclidean transformations.

Lastly, a few researchers studied gestures as an additional avenue to communicate Euclidean transformations (Chu & Kita, 2008; Valenzeno et al., 2003). Chu and Kita's (2008) study examined the gestures people used to determine a rotation of a three-dimensional object. Great Britain adults volunteered to look at several three-dimensional objects. To the best of their abilities, the adults described how one three-dimensional object could be rotated into the position of the others and determined if the three-dimensional objects were indeed the same. Chu and Kita found that adults made two types of representational gestures, hand-object interaction gestures, pretending to manipulate the three-dimensional object in front of them, and object-movement gestures, depicting the axis, angle, and direction of rotation without pretending to hold the three-dimensional object. Chu and Kita's analysis focused on the frequency of these two types of gestures as the experiment progressed. Chu and Kita reported that the participants produced hand–object interaction gestures significantly earlier in the experiment compared to object-movement gestures and over the course of the experiment, the rates of both types of gestures decreased. This finding implied that as the experiment progressed, the adults' simulation of the rotation became more self-contained and the gestures began to represent the movement not the object itself. Finally, Chu and Kita claimed that "co-speech gestures and co-thought gestures may be generated from the same mechanism" (p. 721). This implication supports Alibali et al.'s (2001) comment that representational gestures play an important role in speech production for the speaker themselves.

Valenzeno et al. (2003) investigated whether watching lessons with gestures improved preschool students' understanding of symmetries in images as lines of reflections. In a pre-test, Valenzeno et al. asked the students to define a reflection and to describe how to identify a symmetrical object. The students then watched one of two videos: a video with gestures and a video without gestures. In the gesture video, students watched a teacher produce pointing and representational gestures while explaining the reflection symmetries of five shapes. In the video without gestures, students watched a screen display five shapes while listening to a voice talk about their reflection symmetries. For the post-assessment, Valenzeno et al. scattered six laminated paper figures in front of the student and then asked if each shape contained a reflection symmetry. Analysis of the post-assessment uncovered that students who watched the gesture video correctly categorized the images at a higher rate than students who watched the non-gesture video.

All of the aforementioned studies focused on face-to-face interventions for a more effective teaching strategy of Euclidean transformations. In this dissertation study, I investigated the gestures produced by instructors teaching Euclidean transformations in a synchronous online setting. By documenting the specific gestures that provided students a second modality to access the material, my study began to address a missing piece within the literature, namely how synchronous online instructors could use gestures as an effective tool when teaching Euclidean transformations.

Summary

As access to new technology improves, the option for synchronous learning becomes more realistic. Researchers studied whether utilizing new technology, such as visual and audio capabilities, in synchronous learning improved the students' achievement, the development of class relationships, and the overall satisfaction in the course (Choi & Walters, 2018; Gedeborg, 2016; Golding & Bretscher, 2018; Erixon, 2016; Hadjinicolaou, 2014; Mayer et al., 2017). To varying degrees, the studies concluded that the thoughtful and purposeful inclusion of synchronous learning in online courses was beneficial to students. The current literature surrounding synchronous online learning did not directly address the gestures of these instructors even with overwhelming evidence suggesting an impact of gesturing during communication.

Specifically, research on gesture production in the face-to-face mathematics classroom suggested that the purposeful inclusion of gestures improved student academic achievement (Congdon et al., 2017; Cook et al., 2008, 2013; Fiorella & Mayer, 2016; Goldin-Meadow et al., 2009; Novack et al., 2014). Additionally, numerous researchers looked at the type and frequency of gestures made while teaching a mathematics lesson face-to-face (Alibali & Nathan, 2012; Alibali et al., 2013, 2019; Weinberg et al., 2015). Unlike previous studies, the setting of this dissertation was an online synchronous classroom. The results highlighted the perceived impact of synchronous online setting on the instructor's gestures when teaching Euclidean transformations.

I chose to specifically investigate Euclidean transformations due to the research highlighting students' struggle with the notion of rotations, translations, reflections, and glide reflections (Ada & Kurtuluş, 2010; Aktaş & Ünlü, 2017; Hollebrands, 2003; Özerem, 2012; Yanik, 2014). The teaching experiments employed by scholars to improve student's understanding of Euclidean transformations in a face-to-face classroom included the addition of DGEs, enriched curriculum or activities, and gestures (Bansilal & Naidoo, 2012; Guven, 2012; Hollebrands, 2003; Idris, 2007; Price & Duffy, 2018; Yao & Manouchehri, 2019). Documenting and analyzing the gestures of online synchronous instructors filled an empty cross section of the rich findings surrounding the teaching and learning of Euclidean transformations, the impact of gesturing in the mathematics classroom, and the emerging synchronous online setting. In the next chapter, I discuss my researcher's stance, theoretical perspective, methodological choice, as well as a description of my participants, and the nature of my data collection and analysis.

CHAPTER III

METHODOLOGY

The purpose of my study was to contribute to the literature on the gestures of mathematics instructors as they teach Euclidean transformations in a synchronous online setting. The previous chapter contains recommendations for synchronous online mathematics teaching, the role of instructors' gestures on student learning, as well as students' reasoning about Euclidean transformations. My study sought to answer the following research questions:

Q1 What is the nature of instructors' gestures as they teach Euclidean transformations in a synchronous online setting?

Q2 How, if at all, does a synchronous online setting impact the instructors' intentionality and usage of gestures?

In this chapter, I detail my researcher's stance, theoretical perspective, methodological choice, a description of my participants, and the nature of my data collection and analysis. In the following chapter I describe my results.

Researcher Stance

As an instructor at a four-year doctoral granting institution who has experience teaching, both online and face-to-face, I began this study with my own beliefs about students' learning and effective teaching strategies. I believe learning occurs when students become involved with the mathematics. The level of involvement ranges from actively taking notes and answering online polling questions to discovery learning through group work and rich tasks. My active stance on learning necessitates my classroom to be a healthy blend of interactive lectures and group work activities. Different students have different learning style preferences, so when I teach, the blend of learning modalities is one way that I attempt to reach all my students. Another modality that I consciously bring into the classroom is gesture. I believe that humans communicate not only verbally, but also through gestures. Our gestures have the ability to convey information matching our verbiage but also information that goes beyond what we say. For example, if a teacher says a glide reflection is a flip and a slide, this verbiage does not capture that the translation vector must be parallel to the line of reflection, but potentially the teacher's gesture could. When I teach, I am always thinking of ways to gesture the mathematics because my gestures could be relaying important information and have the potential to serve as another learning modality. I also encourage my students to act out their mathematical ideas because I believe that through purposeful gestures the students have the opportunity to clarify and strengthen their thinking as they actively connect their thoughts to a concrete movement. Hence, not only do I gesture in the classroom, but my students gesture as well. Due to the combination of my view on learning and my belief in the power of gesturing, my theoretical perspective is embodied cognition.

Theoretical Perspective

The nature of my research questions and my personal stance on teaching and learning necessitated my theoretical perspective to connect the instructors' gestures with their mathematical thinking and explanations. In this section, I document the nature of an embodied cognition perspective, as it served as the lens through which I collected, analyzed, and interpreted my data. As part of this section, I provide a brief summary of the role of gesture when adopting an embodied cognition lens.

Embodied Cognition

Embodied cognition is a rapidly emerging area of importance in the philosophy of mind and the social sciences. At its core, embodied cognition represents the belief that human cognition is driven by action and other aspects of our body in conjunction with our brains

(Wilson, 2002). Pouw et al. (2014) wrote cognitive activity involves a "continuous transaction between current states of the brain, body, and the environment" (p. 53). From this perspective, embodied cognition contradicted the idea that humans possessed a mind disconnected from the world that was filled with formal propositions which control our sensory systems' interactions. Gallese and Lakoff (2005) claimed our sensory-motor systems, such as walking, talking, grasping, and standing, not only provided structure to mathematical content but "also characterize the semantic content of concepts in terms of the way that we function with our bodies in the world" (p. 456). Our physical actions served as the foundation for how the mathematics, our bodies, and the environment around us interacted. For example, Pier et al. (2019) investigated students' gestures with an interactive smart board and found that, along with speech, the gestures supported mathematical proof thinking. In small groups, undergraduate students used their upper bodies to gesture on a smartboard enacting the main ideas of mathematical conjectures involving the triangle inequality and the parity in a system of gears. Pier et al. reported that acting out at least one gesture was significantly linked with vocalizing a valid proof for both the triangle inequality and the parity in a system of gears conjectures. Thus, mathematical reasoning and justification skills, students' gesturing bodies, and the environment of the white board worked together to provide students the opportunity to learn the mathematical idea. In other words, Moustakas (1994) pointed out, "Because all knowledge and experience are connected to phenomena . . . inevitably a unity must exist between ourselves as knowers and the things or objects that we come to know" (p. 43). Participants in Pier et al.'s study seemed to demonstrate this unity between their movements, the smartboard, and their mathematical reasoning.

The embodied cognition research community holds two distinct perspectives on how our actions, bodies, and perception of our actions work in tandem with our brains to carry out cognitive function. The main distinction between these two perspectives is the belief of whether or not sensorimotor skills can be exercised both off-line and on-line (Wilson, 2002).

Off-line sensorimotor processing occurs when we disengage from the environment to plan, reminisce, speculate, daydream, or otherwise think beyond the confines of the here-and-now such as in metaphors (Lakoff & Núñez, 2000; Wilson, 2002). Wilson (2002) noted that many mental structures originally developed from actions and appeared to run off-line, decoupled from the physical elements originally used to assist thinking and learning. This off-line processing maintained a body-based perspective because each memory or metaphorical situation upheld a quality of reliving. For example, imagine learning to count. At the beginning of your journey, you may have needed to make large gestures with your fingers to correctly count to ten. Eventually, you may have only needed to move your fingers ever so slightly; others might not have even noticed. Soon thereafter, you may have not needed to use your fingers to count at all. Rather you simply thought about your fingers moving to count and potentially you did not need to think of moving at all. Counting became completely off-line. Thus, even highly abstract mental concepts may be rooted, indirectly, in sensory knowledge. Lakoff and Núñez (2000) proposed humans used cognitive metaphors based upon physical experiences to ground their understanding of mathematical concepts. These metaphors were grounded because we created them from our bodily interactions with the physical world. An intentional experience referred to real entities that we imagined, objects that actually existed. For example, "when somebody is thinking about the moon, it is not just the idea of the moon but of an actual, intentional experience in which the moon is the appearing reference" (Follesdal, 1982, p. 32).

In contrast, on-line sensorimotor processing occurs when we actively engage with the current task environment, taking in sensory input and producing motor output (Nemirovsky et al., 2012; Price & Duffy, 2018). Rather than view our bodies and accessible tools as separate external elements utilized to support our mental activity, on-line sensorimotor processing views them as essential components of cognition. Garrison (2015) argued the quality of an idea relied on the completeness of the sensory, experiential, and bodily foundations that grounded the idea. As an interaction progresses, sensory resources were enlisted for understanding and communicating, and transforming how we utilized the world around us to achieve the desired outcome (Goodwin, 2010). Therefore, the sensory stimuli surrounding us, together with our objectives, dictated what we knew and how we acted. By only discussing the overt experience of the individual, the researcher chose to interpret learning and experience within the environment as fundamentally inseparable (Nemirovsky et al., 2012; Pouw et al., 2014; Price & Duffy, 2018; Soto-Johnson & Troup, 2014). Accordingly, mathematical knowledge was stored in the form of outward responses to a stimulus and acquired by the act of doing (Price & Duffy, 2018). With this perspective, *knowing was doing*, which was observable.

In a literature survey article, Stevens (2012) wrote, "Gestures reveal, manifest, reflect, and provide evidence [for mathematical thoughts]" (p. 340). For example, Soto-Johnson and Troup (2014) utilized their participants' gestures, such as waving fingers to illustrate the motion of a vector z acting on $\bar{z}$ and extending arms to depict a change in magnitude, as evidence of illustrating the analytic-structural and dynamic-synthetic-geometric features of diagrams. These gestures served as a window into the participant's mind, as the gestures seemed to be a result of mathematical thinking. Additionally, gestures could connect or ground abstract ideas or information in the physical world. Grounding depicted a link between the abstract and the more concrete, a familiar object or event, and facilitated meaning making (Alibali, & Nathan, 2012; Koedinger et al., 2008). On-line sensorimotor processing holds the perspective that cognition is more than what occurs in the mind. As Radford (2009) explained:

> The very texture of thinking, I want to suggest, cannot be reduced to that of impalpable mental ideas; it is also made up of speech and our actual actions with objects and all types of signs. Thinking, hence, does not occur solely in the head but in and through language, body, and tools. (p. 113)

This perspective did not argue for the abandonment of individual cognition, rather that cognition could not be minimized to individual mental activity when tools, physical resources, and other individuals created a system more complex than the sum of its parts (Ma, 2017).

Nemirovsky and Ferrara (2009) posited there was no difference between off-line and on-line actions. Rather, the differences between off-line, or mental actions and on-line, or overt actions tended to be that the former ones are condensed enacted versions of the later. Nemirovsky et al. (2012) used the term metaphorical projection which "allows entities that do not have a physical existence to be located in space, move, change over time, be inside each other, and so forth" (p. 289). The idea of metaphorical projection constructed a unique image of bodily activity and abstraction as highly connected, because we portrayed abstract thoughts with having kinetic qualities, while still preserving a clear distinction between them. For example, in mathematics, working with abstract functions in the fourth dimension is common. A fourth-dimension function does not have a graphical representation; however, we could still perform mathematical actions on the function and describe many distinctive features.

My Adoption of Embodied Mathematical Cognition

The use of embodied cognition through gestures in the mathematics classroom is growing and is showing promising results in understanding student learning and research-based teaching practices (Alibali & Nathan, 2012; Alibali et al., 2019; Congdon et al., 2017; Cook et al., 2017). I view mathematical cognition as embodied in two distinct senses, as perceived by Alibali and Nathan (2012). First, mathematical cognition is created and stored in perceptions and actions. Doing and learning mathematics can be thought of as a collection of actions and perceptions of actions we engage in, both overtly and covertly, to justify and understand the environment around us. For example, successful problem solving in a mathematics class often relies on our ability to organize and connect inscriptions created throughout the task such as when we ask students to write down their thoughts and to show all their work. These suggestions demonstrate how mathematical thinking is a combination of our mental activity and the physical inscriptions we create. From this perspective, the physical actions of creating inscriptions and the mental actions of organizing and connecting the inscriptions are integral in the problem-solving process. The creation of inscriptions reduces the cognitive load for students by providing an alternative location to store information rather than in short term memory (Goldin-Meadow, 2005). Núñez (2004) wrote, "From this [embodied cognition] perspective, mathematics is the network of bodily-grounded inferential organization that makes [the mathematical concepts, theorems, definitions, and axioms] possible" (p. 1).

Alibali and Nathan (2012) also claimed mathematical cognition was grounded in the physical environment. This statement remained true regardless of one’s view of embodied cognition as off-line or on-line. From the on-line perspective, we could think of learning with gestures as involving a close pairing of external movements and cognitive processes.

Representational gestures then manifested mental simulations of action and perception (Alibali & Nathan, 2012). These sensory-motor structures grounded understanding of the mathematical relations and operations that they embodied in the physical world, which, in turn, potentially facilitated learning new mathematical concepts. From the off-line perspective, we could think about mental or conceptual metaphors. Lakoff and Núñez (2000) described conceptual metaphors as the mechanism by which we comprehend abstract concepts in terms of our concrete reality by using ideas grounded in our sensory-motor system, like plotting numbers as points on a line. Again, although only in the mind, the metaphors served as the connection to our sensory-motor structures which grounded mathematical concepts.

For my research, I took the on-line perspective of embodied cognition and considered the instructors' gestures taken within the physical environment as evidence for understanding, reasoning, and explaining mathematics. The intentionality of conceptualizing mathematical cognition in this way allowed me to make claims about the instructors' explanations of their own mathematical understandings as a complex system of activity and perception of activity. Individuals communicate their mathematical understanding through their gestures and the words they choose to speak. I believe that we create our knowledge, mathematical or not, through our experience and perceptions of experiences with the world around us. How we demonstrate our knowledge and convince ourselves and others that we understand a concept entails acting upon the knowledge. Thus, from my perspective, knowledge and demonstration of the knowledge is body based and must include some active interaction with the physical world. This view suggests that the instructors' gestures while teaching their synchronous online class in tandem with their verbal descriptions represent their mathematical understandings of Euclidean transformations. Given this view of embodiment, I was interested in how instructors portrayed and communicated their mathematical understanding in a synchronous online classroom setting. I focused my analysis on the instructors' observable verbal language and gestures, particularly those which appeared to help them explain and justify Euclidean transformations in their online classrooms.

Case Study

The overall design of my study was a case study. Merriam (1998) defined a qualitative case study as an "intensive, holistic description and analysis of a single instance, phenomenon, or social unit" (p. 27). Similarly, Patton (2002) described the purpose of a case study as "to gather comprehensive, systematic, and in-depth information about each case [or phenomenon] of interest" (p. 447). The most critical defining characteristic of a case study is delineating the object of study, the case. Merriam viewed "the case as a thing, a single entity, a unit around which there are boundaries" (p. 27). In my dissertation study, a case equated to an instructor. Contrasting with other research designs, a case study is not limited to any particular methods for data collection and analysis. Any methods of gathering data, from interviews to data mining are appropriate for a case study; the important feature of a case study is the case itself. As described in the next section, my data collection included classroom observations and interviews with the instructors. My data analysis involved multiple rounds of coding. By concentrating on the gestures for Euclidean transformations of one instructor at a time, I was more suited to uncover the interaction of significant factors characteristic of the phenomenon such as where a gesture was made, what the gesture was about, how intentional was the gesture. A case study can be further conveyed by its unique features, particularistic – focusing on a particular situation, event, program, or phenomenon, descriptive – meaning the end product is a rich, thick description, and heuristic – illuminating the readers understanding of the phenomenon. My case study had aspects of each feature. For example, this case study was particularistic because it focused on the gestures of synchronous online instructors, a very specific phenomenon. At the same time, my case study was descriptive because answering my first research question required a thick description of the instructors' gestures. Lastly, my case study was heuristic because I attempted to provide a deeper understanding of why the instructors made their gestures, what mathematics they wanted to convey, and how the online environment could have impacted their gestures. As in all research, the choice of a case study design depends on what the researcher wants to know. When answering the how or why questions the case study has a distinct advantage (Merriam, 1998; Patton, 2002). A case study design is particularly well-suited for researchers interested in a process. Case studies may also be described by the overall intent of the study. An educational case study can be descriptive—present a detailed account of the phenomenon, interpretive—used to develop conceptual categories and to illustrate support or challenge theoretical assumptions held prior to gathering data, or evaluative involving description explanation and judgment.

When researchers conduct a study using more than one case and compare the multiple cases, they commonly referred to this comparison as across case study. In a warning, Patton (2002) noted that "the analyst's first and foremost responsibility consists of doing justice to each individual case all else depends on that" (p. 449). In Chapter IV, I use thick descriptions to portray each of the instructors' gestures individually before comparing them. Merriam (1998) wrote that this type of cross case process involves "collecting and analyzing data from several cases that can be distinguished from the single case study that may have subunits or sub cases in bedded within" (p. 40). The case study offers a means of investigating complex social units consisting of multiple variables of potential importance in understanding the phenomenon.

Methods

The following section contains the methods for my dissertation study. First, I describe the participants and setting for my study. Second, I depict the overall structure of the study. Third, I explain my data collection of the classroom observations as well as the procedures for the participant interviews. Lastly, I conclude this section with a discussion of my analysis techniques. Note, before my data collection, I obtained approval from the Institutional Review Board (IRB) for the methods detailed below (see Appendix A).

Participants

Two collegiate mathematics instructors from a doctoral granting, publicly funded institution in the western region of the United States were my participants. In accordance with the IRB, both of my participant's names were changed to pseudonyms to protect their identities. The two instructors taught the third of three mathematics courses, Geometry for Elementary Teachers, designed for prospective elementary teachers in the Fall 2020 semester. Geometry for Elementary Teachers emphasized the development of spatial reasoning in geometry and explored the properties, measurements, constructions, and transformations of two-dimensional shapes. In particular, Geometry for Elementary Teachers had a unit on Euclidean transformations. One instructor, Naomi, served as the coordinator of Geometry for Elementary Teachers and the two instructors met on a weekly basis to maintain consistency between their individual courses. Naomi was a veteran lecturer with 14 years of experience teaching elementary education mathematics courses. She taught Geometry for Elementary Teachers numerous times face-to-face before the Fall of 2020 and was experienced in teaching online courses. Her experience made her a uniquely desirable participant for my study. Naomi’s teaching experience with Geometry for Elementary Teachers was important because she already had the opportunity to critically think about the material of the course as well as the best delivery of that material to enhance learning in an online setting. Her familiarity with the course potentially provided her with better anticipatory knowledge, compared to Edwin who was the other instructor.

Edwin was a graduate teaching assistant with two years of experience teaching elementary education mathematics courses. The Fall of 2020 was his first-time teaching Geometry for Elementary Teachers; however, he taught the preceding two courses in the sequence several times face-to-face in the past. His experience with the sequence of courses and the student made him a suitable participant for my study. Additionally, Edwin was a desirable participant in my study because of his graduate student status. Edwin enrolled in courses that required reading recent research on teaching and learning strategies as well as a wide range of theoretical perspectives. The versatility of his knowledge on the teaching and learning literature potentially provided him with novel tools and strategies to use in the classroom. To recruit the instructors, I sent a personal e-mail asking for their voluntary involvement in my study. In my invitation e-mail, I described the study itself and the time requirement. When the instructors agreed to participate, I sent them an e-mail containing the consent form to read over and sign (see Appendix B).

Overall Study Structure and Setting

My data collection took approximately 60 minutes of the instructors' time beyond their normal teaching responsibilities. I collected two sources of data: video-recordings of the instructors' synchronous online classes as they taught Euclidean transformations and video-recorded interviews. It is important to note that I collected all of my data virtually as to protect the health and safety of the instructors, students, and myself. Table 1 below summarizes the endeavors of my study.

Table 1

Summary of Study

Phases	Phase 1: Online observations of Geometry for Elementary Teachers	Phase 2: Post observation instructor interviews
Collection Method	Video record the synchronous online classes with two cameras	Video record online interviews with each instructor
Purpose	To capture and begin preliminary analyses on the instructors' gestures	To validate the descriptions and perceptions of the instructors' gestures and to describe the intentionality of the gestures

The instructors taught their own section of Geometry for Elementary Teachers which was designed for prospective elementary teachers in the Fall 2020 semester. Each section of Geometry for Elementary Teachers was delivered synchronously online through a university endorsed video conference platform. In a synchronous online session, students and the instructor of each section met live online for 75-minutes, twice a week for 15 weeks throughout the semester. The instructors used strategies for engaging students in discussions inside the video conference platform. Examples of their teaching practices for active learning included the polling feature within the online conference platform and small group work time. As such, students were expected to attend live online classes and authentically participate in virtual conversations. The instructors taught the mathematical topic of interest, Euclidean transformations during the fifth and sixth weeks of the semester. The instructors taught Euclidean transformations for a total of four 75-minute sessions. In three sessions, the instructors introduced and discussed examples of Euclidean transformations. The fourth session was designed to wrap-up or review the materials.

Due to the coordinated nature of Geometry for Elementary Teachers, the instructors used the same activities when they taught the content. See Appendix C for the handouts. However, each instructor had the freedom to choose how they implemented the activities. Thus, each section of Geometry for Elementary Teachers looked different day-to-day. For example, both sections of Geometry for Elementary Teachers began with an opening question of the day, but Edwin put his students into small groups to work while Naomi's students stayed as one large online group. Naomi's students spoke up during her interactive lecture while Edwin's students waited to speak until they were in their own separate, smaller rooms. Both instructors desired student participation. Naomi received formative feedback by using the polling feature built into the online conference platform while Edwin called on individual students to answer questions.

In the first phase of data collection, I virtually attended each of the four synchronous online sessions of Geometry for Elementary Teachers that taught Euclidean transformations to observe and document the instructors' gestures. Disjointed from my study, both instructors recorded their sessions for students to watch at a later time through the screen-capture software built into the online conference platform. In addition, I asked the instructors to record their classes with a separate, auxiliary camera to capture the gestures that the instructors made outside the view of the screen-capture software. The session recordings allowed for an in-depth analysis of the instructors' gestures after the conclusion of the class. After the instructors finished their sessions on Euclidean transformations, I began analysis of data captured on the video recordings to describe the nature of the instructors' gestures as they related to the pre-existing literature on instructor gestures.

In the second phase of data collection, I interviewed each instructor individually to validate my descriptions and perceptions of their gestures as they taught Euclidean transformations and to gather information on the intentionality of their gestures. I conducted the interviews as soon as I completed the analysis process of the video recordings to bolster the chances of the instructors remembering their gestures from the recorded synchronous online sessions. During the interviews, I asked the instructors to make gestures for each of the transformations as if they were teaching the content face-to-face rather than in the synchronous online setting. Next, I mimicked all of the gestures that the instructors made during their online synchronous classes for Euclidean transformations and asked about the consciousness or intentionality of their gestures. I placed a detailed description of the interview protocol later in this section. The interviews not only served as a member check to strengthen the results of my analysis, but they also provided insight into the potential impact of the synchronous online setting on the instructors' gestures.

Phase 1: Euclidean Transformation Class Observations

During the first phase of my study, I joined the instructors' synchronous online sessions virtually. I enlisted the instructors' help to identify which synchronous online sessions most related to Euclidean transformations and attended those specific days. Thus, I did not attend every day of the instructors' class, but only those days the instructors felt were relevant to my topic. While attending, the synchronous online sessions were video recorded in their entirety in two forms and I took handwritten field notes of the gestures the instructors enacted while explaining Euclidean transformations. Patton (2002) described data collection as " more than a single method or technique . . . multiple sources of information are sought and used because no single source of information can be trusted to provide a comprehensive perspective [of the phenomenon]" (p. 306). Thus, the multiple recordings along with my field notes aided in creating the most accurate descriptions of the instructors' enacted gestures. In accordance with my IRB, I obtained permission for attending and recording these classes from both the instructors and their students in the class. I obtained the students' permission to use their recorded dialog in the transcripts of the sessions.

The instructors' synchronous online sessions on Euclidean transformations were recorded in two ways. First, the sessions were recorded through the screen-capture software built into the online conference platform. The instructors self-selected the record option in the online conference platform window setting before the beginning of each session. Each instructor had the ability to highlight and enlarge two different screens. The first screen displayed the view from a document camera where the instructors wrote their hand-written notes and worked through activities. The second screen displayed the instructors' upper body and face. The instructors had the power to switch between these two screens and thus controlled what their students had the opportunity to view. The recording automatically stopped and saved the audio and video to the instructors' personal computers at the end of the session. From this automatically generated video, I had a clear recording of the overall sessions' events from the perspective of the students. It is important to note, that independent from my study, both instructors also recorded their sessions in this manner so that their students had the opportunity to re-watch the session. The second recording method was through an auxiliary camera used to capture any of the instructors' gestures that the online conference platform frame may have missed. I asked the instructors to position the auxiliary camera to frame the space surrounding their upper bodies. The auxiliary camera and its placement were especially necessary because the online conference platform enlarged and recorded the screen of the individual speaking. In addition, the auxiliary camera recorded any gestures that the instructors made outside of the view of the online conference platform window. Thus, the online conference platform could record a student speaking instead of the instructor which might result in an undocumented crucial gesture made by the instructor. The auxiliary camera mitigated this situation. Thus, I had documentation of all of the instructors' gestures, regardless of the speaker within the online conference platform as well as where the instructor enacted a gesture. Video recording the synchronous online classes allowed me to "retain a rich record of behavior that can be re-examined again and again" (Clement, 2000, p. 577) which resulted in strengthening my results.

During the synchronous online classes on Euclidean transformations, I did not ask the instructors to enact specific gestures or to change the frequency in which they gestured. Rather, I documented and described their spontaneous gestures as they taught Euclidean transformations. Note that in phase one, other than recording their sessions with the auxiliary camera, the instructors did not engage in any activities beyond their normal teaching responsibilities.

While attending the synchronous online classes on Euclidean transformations, I took field notes following the recommendations of Patton (2002). Patton wrote that field notes should "consist of what is being experienced and observed, quotations from the people observed, the observer's feelings and reactions to what is observed, and field generated insights and interpretations" (p. 305). For my study, recording the descriptions of the class events and quotes from the instructors in my field notes was not as critical as Patton proclaimed because of my access to the classes at a later date through the video recordings. I had the ability to revisit and re-watch any gesture made by the instructors. Therefore, my field notes centered on my in the moment reflections and preliminary descriptions of the instructors' gestures and what I found interesting or important about them. Patton wrote, "These emergent ideas, themes, concepts, and dimensions generated inductively through fieldwork can also now be deep end, further examined, and verified during the closure period in the field" (p. 323). Later in this section, I detail my process for using my field notes in the analysis process.

Phase 2: Post Observation Instructor Interviews

After analyzing the video recordings of the synchronous online classes, I scheduled a one-on-one, open-ended interview with each instructor through an online conference platform. The interviews lasted around 60 minutes and, in accordance with the IRB, I verbally reminded the instructors that the interview was recorded. I derived the predesigned interview questions from my analysis of the video recorded classrooms. In the next section, I provided details of this analysis process for the video recordings. The interviews consisted of five parts: a collection of questions on each of the four individual Euclidean transformations and then a series of overarching questions on gesturing in general. I ordered the Euclidean transformation questions to match the order in which the instructors taught them in their class, reflection, translation, rotation, glide reflection.

To begin the interviews, I asked how each instructor would gesture for the concept of reflection if they were teaching it in a face-to-face setting. The instructors' answers to this question served as a data point to compare with the gestures that they actually enacted in their online synchronous classes. If the instructor made and described a gesture that they enacted during their online synchronous classes then, potentially, the online environment had a minimal impact on the gestures the instructors created. However, if the instructor made and described a gesture that they did not enact during their online synchronous classes then, potentially, the online environment had a greater impact on the gestures the instructors created. As a follow up to this question, I asked the instructors what mathematical ideas their gestures conveyed. The purpose of this question was to gain insight into how the instructors viewed gestures. If the instructors found many aspects of the formal definition for a reflection in their gesture, it could suggest that they view gestures as conveying mathematical information. At this point, I reenacted all of the gestures the instructors made for a reflection during their online synchronous classes. I mimicked all of the instructors' gestures to efficiently refresh their memory. After observing all of their gestures for a reflection, I asked the instructors if they were aware that they made the different gestures for a reflection. This direct question helped gather evidence for my second research question on the consciousness and intentionality of the instructors' gestures. For the final question, I asked the instructors to reflect on why they made the gestures for a reflection and what could be a potential benefit or pitfall of their gestures. I asked these questions because they allowed the instructors to again describe the mathematics conveyed by their gestures and potentially comment on their consciousness and intentionality while enacting the gestures. According to Patton (2002), the questions I asked were experience and behavior questions. These were questions about what a person did and aimed to elicit behaviors, experiences, actions, and activities of the interviewee. The collection of questions for the remaining three Euclidean transformations, translation, rotation, and glide reflections consisted of the same series of questions.

For each Euclidean transformation, I asked the instructors to highlight gestures which I felt depicted different aspects of the formal definition of the given Euclidean transformation. For a reflection, I asked the instructors to discuss gestures for a reflection that did and did not portray an orientation flip of the image. For a translation, I asked the instructors to discuss gestures that conveyed a translation as a continuous motion and a discrete motion. For a rotation, I asked the instructors to discuss gestures for a rotation where the center of rotation was inside and outside the image. For a glide reflection, I asked the instructors to comment on gestures that demonstrated the translation vector as parallel to the line of reflection and gestures that did not.

After finishing the series of questions on the Euclidean transformations, I began my overarching questions on gestures in general as well as questions specific to each instructor. I asked both instructors to comment on the differences between the gestures they enacted under and away from the document camera. This question elicited answers that provided me with more insight into why the instructors made the gestures where they did and if they were aware or intentional as to where they enacted a gesture. I also asked the instructors why they gestured when they taught and if they were aware of gesture literature. These questions spoke to the intentionality and consciousness of the instructors' gestures. In the individualized questions, I asked the instructors to address unique and peculiar gestures or events from their classes. I attached a description of the interview protocol in Appendix C.

The goal of the interview was not to criticize or judge the instructors' gestures that they made for Euclidean transformations. Rather, the interview validated my descriptions and perceptions of the instructors' gestures and assisted me in gathering information on the intentionality of the instructors' gestures while teaching Euclidean transformations. Overall, sharing my perception of the gestures the instructors produced during their synchronous online sessions served as a member check to strengthen the creditability of my study. Patton (2002) wrote, researchers "can learn a great deal about the accuracy and completeness fairness and perceived the levity of their data analysis by having the people described in that analysis react to what is described and concluded" (p. 560). As previously described, asking questions on the intentionality and purposefulness of the instructors' gestures allowed me to discuss a perceived impact of the synchronous online setting on the instructors' enacted gestures in Chapters IV and V. Lastly, these questions provided the instructors time to critically think about why and how they gestured and potentially afforded them the space to reflect on their gestures.

Using pre-determined open-ended questions provided me the opportunity to compare the instructors' responses to the interview questions and limited the variation between the interviews, thus building reliability (Patton, 2002). Additionally, the interviews took place after the completion of the analysis process of the video recordings. Timely interviews enhanced the chance of the instructors remembering why they made particular gestures from the recorded synchronous online classes. According to Patton, timely interviews with participants provided more accurate descriptions of what happened and the participants' reflections about the phenomenon of interest.

Data Analysis Procedures

My analysis process began during my observations of the synchronous online classes on Euclidean transformations. The units of analysis or cases under investigation were the individual instructors. Adopting Alibali and Nathan's (2012) definition of gestures, I excluded bodily actions, such as posture and gaze, and kept with overt motions of the instructor's arms, hands, and fingers. I did not wish to minimize the importance of the combination of communication modalities, such as gesture, posture, and gaze, however, from a practical standpoint, Alibali and Nathan's interpretation readily allowed for video analysis.

From my field notes and video recordings, I answered my first research question:

Q1 What is the nature of instructors' gestures when teaching Euclidean transformations in a synchronous online setting?

The early analysis occurred within my field notes. Although not a formal part of the analysis process, my field notes provided important preliminary observations as well as identified instances needing further, more in-depth, analysis. For example, in my field notes I documented a particular exchange between Edwin and a student that occurred after one of the recording stopped. If I had not made a note to watch this specific interaction again, I may have missed several novel gestures Edwin enacted for a rotation. My field notes were a place where I collected some general feelings about the instructor's gestures which I then confirmed in the in-depth analysis of the video recording data. For example, I wrote that I felt Edwin used three-dimensional gestures more than Naomi. With the in-depth analysis of the video recording data, this hypothesis was correct.

For the video recordings of the synchronous online classes, I conducted a majority of my analysis on the recording from the separate auxiliary camera and used the automatically generated recording from the online conference platform to supplement my analysis when necessary. The separate camera captured all of the instructors' gestures no matter who spoke on the recording and no matter where the instructor made the gesture. I used the automatically generated recording from the online conference platform when the instructors made a gesture under the document camera while taking notes or working on class examples. It is important to note that Naomi's students could only see the view from the document camera. Even though both Edwin and Naomi had a second screen capable of displaying their upper bodies and faces, Naomi chose to turn this camera off. Therefore, Naomi's students only had the option to view what the document camera captured. This made Naomi's auxiliary camera extremely important because about one out of every five of Naomi's gestures were out of view for her students, but because of the auxiliary camera, I witnessed them.

My analysis began by transcribing all of the instructors and students' verbiage. After completing the detailed verbal transcription, I re-watched all the video recordings and added all of the instructors' gestures to the transcripts. I described and coded all of the gestures made by the instructors. A coded description of the gesture included the time in the recording, the accompanying speech, a written description of the gestures, a series of pictures showing the gesture, the mathematical topic, and the purpose for the gesture. I added photos of the instructors' gestures to the transcripts to provide rich descriptions. The richness of my transcripts promoted "credibility as it helps to convey the actual situations that have been investigated and, to an extent, the contexts that surround them" (Shenton, 2004, p. 69). Table 2 provided an example of this analysis. Additionally, I bolded specific portions of the text within the quote that accompanied the gestures.

I conducted two rounds of first cycle coding, in vivo coding and descriptive coding, and one round of second cycle, coding-pattern coding (Saldaña, 2013). The purpose of the first cycle codes was to divide and describe the video recording into fine-grained pieces, while the second cycle coding reorganized the small data video pieces into meaningful summaries to answer my research questions. The coding displayed in the last two columns of Table 2 were what Saldaña (2013) referred to as descriptive coding or codes that encapsulated the basic topic of a passage of qualitative data in a word or short phrase. These codes identified what was uttered, written, or gestured about in the video recording at that time. From the example below, every time an instructor made a gesture for a rotation, I coded the transcript with the descriptive code "Representing a rotation." According to Saldaña, the third column, describing the gesture itself, was in vivo coding meaning "in that which is alive" (p. 91). This type of code is particularly appropriate for studies that prioritize and honor the participant's voice. For my study, the instructors' voice extended to the gestures they enacted. From the example below, an in vivo code was the juggling motion, raising and lowering his hands in a quick alternating motion on the bolded words. This was the actual gesture that Edwin made one hour 26 minutes and 26 seconds into the session recording.

Table 2

Coded Transcript Example

Time	Accompanying Speech	Description of Gesture	Pictures	Math Topic	Purpose for Gesture

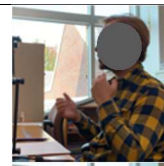
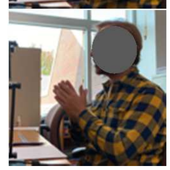
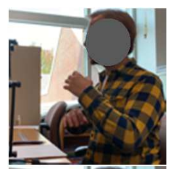
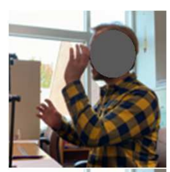
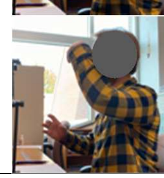
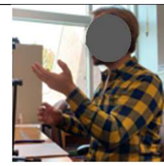
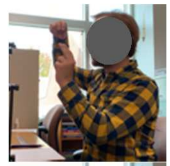
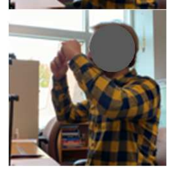
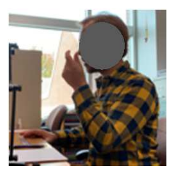
1:21:03	So **similar to** reflection symmetry,	Juggling motion, raising and lowering his hands in a quick alternating motion on each word	 	Rotation	Emphasizing words
1:21:05	you have to find **certain line**s of symmetry to do that with.	Hands make soft blades that he vertically frames his face with, then the blades arc down to table height	 	Rotation	Showing a particular line
1:21:15	A circle has **infinite rotation symmetries** because no matter how you turn a circle, it's still just a circle.	Does a barrel roll with his hands at chest height	 	Rotation	Representing infinite rotations
1:21:19	No matter how you turn a circle, it's still just a circle.	With his left hand he grabs an imaginary circle from the bottom and twists it clockwise 180 degrees then back to the starting place	 	Rotation	Representing a rotation

Table 2 Continued

Time	Accompanying Speech	Description of Gesture	Pictures	Math Topic	Purpose for Gesture
1:21:24	But something like a **square**,	with his left hand still in the air he flicks his wrist open at shoulder height		Rotation	Emphasizing words
1:21:25	if you only rotate it **a little bit,** it becomes a diamond. And so that's not a rotational symmetry.	With both hands he graphs the lower left and upper right corners of an imaginary square and rotates it clockwise and then holds his hands there	 	Rotation	Representing a rotation
1:21:26	So, try to refine your definition so that it's not **infinite rotation symmetries**,	Juggling motion, raising and lowering his hands in a quick alternating motion on the bolded words		Rotation	Emphasizing words
1:21:28	but it's **particular** ones.	With his left index finger and thumb he makes a small gap between them and then holds this small space up to eye level		Rotation	Enacting the notion of specificity

As a way to safeguard the authenticity and quality of my coded descriptions of the instructor's gestures, all of my coded transcripts and the accompanying video were available to one of my committee members to watch and read. At our bi-weekly debriefing sessions, we discussed portions of the coded transcripts and created a plan of how to make the transcripts more precise if my descriptions were not illustrative enough. Shenton (2004) described this tactic as a frequent debriefing session where "through discussion, the vision of the investigator may be widened as others bring to bear their experiences and perceptions" (p. 67).

Next, I preformed second cycle: pattern coding. According to Saldaña (2013) this involved collecting similarly coded passages from the data as a "way of grouping the passage summaries into a smaller number of sets, themes, or constructs" (p. 212). The pattern codes pulled together several fine grain descriptors into a more meaningful unit of analysis. The patterns that I collected were any gestures which represented an aspect of one of the four Euclidean transformations. I chose to look for representational gestures due to the literature that suggested that representational gestures, a combination of McNeill's (1992) iconic and metaphoric gestures, manifested mental simulations of action and perception (Alibali et al., 2001, 2013). After collecting all of the representational gestures for each of the four transformations, I created a master list which contained the representational gesture, the number of times the instructor enacted that gesture, and where the instructor made the gesture. To promote the validity of my second coding process, in a bi-weekly meeting with my committee member, we discussed my second cycle codes and the master list. It was out of this list that I created the interview. On the completion of the provisional coding and analysis, I constructed an answer to my first research question:

Q1 What is the nature of instructors' gestures when teaching Euclidean transformations in a synchronous online setting?

For the recordings of the interviews, my analysis began with transcriptions much like that of the synchronous online classes. First, I transcribed all of the instructors' verbiage. After completing the detailed verbal transcription, I re-watched all the video recorded interviews and added the instructors' gestures to the transcripts. Including the instructors' gestures during the interviews was important because according to my theoretical perspective, the instructors' knowledge and experiences were inherently tied to their actions (Nemirovsky et al. 2012; Pouw et al., 2014; Price & Duffy. 2018; Soto-Johnson & Troup, 2014). Therefore, when the instructors produced gestures during the interview that carried information about the mathematical ideas underpinning a Euclidean transformation, I wanted to have them documented. Similar to my procedures with the synchronous online class recordings, the entire interview and corresponding transcripts were available to a committee member to watch and read.

The coding process of the interviews also consisted of first and second round coding cycles. I used descriptive coding as my first step. Again, Saldaña (2013) posited that descriptive coding helped answer the general question "what is going on here" (p. 88). By asking interview questions on the intentionality and purposefulness of the instructors' enacted gestures, I was able to gain a better understanding of the online setting's impact on the gestures the instructors produced. Next, I moved to the second cycle coding phase, pattern coding. According to Saldaña, pattern coding was a way of grouping several smaller first round codes into "ones that identify an emergent theme, configuration, or explanation" (p. 210). I used the emergent themes to compare the instructors' interviews as well as classroom observations in Chapter IV. This assisted me in answering my second research question: How, if at all, does the synchronous online setting impact the instructors' intentionality and usage of gestures? My committee member and I discussed the emergent themes from the final pattern codes in one of our bi-weekly meetings. According to Patton (2002) each conversation between my committee member and myself "reduces the potential bias that comes from a single person doing all the data collection… and analysis" (p. 560). Again, the unit of analysis or case under investigation was each individual instructor. Therefore, my cross-case analysis consisted of my comparison between the two instructors' the overall gesturing patterns or emerging themes from the online synchronous session observations and interviews (Patton, 2002).

In the next chapter, I describe the instructor's gestures when teaching Euclidean transformations in exquisite detail as well as depict the interview conversations on the intentionality of these gestures to address my research questions.

CHAPTER IV

RESULTS

In this chapter, I first describe the instructors' gestures, when the gestures were used, how the gestures were used, and the purpose of the gestures. From this thick description, the story of the nature of the instructors' gestures while teaching Euclidean transformations evolved. Then, I summarize the post interview conversations on the instructors' intentionality of these gestures. Finally, I compare the stories of the nature of each instructors' gestures to address my research questions.

Q1 What is the nature of instructors' gestures when teaching Euclidean transformations in a synchronous online setting?

Q2 How, if at all, does a synchronous online setting impact the instructors' intentionality and usage of gestures?

I partitioned the results from my data analysis by instructor. The presentation of Edwin and Naomi's separate classroom narratives depicted their instruction surrounding the Euclidean transformations reflection, translation, rotation, and glide reflection. The narrative synthesizing the four-day Euclidean transformation unit followed the order in which the instructor's taught the material in their synchronous online sessions. To seamlessly depict the instructors' simultaneous verbiage and gestures, I included the instructors' gestures in parentheses within the quote. Additionally, I bolded the text of the quote that accompanied the gestures. Then, I crafted a narrative on the intentionality, background, and usage of the instructors' gestures from their post observation interviews. Lastly, in a cross-case analysis, I compared Edwin and Naomi's narratives. I begin with Edwin.

Edwin's Synchronous Online Classroom Narrative

Overall, after analyzing the data from Edwin's class as he taught transformations to his pre-service elementary teachers, I found that he utilized a combination of representational and pointing gestures. He frequently used large three-dimensional gestures when he addressed the web camera and small gestures when he used the space under the document camera. Based on my analysis, many of Edwin's representational and pointing gestures portrayed a holistic picture of the action of the Euclidean transformation rather than the mathematical precision of the formal definition of a given Euclidean transformation. Edwin and his students worked through a class handout consisting of the definitions of each Euclidean transformation and many opportunities to practice the Euclidean transformations (see Appendix D). Through my observations of Edwin's synchronous online classroom, it seemed as though he wanted his students to actively work through the handouts while he demonstrated a Euclidean transformation as well as when they worked in small groups with peers. In this section, I created a narrative synthesizing Edwin's four-day Euclidean transformation unit. I included all of Edwin's gestures for Euclidean transformations.

To start the unit on Euclidean transformations, Edwin described a Euclidean transformation in general terms. He talked about a Euclidean transformation as a mapping which took an image and changed it through some type of process such as sliding, spinning, or flipping. In his interpretation of the definition of a Euclidean transformation he said, "Breaking down that definition, that one-to-one mapping or point by point mapping, means that every point on our figure we are going to pick up and move individually to our new position through some sort of process." The first Euclidean transformation under investigation was a reflection. Before formally defining a reflection, Edwin provided a brief demonstration on how to use a translucent reflection tool called a mira. He described a mira as a tool used to reflect an image to the other side of a line and explained that a mira's weakness was its lack of precision due to human error. First, the students reflected two images across a provided line and found the line of reflection between two pairs of images on the handout with the mira. As the students worked quietly on the reflection problems, Edwin reminded the students that a line of reflection: "Is the line that this image was **reflected over** (lifted his left hand, palm facing up, at shoulder level as if holding a tray and in an arcing motion flipped his hand to the right until the palm of his hand faced down." (see Figure 5)

I coded this gesture as a representational gesture due to the fact that Edwin's hand, which was the object under reflection, flipped across an imaginary line in such a way that mirrored his hand's orientation. His gesture was directly tied to his words. This gesture highlighted for the students that a reflection switched an image from one side of the line of reflection to the other while reversing the orientation. By itself, the gesture did not emphasize that the line segment connecting a point p and its reflected image p' was perpendicular to the line of reflection or that the line of reflection bisected the line segment created between point p and its reflected image p'. However, this did not appear to be the overall goal of the gesture.

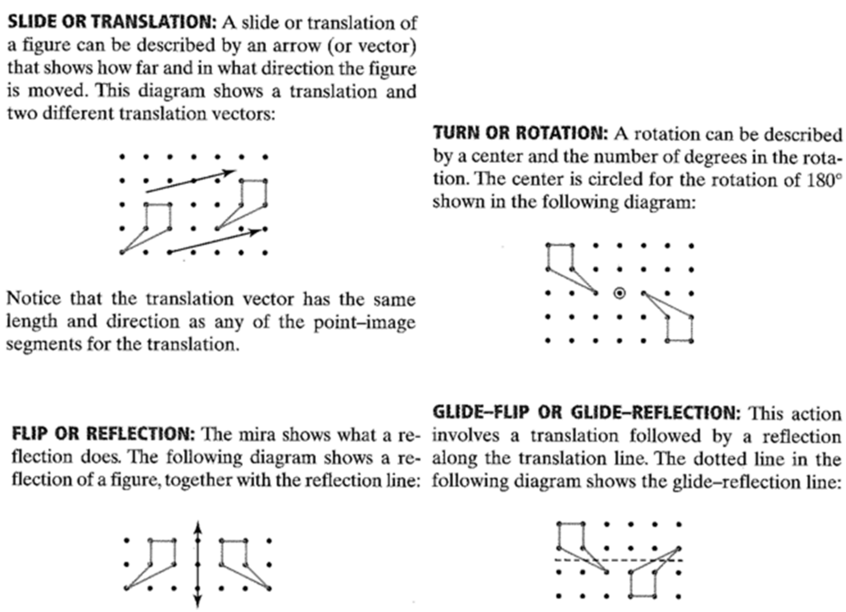
After a few more minutes of quiet individual work time, the students completed the opening questions on the handout. Once the students had the opportunity to reflect objects for themselves, Edwin transitioned into formally defining not only a reflection, but all four Euclidean transformations. An example of the same polygon altered by each of the Euclidean transformations on a grid was part of the handout definitions (see Figure 6). Interestingly, the order of the handout's definitions was not the order in which Edwin presented the content. The grid background highlighted the nuance characteristics of each Euclidean transformation.

Figure 5

Hand Flip from Palm Up to Palm Down at Shoulder Level

Figure 6

The Handout's Definitions of a Euclidean Transformation



SLIDE OR TRANSLATION: A slide or translation of a figure can be described by an arrow (or vector) that shows how far and in what direction the figure is moved. This diagram shows a translation and two different translation vectors:

Notice that the translation vector has the same length and direction as any of the point–image segments for the translation.

TURN OR ROTATION: A rotation can be described by a center and the number of degrees in the rotation. The center is circled for the rotation of 180° shown in the following diagram:

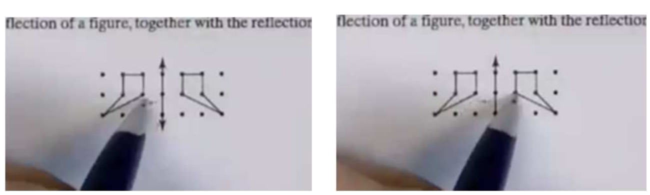
FLIP OR REFLECTION: The mira shows what a reflection does. The following diagram shows a reflection of a figure, together with the reflection line:

GLIDE–FLIP OR GLIDE–REFLECTION: This action involves a translation followed by a reflection along the translation line. The dotted line in the following diagram shows the glide–reflection line:

Edwin started with the definition of a reflection. He noted that on the grid, everyone could clearly identify that the distance between a point and the line of reflection remained unchanged in the reflected image. He said, "We can see that distances are preserved from the line of symmetry. **This dot** (pointed to the left image) is one unit away from our line as is, as is **its post image** (pointed to the right image)" (see Figure 7). I coded this gesture as a pointing gesture, because Edwin used his gesture to draw attention to the image and grounded his speech in a concrete example on the handout. This gesture highlighted that a reflection switched an image from one side of the line of reflection to the other rather than the notion that a line of reflection served as the perpendicular bisector of the line segment connecting a point p and its reflected image p'. Verbally, he emphasized that when investigating a reflection, the line segment between point p and its reflected image p' was important to identify. However, he did not elaborate on why this was the case verbally or with his gesture.

Figure 7

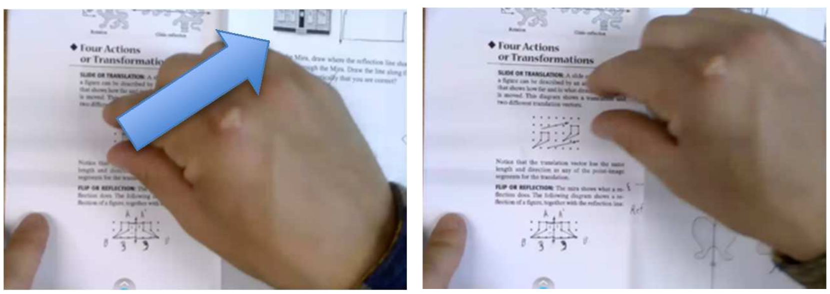
Pen Hopping from Left Point Over the Line of Reflection to the Right Point



Next, Edwin progressed onto the handout's definition for a translation or as he described it, a slide. He said that a translation was: "What we tend to think of as **picking this up** (pretended to grab onto the printed polygon) and **moving it over** (slid his hand in a northeast direction while maintaining the shape of his hand) without really altering its state at all" (see Figure 8). Because the small imaginary line segment between his index finger and thumb symbolized the object that Edwin translated, and his motion embodied the smooth movement of the translation, I coded this a representational gesture. His gesture and the words he spoke to his students conveyed parallel ideas. This gesture appeared to portray the idea of a translation as a rigid motion, because throughout the motion the distance between his finger and thumb remained the same. Additionally, this particular gesture seemed to convey that all points in a translation shifted the same distance in the same direction.

Figure 8

Edwin Sliding Printed Figure Across Paper in Northeast Direction



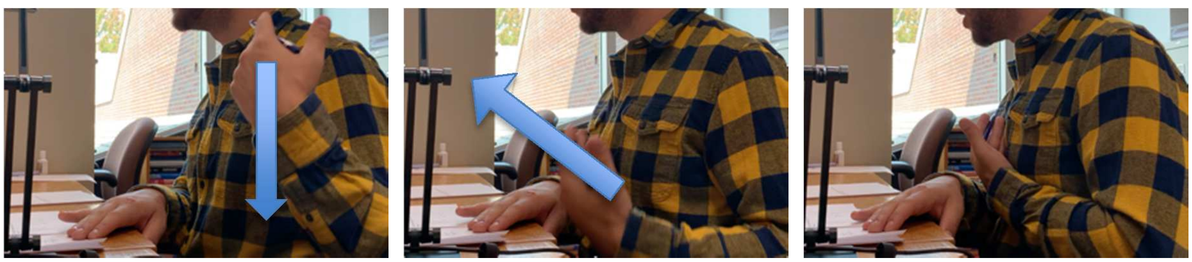
Edwin further explained how to view a translation in terms of the horizontal and vertical components of the translation vector. On the handout's definition for a translation, he pointed to the right polygon, and said,

> If **this** (pointed to the right polygon) was our pre-image and **this** (pointed to the left polygon) was our post image, the translation that would take us there is **down one** (lowered his open-faced, hand down a few inches), **left four** (slid his hand left to the right). (see Figure 9)

I coded these two gestures as pointing and representational respectively. Edwin's first set of gestures connected his words to the printed example on the handout and drew attention to the direction of the translation. From the gesture, the students did not have the opportunity to gain any mathematical information, hence, I coded this first set of gestures as pointing gestures. For the second set of gestures, Edwin's gesture served as a second form of communication; the students had two opportunities to digest what he said, one through speech and one through gesture. Thus, I coded the second set of gestures as representational. The smooth sweeping motion of Edwin's hands potentially conveyed to the students that a translation shifted all points of the object at the same time in the same direction.

Figure 9

Edwin Moving Right Hand Down Then to the Right



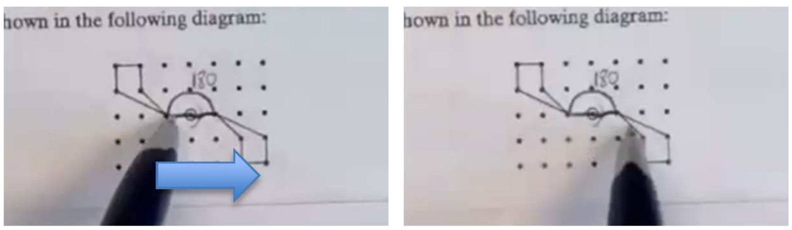
For the handout definition of a rotation, the provided example was a 180-degree rotation about a point not on or within the polygon (see Figure 6). To explain why this particular example was a 180-degree rotation, Edwin said,

> We can tell that that's a 180-degree rotation because the **connection** (drew a line between a point and its rotated image) between the two lines, the connection between the pre and the post image, not only runs through the center**,** but forms a **straight line** (traced the line). (see Figure 10)

I coded this gesture as pointing because his movements annotated the image and brought attention to the features of the printed image while he spoke. The pointing gestures grounded his speech within the definition example on the handout. The gesture did not convey information about a rotation, rather the gesture allowed the students to locate parts of Edwin's speech. Edwin briefly added that the angle formed between any two corresponding points and the center of rotation would always be 180-degrees in this rotation. However, he did not elaborate further.

Figure 10

Drawing and Tracing the Line of Reference

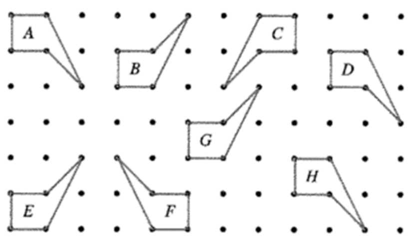


The last definition that Edwin deconstructed for his class was a glide reflection or as the handout labeled it, glide flip (see Figure 6). Edwin explained a glide reflection as a combination of a translation and a reflection. In addition, he emphasized that the order of these two Euclidean transformations did not change the location of the final image. When describing the provided example on the handout, Edwin said, "We could reflect that **image** (touched the left polygon) across **this line** (hopped the pen over the given line of reflection) and then move it **three right** (moved the pen to the right polygon)." I coded this series of gestures as pointing because, similar to the previous gesture for a rotation, each individual gesture drew the student's attention to the image under discussion and grounded what Edwin said within the printed image.

After illuminating the handout's definitions and examples for each of the four Euclidean transformations, Edwin placed his students in small groups to work on the first of many problems. In Edwin's online synchronous classroom he placed students into small virtual groups to work as a team to complete problems. Edwin then joined each of the small virtual groups and provided motivation or assistance when needed. When Edwin determined that the students had a sufficient amount of time to complete the task, he brought the whole class together and asked the students to share their responses. In the first problem (see Figure 11), the students determined which transformation altered the original polygon, image A, to match each of the other polygons shown, images B through H.

Figure 11

First Small Group Problem on the Handout



After allowing the students several minutes to work in their small groups, Edwin brought the whole class back together and asked for student volunteers to share their answers. Several students identified a transformation they found, but the class soon became quiet. It appeared that none of the students wanted to share how to alter image A to make image C. To help advance the class' progress, Edwin said "C you really can't make by **rotating** (traced the circumference of a circle with his right index finger in the air at eye level) A around in a circle" (see Figure 12). I coded this gesture as representational because the tip of his finger symbolized the object that Edwin rotated, and his motion embodied the continuous movement of the rotation. The gesture was a physical illustration of his verbiage. From his finger tracing gesture, the students had the opportunity to view a rotation about a center point. In this gesture, the center of rotation was his wrist, and the tip of his finger was the point under rotation. His center of rotation remained in approximately the same location while his finger traversed the circumference of a circle. The radius of this circle, also the distance from the center of rotation to the point, was the length from the tip of his finger to his wrist and it remained unchanged.

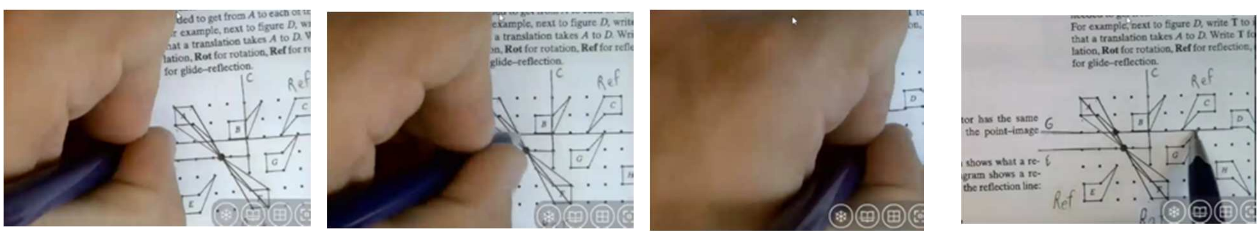
Figure 12

Tracing the Circumference of a Circle in the Air with His Right Index Finger

After eliminating a rotation as the Euclidean transformation necessary to alter image A into image C, a student spoke up and said, “I saw from A to C as a reflection, so a [reflection] line three to the right.” Edwin happily exclaimed “Yes, I would agree with you!” A few more students shared their thoughts on which Euclidean transformation converted image A onto other images until the only unaccounted for image was image G. Edwin shared with the class that the Euclidean transformation connecting image A to image G was a glide reflection. He said, “We're going to have a line of symmetry **here** (drew a line on the paper) ... and so **this image** (touched image A) to G is a translation and then a reflection or just a **glide reflection** (hopped his pen over the line of reflection and hopped to image G)” (see Figure 13). I coded this gesture as a pointing gesture, because Edwin used his gesture to draw attention to the image and grounded his speech in a concrete example on the handout. This gesture provided the students the opportunity to view a glide reflection as a composition of two transformations. Also, the students could see the necessary condition that the translation vector must be parallel to the line of reflection through the movement of his pen. This mathematical idea did not surface in his verbiage, only in his quick hopping gesture.

Figure 13

Sequentially Tapping on the Line of Reflection, Original Image, Translated Image, and Reflected Image



Edwin sent the students back into their small groups to work on the next problem. The second problem provided the students with a pentagon $ABCDE$ and its translated image $A'B'C'D'E'$. In problem two, the students described a translation with an ordered pair. Edwin virtually switched between the groups to assist them with their progress. The following exchanges were from these small group interactions and highlighted four of Edwin's gestures.

> Edwin: Have we said anything about the movement?
>
> Student: I wrote that it was up three and to the right four. So, A was up three right four to make A prime.
>
> Edwin: Okay. So now we're saying something about the actual movement that's taking place. Is that the same movement for all of the points?
>
> Student: Yes?

Edwin: Yeah! And so, we don't really want to say that we're going to take A to A prime because that's what any transformation is going to do. We want to say something about those specific movements. And so, our ordered pair should capture that movement to the **right** (held the sides of an imaginary cube and slid the cube to the right) and our movement **up** (slid the imaginary cube away from his body) specifically. (see Figure 14)

Student: Okay! So, I got three four.

Edwin: Humm, how do we normally talk about things on the **$\boldsymbol{X}$-axis** (moved left hand from left to right) and **$\boldsymbol{Y}$-axis** (moved left hand up and down) do we normally say X then Y or do we normally say Y then X?

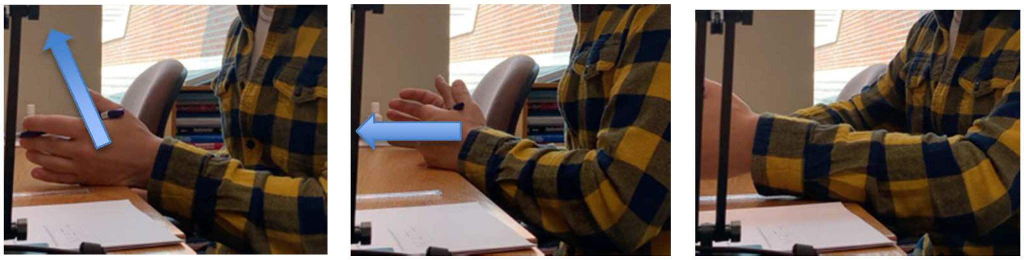
Student: X then Y

Edwin: Right, and so, we want to capture that horizontal movement first and then the vertical movement. Okay. I'll let you guys think about what we can say, and I'll go check in on somebody else.

In the above exchange, Edwin gestured in three dimensions using an imaginary cube to demonstrate a translation. I coded this gesture as representational, because the gesture was closely tied to the words Edwin spoke and the motion demonstrated a translation as the movement of a tangible object. This three-dimensional gesture appeared to convey to the students that the imaginary cube shifted with constant speed in two separate directions as if to decompose the translation vector into its vertical and horizontal components. The gesture seemed to convey to the students that the entire imaginary cube shifted as a whole unit in the same distance and direction. I also coded Edwin’s gestures for the coordinate axis as representational because the vertical and horizontal motion of his hand aligned with his spoken words.

Figure 14

Moving an Imaginary Cube to the Right and Then Away from His Body



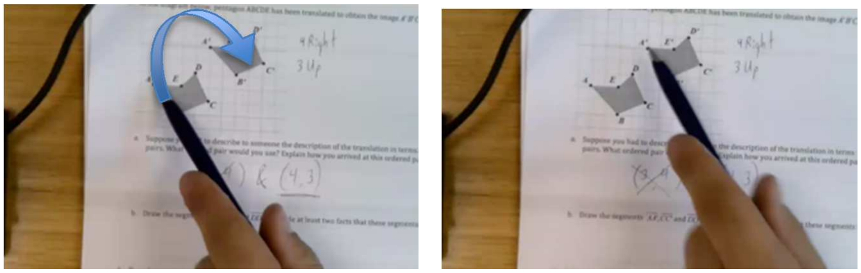
In a different small group interaction, Edwin assisted students who claimed the order of the numbers in their ordered pair describing the translation was unimportant. The students believed $(4, 3)$ was the same translation as $(3, 4)$. To gently push the students to see that the order within the ordered pair carried crucial information, Edwin said,

> We tend to read them [ordered pairs] in the vertical or in the horizontal direction or the x axis first and then vertically in terms of y. So, it makes sense that our mapping, or our directions, from **point A** (tapped point A with is pen) to **point A prime** (hopped his pen to point A') are in that order. (see Figure 15)

I coded this gesture as pointing because it located the points that Edwin talked about on the printed handout. From Edwin's pointing gesture it was clear that a translation needed a starting point or pre-image and resulted with a shifted point or post-image, but this gesture did not emphasize the notion that a translated image as a whole maintained size and shape.

Figure 15

Jumps Pen from Point A to Point A Prime



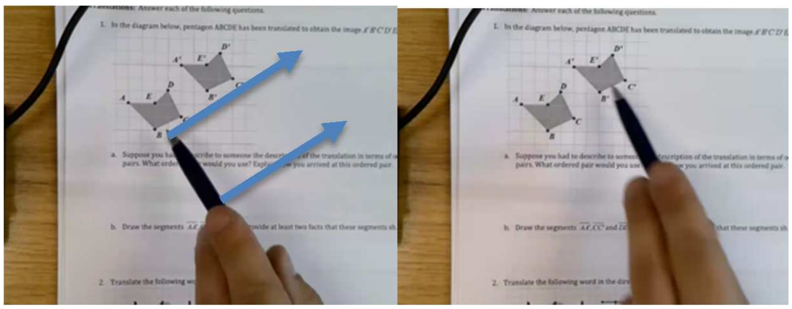
After his statement and accompanying gesture, one of the students in the small group shared, “Oh, so the order of the numbers effect what, like how many it goes up, like wouldn’t following four before the three, make it in a different position”? Edwin excitedly replied, “Yes, it totally is”! Edwin then proceeded into the next virtual small group and discovered that these students believed the ordered pair to describe their particular translation was (B, B'). To help the students recognize their initial guess did not carry enough information, Edwin said,

> Edwin: Before we talk about the ordered pair, what is the motion that is going on between our **image** (tapped the left pentagon) and our **post image** (slid his pen to the right pentagon)? (see Figure 16)
>
> Student: Is it a slide?
>
> Edwin: It is a slide! Can we be a little more specific? That's okay if not. It is a slide or a translation, but I'm looking for numerical values on that translation.
>
> Student: Would it be um, four right and three up?
>
> Edwin: Right!

For the same reasons as the previous gesture, I coded this gesture as pointing. This particular gesture identified the pre- and post-image points Edwin talked about, making this a pointing gesture. The motion between these points carried some mathematical information that his words did not. The sliding action with his pen potentially conveyed to the students that a translation shifted a point in a particular direction for a specified distance.

Figure 16

Sliding Pen Smoothly Along the Paper Between the Pre- and Post-Images



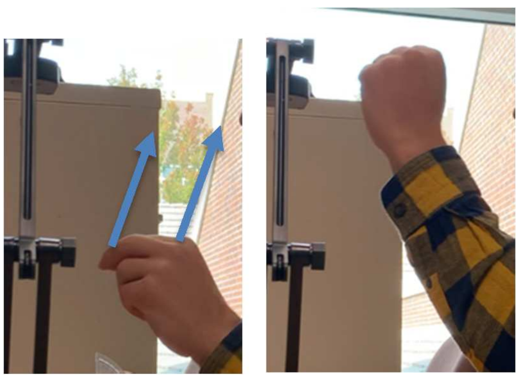
In the last small virtual group interaction, Edwin encountered another group who believed that the ordered pair for the particular translation was (A, A').

> Edwin: How did we characterize our point from, our ordered pair?
>
> Student: I said A, A prime.
>
> Edwin: That's interesting. What is our ordered pair meant to communicate?
>
> Student: I am not sure if it is going to be like the order that you follow [around the shape].
>
> Edwin: Well, part a says that our ordered pair should describe the translation and we can make our **ordered pair** (grabbed onto an imaginary point in front of him) something having to do with three and four, it's a **movement** (moved the imaginary point in a new location). (see Figure 17)
>
> Student: Right! It is like directions.
>
> Edwin: Yes. I'm going to kind of leave you there, but I think there's still a little more to flush out with our definition.

I coded this gesture as a representational gesture because it depicted his verbiage with imaginary referents. Unlike the previous gesture, he did not locate an item within the handout, rather he used an imaginary point he plucked from the space in front of him to aid his speech. This gesture potentially provided the students with a second avenue to grapple with a translation, one through Edwin's words and one through physical action. Much like the previous gestures of jumping or sliding to a new location, this gesture appeared to describe a translation as a mapping of a point to a new location, a pre-image to a post-image.

Figure 17

Edwin Plucking an Imaginary Point from Space and Moving It Up to a New Location



When Edwin brought the class back together, he chose to demonstrate the next problem himself. He could have made this choice because of the common error that several groups made when describing the translation as the ordered pair (A, A'). Problem three on the handout provided students with the word "MATH" printed in large letters on a grid. The task was to translate the word MATH using the ordered pair $(3, -6)$. While demonstrating how to translate one point from the M, he said,

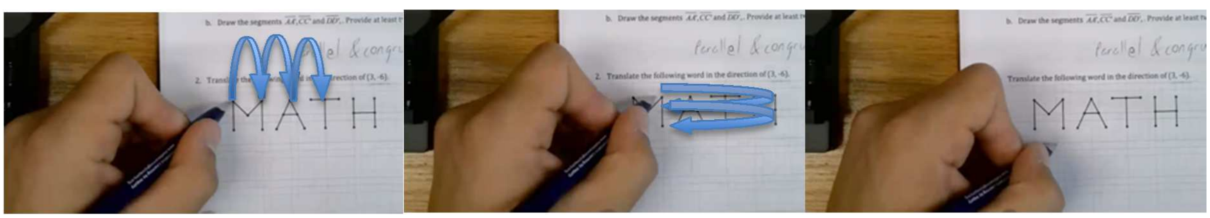
> We're going to move right three and down six. And so, we can take this mapping of **one, two, three** (jumped his pen three spaces to the right)**, one, two, three, four, five, six**,

(jumped his pen six spaces to down) and see that this point gets taken all the way down here. (see Figure 18)

I coded this gesture as a pointing gesture because it drew the students' attention to a location within the handout that complemented Edwin's words. This gesture, of several small jumps along the different components of a translation vector, highlighted a translation as shifting each point of a pre-image in a specified direction and distance rather than the notion of a translation as a rigid motion.

Figure 18

Counting the Individual Unit Movements Within a Larger Translation

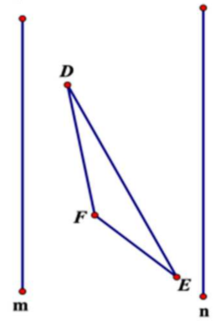


In the next handout problem, the students were required to reflect the triangle DEF across line m to create triangle $D'E'F'$ and then to reflect this new triangle across a parallel line n to make triangle D"E"F" (see Figure 19). Edwin again made the choice to not place the students in small virtual groups.

Figure 19

Double Reflection Problem in the Handout

6. In the figure below, the two lines m and n are parallel to one another.



Instead of putting the students back into their small virtual groups, he professed, “I really want us to focus on this construction and the properties will kind of come out of it in our discussion … [Please be] accurate and precise with our measuring tools. I'm going to mute myself, do some of the action on my side.” For the next five minutes, Edwin completed the double reflection construction silently for himself and expected the students to do the same. When Edwin completed his double reflection construction, he explained his process out loud to his students. Throughout the verbal description of his process, Edwin enacted many pointing gestures which grounded his speech within his inscriptions on the provided diagram. These pointing gestures matched the pointing gestures Edwin made in Figures 7, 13, and 15. He explicitly noted that the line segment created between the original point D and the first reflected point D' measured 3.6 centimeters while the line segment created between the original point D and the second reflected point D'' measured 7.2 centimeters. Edwin then asked the class a follow-up question that resulted in Edwin gesturing. The following was the conversation.

Edwin: So, does anyone see the relationship?

Student: Um, it [line segment DD''] would be double.

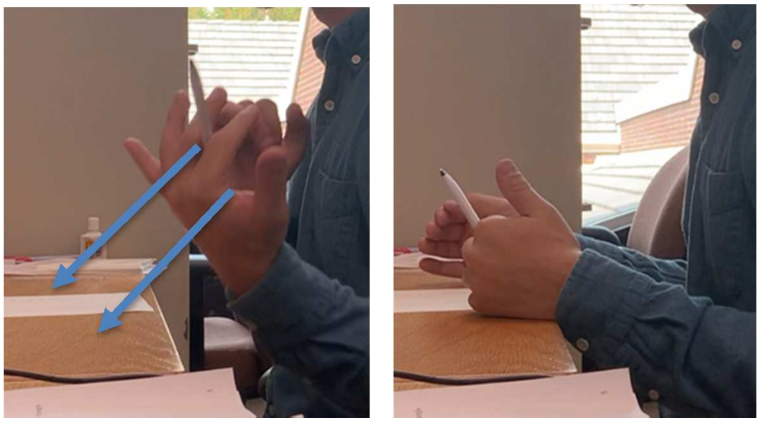
Edwin: I completely agree with your claim. It looks like that 7.2 is double our 3.6. That's going to hold true for any point that we do this with. Do you know, do you think you have a reason why that property is going to hold?

> Student: Um, I don't, but I mean it has something to do with the reflection process.
>
> Edwin: Good, thank you … it's because of the doubling property **through the reflection** (grabbed the sides of an imaginary cube on the left side of his body and smoothly moved the imaginary cube to the right) that is resulting in this kind of nicer thing that's going on. (see Figure 20)

I coded this three-dimensional gesture as representational because the gesture depicted Edwin shifted an imaginary cube from one side of an imaginary line to the other. His movements connected to what he said and potentially displayed his conception of a reflection as a movement with a tangible object. This gesture focused on the notion that a reflection changed the location of an image. However, this particular gesture did not appear to communicate the notion that a reflection flipped the orientation of the object, that the line segment connecting a point p and its reflected image p' was perpendicular to the line of reflection, or that the line of reflection bisected the line segment between point p and its reflected image p'. The perceived goal of this two-handed gesture was on a reflection as starting with a pre-image and resulting with a post-image.

Figure 20

Moving Imaginary Cube from the Left to Right



After emphasizing that the double reflection resulted in doubling the lengths of the line segments connecting the original image to each reflection, Edwin asked if the class had any lingering questions. One student spoke up:

> Student: Yeah, um, I was just wondering, I don't know if I missed it, but how do we know where to put the triangle all the way on the right? … I think I missed you explaining that part, but I'm just confused about why it's like all over there.
>
> Edwin: No, you're good. Thank you for asking that clarifying question. So, without **this entire second half** (covered m line with hand), we measure **these three distances** (pointed to the line segments he added from triangle DEF to line n) and mirror those three distances **on this side** (pointed to the reflected triangle $D'E'F'$). So, the second half of the prompt is saying, okay, we are no longer here we are **here** (pointed to triangle DEF). So, discount all of **these things in the middle** (pretended to circle triangle DEF) discount, our original figure and our M line. We are now going to reflect our **prime image** (pointed to triangle $D'E'F'$) over **[line] N** (pointed line n).
>
> Student: Okay, I got it now. That makes sense!

I coded this series of gestures as a pointing gesture, because Edwin used his finger to draw attention to the image and grounded his speech in a concrete example on the handout. These gestures clarified what triangle to reflect and which line of reflection to reflect over. With no further questions, Edwin shifted the class' attention to the next problem on the handout. He said,

> We're going to transition from kind of **sliding** (grabbed onto the sides of an imaginary cube on the right side of his body and slid the cube to the left; see Figure 21) and kind of **mirroring** (grabbed onto the top and bottom of the imaginary cube on this left side and in a large arching motion, flipped the imaginary cube to the right; see Figure 22) to more of a rotation sense.

I coded both of these three-dimensional gestures as representational because the movements were physical depictions of Edwin's verbiage. The spoken word "sliding" became a physical movement an imaginary cube and the audible word "mirroring" became flipping an imaginary cube. These two gestures potentially conveyed to the students that a translation shifted an object all at once while maintaining orientation and reflection changed the location of an object in such a manner that resulted in reversed orientation of the object. Again, these gestures explicitly communicated to the students the idea of a pre- and post-image of a translation and reflection.

Figure 21

Sliding an Imaginary Cube from Left to Right

Figure 22

Flipping an Imaginary Cube from Right to Left

On the first rotation problem within the handout, the students rotated an image of a car 120 degrees clockwise about a point. Edwin suggested the students choose a "nice point" on the car to work with. After choosing his nice point to be on the front bumper, Edwin said

> I can pick **that point** (pointed to the nice point) up. And **move it** (moved the pen in an arching motion to the right) 120-degrees (see Figure 23) … [And] That is the direction that it's going to be **mapped to** (held his elbow at a fixed location and moved his

forearm, with his hand in a stiff blade, in a large arcing motion from left to right. (see Figure 24)

Figure 23

Tracing the Arrow Representing the Angle of Rotation

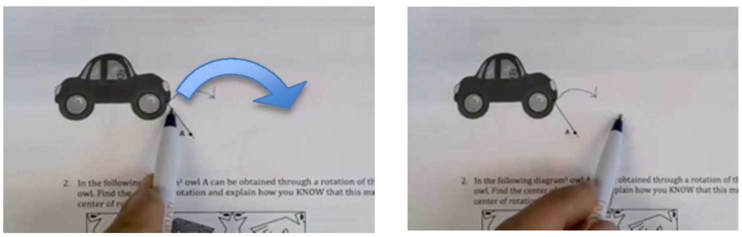


Figure 24

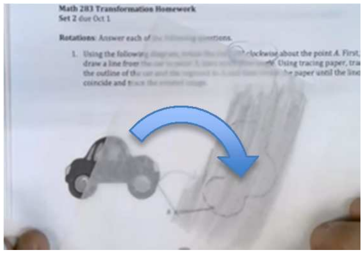
Edwin's Large Arcing Motion from Left to Right

I coded the first set of gestures, the point and movement with the pen, as pointing because the gestures identified which point Edwin talked about and demonstrated that he wanted the point to slide to the right along an arc shaped path. The second set of gestures were representational due to the fact that his hand was the object under rotation. He bent his arm along an imaginary arc created by holding his elbow as the center of rotation. His gesture provided the students the opportunity to see how his hand rotated in three-dimensional space. Additionally, this hand gesture implicitly communicated to the students all of the qualities of a rotation, even though Edwin did not vocalize all the qualities. For example, the gesture with his hand communicated a rotation as a turn around a point. In this gesture, the center of rotation was his elbow and the distance from Edwin's hand to elbow remained unchanged throughout the rotation. Moreover, the shape of Edwin's hand did not change in size or shape, so this gesture also conveyed the idea of a rotation as a rigid motion.

Edwin proceeded to use his protractor to precisely rotate his nice point 120 degrees clockwise on his handout. He then said, "this is a complex picture, and I don't want to do that for all of these critical points around my figure. That's going to be a really slow process." At this point, Edwin introduced tracing paper to his students as a way to efficiently rotate a figure about a specified point. He explained that the students should trace the image on the car on the translucent tracing paper and then while holding the center of rotation stationary "we simply **rotate** (twisted the translucent tracing paper) that to our new spot" (see Figure 25). The use of the tracing paper not only quickened the rotation process, but it preserved the necessary characteristics of a rotation.

Figure 25

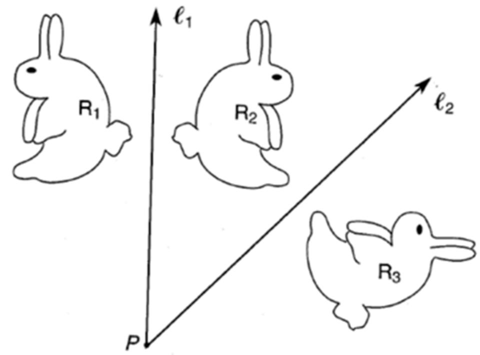
Example of Using Translucent Paper



Tracing paper was the students' tool of choice when preforming a rotation for the remainder of the Euclidean transformation unit. Whenever Edwin asked the students how they wanted to do a rotation, the students voted for using tracing paper. The last problem on the handout connected back to the double reflection construction from earlier. Edwin hoped the previous class discussion would provide the students with some insight into this final problem. In the final problem, shown in Figure 26, the students looked at a picture of a rabbit, labeled R_1, two lines intersecting of reflection l_1 and l_2, and two reflected rabbits labeled R_2 and R_3.

Figure 26

Rabbit to Duck Reflection Problem from the Handout



As Edwin explained reflections that created the picture shown, he said:

> We are going to take **R1** (tapped R1 with his pen) reflected over **L1** (tapped the line L1 with his pen) to get **R2** (tapped R2 with his pen). And then we are going to reflect R2 over **L2** (tapped the line L2 with his pen) to get **R3** (tapped R3 with his pen).

I coded these gestures as pointing gestures because they drew attention to the handout and located the objects that Edwin spoke about within the picture. These pointing gestures matched the gestures in Figures 7, 13, and 15. Edwin asked the students to think about the relationship between R_1 and R_3 in their small virtual groups for a few minutes. During Edwin's visits to the small groups, most of the students identified that between R_1 to R_3 was a rotation of 90 degrees, but many of the small virtual groups struggled to understand why this rotation occurred and how it connected to the earlier double reflection construction. Soon after, Edwin brought the class back together to have one large conversation:

> Edwin: If you look at the angle between L1 and L2 … you'll notice that that angle is 45-degrees. And so, we're seeing that same doubling property that we were seeing from reflections initially … [but] why did these two reflections end up in a rotation?

> Student: Two reflections will end up a rotation. Um, only if you have, if you like, I don't know. Cause like you can reflect it twice, but this one has like a rotation point and it's like, huh? I don't know!
>
> Edwin: I hear a lot of what you're saying. And I think I have a way of kind of proving what you're talking … what you are saying that there exists a rotation point, a point that we're rotating about here, our P. This point exists it's it is a point that we can **rotate about** (grabbed onto an imaginary cube and rotated it about a center near his belly button) (see Figure 27) and do things with … [another] group had a kind of a discussion around the difference between these two problems. Can you tell us a little more about that?
>
> Student: The angle [formed by the lines], it's not parallel!

I coded this gesture with the imaginary cube as representational because the movements were physical demonstrations of Edwin's words. For Edwin's gesture, the center of rotation was an imaginary point somewhere near his belly button. He did not verbally communicate the center of rotation to his students, thus, the students needed to infer the location of the center of rotation on their own. If the students identified the center of rotation, his rotation of the cube conveyed that all the points on the cube moved in the same direction and communicated the idea of a pre-image and post image in a rotation.

Figure 27

Rotating an Imaginary Cube from Left to Right

Edwin used students' comments to stress that when reflecting an image twice over a set of lines, whether the final transformation was a translation or reflection depended on the presence of an intersection point between the lines of reflection. Many students portrayed a look of an epiphany on their faces once Edwin clearly stated this fact. Before moving on, Edwin asked his students "are there any questions about this diagram, any of the properties that we're using are invoking before we kind of move on"? No students responded and Edwin interpreted the silence as the students having no further questions on the double reflection. For the last problem, the students thought critically about the impact of the order of transformations. They determined whether or not translating, reflecting, and rotating an image of an octagon resulted in the same image if they instead rotated, reflected, and then translated the image of an octagon. After providing the students a few minutes of private think time, the following conversation transpired.

> Edwin: I am asking is a translation, reflection, rotation, equivalent to a rotation, reflection, and translation.
>
> Student: I think that it is the same only if you do it in the exact reverse order … I think it is the same, but only if you do it in the exact opposite order, you can't just like mix up the order you do things.

> Edwin: So that's one conjecture. I think that there are cases where that occurs. I'm going to hit back on you and say, I don't think that this is one of them. There are going to be cases where **these transformations** (pretended to circle the printed directions) will work and you will see images that are the exact same, no matter what direction that you're going to do. I'm going to hit back on you a little bit because of the things that are embedded in **the question** (underlined the words reflection and rotation on the problem). All of these things that I've underlined here are different reference points for our movements. Now, one of the reasons why we can do reflections and translations is because these **two things** (pointed to the printed words) have no connection to one another. There's nothing about moving three over and six up that impacts our reflection. It doesn't move **this line** (pointed to the line of reflection) at all. But, depending on some other transformations rotations, those things can adjust where our centers are, where our origins are, where our fixed points are.

I coded this series of gestures as pointing gestures because they drew attention to the directions on the problem and located the objects that Edwin spoke about within the provided diagram. The students did not ask any further questions on the impact of the order of transformation on the final image, and so Edwin released the student. He said, "I'm going to stick around in case there are any questions, but we are out of time for today." One of Edwin's student stayed after class to ask a specific question about symmetries. I noted their exchange because Edwin did not enact these gestures at any other point in his synchronous online sessions. The following was their interaction.

> Student: I am unsure about my definition for rotationally summary, er symmetry … I just said that it's when there's a center point and then the shape holds its form and stays in the same shape anyway it's rotated. Is that right?
>
> Edwin: So that's not quite what rotational symmetry means … For rotation symmetry, you're going to need specific angles to rotate … A **circle** (clutched an imaginary sphere at eye level from the bottom) has infinite rotation symmetries because no matter how you **turn a circle** (rotated the imaginary sphere clockwise by twisting his wrist and arm), it's still just a circle (see Figure 28). Something like a **square** (grabbed onto the lower left and top right corner of an imaginary square), if you only **rotate it a little bit** (rotated the imaginary square 45 degrees clockwise), it becomes a diamond. And so that's not a rotational symmetry. (see Figure 29)

I coded both sets of gestures as representational because his gestures depicted his verbiage with imaginary referents. He grabbed onto an imaginary circle and square when he referred to them and his motion with these imaginary objects embodied a rotation. These gestures highlighted a rotation as a turn producing a pre-image and post-image and that when the center of rotation was the center of the image, the image remained roughly in the same location.

Figure 28

Rotating a Square by the Corners

Figure 29

Rotating an Imaginary Circle in the Air

Overall, Edwin enacted gestures during many classroom situations including during lectures, small group interactions, and one-on-one with a student. Edwin used a healthy combination of representational and pointing gestures. From my observations, it appeared that

Edwin utilized representational gestures at a slightly higher rate than pointing gestures. His gestures ranged from small taps of his pen to large motions involving imaginary cubes both towards the web camera and under the document camera. From my analysis, the gestures which carried the most mathematical information occurred while addressing the web camera. Edwin's gestures always communicated a transformation with a pre- and post- image, but his gestures did not always highlight the all conditions for each individual transformation. Lastly, on several occasions, Edwin's gestures had the opportunity to convey more mathematical information than his verbiage, however, Edwin never instructed his students to specifically look at his gestures. Hence, it was unknown if the students noticed these subtleties and made the appropriate inferences. The next section describes the results from Edwin's post observation interview.

Edwin's Interview Narrative

During the post-observation interview we discussed Edwin's beliefs and prior experience surrounding gesturing in the classroom, the gestures he would make if he taught face-to-face, and his intentionality when gesturing in his synchronous online sessions.

At the start of the interview, Edwin proclaimed to be a "hands-talker" both inside and outside of the mathematics classroom and as such, he worried about his ability to gesture effectively in an online course for my study. He said that the unit on Euclidean transformations was "the one that we're most likely to mess up because we're online." Edwin's apprehension seemed to stem from his beliefs surrounding gestures. In the interview, Edwin proclaimed that gestures communicated mathematical ideas and, more importantly, captured the attention of his students. He said gestures should be "loud and really draw the attention of the class" and can "really bring the concept to life." In the interview, Edwin professed, "I think the reason that I do it [gesture] is I think that it's just, I think it makes me more interesting really … I feel like if it [gesturing] gets people to pay attention to me, it only works better in the classroom." Edwin did not cite evidence that supported his hypothesis that his gestures captured the students' attention or that holding the students' attention made him a more effective instructor, rather they were principles he believed in.

Following this discussion, I asked where, if any place, he learned about gesturing. Edwin said that during his graduate program he learned about the influence of gestures in mathematics classrooms on students' learning. Edwin said that one professor in particular provided him the support to academically analyze the gestures he made as an instructional technique. He explained,

> [She] made me think about the concept [of gesturing] more and made me think about why I was doing more, how I conceptualized it, and how I could translate that to my teaching … a lot of gestures probably would have come out naturally, but I'm definitely attending to them and trying to do them more in my classes.

Along with the knowledge that gestures may serve as an instructional technique, Edwin professed that people learn by engaging with mathematics in the most authentic manner possible. In the interview he said, "For geometric based courses, this [students gesturing] is a hugely important thing to emphasize. Having a visual for the mathematics that they're doing and making the mathematics as real as possible is the best way to do that."

Before showing Edwin the gestures that he made during his online synchronous sessions, I asked Edwin what gestures he would likely make if he taught Geometry for Elementary Teachers in a face-to-face setting. Edwin explained his preferred gesturing style was large, full-bodied, and required the assistance of others. For example, in the interview, he professed that his ideal gesture for a reflection involved pretending a large mirror separated himself and a student. He said, "I would bring up a student and have them be my mirror image and wherever I move, they kind of move as a reflection of that." In the interview, Edwin described this gesture could communicate a reflection as a mapping that preserved size but reversed the orientation of the object. Next, Edwin claimed that his preferred face-to-face gesture for a translation also required the use of his whole body. He said he would want to leverage the floor as a large grid on which his body could be the object translated. He remarked, "I would maintain my shape and size and orientation, and I would just walk somewhere else." Edwin not only claimed he would use his own movement to represent a translation if he taught face-to-face, but he also suggested wanting the students to get up and try to translate themselves. He added, "One of the best things to do would be to come in and put tape on the ground and say, this is a line, this is a line, groups go do this [translation]." When describing the gestures he would make for a rotation in a face-to-face class, Edwin talked about using his whole body to make a full 360-degree spin. By making this gesture, Edwin said he "would really communicate a 360 rotation as ending up back exactly where you are without ever like really changing where you were." In the interview, he said his preferred face-to-face gesture could convey the spirit of a rotation where the center of rotation was the center of his body. Lastly, to demonstrate a glide reflection in a face-to-face setting, Edwin described wanting to place a line of reflection on the floor with tape, walk along a translation vector, and finally have a student be his mirror reflection. In the interview he said, "I would try and get some students to do some of these more embodied activities."

While describing the gestures he would make if teaching face-to-face, Edwin repeatedly commented that his ideal face-to-face gestures were no longer an option in his online synchronous sessions. In a probing question, I asked Edwin to explain how he was able to gesture in the spaces that were available to him. The two spaces were the web camera capturing his upper body and the document camera capturing his written inscriptions. He proclaimed, "I think that on [the web] camera up here, it's a lot easier to talk in three-dimensional space. I have depth behind me … Whereas on the document camera, you're really working more on a plane."

Next, I showed Edwin his gestures for Euclidean transformations from his synchronous online sessions. His initial reaction to watching his own gestures was a mixture of confirmation and surprise. Edwin seemed pleased to see the three-dimensional gestures in front of the web camera and claimed to have consciously enacted them. He said, "I'm really happy that that's coming through with the gestures" and added "I really was trying to emphasize [transforming] the whole object … I knew I made the [gesture of] holding a fixed object and like kind of doing the motion with the whole object." His comments appeared to describe intentionality behind his three-dimensional gestures. He further claimed that while gesturing online he purposefully emphasized a transformation as "picking up [something] and putting it somewhere else." However, following his statement on intentionally making three-dimensional gestures, Edwin described feeling unaware and surprised by his non-three-dimensional gestures. For example, he said, "I did not know I made those" and "I wasn't aware that I did the point with the pen." It was as though he did not enact the non-three-dimensional gestures intentionally.

In the interview, Edwin described his beliefs and prior experience surrounding gesturing, the gestures he would make if face-to-face, and his intentionality when gesturing in his synchronous online sessions. From our discussions and Edwin's proclamations, the synchronous online setting seemed to impact the way in which he gestured about Euclidean transformations. By using his prior knowledge and beliefs surrounding gesturing, he claimed to purposefully produce three-dimensional gestures for Euclidean transformations. He stated that the three-dimensional gestures were important in a synchronous online setting because they captured the attention and helped the students learn, but that he did not remember making the non-three-dimensional gestures. Next, I depicte Naomi's synthesized online synchronous sessions and include all of her gestures as she taught Euclidean transformations.

Naomi's Synchronous Online Classroom Narrative

Overall, based on my analysis of Naomi's synchronous online sessions, she utilized a combination of representational and pointing gestures while teaching transformations to her pre-service elementary teachers. During Naomi's classes, her students only had access to the document camera's view; therefore, any gestures Naomi preformed away from the document camera were not visible to her students. Based on my analysis, Naomi made about one fifth of her gestures out of her students' view and all of these gestures were representational. She utilized the document camera to enact small gestures with her hands and point with her pen or fingers. These condensed gestures seemed to portray a holistic picture of the actions of the Euclidean transformations rather than the mathematical precision of the formal definitions. The overall structure of Naomi's synchronous online sessions consisted of Naomi completing problems from the same handout as Edwin under the document camera, the students silently working in the main online space, and some interactions between Naomi and a handful of students. Again, I included a copy of the handouts in Appendix D. In this section, I created a narrative synthesizing Naomi's four-day Euclidean transformation unit. I included all of Naomi's gestures for Euclidean transformations. Naomi used the same series of handouts as Edwin, however the order in which they progressed through the handouts differed and as such some of the problems appear in a different order in my synthesized narratives. For example, immediately after completing the problem asking the students to identify which Euclidean transformation converted polygon image A into images B through H (see Figure 7), Edwin advanced his class to the practice problems for translations while Naomi required her class to complete two more problems similar to Figure 7.

To begin the unit on Euclidean transformations, Naomi introduced a new tool for constructing reflections called a mira. While showing the mira under the document camera to her students she said,

> The **beveled edge** (pointed to the beveled edge) is always the side we are going to look through, that's the side you place down on your paper … As we look through **this side** (pointed to the beveled edge), we can draw our image on the **back side** (pointed to the non-beveled edge).

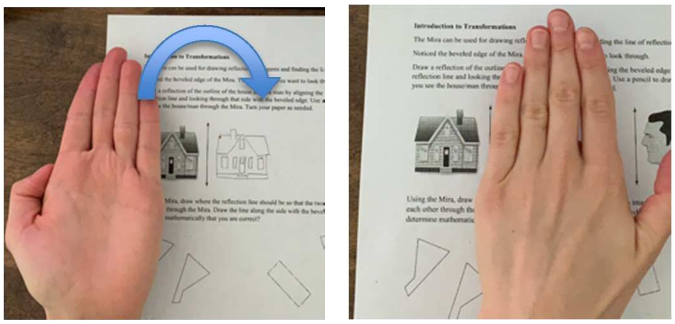
After the brief explanation, Naomi asked the students to use their mira to reflect two images over their respective lines of reflection. A few minutes of silence passed, then Naomi displayed her personal reflected images under the document camera. She said asked what they noticed about the houses.

> Naomi: So, what do you notice about **the houses** (tapped her pen on both houses)?
>
> Student: They're flipped, the chimney is on the right, on the left side and on the left side, it's on the right.
>
> Naomi: We have what they call opposite orientation. Our chimney stays on the inside, it stays on the inside. Your image is getting **flipped** (placed her palm-up hand on the left side of the line of reflection. Then, flipped her hand over the line of reflection so that her hand was palm-down). (see Figure 30)

I coded the hand flipping gesture as a representational gesture due to the fact that Naomi's hand, which was the object under reflection, flipped across the printed line in such a way that reversed her hands orientation. Her gesture was directly tied to her words. The students had the opportunity to see that a reflection shifted an image from one side of the line of reflection to the other in such a manner that reversed the orientation. By itself, the gesture did not communicate that that the line of reflection served as the perpendicular bisector of the segment pp'. However, this notion did not appear to be the goal of the flipping gesture.

Figure 30

Under the Document Camera Hand Flip



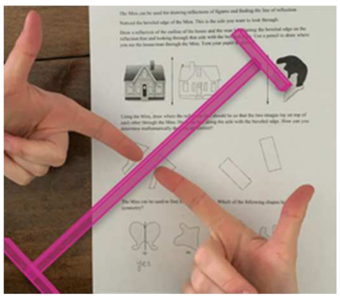
Once the students had the opportunity to create a reflected image with a mira, Naomi demonstrated how to find the line of reflection for a pre-image and post-image pair using a mira. While working under the document camera she said,

> To find the reflection line, we need to make sure we rotate **this** (pointed to the mira with her finger) so we can see this image **laying on top** (formed two small finger guns on either side of the mira) of the other image on the other side. Once it's laying on top, then we'll draw a line. (see Figure 31)

I coded Naomi's pointing towards the mira as a pointing gesture. Naomi used her gesture to locate specific features physically on the mira, rather than convey a mental image of a mathematical idea. Additionally, the opposing finger gun gesture did not match her words and her gesture demonstrated what the students will see when they properly use the mira rather than an action. Hence, I coded the gestures as pointing. This opposing finger gesture implicitly provided the students with mathematical information which her verbiage did not. Her gesture potentially conveyed to the students that a pre-image point and post-image point maintained their distance from the line of reflection because her opposing finger guns appeared to be equal distance from the line of reflection or the mira.

Figure 31

Opposing Finger Gun Gesture



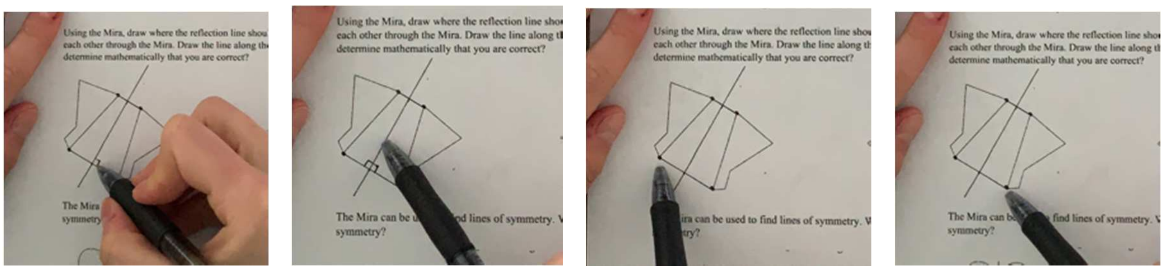
Before advancing onto the next question, Naomi stopped to identify two important relationships between a point p, the line of reflection, and the reflected image of p, called p'. First, that the line segment connecting a point p and its reflected image p' was perpendicular to the line of reflection. Second, that the line of reflection bisected the line segment between point p and its reflected image p'. When highlighting these relationships, Naomi had the following exchange with a student:

> Naomi: If we match up **corresponding points** (connected two corners of the reflected polygons), what do you notice about those statements and your reflection line?
>
> Student: They are perpendicular.
>
> Naomi: Okay, they are always going to be **perpendicular** (added a small right-angle mark to the picture). Anything else you notice? … **This** (tapped the reflection line) is a line of symmetry. That means the distance **from our point to the reflection line** (connected the left point the line of reflection) is the same as **reflection line to the final image point** (connected the line of reflection to the right point). So, distances will be the same. (see Figure 32)

I coded these gestures as pointing gestures because Naomi's movements drew attention to the handout and located where in the diagram the students should look. These gestures grounded Naomi's verbiage within her inscription on the handout.

Figure 32

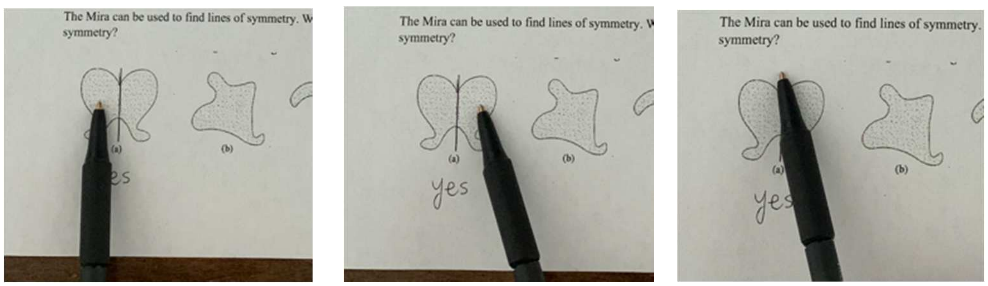
Pointing to Key Features of a Reflection



Advancing to the last handout question that required a mira, the students determined if four given shapes contained a line of symmetry. Naomi modeled the first problem for the students. While working under the document camera, she said, "Hopefully we can see when we have **the shape** (tapped the left and the right side of the butterfly) that we're basically **cutting it** (sliced the butterfly down the middle) in half" (see Figure 33). I coded the first two gestures as pointing because much like the pen pointing gestures from above her actions identified the objects in her verbiage. I coded the cutting gesture as representational because her words and the action she produced aligned. These gestures by themselves did not explicitly highlight the notion that a reflection reversed the orientation of the object and that the line of reflection served as the perpendicular bisector of the segment pp'. However, Naomi's accompanying words seemed to supplement some of the missing information from the gestures because she described the butterfly as cut in half. The students may have extrapolated that the distances from the line of reflection were equal because the left and right side of the butterfly image were the same.

Figure 33

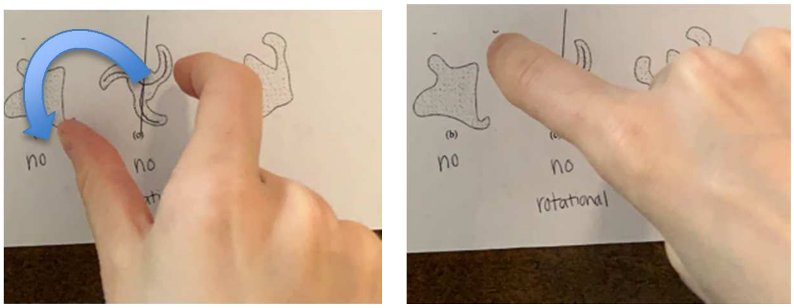
Pen Hopping from the Butterfly Image to the Line of Reflection



Naomi pointed out that the third shape in the problem did not have reflection symmetry, rather, it had rotational symmetry. She said, “We can **turn it** (framed the printed image with her index finger and thumb and rotated her hand counterclockwise) three different times until it lands back on itself” (see Figure 34). I coded this gesture as representational because the space Naomi framed with her hand was the object under rotation. Her gesture was directly tied to her words, the “turn it” was a physical twist of her hand. Hence, the students had the opportunity to view a rotation as a turn. The gesture produced a pre-image and post-image and that when the center of rotation was the center of the image, the image remained roughly in the same location. The way in which Naomi held her thumb and index finger maintained the distance between her pointer finger and thumb hence the gesture portrayed a rotation as a rigid motion.

Figure 34

Rotating the Framed Image Counterclockwise



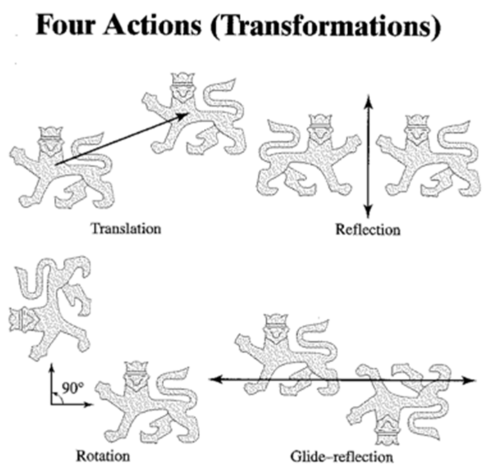
After a few more minutes of quiet work time, a student asked a question on determining the lines of symmetry in the second image.

> Student: So, for B, the reason that it doesn't have a line of symmetry is because it reflects a different image on both sides?
>
> Naomi: Yeah. So, we kind of cut in half **this direction** (pretended to slice the ghost image in half), but we have **this extra bump** (pointed to the right side of the ghost), So, it's not exactly **the same** (pointed to the left side of the ghost). And we kind of have **this hook** (traced the tail) here on the tail. So, it would have to have a hook in **this direction** (drew a reflected hook) as well.
>
> Student: Ahh, okay.

I coded this set of gestures as pointing gestures because Naomi's movements drew attention to particular places on the image of the ghost. Additionally, Naomi identified what needed to change in order for the ghost to have a line of symmetry. These gestures grounded Naomi's verbiage within her inscription on the handout. The class finished the opening questions on the handout and Naomi transitioned into formally defining all four Euclidean transformations. Printed on the handout was an image of a lion altered by each of the Euclidean transformations (see Figure 35). Naomi claimed that the Euclidean transformation definitions should be "more recognizable" to the students because these definitions were "what they have in the elementary school books. So, it's really just recognizing how you're moving your different shapes." Introducing the four Euclidean transformations took less than three minutes of class time.

Figure 35

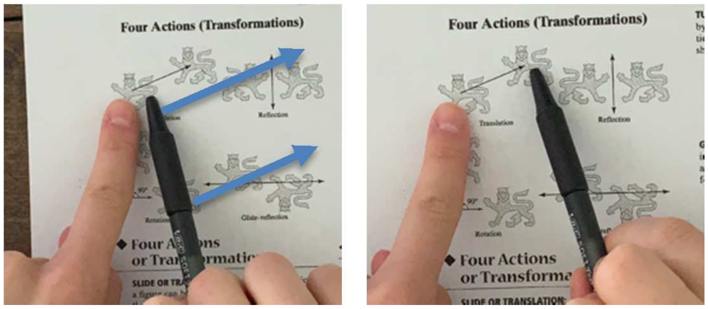
Lion Images from the Handout



The first Euclidean transformation on the page was a translation. Naomi said, "So here is what we refer to as a translation, first taking our **image** (pointed to the left lion) and just **sliding** (slid her pen in the northeast direction) it across" (see Figure 36). I coded this gesture as representational because the tip of her pen was the object under translation. Her action physically depicted her words. This particular gesture seemed to convey to the students that a translation shifted a point in a particular direction for a specified distance because of the smooth motion of the pen. Naomi's words supported this idea because she described a translation as "sliding." The gesture could additionally represent a translation maintaining the size and shape of the object under the transformation because the pen did not change shape along the translation vector.

Figure 36

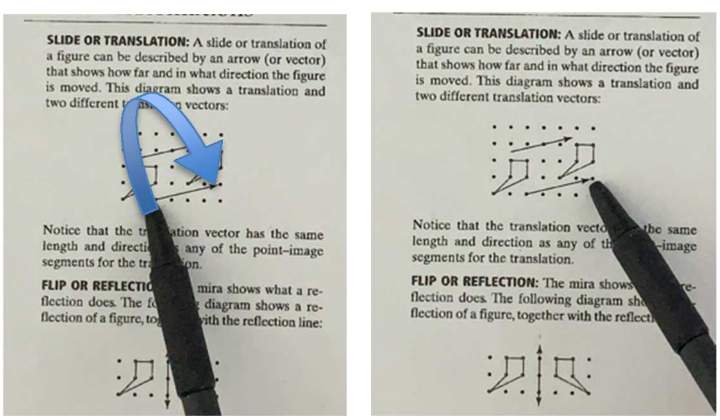
Naomi Sliding Her Pen Across the Paper



Before progressing to the next Euclidean transformation, Naomi added that if "we look at this translation vector it's telling us we're starting at **this point** (pointed to the left end of the vector) and we're going **here** (jumped her pen to the right side of the vector)" (see Figure 37). I coded these gestures as pointing gestures because they drew attention to the handout and located the objects that Naomi spoke about within the printed example.

Figure 37

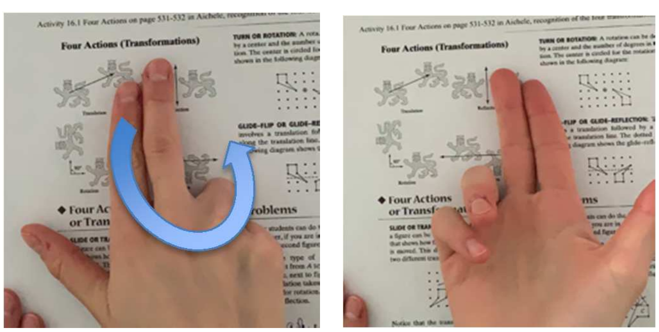
Jumping Pen from One Side of the Translation Vector to the Other



The next definition Naomi discussed was for a reflection, she said, “Here we have a reflection, so, we are taking the **image** (covered the left lion with her pointer and middle finger) and **flipping it over** (flipped her hand over so her fingers covered the right lion)” (see Figure 38). Similar to the gesture she made covering up the house in Figure 30, I coded this gesture as representational because Naomi’s hand, in particular her first two fingers, flipped over the provided line in such a way that it mirrored her hand’s orientation. Her gesture was directly tied to her verbiage. Again, the hand flipping gesture highlighted that a reflection shifted an image from one side of the line of reflection to the other reversing the orientation of the image. The notion that the reflection line served as the perpendicular bisector for the line segment created by connecting a point p and its reflected image p' was not explicitly communicated by this gesture. However, this idea does not appear to be the overall goal of the gesture.

Figure 38

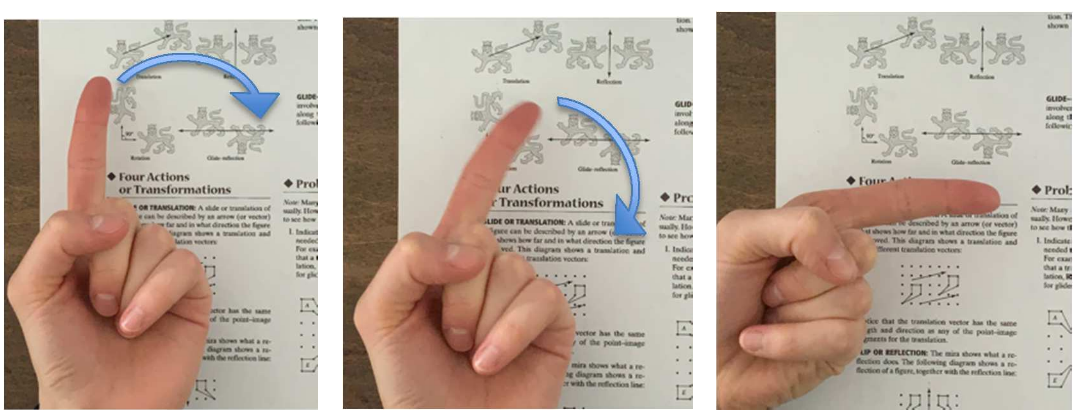
Two Finger Hand Flip



Quickly switching to the next Euclidean transformation, Naomi said, “For a rotation, we are just simply **turning around the point** (extended index finger and then bent her wrist 90 degrees clockwise so that her extended index finger pointed to the right)” (see Figure 39). Due to the fact that the gesture matched Naomi's verbal description of a rotation, I coded this gesture as representational. The tip of her finger was the object under rotation and her bending action represented the smooth movement of a rotation. These gestures implicitly communicated all of the qualities of a rotation to the students, even though Naomi did not vocalize them. For example, her bent wrist gesture communicated a rotation as a turn around a fixed point. In this gesture, the center of rotation was her wrist. Naomi placed her hand on the paper while extending her index finger and bent her wrist to allow the tip of her finger along to traverse an imaginary arc. Hence, in this bent wrist gesture, the distance from the tip of Naomi's finger to her wrist remained unchanged throughout the rotation. Moreover, her finger did not change in size or shape, so this gesture also conveyed the idea of a rotation as a rigid motion.

Figure 39

Extended Left Index Finger Rotates 90 Degrees with a Bend of the Wrist



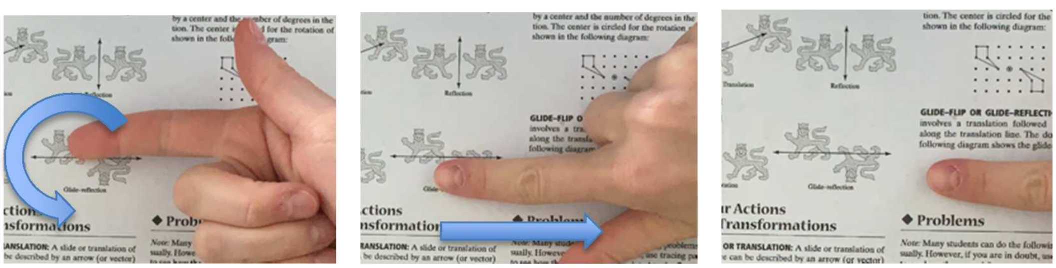
The last Euclidean transformation that Naomi discussed was a glide reflection. To introduce this transformation Naomi said,

The glide reflection is a composition. It is taking this **original image** (placed the fingernail of her right index finger over the image), **flipping it over** (flipped her hand over so that her right fingernail faced up) and then we **slide it across** (slid her finger to the right, parallel to the reflection line). (see Figure 40)

I coded these gestures as representational due to the fact that each of her gestures occurred with a corresponding verbal description. These gestures provided the students the opportunity to view a glide reflection as a composition of two transformations and to see that the vector of translation was parallel to the line of reflection. Additionally, due to Naomi's hand flip, these gestures potentially conveyed the reverse orientation from the reflection piece of the glide reflection.

Figure 40

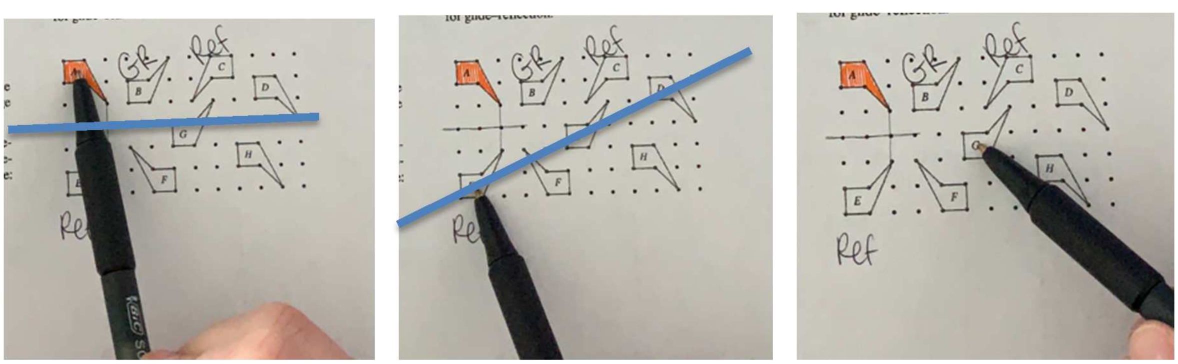
Naomi's Gesture Introducing a Glide Reflection



After highlighting and demonstrating the handout's definitions for each of the four Euclidean transformations, Naomi read the next problem (Figure 11) out loud for the students. In this problem, the students determined which Euclidean transformation altered the original polygon, image A, to each of the other polygons shown, images B through H. Naomi provided the students with quiet individual work time to complete the problem. As a method to formatively assess the students' progress, Naomi requested that they use the online conference platform's non-verbal feedback to click a green check mark when they finished the problem. From my observation, students seemed to follow this classroom norm regularly. Once a substantial number of green check marks appeared; Naomi asked the class for volunteers to share out their answers. The class successfully identified what Euclidean transformations converted image A to most of the other images. However, the class missed the Euclidean transformation which modified image A to image G. Hence, Naomi revealed that to transform image A to image G required a glide reflection. Naomi said, "we go from $\boldsymbol{A}$ (tapped image A), flip over **to** $\boldsymbol{E}$ (tapped image E), and slide straight over **to** $\boldsymbol{G}$ (tapped image G). So, [it is a] glad reflection" (see Figure 41). However, a closer look at Naomi's line of reflection and translation vector she described revealed that the two lines were not parallel. The transformation that maps imaged A to image G was a glide reflection, but she described and gestured at the incorrect line of reflection and translation vector. Naomi did not realize the error live in the synchronous online session. I coded these gestures as pointing because each gesture located an object that she spoke about on the handout. These gestures communicated that a glide reflection was a composition of two transformations.

Figure 41

Reflecting and Translating with Incorrect Lines



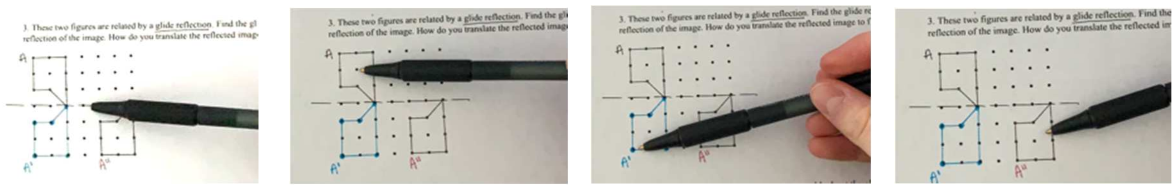
In the next problem from the handout, Naomi described and gestured at the correct line of reflection and translation vector for a glide reflection. Much like the previous problem, this one provided the students with a pre- and post-image of a polygon. This time however, the problem statement included the fact that the Euclidean transformation was a glide reflection. The task was to find the line of reflection and intermediate reflected polygon between the pre- and post-images. As before, Naomi provided the students with quiet individual work time and requested that the students click the green check mark when they finished. Off screen, Naomi completed the problem for herself and, after a wave of green check marks appeared, Naomi transitioned back to under the document camera to explain her work when the following interaction occurred.

> Naomi: Hopefully everyone can see that **this** (tapped the line she added in the grid) is our reflection line. Basically, we're just taking **it** (tapped on the top left polygon) and we're **flipping it over** (tapped on the polygon she created below the line of reflection) and then we can just **translate it** (moved her pen to the right parallel to the line of reflection). (see Figure 42)
>
> Student: You're moving it to the right, four spaces.
>
> Naomi: Right! So, every point gets moved four places to the right.

I coded these gestures as pointing gestures because each gesture located an object that she spoke about on the handout. Once again, these gestures potentially communicated to the students that a glide reflection was a composition of two transformations and this time Naomi shifted her pen along a vector of translation a parallel to the line of reflection. This gesture still did not highlight the opposite orientation from the reflection, however it did not appear to be the goal of the gesture.

Figure 42

Tapping on Polygons Using a Parallel Translation Vector



Prior to beginning the first translation practice problem, Naomi asked the class if they reversed the order of the glide reflection, so instead of reflecting and then translating they first translated and then reflected, would the final image remain the same. The students quickly answered that the final images were the same independent of the order in which they constructed the glide reflection. In response, the following conversation occurred.

> Naomi: Our question is why did we get end up in the same location? We're trying to use those properties for transformations to help us with that. So, what's true about translations in general?
>
> Student: You're repeating the same image a second time.
>
> Naomi: True, but let's go to our **elementary definitions** (displayed the definitions from earlier in class). What's true about translations?
>
> Student: Have the same length and direction?
>
> Naomi: Okay, so every point is moving the same distance in the same direction. Now, what did we say on our front page about a reflection?
>
> Student: The images are the same distance from the line of reflection.
>
> Naomi: Using those two things, what holds for **this one** (displayed the current problem). So, when we did our reflection reflected this over, we kept **these distances the same** (touched the space between A and A prime). And then we **did the translation** (moved pen blue image to final image). We kept our distances the same. So, every point moved the same distance. And because when we translate in **just one direction** (moved pen blue image to final image), we just went to the right four, we kept our translation vector parallel to our reflection line. And that's why they end up being the same location.

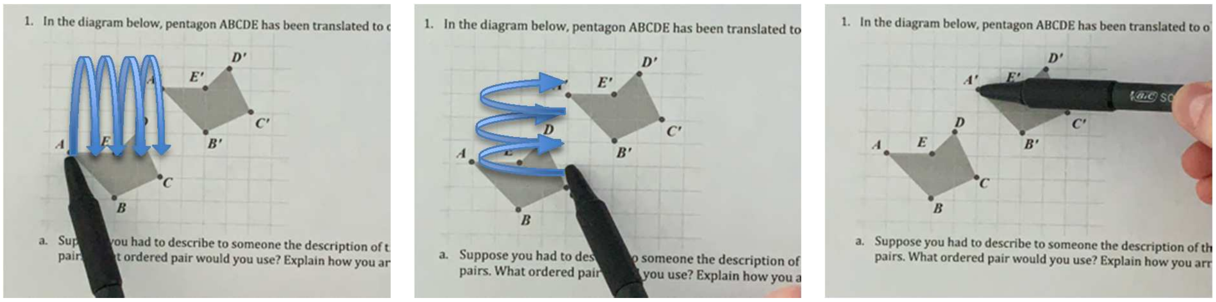
I coded these gestures as pointing gestures because Naomi's movements drew attention to the handout and located where in the diagram the students should look. These gestures grounded Naomi's verbiage within her inscription on the handout. After reinforcing why the order of the transformations in a glide reflection did not impact the final image, Naomi transitioned the class to the practice problems. In next problem, the students investigated a pentagon $ABCDE$ and its translated image $A'B'C'D'E'$. The students described the translation in terms of an ordered pair. The following exchange between Naomi and a student highlighted one of Naomi's gestures.

> Naomi: Let's go in words, what do I do to get from A to A'? What do I do to get from C to C'?
>
> Student: Up three, over four. Or to the right four
>
> Naomi: Does everyone see that? We're going to go to the right **one, two, three, four** (jumped her pen four spaces right) and we'll go up **one, two, three** (jumped her pen three spaces up). That's four right and three up. (see Figure 43)

I coded these gestures as pointing because each gesture located the grid space that Naomi described traversing onto the handout. Much like the single jump gesture from previous examples, these gestures highlighted a translation as shifting each point of a pre-image in a specified direction and distance rather than as a rigid motion.

Figure 43

Small Pen Hops Along the Vertical and Horizontal Components of a Translation Vector



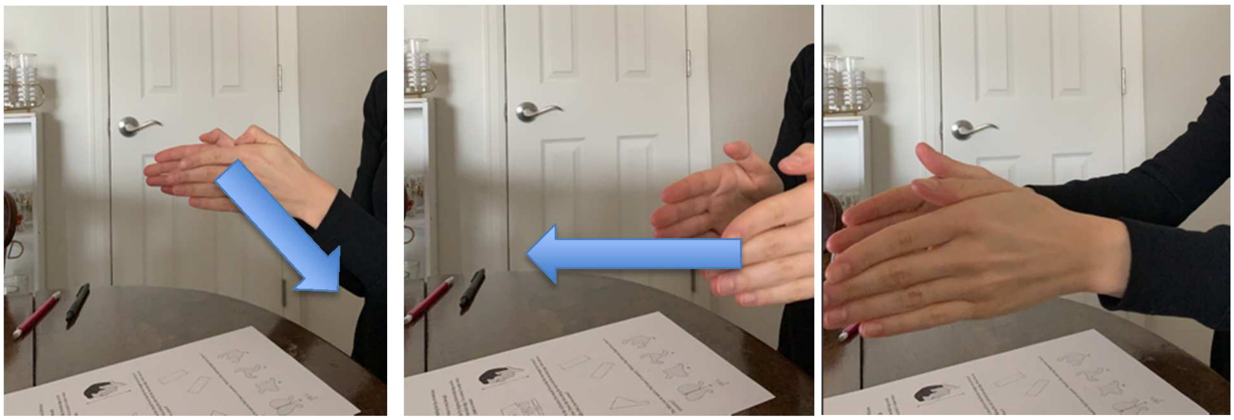
After the student confidently answered the previous question, Naomi told the class a story problem about parking a car at the grocery store that connected to describing a translation in its horizontal and vertical components. Away from the document camera, she said,

> [Suppose] you park your car and you had to walk over five rows **to the left** (moved an imaginary cube to the left holding onto the sides), and then you went **up two rows** (moved the cube further away in front of her) to get to the door … How do you get [back] to your car? (see Figure 44)

I coded this as representational because the gesture was closely tied to the words Naomi spoke and the motion demonstrated a translation as the movement of a tangible object. The students did not have the opportunity to see this gesture because Naomi made this gesture out of the document camera's view, but the smooth sweeping motion of Naomi's hands potentially conveyed that a translation shifted all points of the object at the same distance and direction.

Figure 44

Sliding an Imaginary Cube Along the Vertical and Horizontal Components of a Translation Vector



A student quickly shared that to get back to the car, the person should go "down two and right five, or the opposite directions." Naomi seemed pleased with the student's participation. She ended the discussion on translations by adding that "you can always check yourself, go to that **pre-image** (tapped on the right image) location, do your translation vector and see if you **go** (tapped on the left image) where you need to end up." Throughout the verbal description of her self-checking tip, Naomi enacted pointing gestures which grounded her speech within the provided diagram. These pointing gestures matched the pointing gestures that Naomi made in Figure 33. Transitioning to reflections, Naomi refined the class's previously discussed definition. Away from the document camera she said,

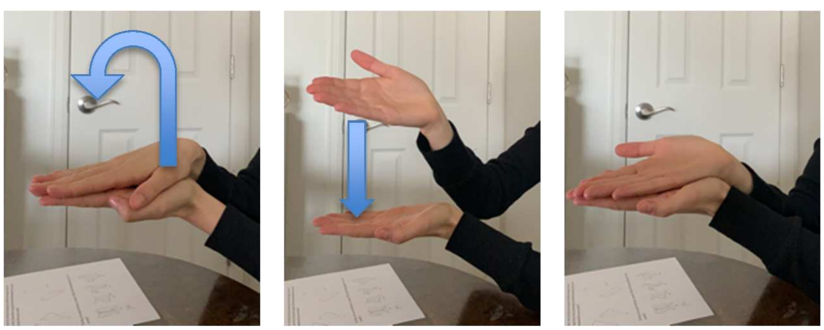
> We're doing kind of the college definitions here. So, we have a reflection, it takes each **point P** (pressed her hands together so that her right hand was on top) in our pre-image and we're basically, **reflecting across line L** (lifted her right hand, rotated her wrist 180 degrees, and placed her right-hand palm up on top of her left hand) to our point, P prime

and that post image location, such that the line L is a perpendicular bisector of segment PP prime. (see Figure 45)

The students did not have the option to view this gesture because Naomi made this gesture out of the document camera's view. However, I coded the gesture as representational because she reflected her right hand across an imaginary line running down her middle finger. With this particular imaginary line in mind, her gesture matched her verbiage. This gesture explicitly communicated that a reflection reversed the orientation of the image and with the particular line of symmetry in mind could communicate that the line of reflection served as the perpendicular bisector of the segment pp'.

Figure 45

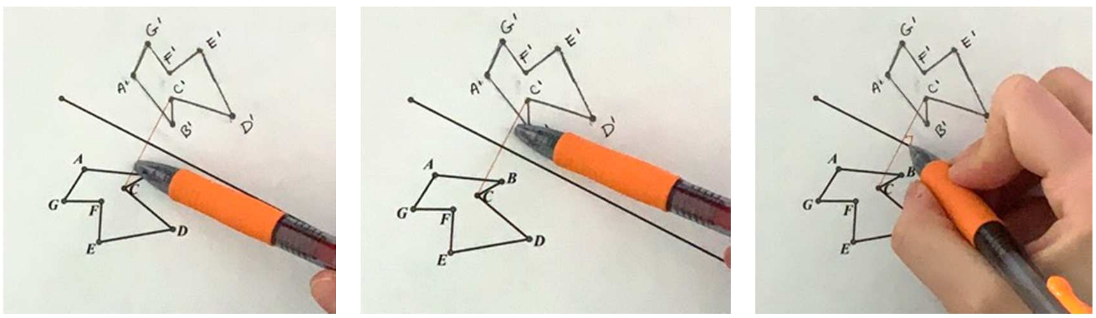
Hands Pressed Together Then Flipped Her Top Hand



In the next problem, the class reflected a polygon across a particular line. Naomi demonstrated this problem for her students, emphasizing the aspects of the "college definition" she gave earlier. Under the document camera she said, "**This distance** (tapped an added line) from C to reflection line is equal to the **reflection line to C prime** (tapped another added line). And it's always **perpendicular** (added a small right-angle mark)" (see Figure 46). Due to the fact that Naomi's actions drew attention to the handout and located the objects she spoke about; I coded these gestures as pointing gestures. These gestures grounded Naomi's verbiage within the inscriptions she added to the handout.

Figure 46

Tapped Inscriptions Naomi Added to the Photo to Accompany Her Words



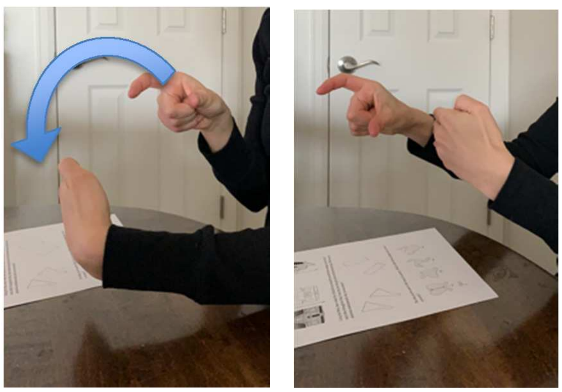
While away from the document camera, she further connected this problem to the college definition by saying,

> If we go back to the definitions it is saying the points are **equidistant** (held a small line segment between her right index finger and thumb) on **either side** (made a small barrier with her left hand and moved the small line segment in her right hand over the barrier and back) of the reflection line. (see Figure 47)

I coded these gestures as representational because her movements aligned with her verbiage and potentially displayed her conception of a reflection as a movement with a tangible object. Like previous gestures for a reflection, the small line segment gesture appeared to convey that a reflection switched an object from one side of the line of reflection to the other as well as the notion that a reflection did not change the size or shape of an object throughout the transformation. With extrapolation from the viewer, this line segment gesture could highlight that the line segment connecting a point p and its reflected image p' was perpendicular to the line of reflection and that the line of reflection bisected the line segment between point pp'.

Figure 47

Moving a Designated Line Segment Across a Line of Reflection and Back



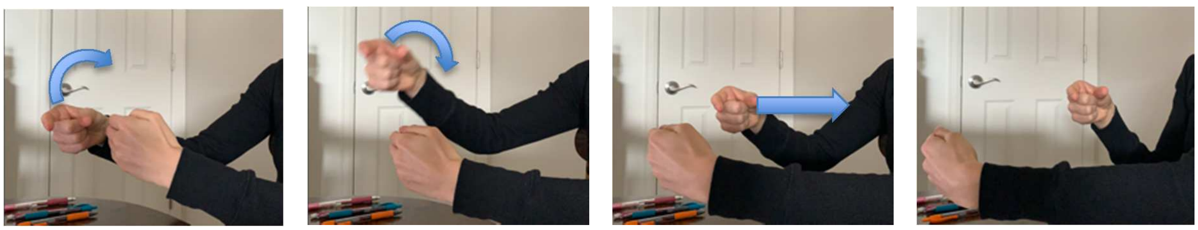
Naomi transitioned the class onto the next problem pertaining to glide reflections. Before she asked the students to work through the problem she refreshed their memory of the definition of a glide reflection. While away from the document camera she said,

> So, a glide reflection is the composition of a translation with a reflection, such that the translation is in the direction of the line of reflection. So that's **why** (specified a distance with her right finger and thumb while making a small wall with her left hand) we **reflect it** (moved the specified distance over the wall toward her body) and **move it** (moved her specified distance closer to her). (see Figure 48)

I coded this gesture as representational because, much like the previous example, her movements depicted her verbiage with imaginary referents. The students did not have the option to see this gesture as it was not under the document camera. This gesture could have communicated a glide reflection as a composition of two transformations, however the vector of translation was not parallel to the line of reflection and the gesture did not convey the opposite orientation from the reflection.

Figure 48

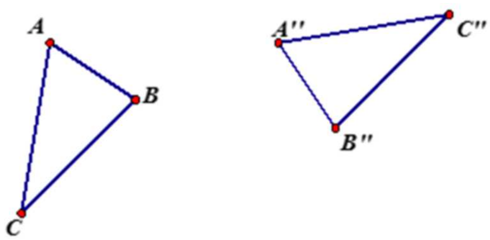
Moving a Specified Distance Over a Line of Reflection Then Parallel to the Line



For the glide reflection problem, the class started with two triangles, ABC and $A''B''C''$ (see Figure 49), and with the knowledge that the two triangles were glide reflections of each other. The directions stated to connect corresponding points and to find the midpoint of each new line segment.

Figure 49

Triangles Which are Glide Reflections of Each Other From the Handout

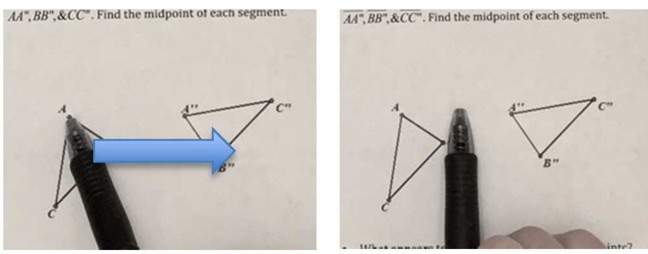


After reading the directions out loud for the class, Naomi said, “we're going to do each segment separately. We construct **segment $\boldsymbol{A}$, $\boldsymbol{A}$ double prime**, (pretended to connect A to A'')

and then we need to **bisect** (tapped the space between A and A'') that segment" (see Figure 50). I coded these gestures as pointing gestures because Naomi's gestures brought attention to particular points on the triangles and grounded her speech with the handout. The students saw which corresponding points to connect and the rough location of the midpoint. Naomi then informed the students that by connecting each of the midpoints, the resulting line was the line of reflection within the glide reflection.

Figure 50

Identifying Corresponding Points and the Estimated Midpoint Location



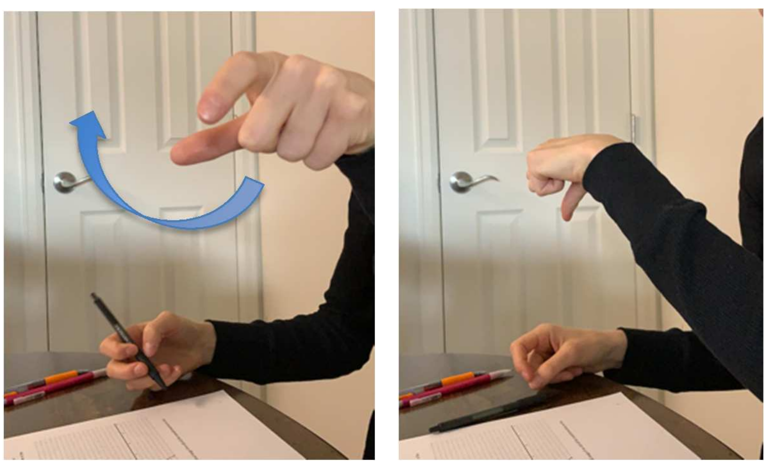
The final Euclidean transformation that Naomi taught was rotations. For the first rotation problem in the handout, the students rotated an image of a car 120 degrees clockwise about a point. Naomi's suggested that the students "choose a friendly point" and label it P. Away from the document camera she said,

> We're going to draw our segment P to A and use that **fixed radius** (designated a specific distance with her index finger and thumb). We just basically do **a partial circle** (bent her wrist back and forth to move this specified distance about an arc in front of her). So, you know somewhere along that arc, we just created will be our point A prime. (see Figure 51)

I coded this gesture as representational because the gestures closely mirrored Naomi's words. The designated distance that she made with her index finger and thumb was the object she rotated and the smooth arcing motion she made represented the rotation about a central fixed point near her chest. Naomi's rotation gesture was not made in a location visible to the students. Still, the gesture could have communicated the specified distance remained unchanged throughout the rotation, hence, demonstrated a rotation as a rigid motion.

Figure 51

Rotating a Specified Distance in Front of Her Body



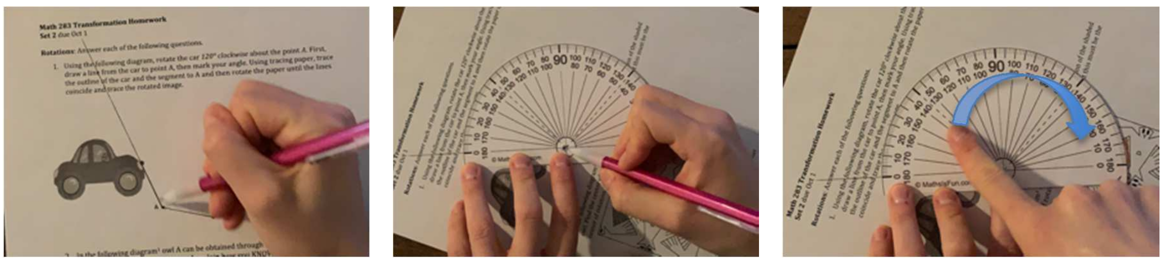
Now under the Document Camera, Naomi asked the students to follow along as she rotated the printed image of the car 120 degrees:

> It is easier to just turn **the paper** (connected points *A* and *P*). Just make sure we're doing 120 degrees, make sure it's nice and big enough. And then we'll use **our protractor** (lined the protractor up on the new line segment) **to copy** (traced the angle measure with her finger) the angle measure. (see Figure 52)

Due to the fact that Naomi's gestures located objects on the paper or protractor and did not always match her words, I coded these gestures as pointing gestures. These gestures drew attention to the handout and the tools used instead of conveying a mathematical concept.

Figure 52

Illustrating Important Steps When Rotating the Car



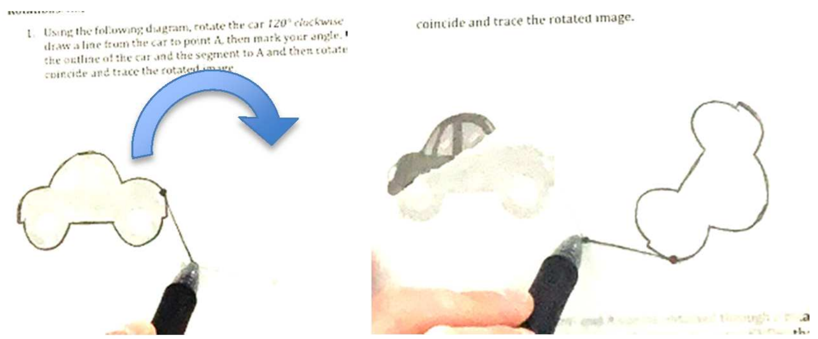
After demonstrating how to rotate her "friendly point" 120 degrees, she showed the class how to use tracing paper to finish rotating the printed image of the car 120 degrees. Naomi said,

> So, keep point A fixed, place your **car on the car** (aligned her tracing of the car on top of the printed car below), you get your segment AP, and we just **turn the car** (turned the tracing paper without moving the handout) until that segment AP co-aligns with what we already have, and then point P lays on P prime. And now we can just trace her car. (see Figure 53)

The use of a physical object helped preserve the necessary characteristics of a rotation. For example, when Naomi rotated the tracing paper, the size and shape of the image remained the same and the distance from any point on their shape to the center of rotation.

Figure 53

Demonstrating How to Rotate the Entire Car Using Tracing Paper



To conclude the unit on Euclidean transformations, the students translated, reflected, and rotated an image of an octagon on a grid. Naomi demonstrated each Euclidean transformation under the document camera for her students. As she translated and reflected the octagon she made several pointing gestures like the ones she previously enacted. However, while rotating the octagon, Naomi made a new gesture with a piece of paper to address a student question about a rotation.

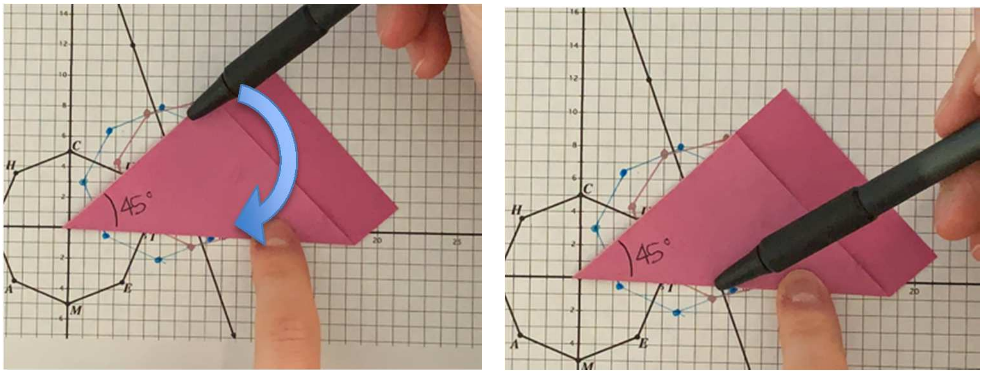
Student: I am confused on how to do the rotation; can you go over it again?

> Naomi: You have your pentagon **here** (points to the image of the pentagon). If you have a **little piece of paper** (aligned the corner of the new paper on the center of the octagon) you can kind of keep **this angle** (pointed to the initial side of her angle and then moved her pen to the terminal angle side) kind of figured out. We're just gonna rotate it until this point hits that ray you created from before. (see Figure 54)

I coded these gestures as pointing gestures because the movements annotated the picture and drew attention to the features of the added paper image. The pointing gestures grounded her speech within the definition example on the handout. The gesture did not convey information about a rotation, rather the gesture allowed the students to see referents Naomi described with her words.

Figure 54

Tapping the Initial Side of the Angle of Rotation and Then the Terminal Side



After translating, reflecting, and rotating the octagon, Naomi asked her class one final question. She asked the class if they started with the same initial octagon and performed the Euclidean transformations in the reverse order, so rotated, reflected then translated, would the final location of the octagon stay the same. Rather than take up class time, Naomi asked the students to visualize where the octagon would go. Once the students had a hypothesis on whether or not the octagon's final location remained unchanged when enacting the Euclidean transformations in the reverse order, they clicked on the green check mark or the red x within the online conference platform. After a majority of the class voted, the students reported a split decision, roughly half felt that the octagon would return to the same location and half thought it would change. One student admitted that she simply had "no idea what to do." To address the question, Naomi said,

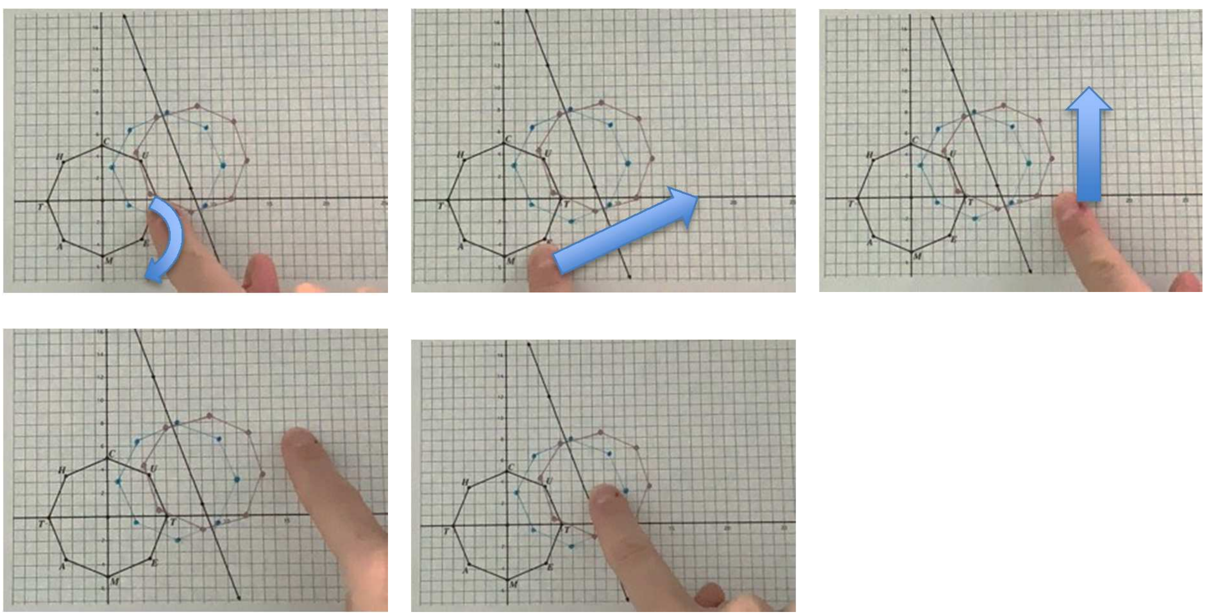
> Let's think about what's going to happen. So, we're starting off with **this guy** (pointed to the initial octagon) … We're to rotate it 45 degrees. So, **pop it** (moved her finger roughly 45 degrees clockwise) down to back here. Now we're going to **reflect it** (jumped her finger across the provided line of reflection) and then we **translate it up** (moved her

finger up). So, it would be over **here** (circled blank space on the grid). So, it would not be in the **same location** (pointed to the other octagon's final location). (see Figure 55)

I coded this gesture as pointing because each movement drew attention to the provided gird following her verbiage. The pointing gestures grounded her words within the grid with the previously transformed octagon on it. The gesture allowed the students to visualize Naomi's words.

Figure 55

Pointing Where Reverse Transformations Should Place the Octagon



Overall, Naomi enacted a combination of representational and pointing gestures when delivering her interactive lectures. From my observations, it appeared that Naomi utilized pointing gestures at a slightly higher frequency than representational gestures. The gestures visible to her students ranged from small taps of her pen to intricate compact movements with her hand and fingers. The gestures away from the document camera consisted of representational gestures which were larger in size and involved imaginary referents. Naomi's gestures always communicated a transformation with a pre- and post- image, but they did not always address the necessary conditions for each individual transformation. Lastly, on several occasions, Naomi's gestures potentially conveyed more mathematical information than her verbiage, however, Naomi never instructed her students to specifically look at her gestures. Hence, it was unknown if the students noticed these subtleties and made the appropriate inferences. In the next section, I describe the results from Naomi's post observation interview.

Naomi's Interview Narrative

During the post-observation interview we discussed Naomi's opinion on the purpose of gesturing, the gestures she would make if teaching face-to-face, her thoughts on her synchronous online gestures, and her reported intentionality while gesturing online.

As an overarching question, I asked Naomi why she gestured while teaching and what was the purpose of gesturing in general. Naomi responded that gesturing while teaching was not only for her students' benefit, but also for her own benefit. She claimed that her gestures for teaching, whether face-to-face or online, helped her remember each of the Euclidean transformations. She said, when I gesture "I'm focusing on the properties that I'm enacting. I think I'm trying to make sure I'm getting everything that I need." In addition to supporting and focusing her own verbiage, Naomi said she believes her gestures provided her students the opportunity to visualize the mathematics and helped them "connect the actual motion with what you're [the instructor] trying to do." In the interview, Naomi hypothesized that by watching her enact gestures for each of the Euclidean transformations, her students were better able to "identify each unit that's being moved [in a translation] …see that it's [a reflection] really flipping it over," and visualize "keeping the size and shape and then turning [for a rotation]." In response to these claims, I asked Naomi if in the past anyone showed her what gestures to make for these transformations. Quickly, she answered, “no, I think I just do them because those students are very visual.” Her statement connected to her belief that gesturing could help students see physical representations of the mathematics.

Before showing Naomi the Euclidean transformation gestures she made during her synchronous online sessions, I asked her what gestures she might make if teaching Geometry for Elementary Teachers in a face-to-face setting. The gestures she described as her likely face-to-face gestures perfectly matched a gesture she actually enacted during her synchronous online sessions. For example, in the interview when describing the translation gesture she would likely make if teaching face-to-face, Naomi held onto the sides of an imaginary cube, slid it in the space in front of her, and claimed that the gesture communicated “each point was being moved the same distance to a new location.” In the online synchronous sessions, Naomi made this exact gesture away from the document camera, out of her students’ view (Figure 44). In the interview, Naomi claimed that when teaching face-to-face she would likely flip her open hand from palm-up to palm down to demonstrate a reflection. She claimed that this potential face-to-face gesture preserved distance because “you're just taking it, you're flipping it … keeping it the same distance away [from the line of reflection].” In the online synchronous sessions, Naomi made this exact gesture under the document camera (Figure 30). Next in the interview, Naomi described that her gesture for rotation if she taught face-to-face was likely to create a blade with her hand and to bend her wrist back and forth. She claimed that this gesture communicated “that each point is moved that same angle measure and you’re kind of keeping the distance from that center the same as you're turning.” In the online synchronous sessions, Naomi made this gesture with her extended finger instead of her blade shaped hand under the document camera (Figure 39). Lastly, in the interview when portraying the gesture for a glide reflection that she would likely make if teaching face-to-face, Naomi flipped her open hand from palm-up to palm down across an imaginary line and then shifted her hand parallel with the imaginary line. She stated that from this gesture students had the opportunity to see a glide reflection as “flipping the orientation and then you're moving the shape the same distance in one direction.” In the online synchronous sessions, Naomi made this gesture with her extended index finger instead of her entire hand under the document camera (Figure 40).

Next, I showed Naomi her gestures for Euclidean transformations from her synchronous online sessions. Naomi appeared to be both intrigued and unaware of the gestures she made in her synchronous online sessions. Initially, she seemed surprised by her gestures, saying “I don't think I realized that I made many of them [the gestures]” and “I just would have thought I did the same thing for all of them [gestures for one transformation].” Her response to seeing her own gestures pointed to a possible lack of intentionality. After the initial shock, Naomi realized that the gestures she described she would likely make if teaching face-to-face matched a gesture she made during her synchronous online sessions. Naomi noted that the synchronous online gestures were “a smaller version” of the gestures she would probably enact teaching face-to-face. She said it seemed as though the potential face-to-face gestures and the synchronous online gestures were “kind of the same, you know just small scale versus big scale” but that “the portion that's being seen is smaller” in the synchronous online setting.

To further investigate this phenomenon, I asked Naomi why she thought she enacted the same gestures during her online synchronous sessions as she might during a face-to-face class. Naomi responded, “I can’t think of any [reasons], I probably just do them [gesture] automatically.” Continuing, Naomi said,

> If I'm in-person, because they [the students] will be looking at me, I might be intentionally doing it [gesturing] as I'm talking about it [the transformations] more. While here [online], it's just, I know they're not seeing me, so I'm not as conscious of making the motions.

Naomi again claimed to make the same gestures online as if she was face-to-face, but added that she lost some of the consciousness and purposefulness driving her gestures when teaching online.

In the interview, Naomi discussed her opinion on the purpose of gesturing, the gestures she would make if teaching face-to-face, her thoughts on her synchronous online gestures, and her reported intentionality while gesturing. Naomi reported to believe that gestures could benefit both her and her students, but this opinion did not seem to prompt any purposeful use of gestures. From our conversation and Naomi’s statements, the synchronous online setting seemed to only change Naomi’s gestures in the sense that the online setting allowed her to relax the conscious intentionality of her gestures. Lastly, the gestures that Naomi described to likely enact while teaching face-to-face were identical to gestures she made in the synchronous online sessions. In the next section, I compare Edwin and Naomi’s classroom narratives and interviews.

Comparing the Narratives

In this section, I compare Edwin and Naomi's narratives from their synchronous online sessions as well as from their interviews. Several aspects of the nature of Edwin and Naomi's gestures for Euclidean transformations were starkly different. The instructors differed on their beliefs on the purpose of gesturing, the location of their gestures, the reported intentionality behind their gestures, and the type of gestures they enacted most frequently. Despite these differences, some similarities between the nature of Edwin and Naomi's gestures occurred. Both instructors used representational gestures to conveyed nuanced characteristics and dynamic aspects of each transformation, over-emphasized all transformations as a pre- and post-image, utilized pointing gestures in response to student contributions, and never articulated to their students to pay attention to their gestures.

The first major difference in the nature of the instructors' gestures was their beliefs on the purpose of gesturing. In Edwin's interview, he explicitly stated that the purpose of gesturing was to make him more interesting in class and to provide his students a new avenue for learning. Additionally, from my analysis, Edwin seemed to believe students should be involved in gesturing about mathematics. He described his ideal gestures for a face-to-face class involved students by asking them to use their whole bodies to engage with the transformations. In Naomi's interview, she explained that the purpose of gesturing was for herself, to better focus her words, and for her students, to provide them an opportunity to see the mathematics. My analysis suggested that Naomi believes students learned best by watching others gesture. In contrast to Edwin, Naomi never advocated for the students to make their own gestures. Rather, in her interview and during the synchronous online sessions, she described wanting her students to passively watch as she gestured.

The second, exceedingly noticeable, difference in the nature of the instructors' gestures was the location where they made their gestures. Edwin's students could see all of his gestures while Naomi produced approximately one out of every five gestures out of her students' view. I captured the gestures Naomi made out of her students' view with the secondary camera. This difference potentially stemmed from the instructor's beliefs on the purpose of gesturing. If the instructors believed that the purpose of gesturing was to attract and keep the attention of students or to assists students' learning, then a logical decision was to gesture within the students' view. However, if the instructors believed that the purpose of gesturing was for their own benefit, then enacting these gestures out of the students' view was valid.

The next noteworthy difference in the nature of the instructors' gestures for Euclidean transformations was the variability in their reported intentionality behind their gestures. During Edwin's interview, he expressed a purposeful attempt to emphasize transformations as moving an entire object. Additionally, he revealed his prior knowledge regarding the potential impact of gestures in the mathematics classroom and his conscious effort to use this information when teaching. However, Edwin reported intentionality behind only one type of gesture, large three-dimensional movements with imaginary objects. In Naomi's interview, she expressed lowering the intentionality of her gesture production in the synchronous online sessions because the students could no longer see her. The gestures she produced were "automatic" or spontaneously created based upon what she believed would be helpful as opposed to deriving from the literature.

The last difference in the nature of the instructors' gestures for Euclidean transformations was the type of gestures they enacted most frequently for their students. From my analysis of the gestures within the students' view, I coded more of Edwin's gestures as representational while I coded more of Naomi's gestures as pointing. Edwin's gestures featured a dynamic movement with a large imaginary referent while Naomi's gestures remained compact and occasionally featured small imaginary referents. For example, as Edwin transitioned from investigating reflections to rotations, he grabbed onto the top and bottom of an imaginary cube on this left side and in a large arching motion, flipped the imaginary cube to the right (Figure 22) as a gesture for the word "mirroring." While Naomi's gesture for a reflection as she introduced the definition was to flip her extended index and middle fingers across a line of reflection (Figure 38).

For all the pronounced differences, the instructors had some similarities in the nature of their gestures for Euclidean transformations. The first similarity was that both instructors used representational gestures to convey nuanced characteristics and dynamic aspects of each transformation. For example, while working with a small group, Edwin smoothly slid an imaginary cube to the right and away from his body (Figure 14) to represent a translation. In this three-dimensional gesture, the entire imaginary cube shifted as a whole unit for the same distance in two separate directions. Hence, the gesture potentially conveyed a translation as a rigid motion sliding all points the same distance and direction. Edwin's representational gesture for a translation was large, dynamic, and provided the students with the opportunity to see specific characteristics of the definition of a translation. When introducing a rotation, Naomi rotated the tip of her extended right index finger by bending her wrist 90 degrees clockwise (Figure 39). In this gesture, the center of rotation was her wrist and the distance from the tip of Naomi's finger to her wrist remained unchanged throughout the rotation. Moreover, her finger did not change in size or shape, so this gesture conveyed the idea of a rotation as a rigid motion. Although Naomi's gesture was small, it was still dynamic and afforded the students the opportunity to see nuanced characteristics of the definition of a rotation.

A closely related similarity between the nature of their gestures was that all gestures for both instructors depicted a transformation as beginning with a pre-image and concluding with a post-image. This pre- and post-image portrayal of a transformation occurred in many forms from the movement of imaginary three-dimensional objects to pointing with a pen. How the transformation mapped the pre-image to the post-image was not always clear. When the instructors' utilized a representational gesture the mapping was unambiguous, but when the instructors utilized a pointing gesture to identify their verbiage within the handout, the gestures did not communicate the nature of the mapping. For example, when describing how image A mapped onto image C in Figure 11on the handout, Edwin sequentially tapped the line of reflection, original image, translated image, and reflected image (Figure 13). His gesture portrayed a glide reflection as a discrete point-by-point movement with individual steps to progress from the starting location to the ending location. By lifting his pen off of the paper to change from image A to image C, the specific characteristics of the mapping became obscured. The orientation, size, and shape of image A could change if the only known information about the mapping was Edwin lifting his pen off of the paper to move between image A and image C. Similarly, when introducing a translation, Naomi touched the left end of a translation vector then jumped her pen to the right end of the translation vector (Figure 37). Again, by lifting her pen off of the paper, how the point traversed the distance of the translation vector was unclear. The mapping could be a translation, a rotation, or a reflection.

The third similarity was that both of the instructors performed more pointing gestures in response to student contributions or questions than representational gestures. These pointing gestures grounded Edwin and Naomi's responses in the class handouts by locating specific items in the directions, diagrams, or added inscriptions under the document camera. For example, one of Edwin's students made a conjecture that the order of translating, rotating, and reflecting an octagon did not change the final location of the octagon if performed in the "exact opposite order." Edwin responded to this student with a series of pointing gestures under the document camera. As he spoke, he circled, underlined, and pointed to different words in the printed directions as well as pointed to the line of reflection in the provided diagram. One of Naomi's students asked a question as to why an image of a ghost did not have reflection symmetry. She asked, "So, for B [the ghost image], the reason that it doesn't have a line of symmetry is because it reflects a different image on both sides?" To confirm the student's thinking, Naomi described and pointed to several features of the ghost that made the image non-symmetrical. Both instructors occasionally enacted representational gestures in response to student contributions and questions, but the overwhelming majority of the gestures following students' input were pointing gestures.

The last similarity was that both instructors neglected to explicitly instruct their students to view gestures as carrying mathematical information and to carefully watch the gestures they produced. Without explicit instruction to look for and utilize their gestures, the students could easily ignore the instructors' gestures and the instructors could lackadaisically produce flawed gestures. For example, when Edwin explained how to view a translation in terms of the horizontal and vertical components of the translation vector in Figure 9, he smoothly slid his open, palm-up hand from shoulder height down and to the right. However, while making the movements, the shape of his hand changed. Thus, this gesture dropped the preciseness of a translation as a rigid motion that preserved the shape of his hand. Edwin did not tell his students to pay attention to his gestures, hence maybe he did not either, which created space for his imprecision. An example of the students likely disregarding a nuanced piece of the instructors' gesture was when Naomi worked with the mira. In her opposing finger gun gesture (Figure 31) her students could interpret that any pre-image point on her hand and its corresponding post-image point maintained their distance from the line of reflection because her opposing finger guns appeared to be equal distance from the line of reflection or the mira. However, at this time she did not vocalize the idea to her students, instead, this mathematical information was only accessible if the student made the unspoken inference. If the students noticed the gestures the instructors made, recognized these gestures as mathematically relevant, accepted the imperfections of the gestures, and made the necessary assumptions and inferences for the gestures, then the students had the opportunity to learn Euclidean transformations in the medium of gestures. However, without explicitly instructing the students to do so, it is unknown if the students used the instructors' gestures as learning opportunities.

Edwin and Naomi taught the same mathematical topic with the same materials in the same synchronous online environment. However, the instructors differed on their beliefs on the purpose of gesturing, the location of their gestures, the reported intentionality behind their gestures, and the type of gestures they enacted most frequently. In a few ways, the nature of the gestures they created for Euclidean transformations remained consistent. They both used representational gestures to convey nuanced characteristics and dynamic aspects of each transformation, over-emphasized all transformations as a pre- and post-image, utilized pointing gestures in response to student contributions, and never articulated to their students to pay attention to their gestures. In the next section, I interpret these findings in light of my research questions. After interpreting these findings, I discuss the implications of the findings for synchronous online teaching and how this study contributed to research surrounding gestures in the mathematics classroom. Finally, I conclude the next section with the limitations of my study and possible directions for future research.

CHAPTER V

DISCUSSION

The purpose of my study was to contribute to the literature on the gestures of mathematics instructors as they taught Euclidean transformations in a synchronous online setting. In particular, my study sought to answer the following research questions:

Q1 What is the nature of instructors' gestures as they teach Euclidean transformations in a synchronous online setting?

Q2 How, if at all, does a synchronous online setting impact the instructors' intentionality and usage of gestures?

In this chapter, I interpret my findings from Chapter IV in light of my research questions. After interpreting these findings, I discuss the implications of the findings for synchronous online teaching and how this study contributed to research surrounding gestures in the mathematics classroom. Finally, I conclude with the limitations of my study and possible directions for future research.

The Nature of the Instructors' Gestures

As a result of my study, I found three attributes of my instructor's synchronous online gestures. These three qualities addressed my first research question. First, I found that Edwin and Naomi made representational gestures and pointing gestures while teaching Euclidean transformations in their synchronous online sessions. On the one hand, the representational gestures potentially served as a second form of communication for the students and as such had the potential to provide the students additional learning opportunities. On the other hand, the pointing gestures grounded Edwin and Naomi's responses to student contributions within the diagrams on the handouts. Second, I discovered that the mathematics conveyed in Edwin and Naomi's gestures did not always communicate all of the mathematical criteria for each Euclidean transformation. Through my analysis, I found that the combination of the instructors' gestures and language provided a more cohesive picture of the Euclidean transformation. Lastly, I found that Edwin and Naomi believe the purpose of their gestures was for the benefit of their students as well as for themselves. Both Edwin and Naomi believe that students could learn from their gestures. In addition, Naomi reported to believe that gesturing also ensured that she communicated the nuanced characteristics of a Euclidean transformation. These three results provided insight into the nature of Edwin and Naomi's gestures when teaching Euclidean transformations.

In their synchronous online sessions, I found that Edwin and Naomi performed Alibali and Nathan's (2012) representational gestures in a variety of movements including, tracing the circumference of a circle for a rotation, sliding an imaginary cube for a translation, and flipping their hand for a reflection. Upon further analysis, I discovered that all of the instructor's representational gestures appeared to capture the continuous movement or a rigid motion of the particular Euclidean transformation. Describing Edwin and Naomi's representational gestures as rigid and continuous motions of mathematical ideas aligned with Chu and Kita's (2016) conclusion that individuals produced more representational gestures when describing a "smooth" object. Much like the work of Chen and Herbst (2013), Edwin and Naomi's gestures and verbiage worked in tandem to bring a picture to life. For example, when Naomi fluidly flipped her hand between two reflected houses while describing the reflection, her gesture and verbiage brought the static picture of the two homes to life. The students had the opportunity to view how the pre-image left the paper and landed on the post-image, which brought the house to life. The continuous rigid motion in each of the instructor's representational gestures potentially provided Edwin and Naomi's students the opportunity to abstract a Euclidean transformation in a blend of speech and gestures. My finding supported Valenzeno et al. (2003), Cook et al. (2013), Congdon et al. (2017), Pi et al. (2017), and Cook et al.'s (2017) results which suggested that representational gestures in the mathematics classroom served as a second modality for students to engage with the material, namely one through auditory mental schema and one through motor mental schema.

Another important quality of the nature of the instructor's representational gestures was where Edwin and Naomi chose to enact these gestures. From my analysis, I uncovered that the instructors' gestures for Euclidean transformations enacted away from the document camera were almost exclusively representational gestures. This finding supports Chu and Kita's (2008) conclusion that gestures that manipulating an imaginary three-dimensional object, or representational gestures, were more likely to be performed in front of the gesturer's body. Edwin utilized both the document camera and a web camera throughout his synchronous online sessions. Hence, his students could view all of his gestures no matter the location. Naomi only utilized a document camera during her synchronous online sessions and so, any gesture made away from the document camera was out of her student's view. Therefore, her students would not have the opportunity to use the representational gestures performed away from the document camera as another modality for learning.

Representational gestures were not the only type of gesture that Edwin and Naomi created, I found that the instructors also enacted many pointing gestures. The instructors performed the pointing gestures with their fingers or pens. As opposed to serving as a second mode of communication, I found that these pointing gestures connected the diagram inscriptions to Edwin and Naomi's verbiage. These pointing gestures frequently portrayed a Euclidean transformation as a disjoint or discrete mapping. Specifically, Edwin and Naomi used a pointing gesture to locate a pre- and post-image within the handout without providing information on how a pre-image transformed into the post-image. This finding corroborated several researchers' findings that pointing gestures grounded or anchored abstract mathematical ideas in the physical classroom by identifying items in the classroom materials to accompany speech, attracting students' attention, and establishing common ground (Abrahamson et al., 2020; Alibali & Nathan, 2012; Valenzeno et al., 2003; Weinberg et al., 2015). My results specifically supported Soto-Johnson and Troup's (2014) claim that gestures were "the link between verbiage and diagrams" (p. 112). Edwin and Naomi used the pointing gestures to connect what they said to the words and pictures on the handouts. Similar to the representational gestures, the location where Edwin and Naomi enacted their pointing gestures was noteworthy. From my analysis, I found that under the document camera the majority of the instructors' gestures were pointing gestures. This finding supported Weinberg et al.'s (2015) results that collegiate mathematics instructors utilized a wide variety of pointing gestures while engaging with their course notes. Lastly, pointing gestures frequently accompanied Edwin and Naomi's responses to student questions and contributions. This finding corroborated Alibali and Nathan's (2007) conclusion that teachers made more gestures grounding their verbiage in response to students' utterance. My discovery that Edwin and Naomi utilized pointing gestures when responding to student questions, specifically supported Alibali et al.'s (2013) conclusion that pointing gestures focused the classes' attention on common referents in situations where students and the teacher did not have a shared understanding. Further, my finding that Edwin and Naomi produced pointing gestures during their responses to students' input aligned with Alibali et al.'s (2019) study on teachers' gestures supporting students' contributions. Much like my findings, Alibali et al. (2019) reported that pointing gestures identify specific referents and that teachers' pointing gestures connect students' verbal contributions to the physical environment, making students' contributions clear for the entire class.

My second finding was that the mathematics conveyed by Edwin and Naomi's representational and pointing gestures did not always communicate all of the nuanced criteria for each Euclidean transformation. Frequently, the gestures by themselves portrayed a more holistic picture of the Euclidean transformation rather than the formal definition. For example, Edwin plucked an imaginary point from the space in front of him and moved it to a new location while describing a translation. This gesture clearly communicated a translation as moving something to a new spot, but did not communicate a translation as a rigid motion moving all points along the same translation vector. Weinberg et al.'s (2015) found similar results when they noticed their advanced collegiate mathematics instructor enacting gestures that did not convey the complete mathematical idea. Additionally, Edwin and Naomi's gestures revealed the intuitive motions associated with the Euclidean transformation. In fact, Edwin described his gestures as the "actionable" and "natural thing" for students to do when first learning about Euclidean transformations. It was as if Edwin wanted to make gestures that his students could relate to when introducing the new material. This finding supported Alibali et al.'s (2013) result that teachers gestured with familiar physical actions to promote a common ground in the classroom.

Furthermore, I uncovered that the combination of Edwin and Naomi's gestures and their accompanying verbiage communicated a more complete mathematical definition of the Euclidean transformation. For example, both Edwin and Naomi, made gestures for a reflection that did not communicate that the line of reflection served as the perpendicular bisector of the line segment created by a point and its image point. However, with words such as “we're basically, reflecting across line l to our point, p prime and that post image location, such that the line l is a perpendicular bisector of segment pp prime” and “This dot (the original point) is one unit away from our [reflection] line as is, as is its post image,” Edwin and Naomi’s gestures and speech created a more robust picture of a reflection. My discovery supported the conclusions of many research teams such as Arzarello et al. (2009), Congdon et al. (2017), and Weinberg et al. (2015) who claimed that when their instructors struck a balance between their speech and gesture, learning opportunities were maximized. Some of Edwin and Naomi’s gestures by themselves fell short of communicating all of the information in the Euclidean transformation definition, however their verbiage complemented their gestures and afforded their student access to a more complete definition.

My last finding, on the nature of Edwin and Naomi’s gestures during their synchronous online sessions, was that Edwin and Naomi reported to believe the purpose of their gestures were both for the benefit of the students and for themselves. Predicated on Edwin’s interview, he believes the purpose of gesturing was for his students, specifically, to capture their attention and to provide them with a new way to engage with Euclidean transformations. Edwin believes that the audience of his gestures was his students. Similarly, from Naomi’s interview, she believes that one purpose of gesturing was to show her students the movement of each Euclidean transformation. My finding that Edwin and Naomi believe gesturing can benefit their students supported the work of Nathan et al. (2019) who surveyed teachers on their opinions towards gesture in the mathematics classroom. The high school teachers in their study reported that they believed their gestures helped students learn, specifically, by making connections between representations and ideas and making abstract concepts more concrete. My finding on Edwin and

Naomi's beliefs complemented the analysis of teachers' gesture production of many past researchers who found that a speaker's gestures benefit the listener. Prior studies concluded that seeing someone gesture helped listeners generalize the message (Congdon et al. 2017; Novack et al., 2014) and establish a common ground (Alibali & Nathan, 2012; Alibali et al., 2019; Weinberg et al., 2015) and prompted listeners to use similar gestures in their own speech (Morett, 2018). My study addressed gestures produced for the benefit of the listener from an alternative, but harmonizing, perspective. The above listed benefits were assumptions of the researchers while my results were the feelings from the speakers themselves.

In addition to benefiting students, Naomi also described the purpose of gesturing as assisting herself, to focus and ensure precision of the material of her lectures. Unlike Edwin, Naomi seemed to believe that the audience of a gesture could be the students or herself. This finding on Naomi's belief for the purpose of gesturing complemented the conclusions of many past researchers who argued that gestures served a functional role for the speaker. Gestures for oneself appeared to help increase the fluency and quality of a speaker's instruction (Yang et al., 2020), focus speakers' attention (Alibali & Kita, 2010; Hostetter & Boncoddo, 2017), and help speakers to remember information (Cook et al., 2012). My study addressed gestures produced for oneself from a different, but complementary, perspective. The aforementioned benefits were assumptions of the researchers while my results were the opinions of the speakers themselves.

Overall, I uncovered that while teaching Euclidean transformations Edwin and Naomi produced a combination of representational and pointing gestures. Further, their representational gestures communicated a Euclidean transformation as a fluid, rigid motion and served as a secondary avenue for explaining the Euclidean transformations. Their pointing gestures grounded their verbiage within the handouts, identified the pre- and post-images of Euclidean transformations, and aided Edwin and Naomi's responses to students. Moreover, I found that under the document camera the instructors made more pointing gestures while away from the document camera the instructors made more representational gestures. I discovered that many of Edwin and Naomi's gestures, both representational and pointing, failed to communicate all the distinctive qualities from the definitions of each Euclidean transformation. Instead, the instructors utilized a familiar motion and their verbiage to communicate a more complete notion of each Euclidean transformation. Lastly, I found that Edwin and Naomi reported to believe the purpose of their gestures were both for the benefit of the students and for themselves. Edwin and Naomi believe that the students could learn from gestures. Additionally, Naomi believes that gesturing could ensure that she communicated the nuanced characteristics of a Euclidean transformation. In the next section, I discuss my second research question on Edwin and Naomi's reported intentionality behind their gestures and their perceived impact of the synchronous online setting on their usage of gestures.

The Intentionality in the Synchronous Online Setting

As a result of my study, I uncovered a connection between Edwin and Naomi's reported prior knowledge of gesturing, desire to adapt their gestures to the online setting, and intentionality surrounding gesturing. This pattern addressed my second research question. I found that Edwin had previous experience with evidenced-based gesturing in the mathematics classroom, needed to adapt his described face-to-face gestures to fit in the restricted synchronous online setting, and described intentionally gesturing with large, three-dimensional, imaginary objects. My finding for Edwin supported the work of Hostetter et al. (2006). Hostetter et al. found that when instructors were briefly introduced to the effectiveness of gestures and to several examples of how to incorporate gestures into their lesson the instructors were able to intentionally alter the gestures they produced during instruction. Edwin, like the participants in

Hostetter et al.'s study, was previously exposed to literature on the benefits of gesturing in the mathematics classroom and he reported intentionality behind some of his gestures. My finding of Edwin's beliefs and opinions on gesturing supported one of Walkington et al.'s (2019) conclusions, namely that teachers who indicated that gesture had a positive effect on instruction were more likely to make gestures designed to be enacted by many members of the class. Edwin displayed a positive attitude towards gesturing, he even went as far as describing himself as a "hands-talker" inside and outside the mathematics classroom. Aligning with Walkington et al.'s conclusion, Edwin professed to want to make gestures that the students could be a part of, saying "I would try and get some students to do some of these" and that "one of the best things to do" would be to get the students involved in gesturing.

On the other hand, Naomi reported that she did not have formal experience with gesturing in the mathematics classroom, did not feel a need to modify her described face-to-face gestures to fit the online setting, and lost some of her consciousness and intentionality behind her online teaching gestures. While Edwin used his formal training with gestures, Naomi relied on her face-to-face experiences teaching Geometry for Elementary Teachers when conducting her synchronous online sessions. Naomi was familiar with the mathematical content and as such reported her gestures in the synchronous online setting were "kind of the same" simply a "smaller version" of the gestures she thought she made in previous face-to-face teaching experiences. This finding supported Walkington et al's. (2019) conclusion that prior teaching experience did not significantly increase gesture production and usage. In fact, Alibali et al. (2014) reported that when a teacher discussed material that they previously taught, they gestured considerably less. Additionally, in Naomi's interview she proclaimed that she was "not as conscious of making the motions" in the synchronous online sessions because her students could not see her, rather her gestures were automatic and spontaneous. This finding corroborated Nagels et al.'s (2015) reported correlation between social perception and gesture production. They found that people reported to use more gestures in interactions with high social pressure to empathetically assist another person. Naomi could not see her students and her students could not see her. Hence, Naomi reported feeling less social pressure and as a result was less conscious and intentional in her gesture production. The proceeding section describes the contributions of my study within the research field of gestures in the mathematics classroom.

Contributions to Gesture Research

My findings contributed to the field of gesture research in two ways. The first contribution was an addition to the usage of representational gestures and pointing gestures in the online mathematics classroom. The second contribution was the perspective from the instructor on what they themselves believe was the purpose of their gestures.

First, as an overarching contribution, my study was the first to investigate the gestures of online mathematics instructors. Until this study, researchers investigated the gestures of face-to-face teachers. As more schools and institutions transition to the online setting, studies like mine will fill the gap in the literature on gestures in the online setting (Bettinger & Loeb, 2017; Black et al., 2020). Specifically, my findings corroborated and expanded Alibali and Nathan's (2012) conclusions that representational gestures reveal the teacher's mental simulations of action. My study added the qualification that when describing Euclidean transformations, the mental simulations of actions appeared to have a smooth quality. In particular, my study demonstrated that representational gestures communicated the continuous movement or the rigid motion of a Euclidean transformation. For example, throughout the online synchronous sessions, Edwin purposefully translated, rotated, and reflected an imaginary cube by moving it in large, continuous motions. Edwin's gesture likely emulated his mental simulation of the fluid action of a transformation. This contribution of smooth continuous actions supported and extended the work of Chu and Kita (2016). They concluded that individuals produced representational gestures when describing a "smooth" object whereas my study extended this notion to mathematical operations or actions such as Euclidean transformations. Edwin and Naomi utilized representational gestures while speaking about "sliding," "flipping" or "moving" a point or object under transformation. Therefore, I argue that instructors utilized representational gestures when describing mappings that they considered to be inherently smooth movements.

Second, my findings corroborated and enhanced Alibali and Nathan's (2012) conclusions that pointing gestures grounded abstract mathematical ideas in the physical classroom. My study added that pointing gestures conveyed Euclidean transformations as a point-by-point movement or discrete mapping. For example, both Edwin and Naomi frequently connected their verbiage to the class handouts by tapping on a pre-image and then its associated post-image under the document camera. This gesture not only grounded the mathematics to the physical diagram, as suggested by Alibali and Nathan, but the pointing gesture also communicated a Euclidean transformation as a procedural movement of one point at a time. Further, my dissertation findings extended the work of Alibali et al. (2013) and Alibali et al. (2019) on teachers' gestures, both representational and pointing, during face-to-face classroom student interactions. My extension was that synchronous online instructors produced primarily pointing gestures during their responses to students' questions and comments. To verbally address a student's contribution, Edwin and Naomi most frequently utilized a series of pointing gestures to guide students' attention and focus on relevant elements of the handouts.

Lastly, my dissertation study began to answer the call from Nathan et al. (2019) who posited that "future research is needed to understand the relations between teachers' beliefs about gestures and their gestural behavior during instruction" (p. 50). My dissertation investigated Edwin and Naomi's beliefs on the purpose of gesturing, their intentionality behind gesturing, and the actual gestures they enacted in their synchronous online sessions. Specifically, my findings highlighted a connection between instructors who believed in the potential impact of gestures in the mathematics classroom and intentionally producing representational gestures. My findings also indicated that instructors who have minimal experience with gestures as communicating mathematical information might spontaneously enact more pointing gestures. This finding corroborated Nagels et al.'s (2015) work that suggested a connection between gesture production and perception of gestures. Nagels et al. completed a correlation analyses between the factors of gestural perception, gesture production, social production, and social perception obtained from their Brief Assessment of Gesture' questionnaire. Perception and production of gestures explained most of the variance in Nagels et al.'s study suggesting that an individual's opinion on the gestures of others and their assumption on how much they gesture likely provided a good indicator for the amount someone gestures. For Edwin, he was a self-proclaimed "hands-talker," professed holding a high opinion of gestures in the mathematics classroom, and enacted many gestures, pointing and representational. For Naomi, she did not place substantial weight into her gestures, as she performed them "automatically" and "less conscious[ly]," did not view herself as someone who made a variety of gestures, and enacted less gestures than Edwin.

Many researchers such as Alibali and Nathan (2012), Weinberg et al. (2015), and Yang et al. (2020) investigated the gestures of teachers and categorized their gestures into who the researchers believed the teachers' gestures were for, namely for the speaker themselves or for their audience. My dissertation study, along with the work of Nathan et al. (2019) approached the idea of the purpose of a gesture from the perspective of the speaker themselves, or in our cases from the perspective of the teacher. Nathan et al. reported that their research participants, consisting of K-12 teachers, genuinely believe that instructional gestures were beneficial for learning. In particular, the K-12 teachers stated that they believe their gestures assisted students by helping them make connections between representations and ideas and by making abstract concepts more concrete. My results mirrored this conclusion because I found that Edwin and Naomi believe gestures that they made benefit their students by providing a visual for the Euclidean transformation, or a redundant message with their speech. My study extended past Nathan et al.'s claims because my results implied another reason why instructors believe they make gestures, to remember and clearly communicate all of the mathematical properties in a lesson. In the next section, I offer recommendations for synchronous online teachers.

Recommendations

Based on my results described above, I suggest several recommendations. First, based upon my findings, I recommend continued education on gesture as an avenue to communicate mathematical ideas. A professional development opportunity may assist collegiate instructors in producing more intentional and mathematically precise gestures. Second, with my findings in mind, I recommend that synchronous online instructors utilize technology which affords students the opportunity to view all of their gestures and instruct their students to pay attention to their gestures. By utilizing technology that always shows the instructor, even if the instructor makes an unconscious gesture, the students have the opportunity to view and use the gestures to advance their understanding. Furthermore, knowing that the students can see them at all times and that the students are looking for gestures might prompt the instructor to gesture with more intentionality.

Continued Professional Development

Prior knowledge of gestures seemed to foreshadow the instructors' intentionality while gesturing in their synchronous online sessions. Based upon this finding, I recommend continued education on gesture as an avenue to communicate mathematics. My study was not the first to recommend continued education for collegiate instructors (Abrahamson et al., 2020; Alibali & Nathan, 2012). In fact, Abrahamson et al. (2020) stressed the importance of professional development that helped increase the instructor's awareness of their own gestures and prompted the instructors to intentionally convey their gesturing to students.

From my findings, attending professional development workshops may assist collegiate instructors in two ways. First, by providing the collegiate instructors the opportunity to attend professional development workshops on the inclusion of gestures they may begin using gestures more intentionally in their classrooms, like Edwin. Based upon Hostetter et al.'s (2006) research, introducing the collegiate instructors to the effectiveness of gestures as well as providing them examples of how to incorporate gestures into a lesson may increase their intentionality when gesturing during their teaching. Along with exposing the collegiate instructors to literature for gesturing in the mathematics classroom, a commonality among successful collegiate professional development workshops is the use of specific inquiry-based, active learning examples which instructors can then directly apply in their own classrooms (Abrahamson et al., 2020; Barton et al., 2015; Hadar & Brody, 2010). Therefore, I recommend that the professional development workshop for gesturing in the mathematics classroom include the opportunity engage with activities that require purposeful gestures such as an activity like Soto's (2019) embodied tarp activities. In Soto's embodied tarp activities, participants acted out transformation tasks on a large grid where the participants themselves were the points and rope served as line segments. By requiring the collegiate instructors to gesture in order to meet the learning objective of the activity, they may find a new appreciation and immediate use for intentional gesturing in their classrooms like in Hostetter et al.'s (2006) study.

Second, as a result of increasing the intentionality of the collegiate instructors' gestures, they may begin to gesture in such a manner that captures more of the mathematical definition for each Euclidean transformation. My dissertation study captured Edwin and Naomi enacting gestures that did not communicate all of the nuanced qualities of a Euclidean transformation. Some of their gestures missed important characteristics, while other gestures communicated incorrect information. Before contemplating the mathematics conveyed in a gesture the collegiate instructors must first think critically about each Euclidean transformation. They must ponder the features of each Euclidean transformation that they want to emphasize to their students and how a gesture can exemplify each quality. This type of knowledge is what Ball et al. (2008) referred to as specialized content knowledge or "the mathematical knowledge and skill unique to teaching…[and] not typically needed for purposes other than teaching" (p. 400). Continuing education through a professional development workshop may provide collegiate instructors the space to deepen their own understanding of Euclidean transformations as well as to connect their mathematical knowledge to physical gestures that ground their understanding in the real world. In fact, Alibali et al. (2014) suggested that teachers viewed learning novel methods for effectively using gestures in communicating mathematical connections to their students as worthwhile and valuable. Next, I describe the implications derived from my results for synchronous online instructors.

Synchronous Online Teaching

My results indicated that synchronous online instructors should utilize technology which captures all of their gestures and should instruct students to pay attention to their gestures. Edwin and Naomi provided their students with different viewing opportunities due to the differing video technologies they used in their synchronous online sessions. Edwin's students could see all of his gestures because he utilized two video technologies, a document camera capturing his written inscriptions on the handouts and a web camera capturing his upper body. Naomi only used a document camera capturing her written inscriptions on the handouts. Her students could not see over one third of her representational gestures because she performed them away from the document camera. As previously stated, Edwin and Naomi's representational gestures communicated the rigid motion of each Euclidean transformation as well as served as a second modality for their students to engage with the material. Therefore, it is significant that Naomi's students missed the opportunity to view these gestures. Keeping in mind the recommendations of Gedeborg (2016) for best practices of online mathematics teaching, my findings imply that to provide students with maximal learning opportunities, instructors should utilize technology which affords students the opportunity to view all of their instructors' gestures. In particular, based upon my results, I recommend synchronous online instructors utilize technology that allows their students to see their upper body at all times in addition to the technology displaying the notes or classroom materials. This will ensure that the students have the ability to view and use all of the instructor's gestures to advance their own understanding, but knowing the students can always see them may promote the instructor to gesture with more intentionality. This was certainly the case with Naomi who, during her interview, disclosed that she would gesture more in a face-to-face setting because her students "will be looking" at her. Again, an increase in intentionality behind gestures could lead to more mathematically precise gestures.

By utilizing technology that displays where synchronous online instructors frequently utilize pointing gestures, as well as where they commonly enact representational gestures, the students receive the social connection recommended by Hostetter (2011) and Pi et al. (2017) as well as the focused instruction advocated by Fiorella and Mayer (2016). Mayer et al. (2017), Erixon (2016), and Gedeborg (2016) noted that technological issues could arise and cause hardship when utilizing multiple audio and visual technologies. However, my findings imply that the benefits of multiple video technologies, namely the learning opportunities provided by representational gestures, outweigh the technical adversities that instructors and their students may face.

Based on my findings, I also recommend that synchronous online instructors should explicitly instruct their students to attend to their gestures. Even if a synchronous online instructor utilized video technologies that afforded their students the opportunity to view all of their gestures, without explicitly acknowledging that gestures communicate mathematical ideas, the students may overlook the gestures as not important. Abrahamson et al. (2020) noted that part of a teachers' roll is to inform students about their pedagogical choices. For gesturing, this is as informal as stating that body movements are a preliminary indicator of learning and that learners should be cognizant of their own gestures as well as the gestures of those around them. Similar to Hostetter et al.'s (2006) study with instructors, by explicitly informing students that gestures convey mathematical ideas and instructing them to carefully watch for gestures, the students may begin to intentionally use their instructor's gestures as a way to think about mathematical concepts. Again, by instructing the students to attend to their gestures, a synchronous online instructor may become more intentional and precise with their gestures. In the next section, I describe the limitations of my dissertation study.

Limitations

As this was a case study, I did not expect my results to generalize beyond the specific circumstances surrounding my instructors and their synchronous online environment. Some constraints of my study involved how my instructors were selected, the timing of my study, and the retrospective nature of the interviews.

The instructors who served as my participants were not randomly selected. I specifically asked Edwin and Naomi to participate in my study because they were instructors of Geometry for Elementary Teachers in the Fall semester of 2020 and had experience teaching the elementary mathematics course sequence. This made my instructors a purposeful sample (Patton, 2002). For my particular study, this limited the amount that I can generalize about the gestures of other instructors at different institutions. However, I believe the purposeful, non-random selection process did not undermine the validity of my results. For my instructors, I created thick and rich descriptions of their gestures and created trustworthy results summarizing and interpreting the intentionality behind their gestures.

The second limitation of my study was the timing of the data collection. I observed the instructors' synchronous online courses during the fifth and sixth weeks of the semester. This potentially was not enough time for the instructors to adequately transition their experiences teaching the elementary mathematics course sequence from a face-to-face setting to the synchronous online environment. There was a possibility that five weeks was not enough time for the instructors to gain a complete understanding of the new synchronous online learning environment and how best to utilize the technology available to them. With this in mind, I did not think that the timing impacted the instructors enough to delegitimize my results. Naomi was an experienced online instructor, so the synchronous online learning environment was not new to her. Edwin described his awareness of the literature surrounding gestures and his natural tendency to gesture when speaking, hence Edwin used all of the technology available to him to ensure his students could see his gestures.

The final limitation of my study was the retrospective nature of the interviews. During the interview, I asked each instructor if they remembered making each of their gestures and if so, I asked them to describe the intent behind the gesture. I also asked them to describe the mathematics that they wished to convey with the gesture. This line of questioning was completely reflective. Edwin and Naomi recalled the gesture and expressed their belief on their intentionality and consciousness when gesturing. With my study design, it was impossible to ask the instructors if they purposefully enacted a gesture live during their synchronous online class. Doing so would have interrupted the class dynamic and distracted both the instructor and the students. Therefore, my results pertaining to the intentionality and consciousness of a gesture were Edwin and Naomi's reflections on their synchronous online instruction. Although my study was not without its limitations, sufficient data were collected to help address the research questions. My conclusions can be considered trustworthy. I conclude with directions for future research.

Directions for Future Research

There were three possible directions for future research stemming from my results and recommendations. First, was to conduct a comparison study of the same instructor teaching Geometry for Elementary Teachers in a synchronous online setting and in a face-to-face setting. Such research would provide strong evidence for the impact of the synchronous online setting on the instructor's usage and intentionality of their gestures. In my dissertation study, I asked Edwin and Naomi to describe what gestures they would likely enact if teaching Geometry for Elementary Teachers face-to-face and I derived my results from their descriptions. This proposed direction would allow a researcher to document the similarities and differences between the instructor's gestures for Euclidean transformations in the two settings. A researcher would then increase the depth of the results in my case study by looking for corroborating evidence. For example, perhaps in the proposed direction, when responding to student comments and questions the instructor still performs more pointing gestures as opposed to representational gestures. Additionally, a follow up interview would afford a researcher the opportunity to investigate the intentionality behind the instructor's gestures in both settings as well as investigate why the instructor enacted gestures which appeared in both or just one of the settings. Knowledge from this interview could illuminate the instructor's intentionality and explanation of gestures in different classroom settings and provide verification of Nagels et al.'s (2015) reported correlation between social perception and gesture production.

The second future direction brought about by my study pertained to professional development on gesture in the mathematics classroom for instructors. Following the recommendations of Abrahamson et al. (2020), a professional development workshop should expose instructors to literature on gesture and embodiment research, show instructors examples of practicing teachers utilizing gestures in their classrooms, and provide instructors the opportunity to authentically engage in activities that require the use of gestures. Before and after the professional development workshop, the instructors could take Nathan et al.'s (2019) survey instrument to measure the instructors' attitudes about gesture in learning and instruction. Knowledge from these surveys may speak to the instructors' beliefs towards gesturing as well as begin to address the effect of attending the professional development workshop. After the workshop, a researcher could observe the instructors teaching, carefully document the instructors' gestures, and conduct follow up interviews to discuss the instructors' intentionality and reasoning behind their gestures. This new corpus of data will again address the effect of attending the professional development workshop and may provide corroborating evidence for my finding of the pattern between prior knowledge of gesturing and intentionality of gestures.

The third future direction inspired by my study was to investigate the gestures of online instructors from the perspective of their students. In my results, I found that the instructors' representational gestures provided a second opportunity to view and engage with the Euclidean transformations. However, I did not collect data from the students and so I did not collect evidence on whether the students noticed or utilized Edwin and Naomi's gestures to advance their mathematical thinking. In the proposed direction, a researcher could interview the students of an instructor who proclaimed to intentionally gesture. The interviews with the students would investigate how, if at all, the students' perceived and utilized their instructor's gestures. With this new collection of data could speak more strongly towards gestures as affording a second modality for learning a new concept. If gestures are to become important teaching tools, the instructors and students' opinion and consciousness of gestures must be investigated.

The studies described above may give rise to additional discoveries about how and why instructors gesture. The comparison study of Geometry for Elementary Teachers instructors in face-to-face and synchronous online classrooms may richen the descriptions of the instructors' gestures presented in this study and improve the results of my study from an inferred impact to data demonstrating an impact on instructional settings. Investigating a professional development workshop through a tested survey instrument, classroom observations, and interviews has the potential to highlight instructors' beliefs towards gesturing, an effect of attending the professional development workshop, and corroborate my pattern between prior knowledge of gesturing and intentionality of gestures. Examining the students' perceptions of their instructors' gestures could reinforce the notion that gestures serve as a second form of communicating an idea and as a teaching tool. The results of the proposed directions may then be leveraged to improve teaching practices, increase student learning opportunities, and inform future research.

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APPENDIX A

INSTITUTIONAL REVIEW BOARD APPROVALS

UNIVERSITY OF
NORTHERN COLORADO

Institutional Review Board

Date: 08/07/2020

Principal Investigator: Andrea Alt

Committee Action: **IRB EXEMPT DETERMINATION – New Protocol**
Action Date: 08/07/2020

Protocol Number: 2007006235
Protocol Title: Online Instructor's Gestures for Euclidean Transformations

Expiration Date:

The University of Northern Colorado Institutional Review Board has reviewed your protocol and determined your project to be exempt under 45 CFR 46.104(d)(701) (702) for research involving

Category 1 (2018): RESEARCH CONDUCTED IN EDUCATIONAL SETTINGS. Research, conducted in established or commonly accepted educational settings, that specifically involves normal educational practices that are not likely to adversely impact students' opportunity to learn required educational content or the assessment of educators who provide instruction. This includes most research on regular and special education instructional strategies, and research on the effectiveness of or the comparison among instructional techniques, curricula, or classroom management methods.

Category 2 (2018): EDUCATIONAL TESTS, SURVEYS, INTERVIEWS, OR OBSERVATIONS OF PUBLIC BEHAVIOR. Research that only includes interactions involving educational tests (cognitive, diagnostic, aptitude, achievement), survey procedures, interview procedures, or observation of public behavior (including visual or auditory recording) if at least one of the following criteria is met: (i) The information obtained is recorded by the investigator in such a manner that the identity of the human subjects cannot readily be ascertained, directly or through identifiers linked to the subjects; (ii) Any disclosure of the human subjects' responses outside the research would not reasonably place the subjects at risk of criminal or civil liability or be damaging to the subjects' financial standing, employability, educational advancement, or reputation; or (iii) The information obtained is recorded by the investigator in such a manner that the identity of the human subjects can readily be ascertained, directly or through identifiers linked to the subjects, and an IRB conducts a limited IRB review to make the determination required by 45 CFR 46.111(a)(7).

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UNIVERSITY OF
NORTHERN COLORADO

Institutional Review Board

You may begin conducting your research as outlined in your protocol. Your study does not require further review from the IRB, unless changes need to be made to your approved protocol.

As the Principal Investigator (PI), you are still responsible for contacting the UNC IRB office if and when:

- You wish to deviate from the described protocol and would like to formally submit a modification request. Prior IRB approval must be obtained before any changes can be implemented (except to eliminate an immediate hazard to research participants).
- You make changes to the research personnel working on this study (add or drop research staff on this protocol).
- At the end of the study or before you leave The University of Northern Colorado and are no longer a student or employee, to request your protocol be closed. *You cannot continue to reference UNC on any documents (including the informed consent form) or conduct the study under the auspices of UNC if you are no longer a student/employee of this university.
- You have received or have been made aware of any complaints, problems, or adverse events that are related or possibly related to participation in the research.

If you have any questions, please contact the Research Compliance Manager, Nicole Morse, at 970-351-1910 or via e-mail at nicole.morse@unco.edu. Additional information concerning the requirements for the protection of human subjects may be found at the Office of Human Research Protection website - http://hhs.gov/ohrp/ and https://www.unco.edu/research/research-integrity-and-compliance/institutional-review-board/.

Sincerely,

Nicole Morse

Nicole Morse
Research Compliance Manager

University of Northern Colorado: FWA00000784

APPENDIX B

PARTICIPANT CONSENT FORM

UNIVERSITY OF NORTHERN COLORADO

Consent Form for Human Participants in Research

Project Title: Online Instructor's Gestures for Euclidean Transformations
Researcher: Andrea Alt, Graduate Student, School of Mathematical Sciences, University of Northern Colorado
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Research Advisor: Dr. Nathaniel Miller, Department of Mathematical Sciences, University of Northern Colorado
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For my dissertation study, I am interested in researching the gestures produced by instructors when they teach Euclidean transformations to pre-service elementary teachers online. In order to explore this phenomenon, I ask your permission to virtually join and video-record your classes when you teach Euclidean transformations in MATH 283. I *do not* wish for you to make any changes to your teaching or to do anything new or different while teaching Euclidean transformations. I simply would appreciate the opportunity to observe the gestures you already enact. The purpose of me joining your class is to provide me the chance to live document any notable or unique gestures you make while covering Euclidean Transformations as well as take note of any emerging patterns in the types of your gestures. By video recording your classes on Euclidean transformations I will be able to use images of your gestures to depict examples of the ways you gesture in exquisite detail. Additionally, I invite you to participate in a one-hour interview post analysis of the recorded classes. During the interview I will ask you to validate my descriptions and perceptions of your gestures on Euclidean transformations and to gather information on the intentionality of you gestures while teaching Euclidean transformations.

The results of this study could inform improved teaching methods of Euclidean transformations, and participation in this study would be a great opportunity to reflect and discuss the ways in which you teach Euclidean transformations.

During the hour-long interview, I will present you with my preliminary findings and ask you to validate my descriptions and perceptions of your gestures on Euclidean transformations. Additionally, I will ask questions on the intentionality of your gestures while teaching Euclidean transformations. I hope to gain a better understanding on whether or not the online setting impacted the gestures you enacted. I am not critiquing your gestures, rather I am describing and categorizing them. There is no incorrect way to move your body, any gestures when teaching Euclidean transformations are both appropriate and valuable to my research whether you purposefully enacted them or not.

Recall from my invitation email:

- I will virtually join all your classes when you teach Euclidean transformations.
- You will use Zoom's built in feature to record these classes.
- You will set up a secondary camera pointed at yourself to record these classes.
- After I analyze the class recordings, we will schedule a time to hold the final hour-long interview on Zoom.

Given the purpose of my research, I would like to incorporate photos that illustrate your gestures and/or diagrams in a publication. Thus, I am requesting permission to do so, but if you would prefer that I protect your identity, then I will honor your request. In such a case, I will only describe your responses rather than use pictures. In any case, I will assign you a pseudonym when reporting any results – care will be taken to protect your identity.

All data will be stored on my (Andrea Alt's) Dropbox account, which is password protected. Thus, no one will have access to this data other than myself and my research advisors.

There are no foreseeable risks to participating in my study other than some discomfort during the observation of your class or if you do not feel comfortable answering a question in the interview. You may benefit from participating in this research if reflecting on your own gestures allows you to gain a new perspective on teaching Euclidean transformations.

Participation is voluntary. You may decide not to participate in this study and if you begin participation you may still decide to stop and withdraw at any time. Your decision will be respected and will not result in loss of benefits to which you are otherwise entitled. Please take your time to read and thoroughly review this document and decide whether you would like to participate in this research study. Please sign below if you would like to participate in this research. Please keep or print this form for your records. If you have any concerns about your selection or treatment as a research participant, please contact Nicole Morse, Office of Research & Sponsored Programs, Carter Hall, University of Northern Colorado Greeley, CO 80639; 970-351-1910.

Please feel free to contact me via phone or email if you have any questions and retain one copy of this letter for your records. Thank you for assisting me with this research!

If willing to participate in the interview and **willing to** disclose your identity i.e., agreeing to have your photo shared with others at conference presentations, publications, etc. please complete the following:

____

Name (please print) Signature Date

If willing to participate in the interview but **do not want** to disclose your identity i.e., do not want to have your photo shared with others at conference presentations, publications, etc. please complete the following:

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Name (please print) Signature Date

____

Researcher's Name Researcher's Signature Date

APPENDIX C

INTERVIEW PROTOCOL

Questions for Edwin

1. If you were teaching transformations face to face how would you gesture the concept of a **translation**?
    a) Explain why you would make that gesture?
        i. What mathematical ideas does that gesture convey?

In class, when talking to the camera while addressing the students, you enacted several different gestures for a translation a) you smoothly move your hand to different locations, b) you hold onto the sides of an imaginary cube and move the cube from one side of your body to the other, and c) one hand is in the shape of a blade and you jump your hand to different locations

Under the DocCam you made the following gestures for a translation a) you frame a printed image with your hands and pretend to slide it to a new location, b) with your pen you point to a single spot and slide you pen to a new location, c) you point to a pre- and post-image, and d) you hop your pen/finger x and y units on your paper.

2. Were you aware that you made these different gestures for a **translation**?
3. What do you think was the reason you gestured in different ways? *(highlight one smooth translation gestures and one non-smooth)*
    a) What are some potential benefits of these different gestures?
    b) What are some potential pitfalls of these different gesture?

4. If you were teaching transformations face to face how would you gesture the concept of a **rotation**?
    a) Explain why you would make that gesture?
        i. What mathematical ideas does that gesture convey?

In class you enacted several gestures for a rotation a) as if you grabbed onto and twisted a lightbulb, b) you held onto corners of a polygon and rotated the polygon, and c) made a circle motion with your finger or pen, d) as if you held onto the sides of an imaginary cube and moved it along an arced path, and e) making you elbow a fixed point and rotating your forearm from left to right

Under the DocCam you made the following gestures for a rotation a) wagged your pen from right to left, b) physically turning the paper, and c) tracing the arc that you are rotating along

5. Were you aware that you made these different gestures for a **rotation**?
6. What do you think was the reason you gestured in different ways? *(highlight one where the center of rotation is inside the shape and one where the center of rotation is not)*
    a) What are some potential benefits of these different gestures?
    b) What are some potential pitfalls of these different gesture?

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7. If you were teaching transformations face to face how would you gesture the concept of a **reflection**?
    a) Explain why you would make that gesture?
        i. What mathematical ideas does that gesture convey?

In class you enacted the following gestures for a reflection a) flip open, upward facing palm to be facing down (or the opposite direction), b) holding onto an imaginary cube on the top and bottom sides and flipping it over in an arcing motion and c) with two hands, you grab two sides of an imaginary object and move from one side of your body to the other

Under the DocCam you made the following gestures for a reflections a) pointed at one side of the line of reflection and then picking up your pen/finger pointing at the other, b) you utilize the Mira to demonstrate a reflection

8. Were you aware that you made these different gestures for a **reflection**?
9. What do you think was the reason you gestured in different ways? *(highlight one with the opposite orientation and one without)*
    a) What are some potential benefits of these different gestures?
    b) What are some potential pitfalls of these different gesture?

---

10. If you were teaching transformations face to face how would you gesture the concept of a **glide reflection**?

a) Explain why you would make that gesture?

i. What mathematical ideas does that gesture convey?

In class you did not enacted a gesture for a glide reflection not under the DocCam

Under the DocCam you made the following gestures for a glide reflection a) you trace a line on your paper, hop your pen to the other side, and then trace a translation vector parallel to your line.

11. Were you aware that you made these different gestures for a **glide reflection**?
12. What do you think was the reason you gestured in this way?
    a) What are some potential benefits of this different gestures?
    b) What are some potential pitfalls of this different gesture?

---

13. The gestures you made under the DocCam were similar, but not the same as the gestures you made when talking to the camera while addressing the students, would you talk about these differences?
    a) For example, what instigated the different gestures with the different modalities?

14. When you discuss transformations in general, you always gesture a translation (gestures b, c, or d from off camera translations). Why do you think you did this?

15. Sometimes you appear to work in 2D and other times in 3D (grab the imaginary box and rotate it on an arc or hold onto two corners of a shape and rotate the flat object, grab the top and bottom of a box and flip it over or flip open, upward facing palm to be facing down), can you comment on what might prompt these gestures?
    a) What could be the take-aways for your students with this different type of gesturing?

16. Did anyone tell/teach/show you about gesturing?
    a) Why do you gesture when you teach?

Questions for Naomi

1. If you were teaching transformations face to face how would you gesture the concept of a **translation**?
    a) Explain why you would make that gesture?
        i. What mathematical ideas does that gesture convey?

Under the DocCam you made the following gestures for a translation a) point to an image and slide pen/finger to new location, b) hop pen horizontal units and then vertical units, and c) using tracing paper to slide an image to a new location

Off camera (where you students cannot see) you made the following gestures for a translation a) you smoothly move your hands to different locations and b) you pick up one imaginary point with your thumb and index finger and move it to a new location

2. Were you aware that you made these different gestures for a **translation**?
3. What do you think was the reason you gestured in different ways? *(highlight one smooth translation gestures and one non-smooth)*
    a) What are some potential benefits of these different gestures?
    b) What are some potential pitfalls of these different gesture?

---

4. If you were teaching transformations face to face how would you gesture the concept of a **reflection**?

a) Explain why you would make that gesture?

i. What mathematical ideas does that gesture convey?

Under the DocCam you made the following gestures for a reflection a) make opposing finger guns around the line of reflection, b) flip open, upward facing palm to be facing down, c) pointing to a pre-image and then to the post image, d) place index and middle finger with nails up oven an image then flip your hand 180 degrees so now your nails are touching the paper, and e) flip an image drawn on tracing paper over

Off camera (where you students cannot see) you made the following gestures for a reflection a) flip open, upward facing palm to be facing down (or the opposite direction), b) palms pressed together starting with the right hand on top, you flip you hand in an arching motion so now the left hand is on top, c) with your index finger and thumb denoting a distance you move you hand over an imaginary line of reflection without changing the distance, and d) your left hand is a flat surface on which your right hand in the shape of a blade makes a chopping motion

5. Were you aware that you made these different gestures for a **reflection**?
6. What do you think was the reason you gestured in different ways? *(highlight one with the opposite orientation and one without)*
    a) What are some potential benefits of these different gestures?
    b) What are some potential pitfalls of these different gesture?

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7. If you were teaching transformations face to face how would you gesture the concept of a **rotation**?
    a) Explain why you would make that gesture?
        i. What mathematical ideas does that gesture convey?

Under the DocCam you made the following gestures for a rotation a) with index finger and thumb framing a printed image pretending to rotate that image clockwise, b) index finger extended and rotating wrist 90 degrees clockwise and back, c) fixing a point on tracing paper and rotating a copied image to a new location, and d) trace finger along the angle of rotation

Off camera (where you students cannot see) you made the following gestures for a rotation a) keeping a fixed distance between your index finger and thumb you rotate your arm using your elbow as the center of rotation and b) with hand stiffly in a blade you rotate your wrist back and forth

8. Were you aware that you made these different gestures for a **rotation**?
9. What do you think was the reason you gestured in different ways? (*highlight one where the rotated image is a point and one it is not)*
    a) What are some potential benefits of these different gestures?
    b) What are some potential pitfalls of these different gesture?

10. If you were teaching transformations face to face how would you gesture the concept of a **glide reflection**?
    a) Explain why you would make that gesture?
        i. What mathematical ideas does that gesture convey?

Under the DocCam you made the following gestures for a glide reflection a) finger gun with palm up flips down over the line of reflection (now is palm down) then slides to the right, and b) same as before but with left index finger and pen framing a single point

Off camera (where you students cannot see) you made the following gestures for a glide reflection a) your left hand makes a line or reflection for your right hand to hop over then you move your right hand further away from your body.

11. Were you aware that you made these different gestures for a **glide reflection**?
12. What do you think was the reason you gestured in different ways?
    a) What are some potential benefits of these different gestures?
    b) What are some potential pitfalls of these different gesture?

---

13. The gestures you made under the DocCam were similar, but not the same compared to when you addressed the students. Would you talk about these differences?
    a) For example, what instigated the different gestures with the different modalities?

14. In what ways did the gestures you made off camera benefit you as an instructor. (*gesture for others vs gestures for self*)

15. Now that I have shared the gestures you made on and off camera, are there any gestures that you made off camera that you wish your students could have seen?
    a) If yes, which ones and why?
    b) If not, why not?

16. Did anyone tell/teach/show you about gesturing?
    a) Why do you gesture when you teach?

APPENDIX D

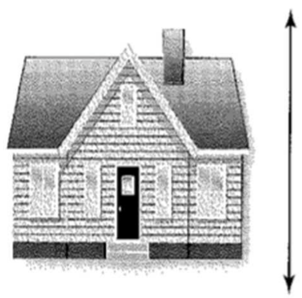
IN CLASS HANDOUTS

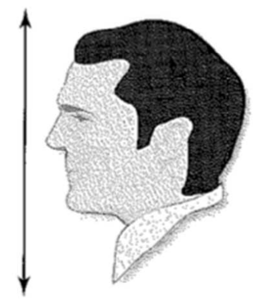
Introduction to Transformations

The Mira can be used for drawing reflections of figures and finding the line of reflection.

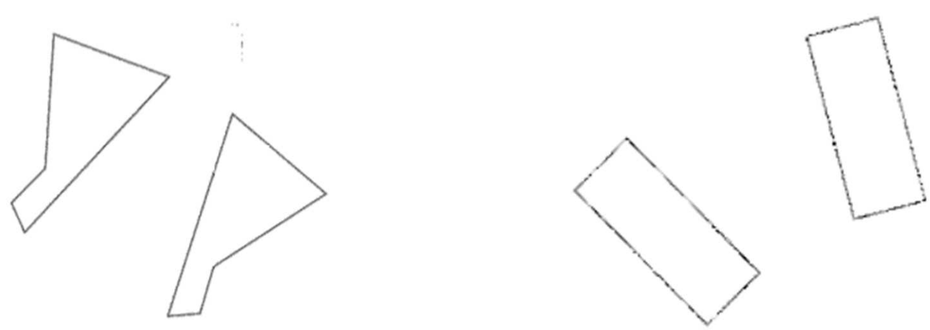
Noticed the beveled edge of the Mira. This is the side you want to look through.

Draw a reflection of the outline of the house and the man by aligning the beveled edge on the reflection line and looking through that side with the beveled edge. Use a pencil to draw where you see the house/man through the Mira. Turn your paper as needed.

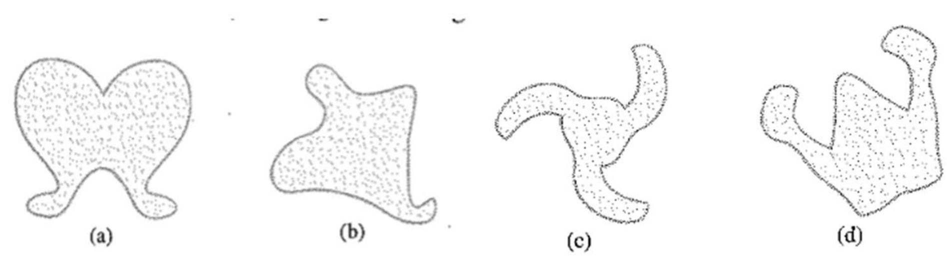




Using the Mira, draw where the reflection line should be so that the two images lay on top of each other through the Mira. Draw the line along the side with the beveled edge. How can you determine mathematically that you are correct?

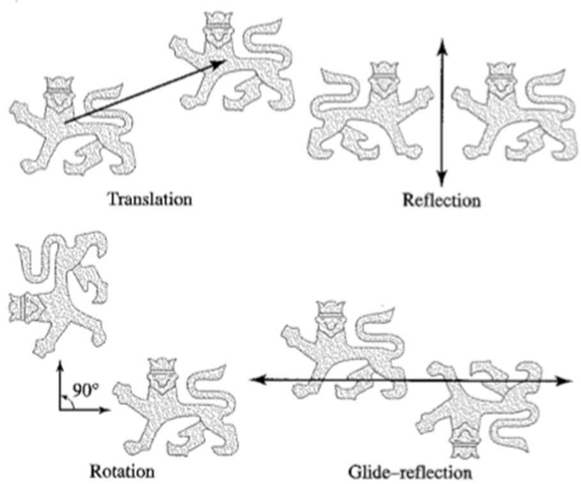


The Mira can be used to find lines of symmetry. Which of the following shapes have a line of symmetry?



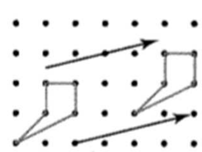
Activity 16.1 Four Actions on page 531-532 in Aichele, recognition of the four transformations.

Four Actions (Transformations)



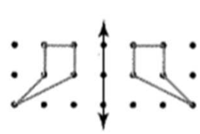
◆ Four Actions or Transformations

SLIDE OR TRANSLATION: A slide or translation of a figure can be described by an arrow (or vector) that shows how far and in what direction the figure is moved. This diagram shows a translation and two different translation vectors:

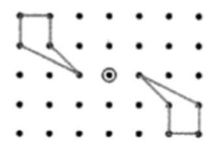


Notice that the translation vector has the same length and direction as any of the point–image segments for the translation.

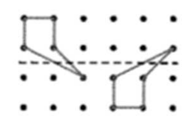
FLIP OR REFLECTION: The mira shows what a reflection does. The following diagram shows a reflection of a figure, together with the reflection line:



TURN OR ROTATION: A rotation can be described by a center and the number of degrees in the rotation. The center is circled for the rotation of 180° shown in the following diagram:



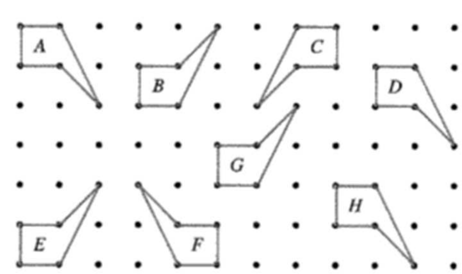
GLIDE–FLIP OR GLIDE–REFLECTION: This action involves a translation followed by a reflection along the translation line. The dotted line in the following diagram shows the glide–reflection line:



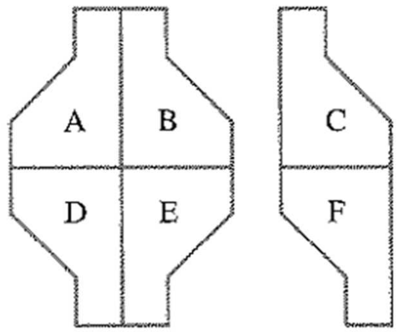
◆ Problems

Note: Many students can do the following problems visually. However, if you are in doubt, use tracing paper to see how the second figure is related to the first.

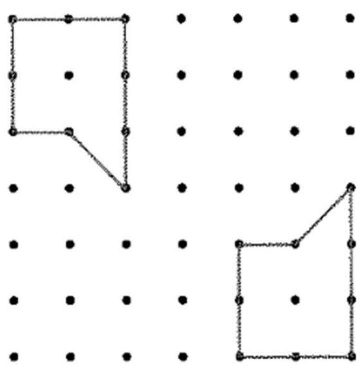
1. Indicate the type of action or transformation needed to get from *A* to each of the figures shown. For example, next to figure *D*, write **T** to indicate that a translation takes *A* to *D*. Write **T** for translation, **Rot** for rotation, **Ref** for reflection, and **GR** for glide–reflection.



2. Five different copies of figure *A* are shown. Next to each copy identify if it is a Translation, rotation, reflection, or glide reflection. You can abbreviate as T, Rot, Ref, or GR.



3. These two figures are related by a glide reflection. Find the glide reflection line. Draw the reflection of the image. How do you translate the reflected image to find the final image?

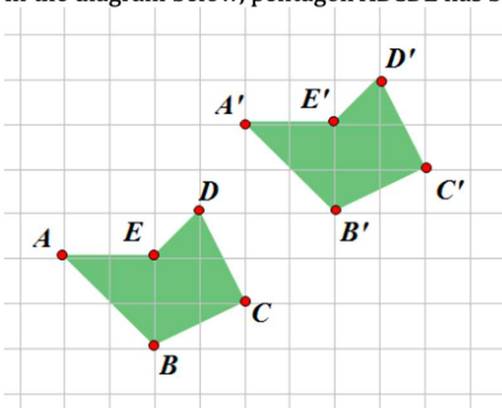


4. If you translate the image first and then reflected it about the glide reflection line, do you get the same final image? Why do you think this occurs?

Math 283 Transformation Homework
Set 1 due Oct 1

Translations: Answer each of the following questions.

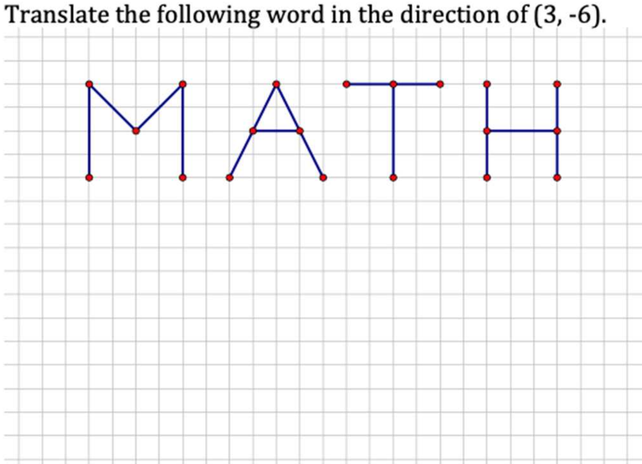
1. In the diagram below, pentagon ABCDE has been translated to obtain the image $A'B'C'D'E'$.



   a. Suppose you had to describe to someone the description of the translation in terms of ordered pairs. What ordered pair would you use? Explain how you arrived at this ordered pair.

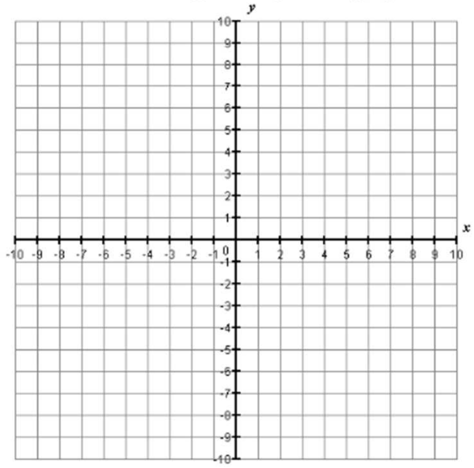
   b. Draw the segments $\overline{AA'}, \overline{CC'}$ and $\overline{DD'}$, . Provide at least two facts that these segments share.

2. Translate the following word in the direction of (3, -6).



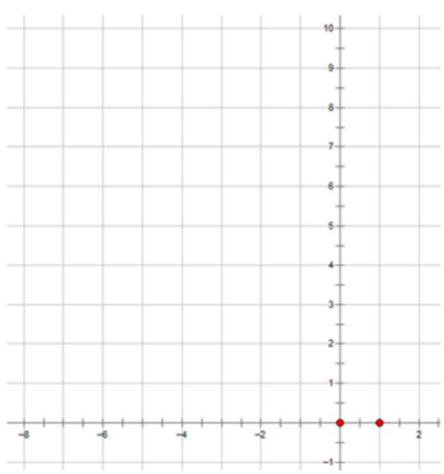
3. Suppose you know that some point (x,y) was translated in the direction of $(-5, 2)$ to obtain the image point $(4, -7)$.
    a. What is the preimage point (x,y)? Explain your answer.

    b. Plot the image and preimage point on the coordinate plane and label each appropriately.



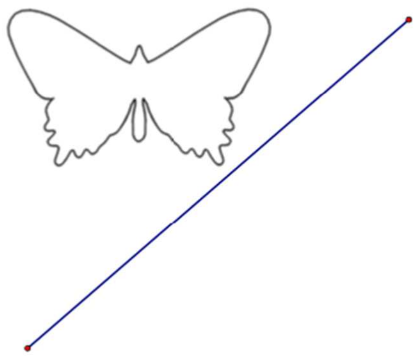
4. Suppose you know that some point (x,y) was translated in the direction of (½ , -2 ¾) to obtain the image (-4, 5).
    a. What is the preimage point (x,y)? Explain your answer.

    b. Plot the image and preimage point on the Cartesian Grid and label each appropriately.

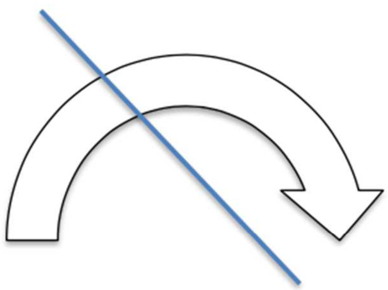


Reflections: Answer each of the following questions.

1. Using a **Mira**, reflect each of the following diagrams about the given line of reflection.
   a.



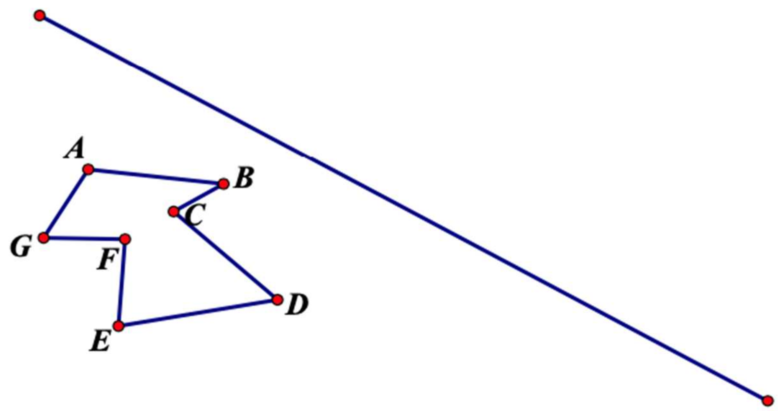
   b.



2. *Fixed points* are points that get sent to themselves under a given transformation.
   a. Do either of the above reflections have a fixed point(s)? Explain your answer.

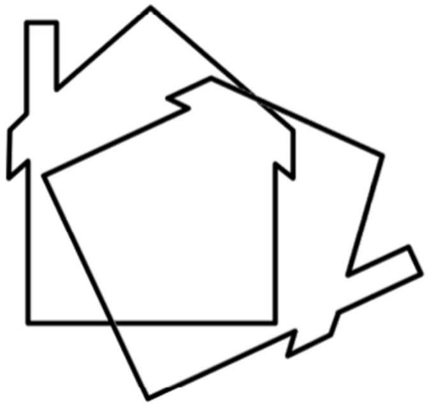
   b. In general, what points are fixed points under a reflection? Why does this occur?

3. Use **tracing paper** to find the reflection of the given figure about the provided line of reflection. Explain your process.

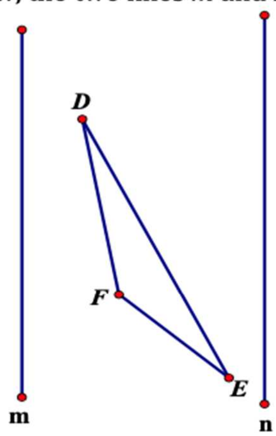


4. In the figure below, one image is a reflection of the other. Find the line of reflection **without** using patty paper or a Mira. Explain how you found it and how you know that this process will always work. Be precise in your explanation. (This may take some thinking, but just think of the definition!)

5. Try your process again with the following figure. Did it work? Why or why not?



6. In the figure below, the two lines *m* and *n* are parallel to one another.



a. Reflect triangle *DEF* about line *m* and label the image as triangle $D'E'F'$, then take triangle $D'E'F'$ and reflect it about line *n* and label the final image as triangle $D''E''F''$.

b. What is the distance between the lines *m* and *n*?

c. Construct the segments $\overline{DD''}, \overline{EE''}, \& \overline{FF''}$.

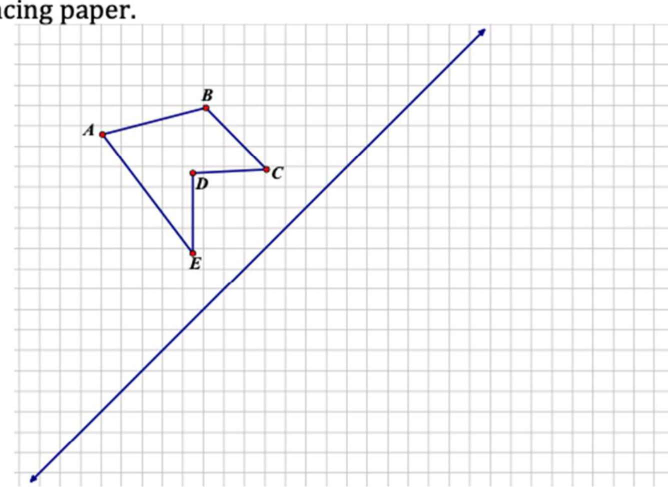
d. What is the distance of each of the segments $\overline{DD''}, \overline{EE''}, \& \overline{FF''}$?

e. What is the relationship between your answers to parts b and d.

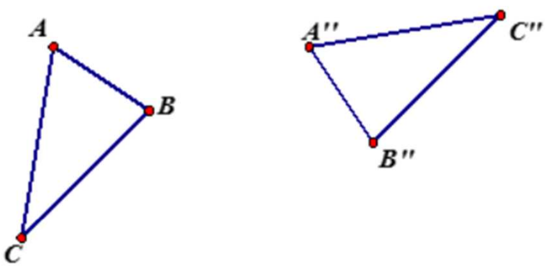
f. Explain why this relationship holds.

Glide Reflections: Answer each of the following questions.

7. Perform a glide reflection of the following figure in the direction of (2, 2). You can use a Mira or tracing paper.



8. The following figures are glide reflections of one another. Construct the segments: $\overline{AA''}, \overline{BB''}, \& \overline{CC''}$. Find the midpoint of each segment.



   a. What appears to be true about all of the midpoints?

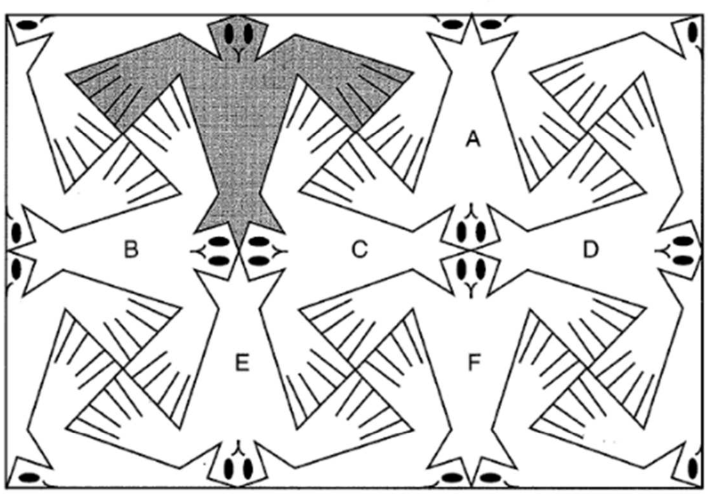
   b. Construct the line that contains all the midpoints. Turns out that this line is your glide reflection line. Verify that this is true by reflecting triangle ABC about this line to obtain triangle $A'B'C'$. How can you verify that this is really the line of reflection? Be specific in your response.

Math 283 Transformation Homework
Set 2 due Oct 1

Rotations: Answer each of the following questions.

1. Using the following diagram, rotate the car *120° clockwise* about the point *A*. First, draw a line from the car to point A, then mark your angle. Using tracing paper, trace the outline of the car and the segment to A and then rotate the paper until the lines coincide and trace the rotated image.

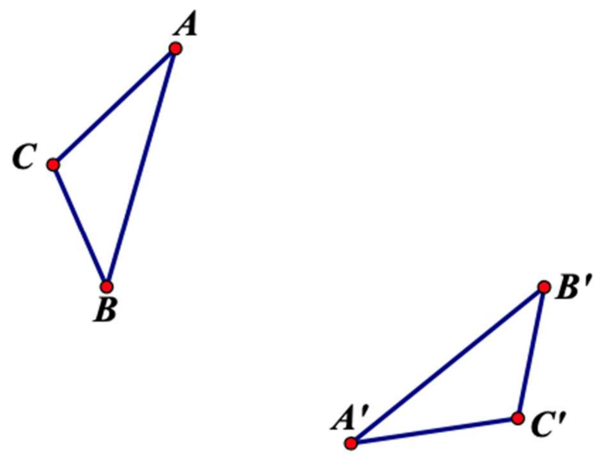
2. In the following diagram[1] owl A can be obtained through a rotation of the shaded owl. Find the center of rotation and explain how you KNOW that this must be the center of rotation.



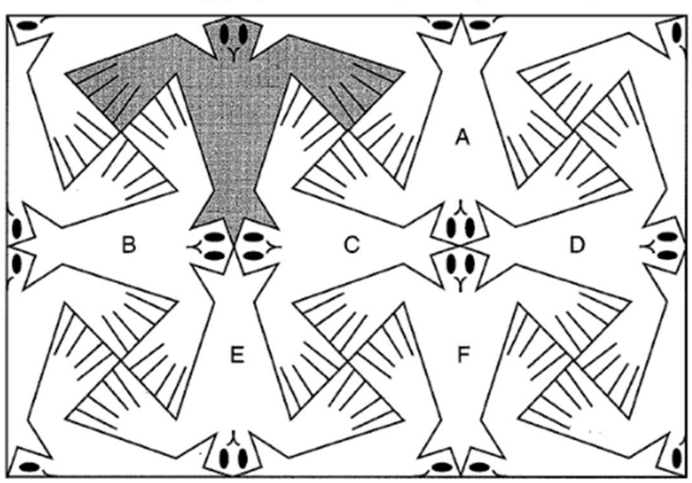
[1] Adopted from *Mathematics Activities* by Dolan, Williamson & Muri

3. <u>Finding the center of rotation</u>: Follow these steps to find the center of rotation between the two figures, shown below.
   - Construct segments $\overline{AA'}, \overline{BB'}, \overline{CC'}$
   - Find the midpoint of each of these segments
   - For each segment, draw the line that is perpendicular to the segment and passes through the midpoint of the segment (this is called the *perpendicular bisector*).
   - Find the point where all three lines intersect, this should be the center of rotation. Test it out using tracing paper.

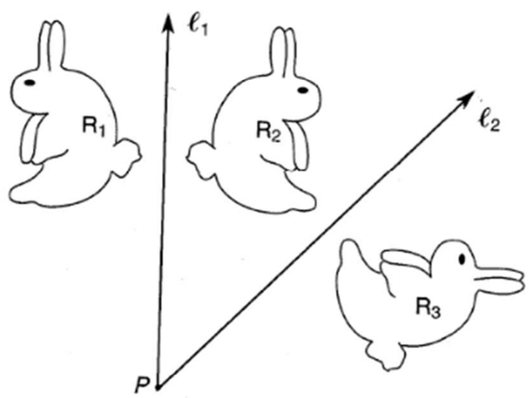
   Was it necessary to find the perpendicular bisector for all three segments? Why or why not?



4. In the figure below, owl D is obtained by rotating the shaded owl.
   a. Find the center of rotation using the methods described above.
   b. Find the angle of rotation and explain how you found it.
   c. Use tracing paper to verify that you really have the center of rotation.



5. Bunny to duck task[2]: Notice that R1 is a bunny. Notice that R3 is a duck. Use your Mira to convince yourself that R2 is the reflection of R1 about line ℓ_1. Use your Mira to convince yourself that R3 is a reflection of R2 about line ℓ_2. Are you convinced? Ok good! Now explain the relationship between R1 and R3. Be thorough.



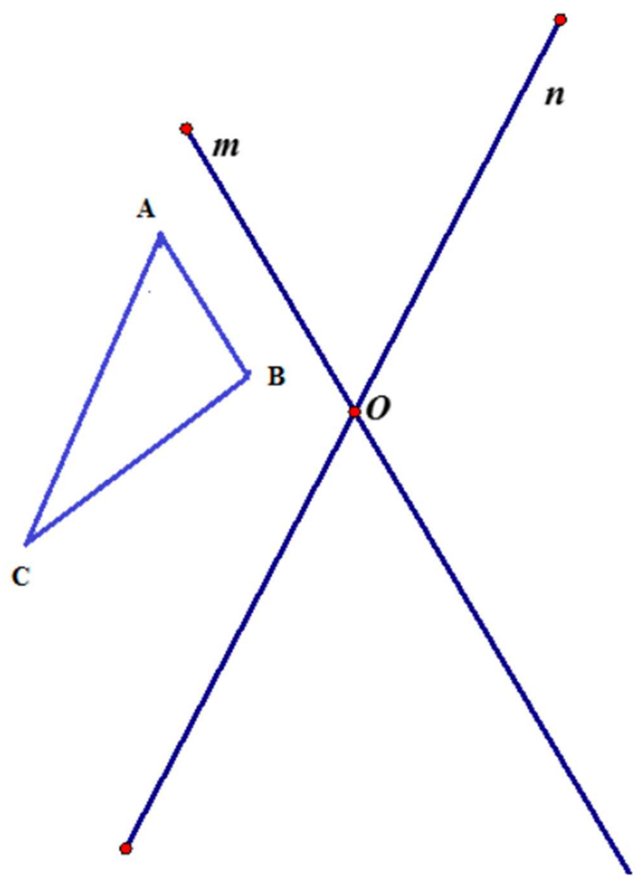
6. In general, a sequence of two reflections about two intersecting lines results in a rotation about the point where the two lines intersect. One question that we might pose is what the angle of rotation might be.
   a. Determine the angle of rotation to get from the bunny to the duck in the exercise above. Explain how you found it.

   b. What is the measure of the angle between ℓ_1 and ℓ_2?

   c. Describe the relationship between your answers to part *a* and *b* above.

[2] Adopted from *Mathematics Activities* by Dolan, Williamson & Muri

7. Consider the figure below and the two intersecting lines *m* and *n* (feel free to use a Mira).
   a. Reflect triangle ABC about line *m* and label it as triangle $A'B'C'$.
   b. Reflect triangle $A'B'C'$ about line *n* and label it as triangle $A''B''C''$.
   c. Using your compass, construct the circle that has center O and radius $\overline{OA}$. Do the same for radius $\overline{OB}$ & $\overline{OC}$.
   d. Describe all the points that pass through each circle and explain why this happens. Be sure to use mathematical ideas about transformations as part of your explanation.



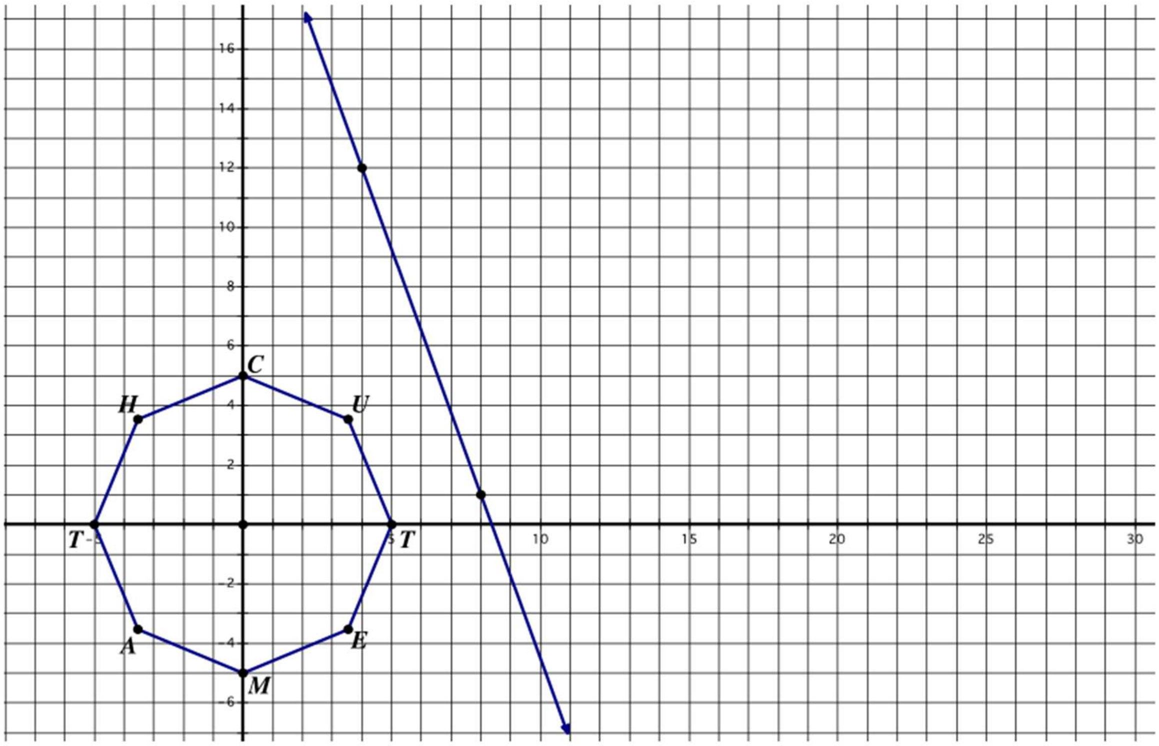
5. **Fun stuff:** Using the figure below do the following transformations in the following order. Do each step in a *different color* so you can see the transformation.

a. Translate the figure in the direction of (3, 6).

b. Take the result from part a and reflect it about the given line.

c. Take the result from part b and rotate it 45 degrees clockwise about the origin.

d. Take the result from part c and dilate it by a factor of ½ about the new center of the octagon.



6. Do you think we will get the same final image if we did all the transformations in the reverse order? Why or why not?

Wrapping it Up: Please answer each of the following questions about each of the four transformations.

a. Of the four transformations which ones have fixed points and how do we determine those fixed points?

b. From the homework we see that reflections are the building blocks for rotations and translations. Explain how reflections can be used to create translations and how they can be used to create rotations.